Computing Subset Vertex Covers in H-Free Graphs

Nick Brettell ¹, Jelle J. Oostveen*², Sukanya Pandey ¹, Daniël Paulusma ¹, and Erik Jan van Leeuwen ¹

¹Victoria University of Wellington, Wellington, New Zealand nick.brettell@vuw.ac.nz ²Utrecht University, Utrecht, The Netherlands {j.j.oostveen,s.pandey1,e.j.vanleeuwen}@uu.nl ³Durham University, Durham, United Kingdom daniel.paulusma@durham.ac.uk

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Abstract

We consider a natural generalization of Vertex Cover: the Subset Vertex Cover problem, which is to decide for a graph G=(V,E), a subset $T\subseteq V$ and integer k, if V has a subset S of size at most k, such that S contains at least one end-vertex of every edge incident to a vertex of T. A graph is H-free if it does not contain H as an induced subgraph. We solve two open problems from the literature by proving that Subset Vertex Cover is NP-complete on subcubic (claw,diamond)-free planar graphs and on 2-unipolar graphs, a subclass of $2P_3$ -free weakly chordal graphs. Our results show for the first time that Subset Vertex Cover is computationally harder than Vertex Cover (under $P \neq NP$). We also prove new polynomial time results. We first give a dichotomy on graphs where G[T] is H-free. Namely, we show that Subset Vertex Cover is polynomial-time solvable on graphs G, for which G[T] is H-free, if $H = sP_1 + tP_2$ and NP-complete otherwise. Moreover, we prove that Subset Vertex Cover is polynomial-time solvable for $(sP_1 + P_2 + P_3)$ -free graphs and bounded mim-width graphs. By combining our new results with known results we obtain a partial complexity classification for Subset Vertex Cover on H-free graphs.

1 Introduction

We consider a natural generalization of the classical Vertex Cover problem: the Subset Vertex Cover problem, introduced in [6]. Let G = (V, E) be a graph and T be a subset of V. A set $S \subseteq V$ is a T-vertex cover of G if S contains at least one end-vertex of every edge incident to a vertex of T. We note that T itself is a T-vertex cover. However, a graph may have much smaller T-vertex covers. For example, if G is a star whose leaves form T, then the center of G forms a T-vertex cover. We can now define the problem; see also Fig. 1.

Subset Vertex Cover

Instance: A graph G = (V, E), a subset $T \subseteq V$, and a positive integer k.

Question: Does G have a T-vertex cover S_T with $|S_T| \leq k$?

If we set T=V, then we obtain the Vertex Cover problem. Hence, as Vertex Cover is NP-complete, so is Subset Vertex Cover.

To obtain a better understanding of the complexity of an NP-complete graph problem, we may restrict the input to some special graph class. In particular, *hereditary* graph classes, which are the classes closed under vertex deletion, have been studied intensively for this purpose. It is readily seen that a graph

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class \mathcal{G} is hereditary if and only if \mathcal{G} is characterized by a unique minimal set of forbidden induced subgraphs \mathcal{F}_G . Hence, for a systematic study, it is common to first consider the case where $\mathcal{F}_{\mathcal{G}}$ has size 1. This is also the approach we follow in this paper. So, for a graph H, we set $\mathcal{F}_{\mathcal{G}} = \{H\}$ for some graph H and consider the class of H-free graphs (graphs that do not contain H as an induced subgraph). We now consider the following research question:

For which graphs H is Subset Vertex Cover, restricted to H-free graphs, still NP-complete and for which graphs H does it become polynomial-time solvable?

We will also address two open problems posed in [6] (see Section 2 for any undefined terminology):

- Q1. What is the complexity of Subset Vertex Cover for claw-free graphs?
- Q2. Is Subset Vertex Cover is NP-complete for P_t -free graphs for some t?

The first question is of interest, as VERTEX COVER is polynomial-time solvable even on $rK_{1,3}$ -free graphs for every $r \geq 1$ [5], where $rK_{1,3}$ is the disjoint union of r claws (previously this was known for rP_3 -free graphs [16] and $2P_3$ -free graphs [17]). The second question is of interest due to some recent quasi-polynomial-time results. Namely, Gartland and Lokshtanov [12] proved that for every integer t, VERTEX COVER can be solved in $n^{O(\log^3 n)}$ -time for P_t -free graphs. Afterwards, Pilipczuk, Pilipczuk and Rzążewski [22] improved the running time to $n^{O(\log^2 n)}$ time. Even more recently, Gartland et al. [13] extended the results of [12, 22] from P_t -free graphs to H-free graphs where every connected component of H is a path or a subdivided claw.

Grötschel, Lovász, and Schrijver [14] proved that VERTEX COVER can be solved in polynomial time for the class of perfect graphs. The class of perfect graphs is a rich graph class, which includes well-known graph classes, such as bipartite graphs and (weakly) chordal graphs.

Before we present our results, we first briefly discuss the relevant literature.

Existing Results and Related Work

Whenever Vertex Cover is NP-complete for some graph class \mathcal{G} , then so is the more general problem Subset Vertex Cover. Moreover, Subset Vertex Cover can be polynomially reduced to Vertex Cover: given an instance (G, T, k) of the former problem, remove all edges not incident to a vertex of T to obtain an instance (G', k) of the latter problem. Hence, we obtain:

Proposition 1. The problems Vertex Cover and Subset Vertex Cover are polynomially equivalent for every graph class closed under edge deletion.

For example, the class of bipartite graphs is closed under edge deletion and VERTEX COVER is polynomial-time solvable on bipartite graphs. Hence, by Proposition 1, Subset Vertex Cover is polynomial-time solvable on bipartite graphs. However, a class of H-free graphs is only closed under edge deletion if H is a complete graph, and Vertex Cover is NP-complete even for triangle-free graphs [23]. This means that there could still exist graphs H such that Vertex Cover and Subset Vertex Cover behave differently if the former problem is (quasi)polynomial-time solvable on H-free graphs. The following well-known result of Alekseev [1] restricts the structure of such graphs H.

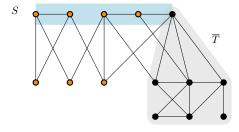


Figure 1: An instance (G, T, k) of Subset Vertex Cover, where T consists of the orange vertices, together with a solution S (a T-vertex cover of size 5). Note that S consists of four vertices of T and one vertex of $\overline{T} = V \setminus T$.

Theorem 2 ([1]). For every graph H that contains a cycle or a connected component with two vertices of degree at least 3, VERTEX COVER, and thus SUBSET VERTEX COVER, is NP-complete for H-free graphs.

Due to Theorem 2 and the aforementioned result of Gartland et al. [13], every graph H is now either classified as a quasi-polynomial case or NP-hard case for VERTEX COVER. For SUBSET VERTEX COVER the situation is much less clear. So far, only one positive result is known, which is due to Brettell et al. [6].

Theorem 3 ([6]). For every $s \ge 0$, SUBSET VERTEX COVER is polynomial-time solvable on $(sP_1 + P_4)$ -free graphs.

Subset variants of classic graph problems are widely studied, also in the context of H-free graphs. Indeed, Brettell et al. [6] needed Theorem 3 as an auxiliary result in complexity studies for Subset Feedback Vertex Set and Subset Odd Cycle Transversal restricted to H-free graphs. The first problem is to decide for a graph G = (V, E), subset $T \subseteq V$ and integer k, if G has a set S of size at most k such that S contains a vertex of every cycle that intersects T. The second problem is similar but replaces "cycle" by "cycle of odd length". Brettell et al. [6] proved that both these subset transversal problems are polynomial-time solvable on $(sP_1 + P_3)$ -free graphs for every $s \ge 0$. They also showed that Odd Cycle Transversal is polynomial-time solvable for P_4 -free graphs and NP-complete for split graphs, which form a subclass of $2P_2$ -free graphs, whereas NP-completeness for Subset Feedback Vertex Set on split graphs was shown by Fomin et al. [11]. Recently, Paesani et al. [20] extended the result of [6] for Subset Feedback Vertex Set from $(sP_1 + P_3)$ -free graphs to $(sP_1 + P_4)$ -free graphs for every integer $s \ge 0$. If H contains a cycle or claw, NP-completeness for both subset transversal problems follows from corresponding results for Feedback Vertex Set [19, 23] and Odd Cycle Transversal [9].

Combining all the above results leads to a dichotomy for Subset Feedback Vertex Set and a partial classification for Subset Odd Cycle Transversal (see also [6, 20]). Here, we write $F \subseteq_i G$ if F is an induced subgraph of G.

Theorem 4. For a graph H, SUBSET FEEDBACK VERTEX SET on H-free graphs is polynomial-time solvable if $H \subseteq_i sP_1 + P_4$ for some $s \ge 0$, and NP-complete otherwise.

Theorem 5. For a graph $H \neq sP_1 + P_4$ for some $s \geq 1$, Subset Odd Cycle Transversal on H-free graphs is polynomial-time solvable if $H = P_4$ or $H \subseteq_i sP_1 + P_3$ for some $s \geq 0$, and NP-complete otherwise.

Our Results

In Section 3 we prove two new hardness results, using the same basis reduction, which may have a wider applicability. We first prove that SUBSET VERTEX COVER is NP-complete for subcubic planar line graphs of triangle-free graphs, or equivalently, subcubic planar (claw, diamond)-free graphs. This answers Q1 in the negative. We then prove that SUBSET VERTEX COVER is NP-complete for a 2-unipolar graphs and thus for $2P_3$ -free graphs. Hence, SUBSET VERTEX COVER is NP-complete for P_7 -free graphs, and we have answered Q2 for t=7.

Our hardness results show a sharp contrast with VERTEX COVER, which can be solved in polynomial time for both weakly chordal graphs [14] and $rK_{1,3}$ -free graphs for every $r \geq 1$ [5]. Hence, SUBSET VERTEX COVER may be harder than VERTEX COVER for a graph class closed under vertex deletion (if $P \neq NP$). This is in contrast to graph classes closed under edge deletion (see Proposition 1).

In Section 3 we also prove that Subset Vertex Cover is NP-complete for inputs (G,T,k) if the subgraph G[T] of G induced by T is P_3 -free. On the other hand, our first positive result, shown in Section 4, shows that the problem is polynomial-time solvable if G[T] is sP_2 -free for any $s \geq 2$. In Section 4 we also prove that Subset Vertex Cover can be solved in polynomial time for $(sP_1+P_2+P_3)$ -free graphs for every $s \geq 1$. Our positive results generalize known results for Vertex Cover. The first result also implies that Subset Vertex Cover is polynomial-time solvable for split graphs, contrasting our NP-completeness result for 2-unipolar graphs, which are generalized split, $2P_3$ -free, and weakly chordal. Combining our new results with Theorem 3 gives us a partial classification and a dichotomy, both of which are proven in Section 5.

Theorem 6. For a graph $H \neq rP_1 + sP_2 + P_3$ for any $r \geq 0$, $s \geq 2$; $rP_1 + sP_2 + P_4$ for any $r \geq 0$, $s \geq 1$; or $rP_1 + sP_2 + P_4$ for any $r \geq 0$, $s \geq 0$, $t \in \{5,6\}$, SUBSET VERTEX COVER on H-free graphs

is polynomial-time solvable if $H \subseteq_i sP_1 + P_2 + P_3$, sP_2 , or $sP_1 + P_4$ for some $s \ge 1$, and NP-complete otherwise.

Theorem 7. For a graph H, SUBSET VERTEX COVER on instances (G, T, k), where G[T] is H-free, is polynomial-time solvable if $H \subseteq_i sP_2$ for some $s \ge 1$, and NP-complete otherwise.

Theorems 4–7 show that Subset Vertex Cover on H-free graphs can be solved in polynomial time for infinitely more graphs H than Subset Feedback Vertex Set and Subset Odd Cycle Transversal. This is in line with the behaviour of the corresponding original (non-subset) problems.

In Section 6 we discuss our final new result, which states that SUBSET VERTEX COVER is polynomialtime solvable on every graph class of bounded mim-width, such as the class of circular-arc graphs. In Section 7 we discuss some directions for future work, which naturally originate from the above results.

2 Preliminaries

Let G = (V, E) be a graph. The degree of a vertex $u \in V$ is the size of its neighbourhood $N(u) = \{v \mid uv \in E\}$. We say that G is subcubic if every vertex of G has degree at most 3. An independent set I in G is maximal if there exists no independent set I' in G with $I \subsetneq I'$. Similarly, a vertex cover S of G is minimal if there no vertex cover S' in G with $S' \subsetneq S$. For a graph H we write $H \subseteq_i G$ if H is an induced subgraph of G, that is, G can be modified into H by a sequence of vertex deletions. If G does not contain H as an induced subgraph, G is H-free. For a set of graphs H, G is H-free if G is G is G in the first G in the first G in the first G is G in the first G in the first G in the first G is G. If G is G is G in the first G is G in the first G is G in the first G in the first G is G. If G is G is G in the first G is G in the first G is G in the first G in the first G is G. If G is G is G is G is G in the first G is G in the first G in the first G is G in the first G in the first G is G in the first G in the first G in the first G is G in the first G is G in the first G in the first G in the first G in the first G is the first G in the first G in the first G in the first G is the first G in the first G in the first G in the first G is the first G in the first G in the first G in the first G is the first G in the first G

The line graph of a graph G=(V,E) is the graph L(G) that has vertex set E and an edge between two vertices e and f if and only if e and f share a common end-vertex in G. The complement \overline{G} of a graph G=(V,E) has vertex set V and an edge between two vertices u and v if and only if $uv \notin E$.

For two vertex-disjoint graphs F and G, the disjoint union F+G is the graph $(V(F) \cup V(G), E(F) \cup E(G))$. We denote the disjoint union of s copies of the same graph G by sG. A linear forest is a disjoint union of one or more paths.

Let C_s be the cycle on s vertices; P_t the path on t vertices; K_r the complete graph on r vertices; and $K_{1,r}$ the star on (r+1) vertices. The graph $C_3 = K_3$ is the triangle; the graph $K_{1,3}$ the claw, and the graph $\overline{2P_1 + P_2}$ is the diamond (so the diamond is obtained from the K_4 after deleting one edge). The subdivision of an edge uv replaces uv with a new vertex w and edges uw, wv. A subdivided claw is obtained from the claw by subdividing each edge zero or more times.

A graph is chordal if it has no induced C_s for any $s \geq 4$. A graph is weakly chordal if it has no induced C_s and no induced $\overline{C_s}$ for any $s \geq 5$. A cycle C_s or an anti-cycle $\overline{C_s}$ is odd if it has an odd number of vertices. By the Strong Perfect Graph Theorem [10], a graph is perfect if it has no odd induced C_s and no odd induced $\overline{C_s}$ for any $s \geq 5$. Every chordal graph is weakly chordal, and every weakly chordal graph is perfect. A graph G = (V, E) is unipolar if V can be partitioned into two sets V_1 and V_2 , where $G[V_1]$ is a complete graph and $G[V_2]$ is a disjoint union of complete graphs. If every connected component of $G[V_2]$ has size at most 2, then G is 2-unipolar. Unipolar graphs form a subclass of generalized split graphs, which are the graphs that are unipolar or their complement is unipolar. It can also readily be checked that every 2-unipolar graph is weakly chordal (but not necessarily chordal, as evidenced by $G = C_4$).

3 NP-Hardness Results

In this section we prove our hardness results for Subset Vertex Cover, using the following notation. Let G be a graph with an independent set I. We say that we augment G by adding a (possibly empty) set F of edges between some pairs of vertices of I. We call the resulting graph an I-augmentation of G. The following lemma forms the basis for our hardness gadgets.

Lemma 8. Every vertex cover of a graph G = (V, E) with an independent set I is a $(V \setminus I)$ -vertex cover of every I-augmentation of G, and vice versa.

Proof. Let G' be an I-augmentation of G. Consider a vertex cover S of G. For a contradiction, assume that S is not a $(V \setminus I)$ -vertex cover of G'. Then G' - S must contain an edge uv with at least one of u, v

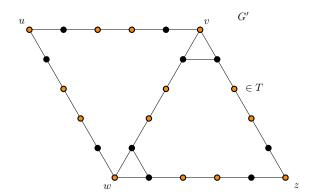


Figure 2: The graph G' from Theorem 10, where T consists of the orange vertices.

belonging to $V \setminus I$. As G - S is an independent set, uv belongs to $E(G') \setminus E(G)$ implying that both u and v belong to I, a contradiction.

Now consider a $(V \setminus I)$ -vertex cover S' of G'. For a contradiction, assume that S' is not a vertex cover of G. Then G - S' must contain an edge uv (so $uv \in E$). As G' is a supergraph of G, we find that G' - S' also contains the edge uv. As S' is a $(V \setminus I)$ -vertex cover of G', both u and v must belong to I. As $uv \in E$, this contradicts the fact that I is an independent set.

To use Lemma 8 we need one other lemma, due to Poljak [23]. A graph G' is a 2-subdivision of a graph G if G' can be obtained from G by subdividing every edge of G twice, that is, by replacing each edge $uv \in E(G)$ with a path $uw_{uv}w_{vu}v$ of length 3.

Lemma 9 ([23]). A graph G with m edges has an independent set of size k if and only if the 2-subdivision of G has an independent set of size k + m.

We are now ready to prove our first two hardness results. Recall that a graph is (claw, diamond)-free if and only if it is a line graph of a triangle-free graph. Hence, the result in particular implies NP-hardness of Subset Vertex Cover for line graphs. Recall also that we denote the claw and diamond by $K_{1,3}$ and $\overline{2P_1 + P_2}$, respectively.

Theorem 10. Subset Vertex Cover is NP-complete for $(K_{1,3}, \overline{2P_1 + P_2})$ -free subcubic planar graphs.

Proof. We reduce from VERTEX COVER, which is NP-complete even for cubic planar graphs [18], and thus for cubic planar graphs that are 4-subdivisions due to two applications of Lemma 9 (note that subdividing an edge preserves both maximum degree and planarity). So, let G = (V, E) be a subcubic planar graph that is a 4-subdivision of some graph G^* , and let k be an integer.

In G, we let $U = V(G^*)$ and W be the subset of $V(G) \setminus V(G^*)$ that consists of all neighbours of vertices of U. Note that W is an independent set in G. We construct a W-augmentation G' as follows; see also Figure 2. For every vertex $u \in U$ of degree 3 in G, we pick two arbitrary neighbours of U (which both belong to U) and add an edge between them. It is readily seen that U is U is U is U in U is U in U in U is U in U in

Theorem 11. Subset Vertex Cover is NP-complete for instances (G, T, k), for which G is 2-unipolar and G[T] is a disjoint union of edges.

Proof. We reduce from VERTEX COVER. Let G = (V, E) be a graph and k be an integer. By Lemma 9, we may assume that G is a 2-subdivision of a graph G^* . In G, we let $U = V(G^*)$, and we let $W = V(G) \setminus V(G^*)$. Note that U is an independent set in G. We construct a U-augmentation G' by changing U into a clique; see also Figure 3. It is readily seen that G' is 2-unipolar. We set T := W, so G[T] is a disjoint union of edges. It remains to apply Lemma 8.

Remark 12. It can be readily checked that 2-unipolar graphs are $(2C_3, C_5, C_6, C_3 + P_3, 2P_3, \overline{P_6}, \overline{C_6})$ -free graphs, and thus are $2P_3$ -free weakly chordal.

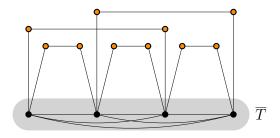


Figure 3: The graph G' from Theorem 11, where T consists of the orange vertices.

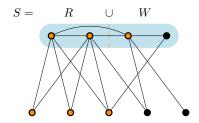


Figure 4: An example of the $2P_2$ -free graph G' of the proof of Theorem 15. Here, T consists of the orange vertices. A solution S can be split up into a minimal vertex cover R of G'[T] and a vertex cover W of $G[V \setminus R]$.

4 Polynomial-Time Results

In this section, we prove our polynomial-time results for instances (G, T, k) where either G is H-free or only G[T] is H-free. The latter type of results are stronger, but only hold for graphs H with smaller connected components. We start with the case where $H = sP_2$ for some $s \ge 1$. For this case we need the following two well-known results. The *delay* is the maximum of the time taken before the first output and that between any pair of consecutive outputs.

Theorem 13 ([2]). For every constant $s \ge 1$, the number of maximal independent sets of an sP_2 -free graph on n vertices is at most $n^{2s} + 1$.

Theorem 14 ([24]). For every constant $s \ge 1$, it is possible to enumerate all maximal independent sets of a graph G on n vertices and m edges with a delay of O(nm).

We now prove that SUBSET VERTEX COVER is polynomial-time solvable for instances (G, T, k), where G[T] is sP_2 -free. The idea behind the algorithm is to remove any edges between vertices in $V \setminus T$, as these edges are irrelevant. As a consequence, we may leave the graph class, but this is not necessarily an obstacle. For example, if G[T] is a complete graph, or T is an independent set, we can easily solve the problem. Both cases are generalized by the result below.

Theorem 15. For every $s \ge 1$, Subset Vertex Cover can be solved in polynomial time for instances (G, T, k) for which G[T] is sP_2 -free.

Proof. Let $s \geq 1$, and let (G,T,k) be an instance of Subset Vertex Cover where G=(V,E) is a graph such that G[T] is sP_2 -free. Let G'=(V,E') be the graph obtained from G after removing every edge between two vertices of $V \setminus T$, so $G'[V \setminus T]$ is edgeless. We observe that G has a T-vertex cover of size at most k if and only if G' has a T-vertex cover of size at most K. Moreover, K if K in K in

We first prove the following two claims, see Figure 4 for an illustration.

Claim 15.1. A subset $S \subseteq V(G')$ is a T-vertex cover of G' if and only if $S = R \cup W$ for a minimal vertex cover R of G'[T] and a vertex cover W of $G'[V \setminus R]$.

Proof. We prove Claim 15.1 as follows. Let $S \subseteq V(G')$. First assume that S is a T-vertex cover of G'. Let $I = V \setminus S$. As S is a T-vertex cover, $T \cap I$ is an independent set. Hence, S contains a minimal

vertex cover R of G'[T]. As $G'[V \setminus T]$ is edgeless, S is a vertex cover of G, or in other words, I is an independent set. In particular, this means that $W \setminus R$ is a vertex cover of $G'[V \setminus R]$.

Now assume that $S = R \cup W$ for a minimal vertex cover R of G'[T] and a vertex cover W of $G'[V \setminus R]$. For a contradiction, suppose that S is not a T-vertex cover of G'. Then G' - S contains an edge $uv \in E'$, where at least one of u, v belongs to T. First suppose that both u and v belong to T. As R is a vertex cover of G'[T], at least one of u, v belongs to $R \subseteq S$, a contradiction. Hence, exactly one of u, v belongs to T, say $u \in T$ and $v \in V \setminus T$, so in particular, $v \notin R$. As $R \subseteq S$, we find that $u \notin R$. Hence, both u and v belong to $V \setminus R$. As W is a vertex cover of $V \setminus R$, this means that at least one of u, v belongs to $W \subseteq S$, a contradiction. This proves the claim.

Claim 15.2. For every minimal vertex cover R of G'[T], the graph $G'[V \setminus R]$ is bipartite.

Proof. We prove Claim 15.2 as follows. As R is a vertex cover of G'[T], we find that $T \setminus R$ is an independent set. As $G'[V \setminus T]$ is edgeless by construction of G', this means that $G'[V \setminus R]$ is bipartite with partition classes $T \setminus R$ and $V \setminus T$.

We are now ready to give our algorithm. We enumerate the minimal vertex covers of G'[T]. For every minimal vertex cover R, we compute a minimum vertex cover W of $G'[V \setminus R]$. In the end, we return the smallest $S = R \cup W$ that we found.

The correctness of our algorithm follows from Claim 15.1. It remains to analyze the running time. As G'[T] is sP_2 -free, we can enumerate all maximal independent sets I of G'[T] and thus all minimal vertex covers $R = T \setminus I$ of G'[T] in $(n^{2s} + 1) \cdot O(nm)$ time due to Theorems 13 and 14. For a minimal vertex cover R, the graph $G'[V \setminus R]$ is bipartite by Claim 15.2. Hence, we can compute a minimum vertex cover R of $G'[V \setminus R]$ in polynomial time by applying König's Theorem. We conclude that the total running time is polynomial.

For our next result (Theorem 18) we need two known results as lemmas.

Lemma 16 ([6]). If Subset Vertex Cover is polynomial-time solvable on H-free graphs for some H, then it is so on $(H + P_1)$ -free graphs.

Lemma 17 ([5]). For every $r \ge 1$, Vertex Cover is polynomial-time solvable on $rK_{1,3}$ -free graphs.

We are now ready to prove our second polynomial-time result.

Theorem 18. For every integer s, Subset Vertex Cover is polynomial-time solvable on $(sP_1 + P_2 + P_3)$ -free graphs.

Proof. Due to Lemma 16, we can take s = 0, so we only need to give a polynomial-time algorithm for $(P_2 + P_3)$ -free graphs. Hence, let (G, T, k) be an instance of SUBSET VERTEX COVER, where G = (V, E) is a $(P_2 + P_3)$ -free graph.

First compute a minimum vertex cover of G. As G is $(P_2 + P_3)$ -free, and thus $2K_{1,3}$ -free, this takes polynomial time by Lemma 17. Remember the solution S_{VC} .

We now compute a minimum T-vertex cover S of G that is not a vertex cover of G. Then G - S must contain an edge between two vertices in G - T. We branch by considering all $O(n^2)$ options of choosing this edge. For each chosen edge uv we do as follows. As both u and v will belong to G - S for the T-vertex cover S of G that we are trying to construct, we first add every neighbour of u or v that belongs to G to G.

Let T' consist of all vertices of T that are neither adjacent to u nor to v. As G is $(P_2 + P_3)$ -free and $uv \in E$, we find that G[T'] is P_3 -free and thus a disjoint union of complete graphs. We call a connected component of G[T'] large if it has at least two vertices; else we call it *small* (so every small component of G[T'] is an isolated vertex). See also Figure 5 for an illustration.

Case 1. The graph G[T'] has at most two large connected components.

Let D_1 and D_2 be the large connected components of G[T'] (if they exist). As $V(D_1)$ and $V(D_2)$ are cliques in G[T], at most one vertex of D_1 and at most one vertex of D_2 can belong to G-S. We branch by considering all $O(n^2)$ options of choosing at most one vertex of D_1 and at most one vertex of D_2 to be these vertices. For each choice of vertices we do as follows. We add all other vertices of D_1 and D_2 to S. Let T^* be the set of remaining vertices of T. Then T^* is an independent set.

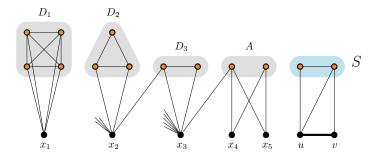


Figure 5: An illustration of the graph G in the proof of Theorem 18, where T consists of the orange vertices, and p=3. Edges in $G[V \setminus T]$ are not drawn, and for x_2 and x_3 some edges are partially drawn. None of x_1, x_4, x_5 satisfy a property; x_2 satisfies Property 1 for D_2 and Property 2 for D_2 and D_3 ; and D_3 ; and D_3 satisfies Property 3 for D_3 .

We delete every edge between any two vertices in G-T. Now the graph G^* induced by the vertices of $T^* \cup (V \setminus T)$ is bipartite (with partition classes T^* and $V \setminus T$). It remains to compute a minimum vertex cover S^* of G^* . This can be done in polynomial time by applying König's Theorem. We let S consist of S^* together with all vertices of T that we had added in S already.

For each branch, we remember the output, and in the end we take a smallest set S found and compare its size with the size of S_{VC} , again taking a smallest set as the final solution.

Case 2. The graph G[T'] has at least three large connected components.

Let D_1, \ldots, D_p , for some $p \geq 3$, be the large connected components of G[T']. Let A consists of all the vertices of the small connected components of G[T'].

We first consider the case where G-S will contain a vertex $w \in V \setminus T$ with one of the following properties:

- 1. for some i, w has a neighbour and a non-neighbour in D_i ; or
- 2. for some i, j with $i \neq j$, w has a neighbour in D_i and a neighbour in D_j ; or
- 3. for some i, w has a neighbour in D_i and a neighbour in A.

We say that a vertex w in G-S is semi-complete to some D_i if w is adjacent to all vertices of D_i except at most one. We show the following claim that holds if the solution S that we are trying to construct contains a vertex $w \in V \setminus (S \cup T)$ that satisfies one of the three properties above. See Figure 5 for an illustration

Claim 18.1. Every vertex $w \in V \setminus (S \cup T)$ that satisfies one of the properties 1-3 is semi-complete to every $V(D_j)$.

Proof. We prove Claim 18.1 as follows. Let $w \in V \setminus (S \cup T)$. First assume w satisfies Property 1. Let x and y be vertices of some D_i , say D_1 , such that $wx \in E$ and $wy \notin E$. For a contradiction, assume w is not semi-complete to some D_j . Hence, D_j contains vertices y' and y'', such that $wy' \notin E$ and $wy'' \notin E$. If $j \geq 2$, then $\{y', y'', w, x, y\}$ induces a $P_2 + P_3$ (as D_1 and D_j are complete graphs). This contradicts that G is $(P_2 + P_3)$ -free. Hence, w is semi-complete to every $V(D_j)$ with $j \geq 2$. Now suppose j = 1. As $p \geq 3$, the graphs D_2 and D_3 exist. As w is semi-complete to every $V(D_j)$ for $j \geq 2$ and every D_j is large, there exist vertices $x' \in V(D_2)$ and $x'' \in V(D_3)$ such that $wx' \in E$ and $wx'' \in E$. However, now $\{y', y'', x', w, x''\}$ induces a $P_2 + P_3$, a contradiction.

Now assume w satisfies Property 2, say w is adjacent to $x_1 \in V(D_1)$ and to $x_2 \in V(D_2)$. Suppose w is not semi-complete to some $V(D_j)$. If $j \geq 3$, then the two non-neighbours of w in D_j , together with x_1, w, x_2 , form an induced $P_2 + P_3$, a contradiction. Hence, w is semi-complete to every $V(D_j)$ for $j \geq 3$. If $j \in \{1, 2\}$, say j = 1, then let y, y' be two non-neighbours of w in D_1 and let x_3 be a neighbour of w in D_3 . Now, $\{y, y', x_2, w, x_3\}$ induces a $P_2 + P_3$, a contradiction. Hence, w is semi-complete to $V(D_1)$ and $V(D_2)$ as well.

Finally, assume w satisfies Property 3, say w is adjacent to $z \in A$ and $x_1 \in V(D_1)$. If w not semi-complete to $V(D_j)$ for some $j \geq 2$, then two non-neighbours of w in D_j , with z, w, x_1 , form an induced

 $P_2 + P_3$, a contradiction. Hence, w is semi-complete to every $V(D_j)$ with $j \geq 2$. As before, by using a neighbour of w in D_2 and one in D_3 , we find that w is also semi-complete to $V(D_1)$. This completes the proof of Claim 18.1.

We now branch by considering all O(n) options for choosing a vertex $w \in V \setminus (S \cup T)$ that satisfies one of the properties 1-3. For each chosen vertex w, we do as follows. We remove all its neighbours in T, and add them to S. By Claim 18.1, the remaining vertices in T form an independent set. We delete any edge between two vertices from $V \setminus T$, so $V \setminus T$ becomes an independent set as well. It remains to compute, in polynomial time by König's Theorem, a minimum vertex cover in the resulting bipartite graph and add this vertex cover to S. For each branch, we store S. After processing all of the O(n) branches, we keep a smallest S, which we denote by S^* .

We are left to compute a smallest T-vertex cover S of G over all T-vertex covers that contain every vertex from $V \setminus T$ that satisfy one of the properties 1–3. We do this as follows. First, we put all vertices from $V \setminus T$ that satisfy one of the three properties 1–3 to the solution S that we are trying to construct. Let G^* be the remaining graph. We do not need to put any vertex from any connected component of G^* that contains no vertex from T in S.

Now consider the connected component D_1' of G^* that contains the vertices from D_1 . As D_1' contains no vertices from $V \setminus T$ satisfying properties 2 or 3, we find that D_1' contains no vertices from A or from any D_j with $j \geq 2$, so $V(D_1') \cap T = V(D_1)$. Suppose there exists a vertex v in $V(D_1') \setminus V(D_1)$, which we may assume has a neighbour in D_1 (as D_1' is connected). Then, v is complete to D_1 as it does not satisfy Property 1. Then, we must put at least $|V(D_1)|$ vertices from D_1' in S, so we might just as well put every vertex of D_1 in S. As $V(D_1') \cap T = V(D_1)$, this suffices. If $D_1' = D_1$, then we put all vertices of D_1 except for one arbitrary vertex of D_1 in S.

We do the same as we did for D_1 for the connected components D'_2, \ldots, D'_p of G^* that contain $V(D_2), \ldots V(D_p)$, respectively.

Now, it remains to consider the induced subgraph F of G^* that consists of connected components containing the vertices of A. Recall that A is an independent set. We delete every edge between two vertices in $V \setminus T$, resulting in another independent set. This changes F into a bipartite graph and we can compute a minimum vertex cover S_F of F in polynomial time due to König's Theorem. We put S_F to S and compare the size of S with the size of S^* and S_{VC} , and pick the one with smallest size as our solution.

The correctness of our algorithm follows from the above description. The number of branches is $O(n^4)$ in Case 1 and $O(n^3)$ in Case 2. As each branch takes polynomial time to process, this means that the total running time of our algorithm is polynomial. This completes our proof.

5 The Proof of Theorems 6 and 7

We first prove Theorem 6, which we restate below.

Theorem 6 (restated). For a graph $H \neq rP_1 + sP_2 + P_3$ for any $r \geq 0$, $s \geq 2$; $rP_1 + sP_2 + P_4$ for any $r \geq 0$, $s \geq 1$; or $rP_1 + sP_2 + P_t$ for any $r \geq 0$, $s \geq 0$, $t \in \{5,6\}$, Subset Vertex Cover on H-free graphs is polynomial-time solvable if $H \subseteq_i sP_1 + P_2 + P_3$, sP_2 , or $sP_1 + P_4$ for some $s \geq 1$, and NP-complete otherwise.

Proof. Let H be a graph not equal to $rP_1+sP_2+P_3$ for any $r\geq 0$, $s\geq 2$; $rP_1+sP_2+P_4$ for any $r\geq 0$, $s\geq 1$; or $rP_1+sP_2+P_t$ for any $r\geq 0$, $s\geq 0$, $t\in \{5,6\}$. If H has a cycle, then we apply Theorem 2. Else, H is a forest. If H has a vertex of degree at least 3, then the class of H-free graphs contains all $K_{1,3}$ -free graphs, and we apply Theorem 10. Else, H is a linear forest. If H contains an induced $2P_3$, then we apply Theorem 11. If not, then $H\subseteq_i sP_1+P_2+P_3$, sP_2 , or sP_1+P_4 for some $s\geq 1$. In the first case, apply Theorem 18; in the second case Theorem 15; and in the third case Theorem 3.

We now prove Theorem 7, which we restate below.

Theorem 7 (restated). For a graph H, SUBSET VERTEX COVER on instances (G, T, k), where G[T] is H-free, is polynomial-time solvable if $H \subseteq_i sP_2$ for some $s \ge 1$, and NP-complete otherwise.

Proof. First suppose $P_3 \subseteq_i H$. As a graph that is a disjoint union of edges is P_3 -free, we can apply Theorem 11. Now suppose H is P_3 -free. Then $H \subseteq_i sP_2$ for some $s \ge 1$, and we apply Theorem 15. \square

6 Graphs of Bounded Mim-width

In this section, we give a polynomial algorithm for Subset Vertex Cover on graphs of bounded mimwidth. Our algorithm is inspired by the algorithm of Bui-Xuan et al. [8] for Independent Set and of Bergougnoux et al. [4] for Subset Feedback Vertex Set. Our presentation of the algorithm follows the presentation form in Bergougnoux et al. [4].

We start by introducing some more terminology. Let G = (V, E) be a graph. For $X \subseteq V$, we use 2^X to denote its power set and \overline{X} to denote $V \setminus X$. A set $M \subseteq E$ is a matching in G if no two edges in M share an end-vertex. A matching M is an induced matching if no end-vertex of an edge $e \in M$ is adjacent to any other end-vertex in M except the other end-vertex of e.

We now introduce the notion of mim-width, which was first defined by Vatshelle [25]. Let G = (V, E) be a graph. A rooted binary tree is a rooted tree of which each node has degree 1 or 3, except for a distinguished node that has degree 2 and is the root of the tree. A rooted layout $\mathcal{L} = (L, \delta)$ of G consists of a rooted binary tree L and a bijection δ between V and the leafs of L. For each node $x \in V(L)$, let L_x be the set of leaves that are a descendant of x (including x if x is a leaf). Then define V_x as the corresponding set of vertices of G, that is, $V_x = \{\delta(y) \mid y \in L_x\}$. For a set $A \subseteq V$, let $\min(A)$ be the size of a maximum induced matching in the bipartite graph obtained from G by removing all edges between vertices of G. In other words, this is the bipartite graph G is the mim-width G of a rooted layout G is the minimum over all G is the minimum mim-width over all rooted layouts of G.

In general, it is not known if there exists a polynomial-time algorithm for computing a rooted layout \mathcal{L} of a graph G, such that $\min(\mathcal{L})$ is bounded by a function in the mim-width of G. However, Belmonte and Vatshelle [3] showed that for several graph classes \mathcal{G} of bounded mim-width, including interval graphs and permutation graphs, it is possible to find in polynomial time a rooted layout of a graph $G \in \mathcal{G}$ with mim-width equal to the mim-width of G.

We now introduce the notion of neighbour equivalence, which was first defined by Bui-Xuan et al. [8]. Let G = (V, E) be a graph on n vertices. Let $A \subseteq V$ and $d \in \mathbb{N}^+$. We say that $X, W \subseteq A$ are d-neighbour equivalent with respect to A, denoted $X \equiv_d^A W$, if $\min\{d, |X \cap N(v)|\} = \min\{d, |Y \cap N(v)|\}$ for all $v \in \overline{A}$. Clearly, this is an equivalence relation. We let $\mathsf{nec}_d(A)$ denote the number of equivalence classes of \equiv_d^A .

For each $X \subseteq A$, let $\operatorname{rep}_d^A(X)$ denote the lexicographically smallest set $R \subseteq A$ such that $R \equiv_d^A X$ and |R| is minimum. This is called the *representative* of X. We use $\mathcal{R}_d^A = \{\operatorname{rep}_d^A(X) \mid X \subseteq A\}$. Note that $|\mathcal{R}_d^A| \ge 1$, as the empty set is always a representative. The following lemma allows us to work efficiently with representatives.

Lemma 19 (Bui-Xuan et al. [8]). It is possible to compute in time $O(\mathsf{nec}_d(A) \cdot n^2 \log(\mathsf{nec}_d(A)))$, the set \mathcal{R}_d^A and a data structure that given a set $X \subseteq A$, computes $\mathsf{rep}_d^A(X)$ in $O(|A| \cdot n \log(\mathsf{nec}_d(A)))$ time.

We are now ready to solve Subset Vertex Cover on graphs of bounded mim-width. In fact, we solve the complementary problem. Given a graph G=(V,E) with a rooted layout $\mathcal{L}=(L,\delta)$, a set $T\subseteq V$, and a weight function ω on its vertices, we find a maximum-weight T-independent set on G. Our goal is to use a standard dynamic programming algorithm. However, the size of the table that we would need to maintain by a naive approach is too large. Instead, we work with representatives of the sets in our table. We show that we can reduce the table size so that it is bounded by the square of the number of 1-neighbour equivalence classes.

First, we define a notion of equivalence between elements of our dynamic programming table. Given a set $T \subseteq V$, a set $X \subseteq V$ is a T-independent set if in G[X] there is no edge incident on any vertex of $T \cap X$. Note that X is a T-independent set if and only if X is a T-vertex cover.

Definition 20. Let $X, W \subseteq V_x$ be T-independent sets. We say that X and W are equivalent, denoted by $X \sim_T W$, if $X \cap T \equiv_1^{V_x} W \cap T$ and $X \setminus T \equiv_1^{V_x} W \setminus T$.

We now prove the following lemma.

Lemma 21. For every $Y \subseteq \overline{V_x}$ and every T-independent sets $X, W \subseteq V_x$ such that $X \sim_T W$, it holds that $X \cup Y$ is a T-independent set if and only if $W \cup Y$ is a T-independent set.

Proof. By symmetry, it suffices to prove one direction. Suppose that $X \cup Y$ is a T-independent set, but $W \cup Y$ is not. Note that X and W are T-independent sets by definition and that Y must be a

T-independent set as well, because $X \cup Y$ is. Hence, the fact that $W \cup Y$ is not a T-independent set implies there is an edge $uv \in E(G)$ for which:

- 1. $u \in W \cap T, v \in Y \cap T$,
- 2. $u \in W \cap T, v \in Y \setminus T$, or
- 3. $u \in W \setminus T, v \in Y \cap T$.

In the first case, since $v \in Y \cap T$ has a neighbour in $W \cap T$, note that $\min\{1, |(W \cap T) \cap N(v)|\} = 1$. Since $X \cap T \equiv_1^{V_x} W \cap T$ by the assumption that $X \sim_T W$, it follows that $\min\{1, |(X \cap T) \cap N(v)|\} = 1$. Hence, there is an edge from $v \in Y \cap T$ to $X \cap T$, contradicting that $X \cup Y$ is a T-independent set.

The second case is analogous to the first case. The third case is also analogous, but uses that $X \setminus T \equiv_1^{V_x} W \setminus T$.

We now introduce a final definition.

Definition 22. For every $A \subseteq 2^{V_x}$ and $Y \subseteq \overline{V_x}$, let

$$\mathsf{best}(\mathcal{A}, Y) = \max\{\omega(X) \mid X \in \mathcal{A} \ and \ X \cup Y \ is \ a \ T\text{-}independent \ set}\}.$$

Given $\mathcal{A}, \mathcal{B} \subseteq 2^{V_x}$, we say that \mathcal{B} represents \mathcal{A} if $\mathsf{best}(\mathcal{A}, Y) = \mathsf{best}(\mathcal{B}, Y)$ for every $Y \subseteq \overline{V_x}$.

We use the above definition in our next lemma.

Lemma 23. Given a set $A \subseteq 2^{V_x}$, we can compute $B \subseteq A$ that represents A and has size at most $|\operatorname{nec}_1(V_x)|^2$ in $O(|A| \cdot n^2 \log(\operatorname{nec}_1(V_x)) + \operatorname{nec}_1(V_x) \cdot n^2 \log(\operatorname{nec}_1(V_x)))$ time.

Proof. We obtain \mathcal{B} from \mathcal{A} as follows: for all sets in \mathcal{A} that are equivalent under \sim_T , maintain only a set X that is a T-independent set for which $\omega(X)$ is maximum. Note that if among a collection of equivalent sets, there is no T-independent set, then no set is maintained. By construction, \mathcal{B} has size at most $|\mathsf{nec}_1(V_x)|^2$.

We now prove that \mathcal{B} represents \mathcal{A} . Let $Y \subseteq \overline{V_x}$. Note that $\mathsf{best}(\mathcal{B},Y) \leq \mathsf{best}(\mathcal{A},Y)$, because $\mathcal{B} \subseteq \mathcal{A}$. Hence, if there is no $X \in \mathcal{A}$ such that $X \cup Y$ is a T-independent set, then $\mathsf{best}(\mathcal{B},Y) = \mathsf{best}(\mathcal{A},Y) = -\infty$. So assume otherwise, and let $W \in \mathcal{A}$ satisfy $\omega(W) = \mathsf{best}(\mathcal{A},Y)$. This means that $W \cup Y$ is a T-independent set and in particular, W is a T-independent set. By the construction of \mathcal{B} , there is a $X \in \mathcal{B}$ that is a T-independent set with $X \sim_T W$ and $\omega(X) \geq \omega(W)$. By Lemma 21, $X \cup Y$ is a T-independent set. Hence, $\mathsf{best}(\mathcal{B},Y) \geq \omega(X) \geq \omega(W) = \mathsf{best}(\mathcal{A},Y)$. It follows that $\mathsf{best}(\mathcal{B},Y) = \mathsf{best}(\mathcal{A},Y)$ and thus \mathcal{B} represents \mathcal{A} .

For the running time, note that we can implement the algorithm by maintaining a table indexed by pairs of representatives of the 1-neighbour equivalence classes. By Lemma 19, we can compute the indices in $O(\mathsf{nec}_1(V_x) \cdot n^2 \log(\mathsf{nec}_1(V_x)))$ time. Then for each $X \in \mathcal{A}$, we can compute its representatives in $O(|V_x| \cdot n \log(\mathsf{nec}_1(V_x)))$ time and check whether it is a T-independent set in $O(n^2)$ time. Hence, the total running time is $O(|\mathcal{A}| \cdot n^2 \log(\mathsf{nec}_1(V_x)) + \mathsf{nec}_1(V_x) \cdot n^2 \log(\mathsf{nec}_1(V_x)))$.

We are now ready to prove the following result.

Theorem 24. Let G be a graph on n vertices with a rooted layout (L, δ) . We can solve Subset Vertex Cover in $O(\sum_{x \in V(L)} (\mathsf{nec}_1(V_x))^4 \cdot n^2 \log(\mathsf{nec}_1(V_x)))$ time.

Proof. It suffices to find a maximum-weight T-independent set of G. For every node $x \in V(L)$, we aim to compute a set $\mathcal{A}_x \subseteq V_x$ of T-independent sets such that \mathcal{A}_x represents 2^{V_x} and has size at most $p(x) := |\mathsf{nec}_1(V_x)|^2 + 1$. Letting r denote the root of L, we then return the set in \mathcal{A}_r of maximum weight. Since \mathcal{A}_r represents 2^{V_r} , this is indeed a maximum-weight T-independent set of G.

We employ a bottom-up dynamic programming algorithm to compute \mathcal{A}_x . If x is a leaf with $V_x = \{v\}$, then set $\mathcal{A}_x = \{\emptyset, \{v\}\}$. Clearly, \mathcal{A}_x represents 2^{V_x} and has size at most p(x). So now suppose x is an internal node with children a, b. For any $\mathcal{A}, \mathcal{B} \subseteq 2^{V(G)}$, let $\mathcal{A} \otimes \mathcal{B} = \{X \cup Y \mid X \in \mathcal{A}, Y \in \mathcal{B}\}$. Now let \mathcal{A}_x be equal to the result of the algorithm of Lemma 23 applied to $\mathcal{A}_a \otimes \mathcal{A}_b$. Then, indeed, $|\mathcal{A}_x| \leq p(x)$. Using induction, it remains to show the following for the correctness proof:

Claim 24.1. If A_a and A_b represent 2^{V_a} and 2^{V_b} respectively, then the computed set A_x represents 2^{V_x} .

Proof. We prove Claim 24.1 as follows. If $\mathcal{A}_a \otimes \mathcal{A}_b$ represents 2^{V_x} , then by Lemma 23 and the transitivity of the 'represents' relation, it follows that \mathcal{A}_x represents 2^{V_x} . So it suffices to prove that $\mathcal{A}_a \otimes \mathcal{A}_b$ represents 2^{V_x} . Let $Y \subseteq \overline{V_x}$. Note that

```
\begin{array}{lcl} \mathsf{best}(\mathcal{A}_a \otimes \mathcal{A}_b, Y) & = & \max\{\omega(X) + \omega(W) \mid X \in \mathcal{A}_a, W \in \mathcal{A}_b, \\ & X \cup W \cup Y \text{ is a $T$-independent set}\} \\ & = & \max\{\mathsf{best}(\mathcal{A}_a, W \cup Y) + w(W) \mid W \in \mathcal{A}_b\}. \end{array}
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Note that $\mathsf{best}(\mathcal{A}_a, W \cup Y) = \mathsf{best}(2^{V_a}, W \cup Y)$, as \mathcal{A}_a represents 2^{V_a} , and thus $\mathsf{best}(\mathcal{A}_a \otimes \mathcal{A}_b, Y) = \mathsf{best}(2^{V_a} \otimes \mathcal{A}_b, Y)$. Using a similar argument, we can then show that $\mathsf{best}(2^{V_a} \otimes \mathcal{A}_b, Y) = \mathsf{best}(2^{V_a} \otimes 2^{V_b}, Y)$. Since $2^{V_x} = 2^{V_a} \otimes 2^{V_b}$, it follows that $\mathsf{best}(\mathcal{A}_a \otimes \mathcal{A}_b, Y) = \mathsf{best}(2^{V_x}, Y)$ and thus $\mathcal{A}_a \otimes \mathcal{A}_b$ represents 2^{V_x} . This completes the proof of Claim 24.1.

Finally, we prove the running time bound. Using induction, it follows that $|\mathcal{A}_a \otimes \mathcal{A}_b| \leq p(x)^2$ for any internal node x with children a, b. Hence, $\mathcal{A}_a \otimes \mathcal{A}_b$ can be computed in $O(p(x)^2 \cdot n)$ time. Then, \mathcal{A}_x can be computed in $O(p(x)^2 \cdot n^2 \log(\mathsf{nec}_1(V_x)) + \mathsf{nec}_1(V_x) \cdot n^2 \log(\mathsf{nec}_1(V_x))) = O((\mathsf{nec}_1(V_x))^4 \cdot n^2 \log(\mathsf{nec}_1(V_x)))$ time by Lemma 23.

It was shown by Belmonte and Vatshelle [3] that $\operatorname{nec}_d(A) \leq |A|^{d \cdot \min(A)}$. Combining their result with Theorem 24 immediately yields the following.

Theorem 25. Let G be a graph on n vertices with a rooted layout $\mathcal{L} = (L, \delta)$. Then Subset Vertex Cover can be solved in $n^{O(\min(\mathcal{L}))}$ time.

The following corollary is now immediate from the fact that interval and circular-arc graphs have constant mim-width and a rooted layout of constant mim-width can be computed in polynomial time [3].

Corollary 26. Subset Vertex Cover can be solved in polynomial time on interval and circular-arc graphs.

7 Conclusions

Apart from giving a dichotomy for Subset Vertex Cover restricted to instances (G, T, k) where G[T] is H-free (Theorem 7), we gave a partial classification of Subset Vertex Cover for H-free graphs (Theorem 6). Our partial classification resolved two open problems from the literature and showed that for some hereditary graph classes, Subset Vertex Cover is computationally harder than Vertex Cover (if $P \neq NP$). This is in contrast to the situation for graph classes closed under edge deletion. Hence, Subset Vertex Cover is worth studying on its own, instead of only as an auxiliary problem (as done in [6]).

Our results raise the question whether there exist other hereditary graph classes on which Subset Vertex Cover is computationally harder than Subset Vertex Cover. Recall that Vertex Cover is polynomial-time solvable for perfect graphs [14], and thus for weakly chordal graphs and chordal graphs. On the other hand, we showed that Subset Vertex Cover is NP-complete for 2-unipolar graphs, a subclass of $2P_3$ -free weakly chordal graphs. Hence, as the first candidate graph class to answer this question, we propose the class of chordal graphs. A standard approach for Vertex Cover on chordal graphs is dynamic programming over the clique tree of a chordal graph. However, a naive dynamic programming algorithm over the clique tree does not work for Subset Vertex Cover, as we may need to remember an exponential number of subsets of a bag (clique) and the bags can have arbitrarily large size. Our polynomial-time algorithms for Subset Vertex Cover for interval and circular-arc graphs, which follow from our result for graph classes of bounded mim-width, makes the open question of the complexity of Subset Vertex Cover on chordal graphs, a superclass of the class of interval graphs, even more pressing. Recall that Subset Feedback Vertex Set, which is also solvable in polynomial time for graphs of bounded mim-width [4], is NP-complete for split graphs and thus for chordal graphs [11].

We also note that our polynomial-time algorithms for Subset Vertex Cover for sP_2 -free graphs and $(P_2 + P_3)$ -free graphs can easily be adapted to work for Weighted Subset Vertex Cover for sP_2 -free graphs and $(P_2 + P_3)$ -free graphs. In this more general problem variant, each vertex u is given some positive weight w(u), and the question is whether there exists a T-vertex cover S with weights

 $w(S) = \sum_{u \in S} w(u) \le k$. In contrast, Papadopoulos and Tzimas [21] proved that WEIGHTED SUBSET FEEDBACK VERTEX SET is NP-complete for $5P_1$ -free graphs, whereas SUBSET FEEDBACK VERTEX SET is polynomial-time solvable even for $(sP_1 + P_4)$ -free graphs for every $s \ge 1$ [20] (see also Theorem 4). The hardness construction of Papadopoulos and Tzimas [21] can also be used to prove that WEIGHTED ODD CYCLE TRANSVERSAL is NP-complete for $5P_1$ -free graphs [7], while SUBSET ODD CYCLE TRANSVERSAL is polynomial-time solvable even for $(sP_1 + P_3)$ -free graphs for every $s \ge 1$ [6] (see also Theorem 5).

Finally, to complete the classification of Subset Vertex Cover for H-free graphs it remains to solve the open cases where

- $H = sP_2 + P_3$ for $s \ge 2$; or
- $H = sP_2 + P_4$ for $s \ge 1$; or
- $H = sP_2 + P_t$ for $s \ge 0$ and $t \in \{5, 6\}$.

Brettell et al. [6] asked what the complexity of Subset Vertex Cover is for P_5 -free graphs. In contrast, Vertex Cover is polynomial-time solvable even for P_6 -free graphs [15]. However, the open cases where $H = sP_2 + P_t$ ($s \ge 1$ and $t \in \{4, 5, 6\}$) are even open for Vertex Cover on H-free graphs (though a quasipolynomial-time algorithm is known [12, 22]). So for those cases we may want to first restrict ourselves to Vertex Cover instead of Subset Vertex Cover.

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