Distance-layer structure of the De Bruijn and Kautz digraphs: analysis and application to deflection routing

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Abstract

In this paper, we present a detailed study of the reach distance-layer structure of the De Bruijn and Kautz digraphs, and we apply our analysis to the performance evaluation of deflection routing in De Bruijn and Kautz networks. Concerning the distance-layer structure, we provide explicit polynomial expressions, in terms of the degree of the digraph, for the cardinalities of some relevant sets of this structure. Regarding the application to defection routing, and as a consequence of our polynomial description of the distance-layer structure, we formulate explicit expressions, in terms of the degree of the digraph, for some probabilities of interest in the analysis of this type of routing.

De Bruijn and Kautz digraphs are fundamental examples of digraphs on alphabet and iterated line digraphs. If the topology of the network under consideration corresponds to a digraph of this type, we can perform, in principle, a similar vertex layer description.

1 Introduction

Deflection routing [1] is a routing scheme for bufferless networks based on the fact that if a packet cannot be sent through a given link due to congestion, it is deflected through any other available link (instead of being buffered in the node queue), and the packet is then rerouted to destination. This kind of routing is nowadays interesting in the context of optical networks [19, 25, 31] and on-chip networks [6, 22]. However, its efficiency depends highly on the network topology (as well as on the decision criteria used to deflect packets when collisions appear [11]). More precisely, the routing efficiency will be determined by how much the distance to the destination increases when a deflection occurs. This question is addressed by considering some probabilities, as studied in Subsection 2.3. Because of this reason, the efficiency in networks with unidirectional links may be worse than in the bidirectional case. Nevertheless, in many cases, directed networks are convenient [22, 29].

Despite being known for a long time, active research is still going on on De Bruijn and Kautz digraphs $B(d,D)$ and $K(d,D)$ [2, 9, 20, 21], both in graph theory [4, 10, 18, 23] and in network engineering [14, 24, 28]. Those digraphs have been proposed as topologies for optical networks (see for instance [5, 7, 30]). This paper is concerned with deflection routing in these kinds of networks.

To study the topological properties of $B(d,D)$ and $K(d,D)$ that we need to evaluate the performance of deflection routing, we provide a detailed study of its reach distance-layer structure. We give explicit polynomial expressions, in terms of the degree of the digraph, for the cardinalities of some relevant sets of this structure. For instance, if $S_i^\star(v)$ denotes the set of vertices at distance i from a given vertex v, we show that $|S_i^\star(v)| = d^i - a_{i-1}d^{i-1} - \cdots - a_1 d - a_0$, where the coefficients a_k are 0 or 1, and are explicitly determined from the sequence representation of v. Moreover, if w is a vertex adjacent from v, we demonstrate that there are at most two integers j such that the intersection $S_i^\star(v) \cap S_j^\star(w)$ is nonempty; we show how to determine such values of j; and we relate the polynomial description of $|S_i^\star(v) \cap S_j^\star(w)|$ with that of $|S_i^\star(v)|$.

We apply our results on the distance-layer structure to provide explicit expressions, in terms of the degree d, of some probabilities of interest in the performance evaluation of deflection routing in $B(d,D)$ and $K(d,D)$. Moreover, the polynomial description of the distance-layer structure is interesting by itself

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from a graph theoretical approach, and it can be helpful in other applications of De Bruijn and Kautz digraphs to networks or other engineering fields.

The paper is organized as follows. In Section 2 we present our results on the distance-layer structure of the set of vertices of $B(d,D)$ and $K(d,D)$ (Subsections 2.1 and 2.2), and on deflection routing (Subsection 2.3). To develop the proofs of our results, we need a collection of technical lemmas and remarks that allow us to understand the distance-layer structure comprehensively. The proofs of these lemmas are put together in an Appendix that also contains a long common proof of two of the propositions formulated in Subsection 2.2.

An extended abstract of a preliminary version of our work appeared in [12]. Moreover, and extended preprint version of this paper, with examples and additional remarks, can be found in [13].

2 Our results

Concerning the distance-layer structure of the set of vertices of the De Bruijn and Kautz digraphs we formulate some polynomial expressions (in terms of the degree d of the digraph) for the cardinalities of some relevant sets of this structure. More precisely, let $S_i^\star(v)$ be the set of vertices at distance i from a given vertex v. We show that $|S_i^\star(v)| = d^i - a_{i-1}d^{i-1} - \cdots - a_1 d - a_0$, and the coefficients $a_k \in \{0,1\}$ are explicitly calculated. Moreover, given v, we show that for each vertex w there exists at most one integer $j \geqslant i$ such that the intersection $S_i^\star(v) \cap S_j^\star(w)$ is nonempty; and in the case that w is adjacent from v, we provide a precise characterization of when $S_i^\star(v) \cap S_j^\star(w) \neq \emptyset$. Furthermore, if w is adjacent from v, we prove that if $S_i^\star(v) \cap S_{i-1}^\star(w) \neq \emptyset$, then $|S_i^\star(v) \cap S_{i-1}^\star(w)| = d^{i-1} - b_{i-2}d^{i-2} - \ldots - b_1 d - b_0$, and that if $S_i^\star(v) \cap S_j^\star(w) \neq \emptyset$, then $|S_i^\star(v) \cap S_j^\star(w)| = d^i - \alpha_{i-1}d^{i-1} - \ldots - \alpha_1 d - \alpha_0$, where the coefficients of these polynomial expressions, $b_k, \alpha_k \in \{0,1\}$, $0 \leqslant k \leqslant i-2$, and $\alpha_{i-1} \in \{0,1,2\}$, are determined from the coefficients a_k of the polynomial expression of $|S_i^\star(v)|$.

2.1 The distance-layer structure of $B(d,D)$ and $K(d,D)$

This subsection and the following one are devoted to presenting our results on the characterization of the distance-layer structure of $B(d,D)$ and $K(d,D)$.

We make use of the well-known sequence representation of the vertices of $B(d,D)$ and $K(d,D)$. Each vertex of the De Bruijn digraph $B(d,D)$ corresponds to a sequence $v = v_1v_2\cdots v_D$ such that each element v_k belongs to a base alphabet A of d symbols, and vertex v is adjacent to the d vertices $w = v_2\cdots v_Dv_{D+1}$, where $v_{D+1} \in A$. Analogously, each vertex of the Kautz digraph $K(d,D)$ corresponds to a sequence $v = v_1v_2\cdots v_D$, where now $v_k \neq v_{k+1}$, $1 \leqslant k < D$, and the base alphabet A has $d+1$ symbols. In $K(d,D)$, vertex v is adjacent to the d vertices $w = v_2\cdots v_Dv_{D+1}$, where $v_{D+1} \in A$ and $v_{D+1} \neq v_D$. The digraphs $B(d,D)$ and $K(d,D)$ are d-regular, $d \geqslant 2$, have diameter D, and number of vertices d^D and $d^D + d^{D-1}$, respectively.

Notice that if $v = v_1v_2\cdots v_iv_{i+1}\cdots v_D$ is the sequence representation of a vertex v, then the sequence representation of a generic vertex u for which there exists a walk from v to u of length i, $0 \leqslant i \leqslant D-1$, is $u = v_{i+1}\cdots v_D * \cdots *$, where the subsequence $* \cdots *$ means that the last i symbols of u can be arbitrarily chosen (in the case $G = K(d,D)$, two consecutive symbols must be different). It is easily checked that between any pair of vertices there exists a walk of length D in $B(d,D)$ and of length $D+1$ in $K(d,D)$. It is also a well-known fact that in $B(d,D)$ and $K(d,D)$ the shortest path between any two vertices is unique. Indeed, let v and z be distinct vertices with a sequence representation $v = v_1v_2\cdots v_D$ and $z = z_1z_2\cdots z_D$, respectively. Then, the distance from v to z is k if and only if k is the smallest integer such that $v = v_1\cdots v_kz_1\cdots z_{D-k}$; that is to say, k is the smallest integer such that the last $D-k$ symbols of the sequence representation of v coincide with the first $D-k$ symbols of the sequence representation of z. Moreover, if $k \geqslant 2$, then the shortest path from v to z is $v, u_1, \ldots, u_{k-1}, z$, where the sequence representation of the intermediate vertex u_i is $u_i = v_{i+1}\cdots v_kz_1\cdots z_{D-k+i}$, $1 \leqslant i \leqslant k-1$.

From now on let G be the digraph under consideration (either $G = B(d,D)$ or $G = K(d,D)$) and let V denote its vertex set.

Given $v \in V$, for $i \geqslant 0$, let $S_i(v)$ be the set of vertices for which there exists a walk from v of length i, and let $S_i^\star(v)$ denote the set of vertices at distance i from v. From the definition it is clear that $S_0(v) = \{v\}$; $S_1(v)$ is the set of vertices adjacent from v, usually denoted as $\Gamma^+(v)$; $S_i^\star(v) = \emptyset$ for $i \geqslant D+1$; and

$$S_i^\star(v) = S_i(v) \setminus \left(\bigcup_{k=0}^{i-1} S_k(v)\right) \text{ for } 0 \leqslant i \leqslant D. \tag{1}$$

Moreover, since in $B(d,D)$ there exists a walk of length D between any pair of vertices, if $G=B(d,D)$ and $i\geqslant D$, then $S_i(v)=V$, and so $|S_i(v)|=d^D$. Analogously, if $G=K(d,D)$ and $i\geqslant D+1$, then $S_i(v)=V$ and $|S_i(v)|=d^D+d^{D-1}$, because in $K(d,D)$ there is a walk of length $D+1$ between any pair of vertices.

2.2 Polynomial description of the distance-layer structure

The first goal of this subsection is to present a polynomial description of the cardinality of the set $S_i^\star(v)$, where the polynomial has degree i, variable d (the degree of the digraph), and coefficients 0 or 1. In order to obtain this description, we introduce the following definition.

Definition 1. Given $v\in V$ and two integers k,i such that $0\leqslant i\leqslant D$ and $0\leqslant k\leqslant i$, let

$$S_{k,i}(v)=\begin{cases} S_k(v), & \text{if } S_k(v)\subseteq S_i(v) \text{ and for all } j,\ k<j<i,\\ & \ \text{such that } S_j(v)\subseteq S_i(v) \text{ we have } S_k(v)\not\subseteq S_j(v);\\ \emptyset, & \text{otherwise.}\end{cases}$$

Remark 1. It follows from Definition 1 that $S_{i,i}(v)=S_i(v)$ for any $v\in V$. Moreover, if $G=K(d,D)$, then $S_{i-1,i}(v)=\emptyset$ for any $v\in V$, because $S_{i-1}(v)\not\subseteq S_i(v)$. In the case $G=B(d,D)$ there are vertices v such that $S_{i-1,i}(v)=S_{i-1}(v)$ and vertices v for which $S_{i-1,i}(v)=\emptyset$.

To prove the results presented in Subsections 2.2 and 2.3, we use several technical lemmas gathered in the Appendix at the end of the document. However, for the sake of readability, we include in Subsection 2.2 the statements of Lemmas 1 to 5, and in Subsection 2.3, the statements of Lemmas 6 to 8.

We will use the following notation. If $v\in V$, then $v_{[i,j]}$ denotes the subsequence $v_iv_{i+1}\cdots v_j$ of the sequence representation $v=v_1v_2\cdots v_D$. In particular, $v_{[i,i]}=v_i$ is the i-th element of this sequence.

Lemma 1. *Let $v\in V$. Then $|S_i(v)|=d^i$ for $0\leqslant i\leqslant D$. Moreover, if $G=B(d,D)$ and $i\geqslant D$, then $S_i(v)=V$; while if $G=K(d,D)$ and $i\geqslant D+1$, then $S_i(v)=V$.*

The main part of the next lemma essentially states that, given any two (no necessarily different) vertices v,v', if $k\leqslant i<D$ or $k<i=D$, then either $S_k(v)\subseteq S_i(v')$ or $S_k(v)\cap S_i(v')=\emptyset$. The precise formulation in terms of the sequence representation of v and v' is as follows.

Lemma 2. *Let $v,v'\in V$ be two vertices with sequence representation $v=v_1v_2\cdots v_D$ and $v'=v'_1v'_2\cdots v'_D$. Let $0\leqslant k\leqslant i$ and assume that $S_i(v')\neq V$. Then $i\leqslant D$, $S_k(v)\neq V$, and the following statements hold:*

1. *If $k\leqslant i<D$, then either $S_k(v)\subseteq S_i(v')$ or $S_k(v)\cap S_i(v')=\emptyset$. Moreover, $S_k(v)\subseteq S_i(v')$ if and only if $v_{[k+1,D-(i-k)]}=v'_{[i+1,D]}$.*
2. *If $k<i=D$, then $G=K(d,D)$ and either $S_k(v)\subseteq S_D(v')$ or $S_k(v)\cap S_D(v')=\emptyset$. Moreover, $S_k(v)\subseteq S_D(v')$ if and only if $v_{k+1}\neq v'_D$.*
3. *If $k=i=D$, then $G=K(d,D)$ and $S_D(v)\cap S_D(v')\neq\emptyset$. Moreover, if $v_D=v'_D$ then $S_D(v)=S_D(v')$, whereas if $v_D\neq v'_D$, then $S_D(v)\neq S_D(v')$ and $|S_D(v)\cap S_D(v')|=d^D-d^{D-1}$.*

Remark 2. Let $0\leqslant k<j<i\leqslant D$. By statement (1) of Lemma 2, $S_j(v)\not\subseteq S_i(v)$ if and only if $S_j(v)\cap S_i(v)=\emptyset$. Therefore, if $S_k(v)\subseteq S_i(v)$ and $S_j(v)\not\subseteq S_i(v)$, then $S_k(v)\not\subseteq S_j(v)$. This observation allows us to reformulate Definition 1 in the following way: Let $v\in V$ and let k,i be two integers such that $0\leqslant i\leqslant D$ and $0\leqslant k\leqslant i$. Then

$$S_{k,i}(v)=\begin{cases} S_k(v), & \text{if } S_k(v)\subseteq S_i(v) \text{ and } S_k(v)\cap S_j(v)=\emptyset \text{ for all } j,\ k<j<i;\\ \emptyset, & \text{otherwise.}\end{cases}$$

Remark 3. By Remark 2 and Lemma 2 we have $S_{k,i}(v)=S_k(v)$ if and only if $v_{[k+1,D-(i-k)]}=v_{[i+1,D]}$ and $v_{[k+1,D-(j-k)]}\neq v_{[j+1,D]}$ for all j, $k<j<i$. In particular, if $k=i-1$, then $S_{i-1,i}(v)=S_{i-1}(v)$ if and only if $S_{i-1}(v)\subseteq S_i(v)$; if and only if $v_{[i,D-1]}=v_{[i+1,D]}$; if and only if $v_i=v_{i+1}=\cdots=v_D$.

Remark 4. Let $0\leqslant k_1,k_2<i\leqslant D$. We claim that if $k_1\neq k_2$, then $S_{k_1,i}(v)\cap S_{k_2,i}(v)=\emptyset$. Indeed, if $S_{k_1,i}(v)\cap S_{k_2,i}(v)\neq\emptyset$, then $S_{k_1,i}(v)\neq\emptyset$ and $S_{k_2,i}(v)\neq\emptyset$. So, $S_{k_1,i}(v)=S_{k_1}(v)\subseteq S_i(v)$ and $S_{k_2,i}(v)=S_{k_2}(v)\subseteq S_i(v)$. Hence $S_{k_1}(v)\cap S_{k_2}(v)=S_{k_1,i}(v)\cap S_{k_2,i}(v)\neq\emptyset$. Now, by applying Lemma 2, either $S_{k_1}(v)\subseteq S_{k_2}(v)$ or $S_{k_2}(v)\subseteq S_{k_1}(v)$. Hence either $S_{k_1}(v)\subseteq S_{k_2}(v)\subseteq S_i(v)$ or $S_{k_2}(v)\subseteq S_{k_1}(v)\subseteq S_i(v)$. In any case, this leads us to a contradiction with the definition of $S_{k,i}(v)$. This completes the proof of our claim.

The following result describes the structure of the distance-layer set $S_i^\star(v)$.

Lemma 3. *Let $v \in V$ and let $0 \leqslant i \leqslant D$. Then* $S_i^\star(v) = S_i(v) \setminus \Big(\bigcup_{k=0}^{i-1} S_{k,i}(v)\Big)$.

Now we can prove our first result concerning $|S_i^\star(v)|$.

Proposition 1. *Let $v \in V$ and $i \leqslant D$. Then,*

$$|S_i^\star(v)| = d^i - a_{i-1}d^{i-1} - \cdots - a_1 d - a_0,$$

where the coefficients a_k are 0 or 1, and $a_k = 1$ if and only if $S_{k,i}(v) \neq \emptyset$. In particular, if $v = v_1 v_2 \cdots v_D$, then $a_{i-1} = 1$ if and only if $G = B(d,D)$ and $v_i = v_{i+1} = \cdots = v_D$.

Proof. On one hand, from Lemma 3 we have $S_i^\star(v) = S_i(v) \setminus \Big(\bigcup_{k=0}^{i-1} S_{k,i}(v)\Big)$. On the other hand, from the definition of $S_{k,i}(v)$ we have $S_{k,i}(v) \subseteq S_i(v)$. Therefore,

$$|S_i^\star(v)| = \Big|S_i(v) \setminus \Big(\bigcup_{k=0}^{i-1} S_{k,i}(v)\Big)\Big| = |S_i(v)| - \Big|\bigcup_{k=0}^{i-1} S_{k,i}(v)\Big|.$$

As shown in Remark 4, if $k_1 \neq k_2$, then $S_{k_1,i}(v) \cap S_{k_2,i}(v) = \emptyset$. Therefore, it follows that $\Big|\bigcup_{k=0}^{i-1} S_{k,i}(v)\Big| = \sum_{k=0}^{i-1} |S_{k,i}(v)|$. Thus, from the definition of $S_{k,i}(v)$ and by Lemma 1, we have

$$|S_i^\star(v)| = |S_i(v)| - \sum_{k=0}^{i-1} |S_{k,i}(v)| = d^i - \sum_{k=0}^{i-1} a_k d^k,$$

where the coefficients a_k are 0 or 1, and $a_k = 1$ if and only if $S_{k,i}(v) \neq \emptyset$. Clearly, if $i = D$, then $S_{i-1,i}(v) \neq \emptyset$ if and only if $G = B(d,D)$. To conclude, assume that $i < D$. In such a case we have $a_{i-1} = 1$ if and only if $S_{i-1,i}(v) \neq \emptyset$; if and only if $S_{i-1}(v) \subseteq S_i(v)$; if and only if $v_{[i,D-1]} = v_{[i+1,D]}$ (the last equivalence follows from statement (1) of Lemma 2 which can be applied because $i < D$). Therefore we conclude that if $i < D$, then $a_{i-1} = 1$ if and only if $G = B(d,D)$ and $v_i = v_{i+1} = \cdots = v_D$. This completes the proof of the proposition. □

To illustrate the use of Proposition 1, let us consider the following two examples.

Example 1. Consider the De Bruijn digraph $G = B(d,7)$ and let $v \in V$ be a vertex which sequence representation is $v = \alpha\beta\beta\alpha\beta\alpha\beta$, where α and β are distinct elements of the symbol alphabet A. Let us determine the number of vertices, $|S_6^\star(v)|$, at distance 6 from such a vertex v.

If the symbol $*$ stands for an arbitrary element of A, we can describe the sets $S_i(v)$ as $S_0(v) = \{u \in V : u = \alpha\beta\beta\alpha\beta\alpha\beta\}$, $S_1(v) = \{u \in V : u = \beta\beta\alpha\beta\alpha\beta*\}$, $S_2(v) = \{u \in V : u = \beta\alpha\beta\alpha\beta**\}$, ..., and $S_6(v) = \{u \in V : u = \beta******\}$. We realize that if $k < 6$, then $S_k(v) \subseteq S_6(v)$ if and only if $k = 1,2,4$. Hence $S_{1,6}(v) = S_1(v)$, because $S_1(v) \nsubseteq S_j(v)$ if $1 < j < 6$; $S_{2,6}(v) = \emptyset$, because $S_2(v) \subseteq S_4(v)$; and $S_{4,6}(v) = S_4(v)$, because $S_4(v) \nsubseteq S_5(v)$. Therefore we have $a_1 = a_4 = 1$ and $a_2 = a_3 = a_5 = 0$, and hence $|S_6^\star(v)| = d^6 - d^4 - d$.

Example 2. In this second example we consider the Kautz digraph $G = K(d,10)$ and let us calculate $|S_8^\star(v)|$, being v a vertex with sequence representation $v = \alpha\beta\gamma\alpha\beta\gamma\alpha\beta\alpha\beta$, where α, β and γ stand for different elements of the symbol alphabet A. As in Example 1, the sets $S_i(v)$ can be described as $S_0(v) = \{u \in V : u = \alpha\beta\gamma\alpha\beta\gamma\alpha\beta\alpha\beta\}$, $S_1(v) = \{u \in V : u = \beta\gamma\alpha\beta\gamma\alpha\beta\alpha\beta*\}$, $S_2(v) = \{u \in V : u = \gamma\alpha\beta\gamma\alpha\beta\alpha\beta**\}$, ..., and $S_8(v) = \{u \in V : u = \alpha\beta********\}$. (Remember that, since G is a Kautz digraph, two successive symbols in the above sequence representations must be different.) We can verify that if $k < 8$, then the only subsets $S_{k,8}(v)$ which are nonempty are $S_{0,8}(v) = S_0(v)$, $S_{3,8}(v) = S_3(v)$, and $S_{6,8}(v) = S_6(v)$. So we conclude that $|S_8^\star(v)| = d^8 - d^6 - d^3 - 1$.

For the application to deflection routing, in addition to $|S_i^\star(v)|$, we are also interested in $|S_i^\star(v) \cap S_j^\star(w)|$ when w is a vertex adjacent from v. Let $v \in V$ and $w \in S_1(v)$, and let $i \geqslant 0$. By the triangular inequality we have $S_i^\star(v) \cap S_j^\star(w) = \emptyset$ if $j < i-1$. Therefore, since $V = \bigcup_{j=0}^{D} S_j^\star(w)$, we conclude that

$$S_i^\star(v) = \bigcup_{j=0}^{D} \left(S_i^\star(v) \cap S_j^\star(w)\right) = \bigcup_{j=i-1}^{D} \left(S_i^\star(v) \cap S_j^\star(w)\right).$$

First, we demonstrate in Proposition 2 that there are at most two integers $j \geqslant i-1$ such that the intersection $S_i^\star(v) \cap S_j^\star(w)$ is nonempty. After this, in Propositions 3 and 4 we show how to determine such values of j. Finally, in Theorems 1, 2 and 3 we relate the polynomial description of $|S_i^\star(v)|$ with that of $|S_i^\star(v) \cap S_j^\star(w)|$.

Proposition 2. *Let $v \in V$ and let $i \leqslant D$. Then for each vertex $w \in V$ there exists at most one integer j_0, $i \leqslant j_0 \leqslant D$, such that $S_i^\star(v) \cap S_{j_0}^\star(w) \neq \emptyset$. In particular, if $w \in S_1(v)$, then*

1. *either there exists a unique integer j_0, $i \leqslant j_0 \leqslant D$, such that the intersection $S_i^\star(v) \cap S_{j_0}^\star(w)$ is nonempty, and so $S_i^\star(v) = \left(S_i^\star(v) \cap S_{i-1}^\star(w)\right) \cup \left(S_i^\star(v) \cap S_{j_0}^\star(w)\right)$;*
2. *or, for all integer j, $i \leqslant j \leqslant D$, the intersection $S_i^\star(v) \cap S_j^\star(w)$ is empty, and so $S_i^\star(v) = S_i^\star(v) \cap S_{i-1}^\star(w)$.*

Proof. We have to prove that there exists at most one integer j_0, $i \leqslant j_0 \leqslant D$, such that $S_i^\star(v) \cap S_{j_0}^\star(w) \neq \emptyset$, because if so, statements (1) and (2) follow. To prove that, let us demonstrate that if $S_i^\star(v) \cap S_j^\star(w) \neq \emptyset$ for some j, $i \leqslant j < D$, then $S_i^\star(v) \cap S_{j'}^\star(w) = \emptyset$ for all integer j' such that $j < j' \leqslant D$. Thus assume $S_i^\star(v) \cap S_j^\star(w) \neq \emptyset$ and let $j < j' \leqslant D$. On one hand, if $S_i^\star(v) \cap S_j^\star(w) \neq \emptyset$, then $S_i(v) \cap S_j(w) \neq \emptyset$ and, since $i < D$, we conclude from Lemma 2 that $S_i(v) \subseteq S_j(w)$. On the other hand, by definition we have $S_{j'}^\star(w) = S_{j'}(w) \setminus \left(\bigcup_{k=0}^{j'-1} S_k(w)\right)$ and, since $j < j'$, we get that $S_j(w) \cap S_{j'}^\star(w) = \emptyset$. Thus we have $S_i(v) \cap S_{j'}^\star(w) = \emptyset$, because $S_i(v) \subseteq S_j(w)$, and therefore we conclude that $S_i^\star(v) \cap S_{j'}^\star(w) = \emptyset$, as we wanted to prove. □

Proposition 3. *Assume $d \geqslant 3$. Let $v \in V$, $w \in S_1(v)$, and let $i \leqslant D$. Then,*

1. *If $j \geqslant i \neq D$, then $S_i^\star(v) \cap S_j^\star(w) \neq \emptyset$ if and only if $S_i(v) \cap S_j(w) \neq \emptyset$ and $S_i(v) \not\subseteq S_{k,j}(w)$ for $i \leqslant k < j$.*
2. *The intersection $S_i^\star(v) \cap S_{i-1}^\star(w)$ is empty if and only if $G = B(d,D)$ and $v_i = v_{i+1} = \cdots = v_D = w_D$. Furthermore, if $S_i^\star(v) \cap S_{i-1}^\star(w) = \emptyset$, then $S_i^\star(v) \cap S_i^\star(w) \neq \emptyset$.*
3. *There exists a unique integer j, $i \leqslant j \leqslant D$, such that the intersection $S_i^\star(v) \cap S_j^\star(w)$ is non-empty.*

The condition $d \geqslant 3$ cannot be removed from the hypothesis of Proposition 3, because if $d = 2$ and $G = B(d,D)$, then statements (1) and (3) do not necessarily hold. So we study completely the case $d = 2$ in the next proposition.

Proposition 4. *Assume $d = 2$. Let $v \in V$, $w \in S_1(v)$, and let $i \leqslant D$. Then,*

1. *If $j \geqslant i \neq D$, then $S_i^\star(v) \cap S_j^\star(w) \neq \emptyset$ if and only if $S_i(v) \cap S_j(w) \neq \emptyset$, $S_i(v) \not\subseteq S_{k,j}(w)$ for $i \leqslant k < j$, and one of the following conditions holds:*
   - *(a) $j < D$;*
   - *(b) $j = D$, and $v_{[i,D-1]} \neq v_{[i+1,D]}$ or $S_{i-1,j}(w) = \emptyset$.*
2. *The intersection $S_i^\star(v) \cap S_{i-1}^\star(w)$ is empty if and only if $G = B(d,D)$ and $v_i = v_{i+1} = \cdots = v_D = w_D$. Furthermore, if $S_i^\star(v) \cap S_{i-1}^\star(w) = \emptyset$, then $S_i^\star(v) \cap S_i^\star(w) \neq \emptyset$.*
3. *The intersection $S_i^\star(v) \cap S_j^\star(w)$ is empty for all integer j, $i \leqslant j \leqslant D$, if and only if $G = B(d,D)$ and $v_i = v_{i+1} = \cdots = v_D \neq w_D$.*

Propositions 3 and 4 are proved together in the Appendix.

The following lemma deals with the set of vertices $w \in S_1(v)$ for which the intersection set $S_i(v) \cap S_j(w)$ is nonempty.

Lemma 4. *Let $v \in V$ and, for $0 \leqslant i \leqslant j < D$, let $\Gamma_{i,j}^+(v) = \{w \in S_1(v) : \ S_i(v) \cap S_j(w) \neq \emptyset\}$. Then*

1. *The set $\Gamma_{i,j}^+(v)$ is nonempty if and only if one the following conditions is fulfilled:*
   - *(a) $i < j < D-1$ and $v_{[i+1,D+i-j-1]} = v_{[j+2,D]}$.*
   - *(b) $i < j = D-1$ and either $G = B(d,D)$, or $G = K(d,D)$ and $v_{i+1} \neq v_D$.*
   - *(c) $i = j$, $G = B(d,D)$ and $v_{[i+1,D]} = v_D \cdots v_D$.*
2. *If the set $\Gamma_{i,j}^+(v)$ is nonempty, then $\Gamma_{i,j}^+(v)$ has a unique element w which sequence representation is $w = v_2 \cdots v_D v_{i+(D-j)}$. Moreover, if $i = j$, then $S_{i-1}(w) \subseteq S_i(v) = S_i(w) \subseteq S_{i+1}(v) = S_{i+1}(w) \subseteq \cdots$.*

In the next result we provide a detailed description of the intersection sets $S_i^\star(v) \cap S_j^\star(w)$ when w is a vertex adjacent from v and for $i, j \leqslant D$ with $i-1 \leqslant j$.

Lemma 5. *Let $v \in V$ and $w \in S_1(v)$. Let $i, j \leqslant D$ with $i-1 \leqslant j$. If $S_i(v) \cap S_j(w) \neq \emptyset$, then the intersection set $S_i^\star(v) \cap S_j^\star(w)$ can be described as follows:*

1. $S_{i-1}(w) \setminus \bigcup_{k=0}^{i-1} S_{k,i}(v)$ *if* $j = i-1$.

2. $\Big(S_D(v) \cap S_D(w)\Big) \setminus \bigcup_{k=0}^{D-2} S_{k,D}(v)$ *if* $i = j = D$ *and* $G = K(d, D)$.

3. $V \setminus \Big(S_{D-1}(v) \cup S_{D-1}(w) \cup \bigcup_{k=0}^{D-2} S_{k,D}(v)\Big)$ *if* $i = j = D$ *and* $G = B(d, D)$.

4. $S_i(v) \setminus \Big(\Big(\bigcup_{k=0}^{i-1} S_{k,i}(v)\Big) \cup \Big(\bigcup_{k=i-1}^{j-1} S_{k,j}(w)\Big)\Big)$ *if* $j \geqslant i$ *and* $i \neq D$.

Remark 5. Observe that in the above expressions we have $S_{k,i}(v) \subseteq S_i(v)$ for $0 \leqslant k \leqslant i-1$, because of the definition of $S_{k,i}(v)$. Furthermore, by Lemma 2, either $S_{k,i}(v) \cap S_{i-1}(w) = \emptyset$ or $S_{k,i}(v) \subseteq S_{i-1}(w)$, $0 \leqslant k \leqslant i-1$.

Next we show that, whenever the intersection $S_i^\star(v) \cap S_j^\star(w)$ is nonempty, its cardinality has a polynomial expression of the form

$$\big|S_i^\star(v) \cap S_j^\star(w)\big| = d^i - b_{i-1}d^{i-1} - \ldots - b_1 d - b_0,$$

where the coefficients b_k are determined from the coefficients a_k of the polynomial expression of $|S_i^\star(v)|$. The coefficients b_k are computed in the following theorems.

Theorem 1. *Let $v \in V$, $w \in S_1(v)$, and let $1 \leqslant i \leqslant D$. Assume that $S_i^\star(v) \cap S_{i-1}^\star(w) \neq \emptyset$ and that $S_i^\star(v) \cap S_{j_0}^\star(w) \neq \emptyset$ for some $i \leqslant j_0 \leqslant D$. Then,*

$$S_i^\star(v) = \big(S_i^\star(v) \cap S_{i-1}^\star(w)\big) \cup \big(S_i^\star(v) \cap S_{j_0}^\star(w)\big),$$

and so

$$\big|S_i^\star(v)\big| = \big|S_i^\star(v) \cap S_{i-1}^\star(w)\big| + \big|S_i^\star(v) \cap S_{j_0}^\star(w)\big|.$$

Moreover, if

$$\big|S_i^\star(v)\big| = d^i - a_{i-1}d^{i-1} - \ldots - a_1 d - a_0$$

is the polynomial expression of $|S_i^\star(v)|$ given in Proposition 1, then $|S_i^\star(v) \cap S_{i-1}^\star(w)\big|$ and $|S_i^\star(v) \cap S_{j_0}^\star(w)\big|$ have polynomial expressions

$$\big|S_i^\star(v) \cap S_{i-1}^\star(w)\big| = d^{i-1} - b_{i-2}d^{i-2} - \ldots - b_1 d - b_0,$$

$$\big|S_i^\star(v) \cap S_{j_0}^\star(w)\big| = d^i - (a_{i-1}+1)d^{i-1} - (a_{i-2} - b_{i-2})d^{i-2} - \ldots - (a_1 - b_1)d - (a_0 - b_0),$$

where $b_k \in \{0, 1\}$ and $b_k = 1$ if and only if $a_k = 1$ and $v_{D-i+k+1} = w_D$.

Remark 6. In Theorem 1, the equality $S_i^\star(v) = \big(S_i^\star(v) \cap S_{i-1}^\star(w)\big) \cup \big(S_i^\star(v) \cap S_{j_0}^\star(w)\big)$ as well as $|S_i^\star(v)| = \big|S_i^\star(v) \cap S_{i-1}^\star(w)\big| + \big|S_i^\star(v) \cap S_{j_0}^\star(w)\big|$ is valid even if $S_i^\star(v) \cap S_{i-1}^\star(w)$ or $S_i^\star(v) \cap S_{j_0}^\star(w)$ were empty, as deduced from Proposition 2.

Remark 7. We know from Proposition 1 that $a_k \in \{0, 1\}$. Since $b_k \in \{0, 1\}$ and $b_k = 1$ only if $a_k = 1$, we conclude that, for $0 \leqslant k \leqslant i-2$, the coefficient $a_k - b_k$ in the polynomial expression of $|S_i^\star(v) \cap S_{j_0}^\star(w)\big|$ is also either 0 or 1. Moreover, the coefficient $a_{i-1}+1$ in this polynomial expression is 1 or 2. More precisely, from Proposition 1, we have $a_{i-1}+1 = 2$ if and only if $G = B(d, D)$ and $v_i = v_{i+1} = \cdots = v_D$.

Proof. From Proposition 2 we conclude that $S_i^\star(v) = \big(S_i^\star(v) \cap S_{i-1}^\star(w)\big) \cup \big(S_i^\star(v) \cap S_{j_0}^\star(w)\big)$, and so, since $\big(S_i^\star(v) \cap S_{i-1}^\star(w)\big) \cap \big(S_i^\star(v) \cap S_{j_0}^\star(w)\big) = \emptyset$, we get $\big|S_i^\star(v)\big| = \big|S_i^\star(v) \cap S_{i-1}^\star(w)\big| + \big|S_i^\star(v) \cap S_{j_0}^\star(w)\big|$. To complete the proof of the theorem we must demonstrate that if $|S_i^\star(v)| = d^i - a_{i-1}d^{i-1} - \ldots - a_1 d - a_0$, then $\big|S_i^\star(v) \cap S_{i-1}^\star(w)\big| = d^{i-1} - b_{i-2}d^{i-2} - \ldots - b_1 d - c_0$, where $b_k = 1$ if and only if $a_k = 1$ and $v_{D-i+k+1} = w_D$. Let us demonstrate this.

First of all observe that $S_i(v) \cap S_{i-1}(w) \neq \emptyset$, because $w \in S_1(v)$. So we can apply statement (1) of Lemma 5 to write $S_i^\star(v) \cap S_{i-1}^\star(w)$ as

$$S_i^\star(v) \cap S_{i-1}^\star(w) = S_{i-1}(w) \setminus \bigcup_{k=0}^{i-1} S_{k,i}(v) = S_{i-1}(w) \setminus \Big(\bigcup_{k=0}^{i-2} S_{k,i}(v) \cup \Big(S_{i-1,i}(v) \cap S_{i-1}(w) \Big) \Big). \qquad (2)$$

We claim that, in the above expression, the intersection $S_{i-1,i}(v) \cap S_{i-1}(w)$ is empty. Indeed, if $G = K(d, D)$, then $S_{i-1,i}(v) = \emptyset$ (see Remark 1) and we are done. So we can assume $G = B(d, D)$. Moreover, since $S_i^\star(v) \cap S_{i-1}^\star(w) \neq \emptyset$, we know by statement (2) of Propositions 3 and 4 that $v_i = v_{i+1} = \cdots = v_D = w_D$ does not hold. Now we can see that the assumption $S_{i-1,i}(v) \cap S_{i-1}(w) \neq \emptyset$ leads to contradiction. On one hand, if $S_{i-1,i}(v) \cap S_{i-1}(w) \neq \emptyset$, then $\emptyset \neq S_{i-1,i}(v) = S_{i-1}(v)$, and so $v_i = v_{i+1} = \cdots = v_D$, by Remark 3. On the other hand, by statement (1) of Lemma 2, if $S_{i-1}(v) \cap S_{i-1}(w) \neq \emptyset$, then $v_{[i,D-1]} = w_{[i,D]}$ and so $v_i = v_{i+1} = \cdots = v_D = w_D$, a contradiction that proves our claim.

Therefore, since $S_{i-1}(v) \cap S_{i-1}(w) = \emptyset$, we get from (2) that

$$S_i^\star(v) \cap S_{i-1}^\star(w) = S_{i-1}(w) \setminus \Big(\bigcup_{k=0}^{i-2} \big(S_{k,i}(v) \cap S_{i-1}(w) \big) \Big),$$

where, by statement (1) of Lemma 2, for $0 \leqslant k \leqslant i-2$, we have either $S_{k,i}(v) \cap S_{i-1}(w) = \emptyset$ or $S_{k,i}(v) \subseteq S_{i-1}(w)$. Thus, recalling that if $k_1 \neq k_2$, then $S_{k_1,i}(v) \cap S_{k_2,i}(v) = \emptyset$ (see Remark 4), we have

$$|S_i^\star(v) \cap S_{i-1}^\star(w)| = |S_{i-1}(v)| - \sum_{k=0}^{i-2} |S_{k,i}(v) \cap S_{i-1}(w)| = d^{i-1} - b_{i-2} d^{i-2} - \ldots - b_1 d - b_0,$$

where the coefficients b_k are 0 or 1. More precisely, for $0 \leqslant k \leqslant i-2$, we have $b_k = 1$ if and only if $S_{k,i}(v) \cap S_{i-1}(w) \neq \emptyset$. That is, $b_k = 1$ if and only if $S_{k,i}(v) \neq \emptyset$ and $S_{k,i}(v) \cap S_{i-1}(w) \neq \emptyset$. Now, by applying Proposition 1 and statement (1) of Lemma 2 we have $b_k = 1$ if and only if $a_k = 1$ and $S_k(v) \subseteq S_{i-1}(w)$; if and only if $a_k = 1$ and $v_{[k+1,D-(i-k)+1]} = w_{[i,D]}$. To finish the proof let us demonstrate that we have $a_k = 1$ and $v_{[k+1,D-(i-k)+1]} = w_{[i,D]}$ if and only if $a_k = 1$ and $v_{D-i+k+1} = w_D$. Clearly, we only must show that if $a_k = 1$ and $v_{D-i+k+1} = w_D$, then $v_{[k+1,D-(i-k)+1]} = w_{[i,D]}$. If $i = D$, there is nothing to prove. So let us prove the implication in the case $i < D$. Hence assume $i < D$, $a_k = 1$, and $v_{D-i+k+1} = w_D$. By Proposition 1 and the definition of $S_{k,i}(v)$, if $a_k = 1$, then $S_{k,i}(v) = S_k(v) \subseteq S_i(v)$. Hence, again by statement (1) of Lemma 2 and since $w \in S_1(v)$, we have $v_{[k+1,D-(i-k)]} = v_{[i+1,D]} = w_{[i,D-1]}$. Therefore the equality $v_{[k+1,D-(i-k)+1]} = w_{[i,D]}$ holds, because we are assuming $v_{D-i+k+1} = w_D$. This finishes the proof of the theorem. □

Theorem 2. *Let $v \in V$, $w \in S_1(v)$, and let $1 \leqslant i \leqslant D$. Assume that $S_i^\star(v) \cap S_{i-1}^\star(w) = \emptyset$. Then $S_i^\star(v) = S_i^\star(v) \cap S_i^\star(w)$, and so*

$$\big| S_i^\star(v) \cap S_i^\star(w) \big| = \big| S_i^\star(v) \big| = d^i - a_{i-1} d^{i-1} - \ldots - a_1 d - a_0.$$

Proof. Let $v \in V$ and $w \in S_1(v)$. Let $1 \leqslant i \leqslant D$ and assume that $S_i^\star(v) \cap S_{i-1}^\star(w) = \emptyset$.

If $S_i^\star(v) \cap S_{i-1}^\star(w) = \emptyset$, then, by statement (2) of Propositions 3 and 4, we have $S_i^\star(v) \cap S_i^\star(w) \neq \emptyset$. Hence the unique integer j_0 given in Proposition 2 is $j_0 = i$. Therefore, by statement (1) of this Proposition 2, we have $S_i^\star(v) = S_i^\star(v) \cap S_i^\star(w)$, and so, if $|S_i^\star(v)| = d^i - a_{i-1} d^{i-1} - \ldots - a_1 d - a_0$ is the polynomial expression of $|S_i^\star(v)|$, then $|S_i^\star(v) \cap S_i^\star(w)| = |S_i^\star(v)| = d^i - a_{i-1} d^{i-1} - \ldots - a_1 d - a_0$. □

Theorem 3. *Let $v \in V$, $w \in S_1(v)$, and let $1 \leqslant i \leqslant D$. Assume that $S_i^\star(v) \cap S_j^\star(w) = \emptyset$ for all $i \leqslant j \leqslant D$. Then $S_i^\star(v) = S_i^\star(v) \cap S_{i-1}^\star(w)$ and*

$$\big| S_i^\star(v) \cap S_{i-1}^\star(w) \big| = \big| S_i^\star(v) \big| = d^i - a_{i-1} d^{i-1} - \ldots - a_1 d - a_0,$$

where, in this case, we have $d = 2$ and $a_{i-1} = 1$. Therefore $\big| S_i^\star(v) \cap S_{i-1}^\star(w) \big|$ can be equivalently expressed as

$$\big| S_i^\star(v) \cap S_{i-1}^\star(w) \big| = d^{i-1} - a_{i-2} d^{i-2} \ldots - a_1 d - a_0.$$

Proof. Let $v \in V$ and $w \in S_1(v)$. Let $1 \leqslant i \leqslant D$ and assume that $S_i^\star(v) \cap S_j^\star(w) = \emptyset$ for all $i \leqslant j \leqslant D$.

First of all notice that we must have $d = 2$, because if $d \geqslant 3$, then, by statement (3) of Proposition 3, there exists a unique integer j, $i \leqslant j \leqslant D$, such that the intersection $S_i^\star(v) \cap S_j^\star(w)$ is non-empty, contradicting the assumption that $S_i^\star(v) \cap S_j^\star(w) = \emptyset$ for all $i \leqslant j \leqslant D$.

If $S_i^\star(v) \cap S_j^\star(w) = \emptyset$ for all $i \leqslant j \leqslant D$, then, by statement (2) of Proposition 2, we have $S_i^\star(v) = S_i^\star(v) \cap S_{i-1}^\star(w)$ and so, if $|S_i^\star(v)| = d^i - a_{i-1}d^{i-1} - \ldots - a_1 d - a_0$ is the polynomial expression of $|S_i^\star(v)|$, then

$$\left|S_i^\star(v) \cap S_{i-1}^\star(w)\right| = \left|S_i^\star(v)\right| = d^i - a_{i-1}d^{i-1} - \ldots - a_1 d - a_0. \tag{3}$$

It remains to prove that, in this case, we have $a_{i-1} = 1$. On one hand, by Proposition 1 we know that $a_{i-1} = 1$ if and only if $G = B(d, D)$ and $v_i = v_{i+1} = \cdots = v_D$. On the other hand, by statement (3) of Proposition 4, if $S_i^\star(v) \cap S_j^\star(w) = \emptyset$ for all $i \leqslant j \leqslant D$, then $G = B(d, D)$ and $v_i = v_{i+1} = \cdots = v_D \neq w_D$. Therefore $a_{i-1} = 1$.

Finally notice that, since $d = 2$ and $a_{i-1} = 1$, we have $d^i - d^{i-1} = d^{i-1}$. Then the polynomial expression $d^i - d^{i-1} - a_{i-2}d^{i-2} \ldots - a_1 d - a_0$ in (3) can be equivalently expressed as $d^{i-1} - a_{i-2}d^{i-2} \ldots - a_1 d - a_0$. $\square$

2.3 Application to deflection routing

The authors proposed in [11] an analytical model for evaluating the performance of deflection routing schemes under different deflection criteria. In that model, a Markov chain is defined with states $0, 1, \ldots, D$, corresponding to the possible distances that a packet may be to its destination (D stands for the diameter of the network), and such that the transition probabilities depend on the deflection criteria and the network topology.

In this paper, we determine for the case of $B(d, D)$ and $K(d, D)$ the following two probabilities that appear in the formulation [11]:

- Input probability $\mathbb{P}_{\mathsf{in}}(i)$: Given a vertex v selected uniformly at random, let $\mathbb{P}_{\mathsf{in}}(i)$ be the probability that another distinct vertex v', also selected uniformly at random, be at distance i from v.

- Transition probability $\mathbb{P}_{\mathsf{t}}(i, j)$: Suppose that a packet with destination vertex z is deflected when visiting an intermediate vertex at a distance i to z. We denote by $\mathbb{P}_{\mathsf{t}}(i, j)$ the probability that the new distance to z (after the deflection has occurred) be j.

This subsection applies our results on the distance-layer structure of $B(d, D)$ and $K(d, D)$ to obtain explicit expressions, in terms of the degree d, for these probabilities.

To calculate $\mathbb{P}_{\mathsf{in}}(i)$ and $\mathbb{P}_{\mathsf{t}}(i, j)$ we need to introduce a suitable partition of the vertex set of the digraph, classifying the vertices according to their sequence representation. In this way, we consider in V an equivalence relation $\sim$ defined by $v = v_1 v_2 \ldots v_D \sim v' = v'_1 v'_2 \ldots v'_D$ if and only if there exists a permutation σ of the symbol alphabet A such that $\sigma(v_k) = v'_k$, $1 \leqslant k \leqslant D$. Notice that two equivalent vertices have a sequence representation with the same number s of distinct symbols, where $1 \leqslant s \leqslant \min(d, D)$ if $G = B(d, D)$ and $2 \leqslant s \leqslant \min(d+1, D)$ if $G = K(d, D)$. Let n_s be the number of equivalence classes in which the number of distinct symbols in the sequence representation of the vertices is s (clearly, $n_1 = 1$ if $G = B(d, D)$, and $n_1 = 0$ and $n_2 = 1$ if $G = K(d, D)$). Thus the partition of V induced by the relation $\sim$ can be written as

$$V = \bigcup_s (V_{s,1} \cup \cdots \cup V_{s,n_s}). \tag{4}$$

Moreover, since the sequence representation of a vertex in $V_{s,j}$ contains s different symbols, then, independently of j, $1 \leqslant j \leqslant n_s$, we have that

$$|V_{s,j}| = \begin{cases} d(d-1)\cdots(d-s+1) & \text{if } G = B(d, D), \\ (d+1)d(d-1)\cdots(d-s+2) & \text{if } G = K(d, D). \end{cases}$$

Furthermore, since $|V| = \sum_s \sum_j |V_{s,j}|$, we get

$$\begin{cases} \displaystyle\sum_{s=1}^{\min(d,D)} n_s\, d(d-1)\cdots(d-s+1) = d^D, & \text{if } G = B(d, D); \\ \displaystyle\sum_{s=2}^{\min(d+1,D)} n_s\, (d+1)d(d-1)\cdots(d-s+2) = d^D + d^{D-1}, & \text{if } G = K(d, D). \end{cases}$$

Evaluating the above identities for $d = 1, 2, 3, \ldots$, the values of n_s can be recursively computed. For instance, if $G = B(d, D)$, then the first non-zero values of n_s are $n_1 = 1$, $n_2 = 2^{D-1} - 1$ for all $D \geqslant 1$ and all $d \geqslant 2$, $n_3 = (3^{D-1} - 2^D + 1)/2$ for all $D \geqslant 1$ and all $d \geqslant 3$, …; and if $G = K(d, D)$, then $n_2 = 1$, $n_3 = 2^{D-2} - 1$ for all $D \geqslant 2$ and all $d \geqslant 2$, $n_4 = (3^{D-2} - 2^{D-1} + 1)/2$ for all $D \geqslant 2$ and all $d \geqslant 3$, ….

The total number of classes in the partition (4) is $l = \sum_s n_s$. Therefore, if $D \leqslant d$, then $l = n_1 + \cdots + n_D$ is independent of the degree d. However, if $D > d$ this is not true. For example, if $G = B(2, D)$, then $l = n_1 + n_2 = 2^{D-1}$, whereas if $G = (3, D)$, then $l = n_1 + n_2 + n_3 = (3^{D-1} + 1)/2$.

From now on, for simplicity, we will also denote the partition (4) as

$$V = \mathcal{V}_1 \cup \cdots \cup \mathcal{V}_l, \tag{5}$$

where each term $\mathcal{V}_i$ in (5) corresponds to one of the sets $V_{s,j}$ in (4), and we will use both (4) and (5) as illustrated in the following example.

Example 3. Let $G = K(d, 4)$, $d \geqslant 3$, and let α, β, γ and δ stand for different elements of the base alphabet A. The numbers n_s of equivalence classes for which the the sequence representation of its vertices contains s distinct symbols is $n_2 = 1$, $n_3 = 2^{D-2} - 1 = 3$, and $n_4 = (3^{D-2} - 2^{D-1} + 1)/2 = 1$. So we have $l = 5$ different vertex classes, namely $\mathcal{V}_1 = V_{2,1} = \{v \in V : \ v = \alpha\beta\alpha\beta\}$, $\mathcal{V}_2 = V_{3,1} = \{v \in V : \ v = \alpha\beta\alpha\gamma\}$, $\mathcal{V}_3 = V_{3,2} = \{v \in V : \ v = \alpha\beta\gamma\alpha\}$, $\mathcal{V}_4 = V_{3,3} = \{v \in V : \ v = \alpha\beta\gamma\beta\}$, and $\mathcal{V}_5 = V_{4,1} = \{v \in V : \ v = \alpha\beta\gamma\delta\}$, with respective cardinalities $|\mathcal{V}_1| = (d+1)d$, $|\mathcal{V}_2| = |\mathcal{V}_3| = |\mathcal{V}_4| = (d+1)d(d-1)$, and $|\mathcal{V}_5| = (d+1)d(d-1)(d-2)$.

Now we introduce some additional technical lemmas that we need in this section and that will be proven in the Appendix.

Lemma 6. *Let $v \in V$ and, for $0 \leqslant i \leqslant j < D$, let $\Gamma^\star_{i,j}(v) = \{w \in S_1(v) : \ S^\star_i(v) \cap S^\star_j(w) \neq \emptyset\}$. Then $\Gamma^\star_{i,j}(v) \neq \emptyset$ if and only if there exists a vertex w such that $\Gamma^\star_{i,j}(v) = \{w\}$; if and only if there exists a vertex w such that $\Gamma^\star_{i,j}(v) = \Gamma^+_{i,j}(v) = \{w\}$. Moreover, $w \in \Gamma^\star_{i,j}(v)$ if and only if $w \in \Gamma^+_{i,j}(v)$ and $S_i\,(v) \not\subseteq S_{t,j}\,(w)$ for $i \leqslant t < j$.*

The following lemma discusses the set of vertices $w \in S_1(v)$ for which the intersection set $S_i(v) \cap S_D(w)$ is nonempty. (The case $S_i(v) \cap S_j(w) \neq \emptyset$, for $j < D$, was considered in Lemma 4.)

Lemma 7. *Let $v \in V$ and, for $0 \leqslant i < D$, let $\Gamma^+_{i,D}(v) = \{w \in S_1(v) : \ S_i(v) \cap S_D(w) \neq \emptyset\}$. The following statements hold:*

1. *If $G = B(d, D)$, or $G = K(d, D)$ and $v_{i+1} = v_D$, then $\Gamma^+_{i,D}(v) = S_1(v)$.*
2. *If $G = K(d, D)$ and $v_{i+1} \neq v_D$, then $w \in \Gamma^+_{i,D}(v)$ if and only if $w \in S_1(v)$ and $v_{i+1} \neq w_D$. Moreover, $|\Gamma^+_{i,D}(v)| = d - 1$.*

The last technical lemma deals with the partition $\{\mathcal{V}_1, \ldots, \mathcal{V}_l\}$ of the vertex set V. Notice that if σ is a permutation of the symbol alphabet and $\sigma(v)$ is the vertex whose sequence representation is $\sigma(v_1)\sigma(v_2)\ldots\sigma(v_D)$, then v and $\sigma(v)$ belong to a same vertex class $\mathcal{V}_r$ (that is to say, the sequence representations of v and $\sigma(v)$ have an equivalent structure). The proof of the lemma is an immediate consequence of the definitions and of the fact that σ is a bijection.

Lemma 8. *Let σ be a permutation of the symbol alphabet and, given $v \in V$, let $\sigma(v) = \sigma(v_1)\sigma(v_2)\ldots\sigma(v_D)$. Then the following statements hold:*

1. *$|S^\star_i(v)| = |S^\star_i(\sigma(v))|$.*
2. *If $w \in S_1(v)$, then $|S^\star_i(v) \cap S^\star_j(w)| = |S^\star_i(\sigma(v)) \cap S^\star_j(\sigma(w))|$.*

At this point, using the partition and the layer structure of the digraph, we present our results on input and transition probabilities.

Expressing the input probability as $\sum_r \mathbb{P}_{\mathsf{in}}(i \mid v \in \mathcal{V}_r)\,\mathbb{P}(v \in \mathcal{V}_r)$ we obtain the following result that provides a description of $\mathbb{P}_{\mathsf{in}}(i)$ in terms of the degree d of the digraph.

Theorem 4. *For any choice of the vertices $v^{(1)}, \ldots, v^{(l)}$, where $v^{(r)} \in \mathcal{V}_r$, the input probability $\mathbb{P}_{\mathsf{in}}(i)$ is given by*

$$\mathbb{P}_{\mathsf{in}}(i) = \sum_{r=1}^{l} \frac{|S^\star_i\left(v^{(r)}\right)|}{(|V|-1)} \cdot \frac{|\mathcal{V}_r|}{|V|},$$

and has the following expression:

$$\mathbb{P}_{\mathsf{in}}(i) = \sum_{r=1}^{l} \frac{|\mathcal{V}_r|}{|V|(|V|-1)} \left(d^i - a^{(r,i)}_{i-1} d^{i-1} - \cdots - a^{(r,i)}_1 d - a^{(r,i)}_0\right),$$

where $a^{(r,i)}_k \in \{0, 1\}$. More precisely, $a^{(r,i)}_k = 1$ if and only if $S_{k,i}\left(v^{(r)}\right) \neq \emptyset$; if and only if $v^{(r)}_{[k+1,D-(i-k)]} = v^{(r)}_{[i+1,D]}$ and $v^{(r)}_{[k+1,D-(j-k)]} \neq v^{(r)}_{[j+1,D]}$ for all j, $k < j < i$.

Proof. Let v be a vertex selected uniformly at random from the vertex set V. By definition, the input probability $\mathbb{P}_{\mathsf{in}}(i)$ is the probability of selecting uniformly at random from $V \setminus \{v\}$ a vertex v' which is at distance i from v. For a fixed v, the probability of selecting such a v' is clearly $\mathbb{P}_{\mathsf{in}}(i \mid v) = |S_i^\star(v)|/(|V|-1)$. Moreover, by Lemma 8, this probability is the same for any vertex $v \in \mathcal{V}_r$ in a same vertex class $\mathcal{V}_r$, $1 \leqslant r \leqslant l$. Moreover, since v is chosen uniformly at random from V, we have $\mathbb{P}(v \in \mathcal{V}_r) = |\mathcal{V}_r|/|V|$. Thus, for any choice of $v^{(r)} \in \mathcal{V}_r$, the input probability $\mathbb{P}_{\mathsf{in}}(i)$ can be expressed as

$$\mathbb{P}_{\mathsf{in}}(i) = \sum_r \mathbb{P}_{\mathsf{in}}\left(i \mid v^{(r)} \in \mathcal{V}_r\right) \mathbb{P}\left(v^{(r)} \in \mathcal{V}_r\right) = \sum_{r=1}^{l} \frac{\left|S_i^\star\left(v^{(r)}\right)\right|}{(|V|-1)} \cdot \frac{|\mathcal{V}_r|}{|V|}.$$

The proof is completed by using the polynomial description of $\left|S_i^\star\left(v^{(r)}\right)\right|$ given in Proposition 1 and Remark 3.

Finally, notice that $\mathbb{P}_{\mathsf{in}}(i) = \Theta\left(1/d^{D-i}\right)$, because $|S_i^\star(v)| = \Theta\left(d^i\right)$ (independently of v) and $|V| = \Theta\left(d^D\right)$. □

The transition probability $\mathbb{P}_{\mathsf{t}}(i,j)$ can also be calculated as $\mathbb{P}_{\mathsf{t}}(i,j) = \sum_r \mathbb{P}_{\mathsf{t}}(i,j \mid v \in \mathcal{V}_r)\,\mathbb{P}(v \in \mathcal{V}_r)$. In this sum, $\mathbb{P}_{\mathsf{t}}(i,j \mid v \in \mathcal{V}_r)$ is the conditional probability that the new distance to destination be j, given that a deflection occurs when visiting a vertex v at distance i to the destination and belonging to the class $\mathcal{V}_r$; whereas $\mathbb{P}(v \in \mathcal{V}_r)$ is the probability that the vertex at which deflection occurs be in $\mathcal{V}_r$. In this way, we obtain the following result.

Theorem 5. *The transition probabilities $\mathbb{P}_{\mathsf{t}}(i,j)$, $1 \leqslant i \leqslant j < D$, are given by*

$$\mathbb{P}_{\mathsf{t}}(i,j) = \frac{1}{(d-1)|V|} \sum_r |\mathcal{V}_r|\, p^{(r,i,j)} \left(1 - q^{(r,i)}\right),$$

where $p^{(r,i,j)}$ and $q^{(r,i)}$ are of the form

$$p^{(r,i,j)} = k^{(r,i,j)} \cdot \frac{d^i - \alpha_{i-1}^{(r,i)} d^{i-1} - \cdots - \alpha_1^{(r,i)} d - \alpha_0^{(r,i)}}{d^i - a_{i-1}^{(r,i)} d^{i-1} - \cdots - a_1^{(r,i)} d - a_0^{(r,i)}}$$

and

$$q^{(r,i)} = \kappa^{(r,i)} \cdot \frac{d^{i-1} - b_{i-2}^{(r,i)} d^{i-2} - \ldots - b_1^{(r,i)} d - b_0^{(r,i)}}{d^i - a_{i-1}^{(r,i)} d^{i-1} - \cdots - a_1^{(r,i)} d - a_0^{(r,i)}},$$

and the coefficients of these fractions are 0, 1 *or* 2*. Namely, $k^{(r,i,j)}, \kappa^{(r,i)} \in \{0,1\}$; $\alpha_{i-1}^{(r,i)} \in \{0,1,2\}$; $a_{i-1}^{(r,i)} \in \{0,1\}$; and $\alpha_l^{(r,i)}, a_l^{(r,i)}, b_l^{(r,i)} \in \{0,1\}$ for $0 \leqslant l \leqslant i-2$.*

Remark 8. In the proof of this theorem it will be shown how to determine the coefficients $k^{(r,i,j)}$, $\kappa^{(r,i)}$, $a_k^{(r,i)}$, $b_k^{(r,i)}$, and $\alpha_k^{(r,i)}$. More precisely, we will show that if $v^{(r)}$ is any representative vertex in the class $\mathcal{V}_r$, and if $w^{(r)}$ is the vertex adjacent from $v^{(r)}$ given by $w^{(r)} = v_2^{(r)} \cdots v_D^{(r)} v_{i+(D-j)}^{(r)}$, then

(a) $k^{(r,i,j)} = 1$ if and only if $S_i^\star\left(v^{(r)}\right) \cap S_j^\star\left(w^{(r)}\right) \neq \emptyset$, as determined by statement (1) of Propositions 3 and 4;

(b) $\kappa^{(r,i)} = 1$ if and only if $S_i^\star\left(v^{(r)}\right) \cap S_{i-1}^\star\left(w^{(r)}\right) \neq \emptyset$, as determined by statement (2) of Propositions 3 and 4;

(c) the coefficients $a_k^{(r,i)} \in \{0,1\}$ are determined from $v^{(r)}$ as in Proposition 1;

(d) the coefficients $\alpha_k^{(r,i)}$ and $b_k^{(r,i)}$ are determined from $v^{(r)}$ and $w^{(r)}$ as in Theorems 1 and 2.

We stress that the values of all these coefficients are independent of the choice of $v^{(r)}$ in the class $\mathcal{V}_r$.

Proof. Let v be the vertex at which deflection occurs and suppose that the destination vertex z is at distance i from v. Let $w \in S_1(v)$ be the vertex through which deflection takes place. In other words, we are supposing that a packet circulating within the network (which has to arrive to z) is currently in v and cannot proceed through the shortest path from v to z; and hence it is deflected to vertex w.

Hence the probability that the new distance from w to the destination vertex z is j, given that a deflection occurs in v and that the deflection takes place through w, is just the probability that, conditional on the event $z \in S_i^\star(v)$, the destination vertex z belongs to $S_i^\star(v) \cap S_j^\star(w)$. In this way, denoting this conditional probability as $\mathbb{P}_{\mathsf{t}}(i, j \mid v, w)$, we have

$$\mathbb{P}_{\mathsf{t}}(i, j \mid v, w) = \frac{\left|S_i^\star(v) \cap S_j^\star(w)\right|}{\left|S_i^\star(v)\right|}.$$

It follows that $\mathbb{P}_{\mathsf{t}}(i, j \mid v, w) \neq 0$ if and only if $S_i^\star(v) \cap S_j^\star(w) \neq \emptyset$.

Let $w'_{v,z}$ be the vertex adjacent from v in the unique shortest path from v to z. The vertex w through which deflection takes place is selected uniformly at random from $S_1(v) \setminus \{w'_{v,z}\}$. Hence the probability $\mathbb{P}(w \mid v)$ that, given that deflection occurs, it takes place through $w \in S_1(v)$ can be calculated as

$$\mathbb{P}(w \mid v) = \sum_{z \in S_i^\star(v)} \mathbb{P}(w \mid v, z)\mathbb{P}(z \mid v),$$

where $\mathbb{P}(w \mid v, z) = 0$ if $w = w'_{v,z}$ and $\mathbb{P}(w \mid v, z) = 1/(d-1)$ if $w \neq w'_{v,z}$. Moreover, $w = w'_{v,z}$ if and only if $z \in S_i^\star(v) \cap S_{i-1}^\star(w)$. Therefore, $\mathbb{P}(w \mid v, z) = 0$ if and only if $z \in S_i^\star(v) \cap S_{i-1}^\star(w)$. Furthermore, the probability that the destination vertex is a given vertex z belonging to $S_i^\star(v)$ is simply

$$\mathbb{P}(z \mid v) = \frac{1}{|S_i^\star(v)|}.$$

Then, since $|S_i^\star(v)| - |S_i^\star(v) \cap S_{i-1}^\star(w)|$ is the number of vertices $z \in S_i^\star(v)$ for which $w \neq w'_{v,z}$, we have

$$\begin{aligned}
\mathbb{P}(w \mid v) &= \frac{1}{|S_i^\star(v)|} \sum_{z \in S_i^\star(v)} \mathbb{P}(w \mid v, z) \\
&= \frac{1}{|S_i^\star(v)|} \frac{|S_i^\star(v)| - |S_i^\star(v) \cap S_{i-1}^\star(w)|}{d-1} = \frac{1}{d-1}\left(1 - \frac{|S_i^\star(v) \cap S_{i-1}^\star(w)|}{|S_i^\star(v)|}\right).
\end{aligned}$$

Let $\Gamma_{i,j}^\star(v) = \{w \in S_1(v) : S_i^\star(v) \cap S_j^\star(w) \neq \emptyset\}$. Clearly, we have $\mathbb{P}_{\mathsf{t}}(i, j \mid v, w) \neq 0$ if and only if $w \in \Gamma_{i,j}^\star(v)$. Moreover, it is proved in Lemmas 4 and 6 that if $v = v_1 v_2 \cdots v_D$ and $\Gamma_{i,j}^\star(v) \neq \emptyset$, then $\Gamma_{i,j}^\star(v)$ contains a single vertex w_v which sequence representation is uniquely determined from v, i and j, namely

$$w_v = v_2 \cdots v_D v_{i+(D-j)}. \tag{6}$$

Taking all these considerations into account we can express the transition probability that the new distance to the destination is j, conditional on the event that deflection occurs at v, as

$$\begin{aligned}
\mathbb{P}_{\mathsf{t}}(i, j \mid v) &= \sum_{w \in \Gamma_{i,j}^\star(v)} \mathbb{P}_{\mathsf{t}}(i, j \mid v, w) \cdot \mathbb{P}(w \mid v) = \mathbb{P}_{\mathsf{t}}(i, j \mid v, w_v)\, \mathbb{P}(w_v \mid v) \\
&= \frac{1}{d-1} \cdot \frac{\left|S_i^\star(v) \cap S_j^\star(w_v)\right|}{|S_i^\star(v)|} \left(1 - \frac{|S_i^\star(v) \cap S_{i-1}^\star(w_v)|}{|S_i^\star(v)|}\right),
\end{aligned} \tag{7}$$

if $\Gamma_{i,j}^\star(v) \neq \emptyset$; and $\mathbb{P}_{\mathsf{t}}(i, j \mid v) = 0$ otherwise.

Furthermore, if σ is a permutation of the symbol alphabet A, then, using the notation introduced in Lemma 8, we can check that $\Gamma_{i,j}^\star(\sigma(v)) \neq \emptyset$ if and only if $\Gamma_{i,j}^\star(v) \neq \emptyset$, and that if $\Gamma_{i,j}^\star(\sigma(v)) \neq \emptyset$, then $\Gamma_{i,j}^\star(\sigma(v)) = \{\sigma(w_v)\}$. Moreover, $\sigma(w'_{v,z}) = w'_{\sigma(v),\sigma(z)}$ is the vertex adjacent from $\sigma(v)$ in the shortest path to $\sigma(z)$. This facts, together with the statements of Lemma 8, imply that the probability calculated in (7) is the same for any vertex $v^{(r)}$ in a given vertex class $\mathcal{V}_r$. (Recall that $\mathcal{V}_r$ is the class of vertices to which $v^{(r)}$ belongs according to the structure of its sequence representation.)

Now, by adding for all the classes $\mathcal{V}_r$ and taking into account that $\mathbb{P}\left(v^{(r)} \in \mathcal{V}_r\right) = |\mathcal{V}_r|/|V|$ we obtain the transition probability $\mathbb{P}_{\mathsf{t}}(i, j)$ that, conditional on the event that the deflection occurs in a vertex which is at distance i to the destination vertex, the new distance to this destination is j. In this way, by setting $w^{(r)} = w_{v^{(r)}}$ we have

$$\begin{aligned}
\mathbb{P}_{\mathsf{t}}(i, j) &= \sum_r \mathbb{P}_{\mathsf{t}}\left(i, j \mid v^{(r)} \in \mathcal{V}_r\right) \mathbb{P}\left(v^{(r)} \in \mathcal{V}_r\right) \\
&= \frac{1}{(d-1)|V|} \sum_r |\mathcal{V}_r| \frac{\left|S_i^\star\left(v^{(r)}\right) \cap S_j^\star\left(w^{(r)}\right)\right|}{\left|S_i^\star\left(v^{(r)}\right)\right|} \left(1 - \frac{|S_i^\star\left(v^{(r)}\right) \cap S_{i-1}^\star\left(w^{(r)}\right)|}{|S_i^\star\left(v^{(r)}\right)|}\right).
\end{aligned}$$

Furthermore, if the intersection $S_i^\star\left(v^{(r)}\right) \cap S_j^\star\left(w^{(r)}\right)$ is nonempty, we conclude from Theorems 1 and 2 that $\left|S_i^\star\left(v^{(r)}\right) \cap S_j^\star\left(w^{(r)}\right)\right|$ has a polynomial expression given by $d^i - \alpha_{i-1}^{(r,i)} d^{i-1} - \cdots - \alpha_1^{(r,i)} d - \alpha_0^{(r,i)}$. Moreover, by Theorem 1, if $S_i^\star\left(v^{(r)}\right) \cap S_{i-1}^\star\left(w^{(r)}\right) \neq \emptyset$, then $\left|S_i^\star\left(v^{(r)}\right) \cap S_{i-1}^\star\left(w^{(r)}\right)\right|$ has also a polynomial expression of the form $d^{i-1} - b_{i-2}^{(r,i)} d^{i-2} - \ldots - b_1^{(r,i)} d - b_0^{(r,i)}$. Therefore, by taking also into account Proposition 1, we have

$$\mathbb{P}_{\mathsf{t}}(i,j) = \frac{1}{(d-1)|V|} \sum_r |\mathcal{V}_r|\, p^{(r,i,j)} \left(1 - q^{(r,i)}\right),$$

where $p^{(r,i,j)}$ and $q^{(r,i)}$ are of the form

$$p^{(r,i,j)} = k^{(r,i,j)} \cdot \frac{d^i - \alpha_{i-1}^{(r,i)} d^{i-1} - \cdots - \alpha_1^{(r,i)} d - \alpha_0^{(r,i)}}{d^i - a_{i-1}^{(r,i)} d^{i-1} - \cdots - a_1^{(r,i)} d - a_0^{(r,i)}}$$

and

$$q^{(r,i)} = \kappa^{(r,i)} \cdot \frac{d^{i-1} - b_{i-2}^{(r,i)} d^{i-2} - \ldots - b_1^{(r,i)} d - b_0^{(r,i)}}{d^i - a_{i-1}^{(r,i)} d^{i-1} - \cdots - a_1^{(r,i)} d - a_0^{(r,i)}},$$

and $k^{(r,i,j)}, \kappa^{(r,i)} \in \{0,1\}$. Furthermore, we have $k^{(r,i,j)} = 1$ if and only if $S_i^\star\left(v^{(r)}\right) \cap S_j^\star\left(w^{(r)}\right) \neq \emptyset$, as determined by statement (1) of Propositions 3 and 4; and we have $\kappa^{(r,i)} = 1$ if and only if $S_i^\star\left(v^{(r)}\right) \cap S_{i-1}^\star\left(w^{(r)}\right) \neq \emptyset$, as determined by statement (2) of Propositions 3 and 4.

Finally, observe that the coefficients $a_k^{(r,i)}$ are determined from $v^{(r)}$ by using Proposition 1, and the coefficients $\alpha_k^{(r,i)}, b_k^{(r,i)} \in \{0,1\}$ are determined from $v^{(r)}$ and $w^{(r)}$ by using Theorems 1 and 2. So we conclude that:

1. $a_k^{(r,i)} \in \{0,1\}$;
2. $\alpha_{i-1}^{(r,i)} \in \{0,1,2\}$, and $\alpha_k^{(r,i)}, b_k^{(r,i)} \in \{0,1\}$ for $0 \leqslant k \leqslant i-2$.

This completes the proof of the theorem. □

The following example illustrates the fractions $p^{(r,i,j)}$ and $q^{(r,i)}$, as well as the expressions of the transition probabilities $\mathbb{P}_{\mathsf{t}}(i,j)$ formulated in Theorem 5.

Example 4. Let $G = K(d,12)$, $d \geqslant 3$, and consider the class of vertices $\mathcal{V}_r$ which sequence representation is of the form $\alpha\beta\gamma\alpha\beta\gamma\alpha\beta\gamma\alpha\beta\gamma$, where α, β and γ stand for different symbols of the alphabet A. Suppose that $v \in \mathcal{V}_r$ is the vertex at which the deflection occurs and let $w = \beta\gamma\alpha\beta\gamma\alpha\beta\gamma\alpha\beta\gamma w_{12}$ be the vertex adjacent from v through which this deflection takes place. Let us calculate, for instance, the transition probabilities $\mathbb{P}_{\mathsf{t}}(4,6 \mid v \in \mathcal{V}_r)$ and $\mathbb{P}_{\mathsf{t}}(1,6 \mid v \in \mathcal{V}_r)$.

Firstly, let us consider $\mathbb{P}_{\mathsf{t}}(4,6 \mid v \in \mathcal{V}_r)$. Observe that if this probability is not zero, then the destination vertex z must belong to $S_4^\star(v)$ and also to $S_6^\star(w)$. Therefore, since $S_4^\star(v) \subseteq S_4(v)$ and $S_6^\star(v) \subseteq S_6(v)$, we conclude that $S_4(v) \cap S_6(w) \neq \emptyset$ is a necessary condition for $\mathbb{P}_{\mathsf{t}}(4,6 \mid v \in \mathcal{V}_r) \neq 0$. Using the notation of Examples 1 and 2, we have $S_4(v) = \{u \in V : u = \beta\gamma\alpha\beta\gamma\alpha\beta\gamma * * * *\}$ and $S_6(w) = \{u \in V : u = \beta\gamma\alpha\beta\gamma w_{12} * * * * * *\}$. From these sequence representations we can check that $S_4(v) \cap S_6(w) \neq \emptyset$ if and only if $S_4(v) \subseteq S_6(w)$; if and only if $w_{12} = \alpha$. We conclude that if $\mathbb{P}_{\mathsf{t}}(4,6 \mid v \in \mathcal{V}_r) \neq 0$, then there is only one precise vertex w adjacent from v such that $d(w,z) = 6$, namely $w = \beta\gamma\alpha\beta\gamma\alpha\beta\gamma\alpha\beta\gamma\alpha$.

Using Propositions 1 and 3, and Theorem 1 we deduce that $|S_4^\star(v)| = d^4 - d$, $|S_4^\star(v) \cap S_6^\star(w)| = d^4 - d^3$, and $|S_4^\star(v) \cap S_3^\star(w)| = d^3 - d$. Therefore we get from (7) that the value of the transition probability $\mathbb{P}_{\mathsf{t}}(4,6 \mid v \in \mathcal{V}_r)$ is

$$\begin{aligned}
\mathbb{P}_{\mathsf{t}}(4,6 \mid v \in \mathcal{V}_r) &= \frac{1}{d-1} \cdot \frac{|S_4^\star(v) \cap S_6^\star(w)|}{|S_4^\star(v)|} \left(1 - \frac{|S_4^\star(v) \cap S_3^\star(w)|}{|S_4^\star(v)|}\right) \\
&= \frac{1}{d-1} \cdot \frac{d^4 - d^3}{d^4 - d} \left(1 - \frac{d^3 - d}{d^4 - d}\right) = \frac{d^4}{d^5 + d^4 + d^3 - d^2 - d - 1}. \qquad (8)
\end{aligned}$$

Observe that in (8), the expressions of $|S_4^\star(v) \cap S_6^\star(w)|/|S_4^\star(v)|$ and $|S_4^\star(v) \cap S_3^\star(w)|/|S_4^\star(v)|$ correspond in Theorem 5 to $p^{(r,4,6)} = (d^4 - d^3)/(d^4 - d)$ and $q^{(r,4)} = (d^3 - d)/(d^4 - d)$, respectively.

Secondly, let us determine $\mathbb{P}_{\mathsf{t}}(1,6 \mid v \in \mathcal{V}_r)$. Reasoning as before we deduce that if $\mathbb{P}_{\mathsf{t}}(1,6 \mid v \in \mathcal{V}_r) \neq 0$, then we must have $S_1(v) \cap S_6(w) \neq \emptyset$, being w the vertex adjacent from v through which the deflection takes place. We can check that this necessary condition holds if and only if $w_{12} = \alpha$; that is, w must be again the vertex $w = \beta\gamma\alpha\beta\gamma\alpha\beta\gamma\alpha\beta\gamma\alpha$. But now we deduce from Proposition 3 that $S_1^\star(v) \cap S_6^\star(w) = \emptyset$. Therefore, although the necessary condition $S_1(v) \cap S_6(w) \neq \emptyset$ for having $\mathbb{P}_{\mathsf{t}}(1,6 \mid v) \neq 0$ holds, we have in this case $\mathbb{P}_{\mathsf{t}}(1,6 \mid v) = 0$. This fact is captured in Theorem 5 by setting $k^{(r,1,6)} = 0$.

The previous theorem provides a description of the probabilities $\mathbb{P}_{\mathsf{t}}(i,j)$ in the case $j < D$. Next we discuss the case $j = D$. Clearly we have $\mathbb{P}_{\mathsf{t}}(D,D) = 1$, because D is the maximum possible distance between the vertices of the digraph. Moreover, the transition probabilities $\mathbb{P}_{\mathsf{t}}(i,D)$, $1 \leqslant i < D$, can be obtained from Theorem 5 because, for each i, we have $\mathbb{P}_{\mathsf{t}}(i,D) = 1 - \sum_{j=i}^{D-1} \mathbb{P}_{\mathsf{t}}(i,j)$. However, for the sake of completeness, we present in the following theorem a description of the transition probabilities $\mathbb{P}_{\mathsf{t}}(i,D)$, analogous to those provided in Theorem 5 for $\mathbb{P}_{\mathsf{t}}(i,j)$.

Theorem 6. *The transition probabilities $\mathbb{P}_{\mathsf{t}}(i,D)$, $1 \leqslant i < D$, are given by*

$$\mathbb{P}_{\mathsf{t}}(i,D) = \frac{1}{(d-1)|V|} \sum_r \sum_{s=1}^{m_r} |\mathcal{V}_r| \, p^{(r,s,i)} \left(1 - q^{(r,s,i)}\right),$$

where $m_r \in \{d-1, d\}$ if $G = K(d,D)$ and $m_r = d$ if $G = B(d,D)$, and where $p^{(r,s,i)}$ and $q^{(r,s,i)}$ are of the form

$$p^{(r,s,i)} = k^{(r,s,i)} \cdot \frac{d^i - \alpha_{i-1}^{(r,s,i)} d^{i-1} - \cdots - \alpha_1^{(r,s,i)} d - \alpha_0^{(r,s,i)}}{d^i - a_{i-1}^{(r,s,i)} d^{i-1} - \cdots - a_1^{(r,s,i)} d - a_0^{(r,s,i)}}$$

and

$$q^{(r,s,i)} = \kappa^{(r,s,i)} \cdot \frac{d^{i-1} - b_{i-2}^{(r,s,i)} d^{i-2} - \ldots - b_1^{(r,s,i)} d - b_0^{(r,s,i)}}{d^i - a_{i-1}^{(r,s,i)} d^{i-1} - \cdots - a_1^{(r,s,i)} d - a_0^{(r,s,i)}},$$

where all the coefficients are 0, 1 or 2. Namely, $k^{(r,s,i)}$, $\kappa^{(r,s,i)} \in \{0,1\}$; $\alpha_{i-1}^{(r,s,i)} \in \{0,1,2\}$; $a_{i-1}^{(r,s,i)} \in \{0,1\}$; and $\alpha_l^{(r,s,i)}, a_l^{(r,s,i)}, b_l^{(r,s,i)} \in \{0,1\}$ for $0 \leqslant l \leqslant i-2$.

Proof. We use the same notation and an analysis similar to that in the proof of Theorem 5. The probability that the new distance from w to the destination vertex is D, given that a deflection occurs in v (which is at distance i to the destination) and that the deflection takes place through $w \in S_1(v)$ is

$$\mathbb{P}_{\mathsf{t}}\left(i, D \mid v, w\right) = \frac{|S_i^\star(v) \cap S_D^\star(w)|}{|S_i^\star(v)|}.$$

Let $\Gamma_{i,D}^+(v) = \{w \in S_1(v) : S_i(v) \cap S_D(w) \neq \emptyset\}$ be the set defined in Lemma 7. Clearly, if $w \in S_1(v) \setminus \Gamma_{i,D}^+(v)$, then for such a vertex w we have $\mathbb{P}_{\mathsf{t}}(i, D \mid v, w) = 0$. In Lemma 7 it is proved that $\Gamma_{i,D}^+(v)$ is always nonempty. Indeed, if $G = B(d,D)$, or $G = K(d,D)$ and $v_{i+1} = v_D$, then $\Gamma_{i,D}^+(v) = S_1(v)$; whereas if $G = K(d,D)$ and $v_{i+1} \neq v_D$, then $\Gamma_{i,D}^+(v) = \{w \in S_1(v) : w = v_2 \cdots v_D w_D,\ w_D \neq v_{i+1}, v_D\}$, and hence $|\Gamma_{i,D}^+(v)| = d-1$. Therefore, the transition probability that the new distance to the destination is D, given the event that deflection occurs at v, can be expressed as in (7); that is,

$$\begin{aligned}
\mathbb{P}_{\mathsf{t}}\left(i, D \mid v\right) &= \sum_{w \in \Gamma_{i,D}^+(v)} \mathbb{P}_{\mathsf{t}}\left(i, D \mid v, w\right) \cdot \mathbb{P}\left(w \mid v\right) \\
&= \frac{1}{d-1} \sum_{s=1}^{m} \frac{|S_i^\star(v) \cap S_D^\star(w_{v,s})|}{|S_i^\star(v)|} \left(1 - \frac{|S_i^\star(v) \cap S_{i-1}^\star(w_{v,s})|}{|S_i^\star(v)|}\right),
\end{aligned} \tag{9}$$

where $m = d-1$ if $G = K(d,D)$ and $v_{i+1} \neq v_D$ and $m = d$ otherwise; and $w_{v,s}$, $1 \leqslant s \leqslant m$, are the vertices belonging to $\Gamma_{i,D}^+(v)$.

Furthermore, if σ is a permutation of the symbol alphabet A, then the elements of $\Gamma_{i,j}^+(\sigma(v))$ are $w_{\sigma(v),s} = \sigma(w_{v,s})$, $1 \leqslant s \leqslant m$. Hence, by taking into account Lemma 8, we conclude that the probability (9) is the same for any vertex $v^{(r)}$ in a given vertex class $\mathcal{V}_r$. By adding for all the classes $\mathcal{V}_r$ and taking into account that $\mathbb{P}\left(v^{(r)} \in \mathcal{V}_r\right) = |\mathcal{V}_r|/|V|$ we obtain the transition probability $\mathbb{P}_{\mathsf{t}}(i,D)$ that, conditional on the event that the deflection occurs in a vertex which is at distance i to the destination vertex, the new distance to this destination is D. In this way, by setting $w^{(r,s)} = w_{v^{(r)},s}$, $1 \leqslant s \leqslant m$, we have

$$\begin{aligned}
\mathbb{P}_{\mathsf{t}}(i,D) &= \sum_r \mathbb{P}_{\mathsf{t}}\left(i, D \mid v^{(r)} \in \mathcal{V}_r\right) \mathbb{P}\left(v^{(r)} \in \mathcal{V}_r\right) \\
&= \frac{1}{(d-1)|V|} \sum_r \sum_{s=1}^{m} |\mathcal{V}_r| \frac{\left|S_i^\star\left(v^{(r)}\right) \cap S_D^\star\left(w^{(r,s)}\right)\right|}{\left|S_i^\star\left(v^{(r)}\right)\right|} \cdot \left(1 - \frac{\left|S_i^\star\left(v^{(r)}\right) \cap S_{i-1}^\star\left(w^{(r,s)}\right)\right|}{\left|S_i^\star\left(v^{(r)}\right)\right|}\right),
\end{aligned}$$

Moreover, by considering the polynomial expressions of $\left|S_i^\star\left(v^{(r)}\right)\cap S_D^\star\left(w^{(r,s)}\right)\right|$ and $\left|S_i^\star\left(v^{(r)}\right)\cap S_{i-1}^\star\left(w^{(r,s)}\right)\right|$, we have

$$\mathbb{P}_{\mathsf{t}}(i,D)=\frac{1}{(d-1)|V|}\sum_r\sum_{s=1}^{m_r}|\mathcal{V}_r|\,p^{(r,s,i)}\left(1-q^{(r,s,i)}\right),$$

where $m_r\in\{d-1,d\}$ if $G=K(d,D)$ and $m_r=d$ if $G=B(d,D)$, and where $p^{(r,s,i)}$ and $q^{(r,s,i)}$ are expressed as

$$p^{(r,s,i)}=k^{(r,s,i)}\cdot\frac{d^i-\alpha_{i-1}^{(r,s,i)}d^{i-1}-\cdots-\alpha_1^{(r,s,i)}d-\alpha_0^{(r,s,i)}}{d^i-a_{i-1}^{(r,s,i)}d^{i-1}-\cdots-a_1^{(r,s,i)}d-a_0^{(r,s,i)}}$$

and

$$q^{(r,s,i)}=\kappa^{(r,s,i)}\cdot\frac{d^{i-1}-b_{i-2}^{(r,s,i)}d^{i-2}-\ldots-b_1^{(r,s,i)}d-b_0^{(r,s,i)}}{d^i-a_{i-1}^{(r,s,i)}d^{i-1}-\cdots-a_1^{(r,s,i)}d-a_0^{(r,s,i)}},$$

and $k^{(r,s,i)},\kappa^{(r,s,i)}\in\{0,1\}$. More precisely, $k^{(r,s,i)}=1$ if and only if $S_i^\star\left(v^{(r)}\right)\cap S_D^\star\left(w^{(r,s)}\right)\neq\emptyset$, as determined by statement (1) of Propositions 3 and 4, and $\kappa^{(r,s,i)}=1$ if and only if $S_i^\star\left(v^{(r)}\right)\cap S_{i-1}^\star\left(w^{(r,s)}\right)\neq\emptyset$, as determined by statement (2) of Propositions 3 and 4.

As in Theorem 5, the coefficients $a_k^{(r,s,i)}$, $\alpha_k^{(r,s,i)}$, $b_k^{(r,s,i)}$ are determined from $v^{(r)}$ and $w^{(r,s)}$ by using Proposition 1 and Theorems 1 and 2. Furthermore, $a_k^{(r,s,i)}\in\{0,1\}$, $\alpha_{i-1}^{(r,s,i)}\in\{0,1,2\}$, and $\alpha_k^{(r,s,i)},b_k^{(r,s,i)}\in\{0,1\}$ for $0\leqslant k\leqslant i-2$. □

Using the Markov model [11] mentioned in Section 1, we can apply the probabilities given in Theorems 4, 5, and 6 to measure the efficiency of deflection routing in De Bruijn and Kautz networks.

We conclude this subsection with two corollaries that are straightforward consequences of our previous results. The first one deals with the asymptotic behaviour of the input and transition probabilities. The second one is about the computation of the mean distance in the De Bruijn and Kautz digraphs (some related results can be found in [3, 26, 27]).

Corollary 1.

1. *$\mathbb{P}_{\mathsf{in}}(i)\sim 1/d^{D-i}$ as $d\to\infty$.*
2. *If $j<D$ and d is large enough, then $\mathbb{P}_{\mathsf{t}}(i,j)\leqslant 1/d$.*
3. *If $j=D$, then $\mathbb{P}_{\mathsf{t}}(i,j)\sim 1$ as $d\to\infty$.*

Corollary 2. *If G is the De Bruijn digraph $B(d,D)$ or the Kautz digraph $K(d,D)$, then the mean distance of G is given by $\sum_{i=1}^{D} i\cdot\mathbb{P}_{\mathsf{in}}(i)$.*

3 Final remarks

The digraphs $B(d,D)$ and $K(d,D)$ are fundamental examples of digraphs on alphabets [16] as well as iterated line digraphs [8, 15]. Indeed, in the line digraph $L(G_0)$ of a digraph G_0 each vertex represents an arc (x,y) of G_0; and a vertex (x,y) is adjacent to a vertex (z,t) if and only if $y=z$. For any $k>1$, the k-iterated line digraph, $L^k(G_0)$, is defined recursively by $L^k(G_0)=L(L^{k-1}(G_0))$ (see for instance [15]). In particular, if G_0 is the complete symmetric digraph on d vertices with a loop in each vertex, then $B(d,D)=L^{D-1}(G_0)$; and if G_0 is the complete symmetric digraph on $d+1$ vertices without loops, then $K(d,D)=L^{D-1}(G_0)$. Other used network topologies correspond to iterated line digraphs as, for instance, the generalized De Bruijn cycles [17]. So, we point out that an analysis of the distance-layer structure (and hence the evaluation of the efficiency of deflection routing in the corresponding network topology), similar to the one presented in this paper, could be done in other families of digraphs on alphabets or of iterated line digraphs.

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Appendix

We remind that V denotes the vertex set of the digraph, either $G = B(d, D)$ or $G = K(d, D)$, and that, for $v = v_1 v_2 \cdots v_D \in V$, we denote the subsequence $v_i v_{i+1} \cdots v_j$ by $v_{[i,j]}$.

Proofs of the technical lemmas

Proof of Lemma 1

The result follows directly from the sequence representation of the vertices and the adjacency rules. Moreover, we must consider that in $K(d, D)$ there exists a walk of length $D + 1$ from a given vertex v to any other one. □

Proof of Lemma 2

It follows from Lemma 1 that if $S_i(v') \neq V$, then $i \leqslant D - 1$ if $G = B(d, D)$, $i \leqslant D$ if $G = K(d, D)$, and $S_k(v) \neq V$ for $k \leqslant i$.

Suppose $i < D$. The sequences corresponding to vertices $w \in S_k(v)$ and $w' \in S_i(v')$ are of the form $w = v_{k+1} \cdots v_D * \cdots *$ and $w' = v'_{i+1} \cdots v'_D * \cdots *$, respectively. Since $k \leqslant i$, we get from these sequence representations that $S_k(v) \cap S_i(v') \neq \emptyset$ if and only if

$$v_{k+1} = v'_{i+1}, \ldots, v_{k+(D-i)} = v'_D, \tag{10}$$

that is, the subsequences $v_{[k+1,D-(i-k)]}$ and $v'_{[i+1,D]}$ coincide. Furthermore, $S_k(v) \subseteq S_i(v')$ if and only if condition (10) holds. (Notice that for $k = i$ we have $S_k(v) = S_i(v')$.) Hence if $k \leqslant i < D$, then either $S_k(v) \subseteq S_i(v')$ or $S_k(v) \cap S_i(v') = \emptyset$ and statement (1) is proved.

Now assume that $k < i = D$ and $G = K(d,D)$. Then $S_k(v) = \{w \in V : w = v_{k+1} \cdots v_D * \cdots *\}$ and $S_D(v') = \{w \in V : w = w_1 w_2 \cdots w_D,\ w_1 \neq v'_D\}$. Therefore, if $v_{k+1} \neq v'_D$ then $S_k(v) \subseteq S_D(v')$, whereas if $v_{k+1} = v'_D$ then $S_k(v) \cap S_D(v') = \emptyset$.

Finally assume that $k = i = D$ and $G = K(d,D)$. Then $S_D(v) = \{w \in V : w = w_1 w_2 \cdots w_D,\ w_1 \neq v_D\}$ and $S_D(v') = \{w \in V : w = w_1 w_2 \cdots w_D,\ w_1 \neq v'_D\}$. Hence $S_D(v) \cap S_D(v') \neq \emptyset$ because the alphabet A has $d+1$ symbols and $d \geqslant 2$. Moreover, $S_D(v) \subseteq S_D(v')$ if and only if $S_D(v) = S_D(v')$, if and only if $v_D = v'_D$. If $v_D \neq v'_D$, then $|S_D(v) \cap S_D(v')| = (d-1)d^{D-1} = d^D - d^{D-1}$, because $w_1 \in A \setminus \{v_D, v'_D\}$. $\square$

Proof of Lemma 3

From the definitions it is clear that

$$S_i^\star(v) = S_i(v) \setminus \Big(\bigcup_{k=0}^{i-1} S_k(v) \Big) = S_i(v) \setminus \Big(\bigcup_{k=0}^{i-1} (S_k(v) \cap S_i(v)) \Big).$$

By Lemma 2, either $S_k(v) \cap S_i(v) = \emptyset$ or $S_k(v) \subseteq S_i(v)$. Therefore, $S_i^\star(v) = S_i(v) \setminus \Big(\bigcup_{k=0}^{i-1} S_{k,i}(v) \Big)$. $\square$

Proof of Lemma 4

Let $w \in S_1(v)$. Hence $w_{[1,D-1]} = v_{[2,D]}$. Assume first that $j < D-1$. By statement (1) of Lemma 2, $S_i(v) \cap S_j(w) \neq \emptyset$ if and only if $v_{[i+1,D-(j-i)]} = w_{[j+1,D]}$; that is, $S_i(v) \cap S_j(w) \neq \emptyset$ if and only if $v_{[i+1,D-(j-i)-1]} = v_{[j+2,D]}$ and $v_{D-(j-i)} = w_D$. In particular, if $i = j$, then $v_{i+1} = v_{i+2} = \cdots = v_D = w_D$, and hence $G = B(d,D)$.

Now suppose $j = D-1$. By Lemma 2, $S_i(v) \cap S_{D-1}(w) \neq \emptyset$ if and only if $v_{i+1} = w_D$. Therefore, if $G = B(d,D)$ there is always a vertex $w \in \Gamma^+_{i,j}(v)$, while if $G = K(d,D)$, then there exists $w \in \Gamma^+_{i,j}(v)$ if and only if $v_{i+1} \neq v_D$. In particular, if $G = K(d,D)$ and $\Gamma^+_{i,j}(v) \neq \emptyset$, then $D \neq i+1$.

Until now we have proved statement (1). Next, to prove statement (2) first let us assume that $\Gamma^+_{i,j}(v) \neq \emptyset$. Observe from the above that if $w \in \Gamma^+_{i,j}(v)$, then w_D is uniquely determined and it is equal to $v_{i+(D-j)}$ both for $j < D-1$ as for $j = D-1$. Hence $\Gamma^+_{i,j}(v)$ has a unique element w which sequence representation is $w = v_2 \cdots v_D v_{i+(D-j)}$. It is clear from the previous statements that if $\Gamma^+_{i,j}(v) \neq \emptyset$ and $i = j$, then G cannot be $K(d,D)$. Since $w \in S_1(v)$ we have $S_l(w) \subseteq S_{l+1}(v)$ for all $l \geqslant 0$. Hence to finish the proof of statement (2) we only need to show that if $i = j$, then $S_k(v) = S_k(w)$ for all $k \geqslant i$. But this is clear because, from the above discussion, if $i = j$ and $\Gamma^+_{i,j}(v) \neq \emptyset$, then there exists α such that the sequence representations of v and w are of the form $v = v_1 \cdots v_i \alpha \cdots \alpha$, $w = w_1 \cdots w_i \alpha \cdots \alpha$. $\square$

We include in this appendix three additional results, namely Lemmas A.1, A.2, and A.3, which are used in the remaining proofs. In the following two, we consider some valuable properties of the sets $S_{k,j}(w)$ in the case that w is a vertex adjacent from v. Lemma A.3, which is a refinement of Lemma 5 in the case $j \geqslant i$ and $i \neq D$, is formulated after the proof of Lemma 5.

Lemma A.1. *Let $v \in V$ and $w \in S_1(v)$. Let $0 \leqslant i,j \leqslant D$ and let k, $0 \leqslant k \leqslant j$. Assume that $S_{k,j}(w) \cap S_i(v) \neq \emptyset$. Then*

1. *If $k = D$, then $S_{k,j}(w) = S_D(w)$. Moreover,*
   - *(a) If $i = D$, then $S_{k,j}(w) = S_i(v)$ if $G = B(d,D)$, while $S_{k,j}(w) \neq S_i(v)$ if $G = K(d,D)$.*
   - *(b) If $i < D$, then $S_i(v) \subseteq S_{k,j}(w)$.*
2. *If $k \neq D$, then $S_i(v) \subseteq S_{k,j}(w) = S_k(w)$ if $k \geqslant i$, while $S_k(w) = S_{k,j}(w) \subseteq S_i(v)$ if $k < i$. Moreover,*
   - *(a) If $k = i$, then $G = B(d,D)$, $S_i(v) = S_i(w)$, and either $j = i$ or $j = i+1$.*
   - *(b) If $k = i-1 = j$, then $S_{k,j}(w) \cap S_i(v) = S_{i-1}(w)$.*
   - *(c) If $k = i-1 < j < D$, then either $S_{i-1,i}(v) = \emptyset$ or $G = B(d,D)$ and $j = i$. Moreover, if $S_{i-1,i}(v) \neq \emptyset$, then $S_{i-1,i}(v) = S_{i-1,i}(w) = S_{i-1}(v) = S_{i-1}(w)$.*
   - *(d) If $k = i-1 < j = D$, then either $S_{i-1,i}(v) = \emptyset$ or $G = B(d,D)$. Moreover, if $S_{i-1,i}(v) \neq \emptyset$ and $w_D = v_D$, then $i = j = D$ and $S_{D-1,D}(v) = S_{D-1,D}(w) = S_{D-1}(v) = S_{D-1}(w)$.*
   - *(e) If $k < i-1$, then there exists $k' \leqslant i-1$ such that $S_{k,j}(w) \subseteq S_{k',i}(v)$.*

Proof. Let us prove statement (1). If $k = D$, then $j = D$ and clearly $S_{k,j}(w) = S_D(w)$. If $G = B(d,D)$ we have $S_D(v) = S_D(w) = V$. If $G = K(d,D)$, then $S_D(v) \neq V$ and $S_D(w) \neq V$. Furthermore, since $w \in S_1(v)$, we have $v_D = w_{D-1} \neq w_D$. Hence, by applying Lemma 2, $S_D(v) \neq S_D(w)$. So we have proved (1.a). To prove (1.b) observe that if $G = B(d,D)$, then $S_i(v) \subseteq S_D(w)$, because $S_D(w) = V$; whereas if $G = K(d,D)$ and $S_i(v) \cap S_D(w) \neq \emptyset$, then $S_i(v) \subseteq S_D(w)$ by statement (2) in Lemma 2.

Next we are going to prove statement (2). From now on assume that $k \neq D$.

If $S_{k,j}(w) \cap S_i(v) \neq \emptyset$, then by the definition of $S_{k,j}(w)$ we get $S_{k,j}(w) = S_k(w)$, and hence, by applying Lemma 2, $S_k(w) \subseteq S_i(v)$ if $k < i$, while $S_i(v) \subseteq S_k(w)$ if $i \leqslant k$, because $k < D$.

First let us consider statement (2.a). If $k = i$, then from our assumptions we get that $S_i(v) \cap S_i(w) \neq \emptyset$. Thus the set $\Gamma^+_{i,i}(v)$ defined in Lemma 4 is nonempty. Therefore, if $i < D$, then from Lemma 4 we have $G = B(d,D)$ and $S_i(v) = S_i(w)$. To conclude the proof of statement (2.a) we must demonstrate that either $i = j$ or $i + 1 = j$. On one hand we have $k \leqslant j$. On the other hand we are assuming $k = i$. So, $i \leqslant j$. Thus it only remains to prove that $j \leqslant i + 1$. Assume on the contrary that $i + 1 < j$. Since $\Gamma^+_{i,i}(v) \neq \emptyset$, by applying again Lemma 4 we have $S_{i-1}(w) \subseteq S_i(v) = S_i(w) \subseteq S_{i+1}(v) = S_{i+1}(w) \subseteq \cdots \subseteq S_j(v) = S_j(w)$. Therefore, $S_i(w) \subseteq S_{i+1}(w) \subseteq S_j(w)$ and thus, by Definition 1, we have $S_{i,j}(w) = \emptyset$. This contradicts the assumption $S_{k,j}(w) \cap S_i(v) \neq \emptyset$, because $k = i$.

Now let us prove (2.b). From Remark 1 we have $S_{i-1,i-1}(w) = S_{i-1}(w)$. Moreover, $S_{i-1}(w) \subseteq S_i(v)$ because $w \in S_1(v)$. So, statement (2.b) follows.

Next we demonstrate statements (2.c) and (2.d). Notice that $i \leqslant j$ because $k = i - 1$ and $k < j$. By the assumptions of the lemma we have $w \in S_1(v)$ and $S_{i-1,j}(w) \cap S_i(v) \neq \emptyset$. Hence $S_{i-1,j}(w) = S_{i-1}(w) \subseteq S_i(v)$. Suppose that $S_{i-1,i}(v) \neq \emptyset$. Recall that this assumption implies $G = B(d,D)$ (see Remark 1) and, moreover, from the definition of $S_{i-1,i}(v)$ it follows that $S_{i-1,i}(v) = S_{i-1}(v) \subseteq S_i(v)$.

Let us consider first statement (2.c). So now we are assuming $j < D$. We have to prove that $j = i$ and that $S_{i-1,i}(v) = S_{i-1,i}(w)$ (because this last equality and the assumption $S_{i-1,i}(v) \neq \emptyset$ imply $S_{i-1,i}(w) \neq \emptyset$, and hence $S_{i-1,i}(v) = S_{i-1}(v)$ and $S_{i-1,i}(w) = S_{i-1}(w)$). Since $S_{i-1}(v) \subseteq S_i(v)$ and $i - 1 \leqslant i < D$, we can apply statement (1) of Lemma 2 and we get that $v_{[i,D-1]} = v_{[i+1,D]}$. Moreover, $v_{[2,D]} = w_{[1,D-1]}$ because $w \in S_1(v)$. Hence on one hand we have $v_i = w_i = v_{i+1} = \cdots = w_{D-1} = v_D$. On the other hand, from the definition of $S_{i-1,j}(w)$, in any case we have $S_{i-1,j}(w) \subseteq S_j(w)$. So, $S_{i-1,j}(w) \cap S_i(v) \subseteq S_i(v) \cap S_j(w)$. Therefore, $S_i(v) \cap S_j(w) \neq \emptyset$ because we are assuming $S_{i-1,j}(w) \cap S_i(v) \neq \emptyset$. Thus, since $i \leqslant j < D$, we can apply once more Lemma 2, and now we get that $S_i(v) \subseteq S_j(w)$ and that $v_{[i+1,D-(j-i)]} = w_{[j+1,D]}$. In particular, $w_D = v_{D-(j-i)} = w_{D-(j-i)-1}$ and therefore

$$v_i = w_i = v_{i+1} = \cdots = w_{D-1} = v_D = w_D. \tag{11}$$

Observe that for $i \leqslant l \leqslant j < D$, equality (11) implies $w_{[i,D-l+i-1]} = w_{[l+1,D]}$ and $w_{[j+1,D-(j-l)]} = w_{[l+1,D]}$. Thus (again by Lemma 2) we have $S_{i-1}(w) \subseteq S_l(w) \subseteq S_j(w)$ for $i \leqslant l \leqslant j < D$. But if $j > i$ the definition of $S_{i-1,j}(w)$ would imply $S_{i-1,j}(w) = \emptyset$, a contradiction. Therefore, it must be $j = i$, as we wanted to show. To complete the proof of statement (2.c) in the case $j < D$, it only remains to show that $S_{i-1,i}(v) = S_{i-1,i}(w)$. But this is straightforward because by our assumptions we know that $S_{i-1}(w) = S_{i-1,j}(w) = S_{i-1,i}(w)$ and $S_{i-1}(v) = S_{i-1,i}(v)$, and, moreover, by (11) we have $v_{[i,D]} = w_{[i,D]}$, and so $S_{i-1}(v) = S_{i-1}(w)$. This completes the proof of (2.c).

Now we are going to prove (2.d). So, let us assume $j = D$. In this case we want to prove that if $w_D = v_D$, then $i = D$ and $S_{D-1}(v) = S_{D-1}(w)$ (as in the proof of (2.c) this last equality implies $S_{D-1,D}(v) = S_{D-1,D}(w) = S_{D-1}(v) = S_{D-1}(w)$). First let us show that $i = D$. On the contrary, assume that $i < D$. In that case, since $S_{i-1}(v) \subseteq S_i(v)$ and $i - 1 \leqslant i < D$, we can apply again statement (1) of Lemma 2 to get $v_{[i,D-1]} = v_{[i+1,D]}$ and thus $v_{[2,D]} = w_{[1,D-1]}$, because $w \in S_1(v)$. Thus, since $w_D = v_D$, equality (11) also holds. Therefore, we have $w_{[i,D-l+i-1]} = w_{[l+1,D]}$ and $w_{[j+1,D-(j-l)]} = w_{[l+1,D]}$ for $i \leqslant l < D$. Thus (again by Lemma 2) we have $S_{i-1}(w) \subseteq S_l(w) \subseteq S_D(w) = V$ for $i \leqslant l < D$. Since $i < D$ the definition of $S_{i-1,D}(w)$ implies $S_{i-1,D}(w) = \emptyset$, a contradiction. Therefore, it must be $i = D$, as we wanted to prove. It remains to show that $S_{D-1}(v) = S_{D-1}(w)$. But this is straightforward because $w_D = v_D$ and so $S_{D-1}(v) = S_{D-1}(w)$.

Finally, let us prove statement (2.e). Let $k < i - 1$ and assume that $S_{k,j}(w) \cap S_i(v) \neq \emptyset$. From $S_{k,j}(w) \cap S_i(v) \neq \emptyset$ it follows that $S_{k,j}(w) = S_k(w)$ and $S_k(w) \cap S_i(v) \neq \emptyset$. Notice that $S_k(w) \subseteq S_{k+1}(v)$ because $w \in S_1(v)$. Hence $S_{k+1}(v) \cap S_i(v) \neq \emptyset$ and so, by applying Lemma 2, it follows that $S_{k+1}(v) \subseteq S_i(v)$ because $k + 1 < i$. Let $k' = \max\{l : k + 1 \leqslant l < i \text{ and } S_{k+1}(v) \subseteq S_l(v) \subseteq S_i(v)\}$. From the definition we get $k' \leqslant i - 1$ and $S_{k+1}(v) \subseteq S_{k'}(v) = S_{k',i}(v)$. Therefore, $S_{k,j}(w) = S_k(w) \subseteq S_{k+1}(v) \subseteq S_{k',i}(v)$. □

Lemma A.2. *Let $v \in V$ and $w \in S_1(v)$. If $0 \leqslant k < i - 1 < D$ and $S_{k,i-1}(w) \neq \emptyset$, then there exists $k' \leqslant i - 1$ such that $S_{k,i-1}(w) \subseteq S_{k',i}(v)$.*

Proof. Let $k < i-1$ and assume that $S_{k,i-1}(w) \neq \emptyset$. It follows that $S_{k,i-1}(w) = S_k(w) \subseteq S_{i-1}(w)$. Hence $S_{k+1}(v) \cap S_{i-1}(w) \neq \emptyset$ because $S_k(w) \subseteq S_{k+1}(v)$. So, since $k+1 \leqslant i-1$, by applying Lemma 2 we get $S_{k+1}(v) \subseteq S_{i-1}(w)$. Therefore, $S_{k+1}(v) \subseteq S_i(v)$ because $S_{i-1}(w) \subseteq S_i(v)$. Let $k' = \max\{l : k+1 \leqslant l < i \text{ and } S_{k+1}(v) \subseteq S_l(v) \subseteq S_i(v)\}$. From the definition we get $k' \leqslant i-1$ and $S_{k+1}(v) \subseteq S_{k'}(v) = S_{k',i}(v)$. Thus $S_{k,i-1}(w) \subseteq S_{k+1}(v) \subseteq S_{k',i}(v)$. $\square$

Proof of Lemma 5

By Lemma 3 we have $S_i^\star(v) = S_i(v) \setminus \left(\bigcup_{k=0}^{i-1} S_{k,i}(v)\right)$ and $S_j^\star(w) = S_j(w) \setminus \left(\bigcup_{k=0}^{j-1} S_{k,j}(w)\right)$. Therefore,

$$S_i^\star(v) \cap S_j^\star(w) = \big(S_i(v) \cap S_j(w)\big) \setminus \left(\Big(\bigcup_{k=0}^{i-1} S_{k,i}(v)\Big) \cup \Big(\bigcup_{k=0}^{j-1} S_{k,j}(w)\Big)\right).$$

By applying Lemma 2, it follows that $S_i(v) \subseteq S_j(w)$ if $i \leqslant j \leqslant D$ and $i < D$, while $S_j(w) \subseteq S_i(v)$ if $j = i-1$. Thus

$$S_i^\star(v) \cap S_j^\star(w) = \begin{cases} S_{i-1}(w) \setminus \left(\Big(\bigcup_{k=0}^{i-1} S_{k,i}(v)\Big) \cup \Big(\bigcup_{k=0}^{i-2} S_{k,i-1}(w)\Big)\right) & \text{if } j = i-1; \\ \Big(S_D(v) \cap S_D(w)\Big) \setminus \left(\Big(\bigcup_{k=0}^{D-1} S_{k,D}(v)\Big) \cup \Big(\bigcup_{k=0}^{D-1} S_{k,D}(w)\Big)\right) & \text{if } j = i = D; \\ S_i(v) \setminus \left(\Big(\bigcup_{k=0}^{i-1} S_{k,i}(v)\Big) \cup \Big(\bigcup_{k=0}^{j-1} S_{k,j}(w)\Big)\right) & \text{if } j \geqslant i \text{ and } i \neq D. \end{cases}$$

By Lemma A.2 we have $\bigcup_{k=0}^{i-2} S_{k,i-1}(w) \subseteq \bigcup_{k=0}^{i-1} S_{k,i}(v)$, and hence the description of $S_i^\star(v) \cap S_j^\star(w)$ formulated in statement (1) follows.

Next we are going to prove the expression of $S_i^\star(v) \cap S_j^\star(w)$ given in statements (2) and (3). So assume that $i = j = D$. Hence

$$S_D^\star(v) \cap S_D^\star(w) = \Big(S_D(v) \cap S_D(w)\Big) \setminus \left(\Big(\bigcup_{k=0}^{D-1} S_{k,D}(v)\Big) \cup \Big(\bigcup_{k=0}^{D-1} \big(S_{k,D}(w) \cap S_D(v)\big)\Big)\right).$$

By statement (2.e) of Lemma A.1, if $0 \leqslant k \leqslant D-2$ and $S_{k,D}(w) \cap S_D(v) \neq \emptyset$, then there exists $k' \leqslant D-1$ such that $S_{k,D}(w) \subseteq S_{k',D}(v)$. Hence $S_{k,D}(w) \cap S_D(v) \subseteq S_{k',D}(v) \cap S_D(v) \subseteq S_{k',D}(v)$, and so

$$S_D^\star(v) \cap S_D^\star(w) = \Big(S_D(v) \cap S_D(w)\Big) \setminus \left(\Big(\bigcup_{k=0}^{D-1} S_{k,D}(v)\Big) \cup \big(S_{D-1,D}(w) \cap S_D(v)\big)\right).$$

If $G = K(d,D)$, then from Remark 1 we have $S_{D-1,D}(v) = S_{D-1,D}(w) = \emptyset$, and so the description given in statement (2) follows. If $G = B(d,D)$, then from Remark 1 we have $S_{D-1,D}(v) = S_{D-1}(v)$, $S_{D-1,D}(w) = S_{D-1}(w)$, and from Lemma 1 we have $S_D(v) = S_D(w) = V$. Hence statement (3) follows.

Now we are going to prove that if $j \geqslant i \neq D$, then the set $S_i^\star(v) \cap S_j^\star(w)$ can be expressed as stated in statement (4). So, assume that $j \geqslant i \neq D$. Hence

$$\begin{aligned} & S_i^\star(v) \cap S_j^\star(w) \\ & = S_i(v) \setminus \left(\Big(\bigcup_{k=0}^{i-1} S_{k,i}(v)\Big) \cup \Big(\bigcup_{k=0}^{i-2} S_{k,j}(w)\Big) \cup \Big(\bigcup_{k=i-1}^{j-1} S_{k,j}(w)\Big)\right) \\ & = S_i(v) \setminus \left(\Big(\bigcup_{k=0}^{i-1} S_{k,i}(v)\Big) \cup \Big(\bigcup_{k=0}^{i-2} \big(S_{k,j}(w) \cap S_i(v)\big)\Big) \cup \Big(\bigcup_{k=i-1}^{j-1} S_{k,j}(w)\Big)\right). \end{aligned}$$

By statement (2.e) of Lemma A.1, if $0 \leqslant k \leqslant i-2$ and $S_{k,j}(w) \cap S_i(v) \neq \emptyset$, then there exists $k' \leqslant i-1$ such that $S_{k,j}(w) \subseteq S_{k',i}(v)$. Hence $S_{k,j}(w) \cap S_i(v) \subseteq S_{k',i}(v) \cap S_i(v) \subseteq S_{k',i}(v)$, and so

$$S_i^\star(v) \cap S_j^\star(w) = S_i(v) \setminus \left(\Big(\bigcup_{k=0}^{i-1} S_{k,i}(v)\Big) \cup \Big(\bigcup_{k=i-1}^{j-1} S_{k,j}(w)\Big)\right),$$

as claimed in statement (4). $\square$

Lemma A.3. *Let $v \in V$ and $w \in S_1(v)$. Let $j \geqslant i$ and $i \neq D$. Let $S_i(v) \cap S_j(w) \neq \emptyset$ and $S_i(v) \nsubseteq S_{k,j}(w)$ for $i \leqslant k \leqslant j-1$. Then $S_i^\star(v) \cap S_j^\star(w)$ can be described as*

$$S_i^\star(v) \cap S_j^\star(w) = S_i(v) \setminus \Big(\Big(\bigcup_{k=0}^{i-2} S_{k,i}(v) \Big) \cup S' \Big),$$

where

$$S' = \begin{cases} \emptyset & \text{if } \; S_{i-1,j}(w) = \emptyset \text{ and } v_{[i,D-1]} \neq v_{[i+1,D]}, & (1) \\ S_{i-1}(v) & \text{if } \; S_{i-1,j}(w) = \emptyset \text{ and } v_{[i,D-1]} = v_{[i+1,D]}, & (2) \\ S_{i-1}(w) & \text{if } \; S_{i-1,j}(w) \neq \emptyset \text{ and } v_{[i,D-1]} \neq v_{[i+1,D]}, & (3) \\ S_{i-1}(v) = S_{i-1}(w) & \text{if } \; S_{i-1,j}(w) \neq \emptyset \text{ and } v_{[i,D-1]} = v_{[i+1,D]} \text{ and } j < D, & (4) \\ S_{i-1}(v) \cup S_{i-1}(w) & \text{if } \; S_{i-1,j}(w) \neq \emptyset \text{ and } v_{[i,D-1]} = v_{[i+1,D]} \text{ and } j = D, & (5) \end{cases}$$

where in the last case we have $S_{i-1}(v) \cap S_{i-1}(w) = \emptyset$.

Proof. From statement (4) of Lemma 5, the intersection set $S_i^\star(v) \cap S_j^\star(w)$ can be written as

$$\begin{aligned} & S_i^\star(v) \cap S_j^\star(w) = \\ & = S_i(v) \setminus \Big(\Big(\bigcup_{k=0}^{i-1} S_{k,i}(v) \Big) \cup \Big(S_{i-1,j}(w) \cap S_i(v) \Big) \cup \Big(\bigcup_{k=i}^{j-1} \big(S_{k,j}(w) \cap S_i(v) \big) \Big) \Big) \end{aligned} \tag{12}$$

if $i < j$, and

$$S_i^\star(v) \cap S_j^\star(w) = S_i(v) \setminus \Big(\Big(\bigcup_{k=0}^{i-1} S_{k,i}(v) \Big) \cup \Big(S_{i-1,j}(w) \cap S_i(v) \Big) \Big) \tag{13}$$

if $i = j$. If $S_i(v) \nsubseteq S_{k,j}(w)$ for $i \leqslant k \leqslant j-1$, then from statement (2.e) of Lemma A.1 we get that $S_{k,j}(w) \cap S_i(v) = \emptyset$ for $i \leqslant k \leqslant j-1$. Therefore, our assumptions imply that the following equality holds both for $i < j$ and for $i = j$:

$$S_i^\star(v) \cap S_j^\star(w) = S_i(v) \setminus \Big(\Big(\bigcup_{k=0}^{i-1} S_{k,i}(v) \Big) \cup \Big(S_{i-1,j}(w) \cap S_i(v) \Big) \Big) = S_i(v) \setminus \Big(\Big(\bigcup_{k=0}^{i-2} S_{k,i}(v) \Big) \cup S' \Big),$$

where $S' = S_{i-1,i}(v) \cup (S_{i-1,j}(w) \cap S_i(v))$.

Observe that if $S_{i-1,j}(w) = \emptyset$, then $S' = S_{i-1,i}(v)$. Therefore, statements (1) and (2) follow from Remark 3. Moreover, by the definition of $S_{i-1,i}(v)$, we have $S' = S_{i-1,i}(v) \subseteq S_i(v)$.

From now on we assume that $S_{i-1,j}(w) \neq \emptyset$. In this case we have $S' = S_{i-1,i}(v) \cup (S_{i-1}(w) \cap S_i(v))$, and, so, $S' = S_{i-1,i}(v) \cup S_{i-1}(w) \subseteq S_i(v)$ because $S_{i-1}(w) \subseteq S_i(v)$ (recall that $w \in S_1(v)$).

By Remark 3, if $v_{[i,D-1]} \neq v_{[i+1,D]}$, then $S' = S_{i-1}(w)$, proving statement (3). Therefore, the proof of the lemma will be completed by demonstrating statements (4) and (5).

Thus assume that $S_{i-1,j}(w) \neq \emptyset$ and $v_{[i,D-1]} \neq v_{[i+1,D]}$, or, equivalently, assume that $S_{i-1,j}(w) \neq \emptyset$ and $S_{i-1,i}(v) \neq \emptyset$. Hence we have $S' = S_{i-1,i}(v) \cup S_{i-1}(w) = S_{i-1}(v) \cup S_{i-1}(w)$. To prove (4) and (5) we are going to apply Lemma A.1 with $k = i-1$. Observe that we are under the assumptions of this lemma because, since $S_{i-1,j}(w) \neq \emptyset$ and $w \in S_1(v)$, then $S_{i-1,j}(w) \cap S_i(v) = S_{i-1}(w) \cap S_i(v) = S_{i-1}(w) \neq \emptyset$.

Let us prove (4). If $j < D$, since we are assuming $S_{i-1,i}(v) \neq \emptyset$, from statement (2.c) of Lemma A.1, we conclude that $j = i$ and $S_{i-1,i}(v) = S_{i-1,i}(w) = S_{i-1}(v) = S_{i-1}(w)$. Thus $S' = S_{i-1}(v) = S_{i-1}(w)$, as we wanted to prove.

Finally, let us prove (5). If $j = D$, since we are assuming $S_{i-1,i}(v) \neq \emptyset$, now from statement (2.d) of Lemma A.1, we get $w_D \neq v_D$. In particular we have $v_{[i,D]} \neq w_{[i,D]}$. So, by statement (1) of Lemma 2, we have $S_{i-1}(v) \cap S_{i-1}(w) = \emptyset$. This completes the proof of the lemma. □

Proof of Lemma 6

Since $S_i^\star(v) \cap S_j^\star(w) \subseteq S_i(v) \cap S_j(w)$ we have $\Gamma_{i,j}^\star(v) \subseteq \Gamma_{i,j}^+(v)$. Let us assume $\Gamma_{i,j}^+(v) \neq \emptyset$, in which case $\Gamma_{i,j}^+(v) = \{w\}$, where w is the unique element of $\Gamma_{i,j}^+(v)$ given in statement (2) of Lemma 4. Therefore we have $\Gamma_{i,j}^\star(v) \neq \emptyset$ if and only if $\Gamma_{i,j}^\star(v) = \Gamma_{i,j}^+(v) = \{w\}$; if and only if $S_i^\star(v) \cap S_j^\star(w) \neq \emptyset$. Finally, since $S_i(v) \cap S_j(w) \neq \emptyset$ if $S_i^\star(v) \cap S_j^\star(w) \neq \emptyset$, we conclude from statement (1) of Propositions 3 and 4 (which proof depends only on the preceding technical lemmas) that $S_i^\star(v) \cap S_j^\star(w) \neq \emptyset$ if and only if $w \in \Gamma_{i,j}^+(v)$ and $S_i(v) \nsubseteq S_{t,j}(w)$ for $i \leqslant t < j$; that is, we have $w \in \Gamma_{i,j}^\star(v)$ if and only if $w \in \Gamma_{i,j}^+(v)$ and $S_i(v) \nsubseteq S_{t,j}(w)$ for $i \leqslant t < j$. □

Proof of Lemma 7

If $G = B(d,D)$, then $S_D(w) = V$ and hence $S_i(v) \cap S_D(w) = S_i(v) \neq \emptyset$ for all $w \in S_1(v)$. Therefore if $G = B(d,D)$, then $\Gamma^+_{i,D}(v) = S_1(v)$. Now let $G = K(d,D)$. If $w \in S_1(v)$, then $w_{[1,D-1]} = v_{[2,D]}$ and, in particular, $w_{D-1} = v_D$. Thus we have $w_D \neq v_D$, because two consecutive symbols in the sequence representation of the vertices of $K(d,D)$ are different. By statement (2) of Lemma 2, we have $S_i(v) \cap S_D(w) \neq \emptyset$ if and only if $v_{i+1} \neq w_D$. Therefore if $v_{i+1} = v_D$, then $v_{i+1} \neq w_D$ holds for any $w \in S_1(v)$. We conclude that if $G = K(d,D)$ and $v_{i+1} = v_D$, then $\Gamma^+_{i,D}(v) = S_1(v)$. This completes the proof of statement (1).

Let us demonstrate statement (2). So assume $G = K(d,D)$ and $v_{i+1} \neq v_D$. By the previous considerations, we have $w \in \Gamma^+_{i,D}(v)$ if and only if $w \in S_1(v)$ and $v_{i+1} \neq w_D$; if and only if $w \in S_1(v)$ and $w_D \notin \{v_{i+1}, v_D\}$. Then, since the symbol alphabet has $d+1 \geqslant 3$ symbols, we have $\Gamma^+_{i,D}(v) \neq \emptyset$ and, moreover, $|\Gamma^+_{i,D}(v)| = d-1$. This completes the proof of the lemma. □

Proof of Propositions 3 and 4

We prove together the two propositions. Firstly, we prove the case $i = j = D$ of both statement (3) of Proposition 3 and statement (3) of Proposition 4; secondly, we prove statement (1) of Proposition 3 and statement (1) of Proposition 4; next we consider the common statement (2) of both propositions; and finally, for $i \leqslant j < D$, we complete the demonstration of statement (3) of Proposition 3 and statement (3) of Proposition 4.

Case $i = j = D$ of statement (3) of Propositions 3 and 4

Let $i = j = D$. We have to prove that if $d \geqslant 3$, then $S^\star_i(v) \cap S^\star_j(w) \neq \emptyset$; whereas if $d = 2$, then $S^\star_i(v) \cap S^\star_j(w) \neq \emptyset$ if and only if $G = K(d,D)$ or $v_D = w_D$. Equivalently, we must demonstrate that if $G = K(d,D)$, then $S^\star_i(v) \cap S^\star_j(w) \neq \emptyset$; while if $G = B(d,D)$, then $S^\star_i(v) \cap S^\star_j(w) \neq \emptyset$ if and only if $d \geqslant 3$ or $v_D = w_D$.

If G is the Kautz digraph $K(d,D)$, then, by statement (3) of Lemma 2, we have $S_D(v) \cap S_D(w) \neq \emptyset$ and $|S_D(v) \cap S_D(w)| = d^D - d^{D-1}$. Since $S_D(v) \cap S_D(w) \neq \emptyset$, by statement (2) of Lemma 5, the intersection $S^\star_D(v) \cap S^\star_D(w)$ can be expressed as

$$S^\star_D(v) \cap S^\star_D(w) = \Big(S_D(v) \cap S_D(w)\Big) \setminus \bigcup_{k=0}^{D-2} S_{k,D}(v).$$

Moreover, by Definition 1 and Lemma 1, we have either $S_{k,D}(v) = \emptyset$ or $|S_{k,D}(v)| = |S_k(v)| = d^k$. Therefore the cardinality of the union $\bigcup_{k=0}^{D-2} S_{k,D}(v)$ can be bounded as follows:

$$\left|\bigcup_{k=0}^{D-2} S_{k,D}(v)\right| \leqslant \sum_{k=0}^{D-2} d^k = \frac{d^{D-1}-1}{d-1},$$

and hence

$$|S^\star_D(v) \cap S^\star_D(w)| \geqslant \left(d^D - d^{D-1}\right) - \frac{d^{D-1}-1}{d-1} = \frac{(d-2)d^D + 1}{d-1} > 0.$$

In particular, we get that if $G = K(d,D)$, then $S^\star_D(v) \cap S^\star_D(w) \neq \emptyset$, as we wanted to prove.

Now let us assume $G = B(d,D)$. We must demonstrate that, in this case, we have $S^\star_D(v) \cap S^\star_D(w) \neq \emptyset$ if and only if $d \geqslant 3$ or $v_D = w_D$.

If $G = B(d,D)$, then $S_D(v) \cap S_D(w) = V$ and, by statement (3) of Lemma 5, we can write $S^\star_D(v) \cap S^\star_D(w)$ as

$$S^\star_D(v) \cap S^\star_D(w) = V \setminus \Big(S_{D-1}(v) \cup S_{D-1}(w) \cup \bigcup_{k=0}^{D-2} S_{k,D}(v)\Big). \tag{14}$$

By taking into account again that either $S_{k,D}(v) = \emptyset$ or $|S_{k,D}(v)| = |S_k(v)| = d^k$, we have

$$\begin{aligned}&\left|S_{D-1}(v) \cup S_{D-1}(w) \cup \bigcup_{k=0}^{D-2} S_{k,D}(v)\right| \\ &\qquad \leqslant |S_{D-1}(v) \cup S_{D-1}(w)| + \sum_{k=0}^{D-2} |S_{k,D}(v)| \leqslant |S_{D-1}(v) \cup S_{D-1}(w)| + \frac{d^{D-1}-1}{d-1},\end{aligned}$$

and thus

$$|S_D^\star(v) \cap S_D^\star(w)| \geqslant d^D - |S_{D-1}(v) \cup S_{D-1}(w)| - \frac{d^{D-1}-1}{d-1}. \tag{15}$$

At this point we distinguish two cases: $d \geqslant 3$ and $d = 2$.

First assume $d \geqslant 3$. Since $|S_{D-1}(v)| = |S_{D-1}(w)| = d^{D-1}$, we have the bound $|S_{D-1}(v) \cup S_{D-1}(w)| \leqslant 2d^{D-1}$. Hence it follows from (15) that

$$|S_D^\star(v) \cap S_D^\star(w)| \geqslant d^D - 2d^{D-1} - \frac{d^{D-1}-1}{d-1} = \frac{(d-3)d^D + d^{D-1} + 1}{d-1} > 0,$$

because $d \geqslant 3$. Therefore we have proved that if $G = B(d, D)$ and $d \geqslant 3$, then $S_D^\star(v) \cap S_D^\star(w) \neq \emptyset$.

Finally, assume $d = 2$. In this case we must demonstrate that if $v_D = w_D$, then $S_D^\star(v) \cap S_D^\star(w) \neq \emptyset$; while if $v_D \neq w_D$, then $S_D^\star(v) \cap S_D^\star(w) = \emptyset$.

If $v_D = w_D$, then, by statement (1) of Lemma 2, we have $S_{D-1}(v) = S_{D-1}(w)$. So if $v_D = w_D$, from (15) and by taking into account that $d = 2$, $|V| = 2^D$, and $|S_{D-1}(v)| = 2^{D-1}$, we get

$$|S_D^\star(v) \cap S_D^\star(w)| \geqslant 2^D - 2^{D-1} - \left(2^{D-1} - 1\right) = 1,$$

which demonstrates that $S_D^\star(v) \cap S_D^\star(w) \neq \emptyset$.

To end suppose $v_D \neq w_D$. Since we are assuming $d = 2$, we conclude that v_D and w_D are the two different symbols of the base alphabet A for the sequence representation of the vertices. Moreover, by using this sequence representation, it is easy to check that in this case we have $S_{D-1}(v) \cup S_{D-1}(w) = V$. Therefore, if $v_D \neq w_D$ we conclude from (14) that $S_D^\star(v) \cap S_D^\star(w) = \emptyset$.

This completes the proof of statement (3) of Proposition 3 and statement (3) of Proposition 4 in the case $i = j = D$.

Statement (1) of Propositions 3 and 4

Let $j \geqslant i \neq D$. We have to prove that if $d \geqslant 3$, then $S_i^\star(v) \cap S_j^\star(w) \neq \emptyset$ if and only if $S_i(v) \cap S_j(w) \neq \emptyset$ and $S_i(v) \not\subseteq S_{k,j}(w)$ for $i \leqslant k < j$; whereas if $d = 2$, then $S_i^\star(v) \cap S_j^\star(w) \neq \emptyset$ if and only if $S_i(v) \cap S_j(w) \neq \emptyset$, $S_i(v) \not\subseteq S_{k,j}(w)$ for $i \leqslant k < j$, and one of the following conditions holds:

1. $j < D$;
2. $j = D$, and $v_{[i,D-1]} \neq v_{[i+1,D]}$ or $S_{i-1,j}(w) = \emptyset$.

Firstly we claim that, for any $d \geqslant 2$, if $j \geqslant i$ and $S_i^\star(v) \cap S_j^\star(w) \neq \emptyset$, then $S_i(v) \cap S_j(w) \neq \emptyset$ and $S_i(v) \not\subseteq S_{k,j}(w)$ for $i \leqslant k < j$. Clearly, if $S_i^\star(v) \cap S_j^\star(w) \neq \emptyset$, then $S_i(v) \cap S_j(w) \neq \emptyset$. If $j = i$ we are done, because in this case the condition $S_i(v) \not\subseteq S_{k,j}(w)$ for $i \leqslant k < j$ is empty. So, let us assume $j > i$. Since $S_i(v) \cap S_j(w) \neq \emptyset$, by statement (4) of Lemma 5, the intersection set $S_i^\star(v) \cap S_j^\star(w)$ can be written as

$$S_i^\star(v) \cap S_j^\star(w) = S_i(v) \setminus \left(\left(\bigcup_{k=0}^{i-1} S_{k,i}(v)\right) \cup \left(\bigcup_{k=i-1}^{j-1} S_{k,j}(w)\right)\right).$$

So if $S_i(v) \subseteq S_{k,j}(w)$ for some k, $i \leqslant k < j$, then we conclude from the above expression that $S_i^\star(v) \cap S_j^*(w) = \emptyset$. This finishes the proof of our claim.

Now we are going to demonstrate that if $d = 2$, $j = D$, and $S_i^\star(v) \cap S_j^\star(w) \neq \emptyset$, then $v_{[i,D-1]} \neq v_{[i+1,D]}$ or $S_{i-1,j}(w) = \emptyset$.

We claim that if $d = 2$, $j = D$, $v_{[i,D-1]} = v_{[i+1,D]}$, and $S_{i-1,j}(w) \neq \emptyset$, then $S_{i-1}(v) \cup S_{i-1}(w) = S_i(v)$. Indeed, on one hand, since $v_{[i,D-1]} = v_{[i+1,D]}$, from statement (1) of Lemma 2 we conclude that $S_{i-1}(v) \subseteq S_i(v)$. So, by Definition 1, we have $S_{i-1,i}(v) \neq \emptyset$. On the other hand, since $S_{i-1,D}(w) \neq \emptyset$, then $S_{i-1,D}(w) = S_{i-1}(w)$, and hence $S_{i-1,D}(w) \cap S_i(v) = S_{i-1}(w) \cap S_i(v) = S_{i-1}(w)$, because $w \in S_1(v)$. In particular, $S_{i-1,D}(w) \cap S_i(v) \neq \emptyset$. Therefore, if $i \neq D = j$, $S_{i-1,j}(w) \neq \emptyset$, and $v_{[i,D-1]} = v_{[i+1,D]}$, then we can apply statement (2.d) of Lemma A.1 to conclude that $w_D \neq v_D$.

To finish the proof of our claim, let us use the sequence representation of the vertices to check that if $d = 2$, $i < D$, $v_{[i,D-1]} = v_{[i+1,D]}$, and $w_D \neq v_D$, then $S_{i-1}(v) \cup S_{i-1}(w) = S_i(v)$.

In fact, if $i < D$ and $v_{[i,D-1]} = v_{[i+1,D]}$, then $v_i = v_{i+1} = \cdots = v_D = \alpha$ for some symbol α of the base alphabet A. In particular we have $G = B(d, D)$. Hence a vertex z belongs to $S_{i-1}(v)$ if and only if $z_{[1,D-i+1]} = v_{[i,D]} = \alpha \cdots \alpha$. Analogously, since $w = v_2 \cdots v_D w_D$ (because $w \in S_1(v)$), a vertex z' is in $S_{i-1}(w)$ if and only if $z'_{[1,D-i+1]} = w_{[i,D]} = \alpha \cdots \alpha w_D$. Moreover, we have $z'' \in S_i(v)$ if and only if $z''_{[1,D-i]} = w_{[i+1,D]} = \alpha \cdots \alpha$. If we assume $d = 2$ and $v_D \neq w_D$, then $v_D = \alpha$ and w_D are the two symbols of the base alphabet A, that is, we have $A = \{\alpha, w_D\}$. Now it is clear that a vertex z belongs to $S_{i-1}(v) \cup S_{i-1}(w)$ if and only if $z_{[1,D-i]} = \alpha \cdots \alpha$; if and only if $z \in S_i(v)$. Hence we have $S_{i-1}(v) \cup S_{i-1}(w) = S_i(v)$, as we wanted to check.

This completes the proof of our claim.

From our claim and by statement (5) of Lemma A.3, we conclude that if $d = 2$, $j = D$, $S_i^\star(v) \cap S_j^\star(w) \neq \emptyset$, $S_{i-1,D}(w) \neq \emptyset$, and $v_{[i,D-1]} = v_{[i+1,D]}$, then

$$S_{i-1}(v) \cup S_{i-1}(w) = S_i(v) \text{ and } S_i^\star(v) \cap S_D^\star(w) = S_i(v) \setminus \Big(\Big(\bigcup_{k=0}^{i-2} S_{k,i}(v) \Big) \cup S' \Big),$$

where $S' = S_{i-1}(v) \cup S_{i-1}(w)$ and $S_{i-1}(v) \cap S_{i-1}(w) = \emptyset$. Since $S_{i-1}(v) \cup S_{i-1}(w) = S_i(v)$, we get $S_i^\star(v) \cap S_D^\star(w) = \emptyset$, which is a contradiction.

At this point we have proved the direct implication of statement (1) of Proposition 3 and statement (1) of Proposition 4.

To complete the proof we are going to show that if $S_i(v) \cap S_j(w) \neq \emptyset$, $S_i(v) \not\subseteq S_{k,j}(w)$ for $i \leqslant k < j$, and if one of the following conditions hold:

(i) $d \geqslant 3$;

(ii) $j < D$;

(iii) $j = D$, and $v_{[i,D-1]} \neq v_{[i+1,D]}$ or $S_{i-1,j}(w) = \emptyset$,

then $S_i^\star(v) \cap S_j^\star(w) \neq \emptyset$.

Let us assume $S_i(v) \cap S_j(w) \neq \emptyset$, $S_i(v) \not\subseteq S_{k,j}(w)$ for $i \leqslant k < j$, and that either condition (i), or (ii), or (iii) is fulfilled.

Since $S_i(v) \cap S_j(w) \neq \emptyset$ and $S_i(v) \not\subseteq S_{k,j}(w)$ for $i \leqslant k < j$, by Lemma A.3 we deduce that the intersection set $S_i^\star(v) \cap S_j^\star(w)$ can be described as

$$S_i^\star(v) \cap S_j^\star(w) = S_i(v) \setminus \Big(\Big(\bigcup_{k=0}^{i-2} S_{k,i}(v) \Big) \cup S' \Big), \tag{16}$$

where $S' \subseteq S_{i-1}(v) \cup S_{i-1}(w)$.

First assume that we are under condition (i); that is, $d \geqslant 3$. Since either $S_{k,i}(v) = \emptyset$ or $|S_{k,i}(v)| = |S_k(v)| = d^k$, and $|S_{i-1}(v)| = |S_{i-1}(v)| = d^{i-1}$, we have

$$\left| \bigcup_{k=0}^{i-2} S_{k,i}(v) \cup S' \right| \leqslant \sum_{k=0}^{i-2} |S_{k,i}(v)| + |S_{i-1}(v)| + |S_{i-1}(w)| \leqslant \frac{d^{i-1}-1}{d-1} + 2d^{i-1} = \frac{2d^i - d^{i-1} - 1}{d-1}, \tag{17}$$

and hence

$$|S_i^\star(v) \cap S_j^\star(w)| \geqslant d^i - \frac{2d^i - d^{i-1} - 1}{d-1} = \frac{(d-3)d^i + d^{i-1} + 1}{d-1} > 0, \tag{18}$$

because $d \geqslant 3$. Therefore, if condition (i) holds, then $S_i^\star(v) \cap S_j^\star(w) \neq \emptyset$, as we wanted to prove.

Now we must demonstrate that $S_i^\star(v) \cap S_j^\star(w) \neq \emptyset$ if either condition (ii) or (iii) is satisfied.

First observe that if either condition (ii) or (iii) is satisfied, then, by statements (1), (2), (3), or (4) of Lemma A.3, the set S' in (16) is either $S' = \emptyset$, or $S' = S_{i-1}(v)$, or $S' = S_{i-1}(w)$. Therefore, in any case we have $|S'| \leqslant d^{i-1}$, and so we have the bound

$$\left| \bigcup_{k=0}^{i-2} S_{k,i}(v) \cup S' \right| \leqslant \sum_{k=0}^{i-2} |S_{k,i}(v)| + |S'| \leqslant \frac{d^{i-1}-1}{d-1} + d^{i-1} = \frac{d^i - 1}{d-1}, \tag{19}$$

and hence

$$|S_i^\star(v) \cap S_j^\star(w)| \geqslant d^i - \frac{d^i - 1}{d-1} = \frac{(d-2)d^i + 1}{d-1}. \tag{20}$$

So $|S_i^\star(v) \cap S_j^\star(w)| > 0$ for any $d \geqslant 2$. Therefore we have $S_i^\star(v) \cap S_j^\star(w) \neq \emptyset$.

Statement (2) of both propositions

Here we assume $d \geqslant 2$. We must prove that:

(a) the intersection $S_i^\star(v) \cap S_{i-1}^\star(w)$ is empty if and only if $G = B(d,D)$ and $v_i = v_{i+1} = \cdots = v_D = w_D$.

(b) if $S_i^\star(v) \cap S_{i-1}^\star(w) = \emptyset$, then $S_i^\star(v) \cap S_i^\star(w) \neq \emptyset$.

Let us prove statement (a).

First of all observe that $S_i(v) \cap S_{i-1}(w) \neq \emptyset$, because $w \in S_1(v)$. So we can apply statement (1) of Lemma 5 to write $S_i^\star(v) \cap S_{i-1}^\star(w)$ as

$$
\begin{aligned}
S_i^\star(v) \cap S_{i-1}^\star(w) &= S_{i-1}(w) \setminus \bigcup_{k=0}^{i-1} S_{k,i}(v) \\
&= S_{i-1}(w) \setminus \Big(\bigcup_{k=0}^{i-2} S_{k,i}(v) \cup \big(S_{i-1,i}(v) \cap S_{i-1}(w) \big) \Big) \\
&= S_{i-1}(w) \setminus \Big(\bigcup_{k=0}^{i-2} S_{k,i}(v) \cup S' \Big),
\end{aligned} \tag{21}
$$

where $S' = S_{i-1,i}(v) \cap S_{i-1}(w)$. Next we are going to prove that either $S' = \emptyset$ or $S' = S_{i-1}(v) = S_{i-1}(w)$. To this end, we only must prove that if $S_{i-1,i}(v) \cap S_{i-1}(w) \neq \emptyset$, then $S_{i-1,i}(v) \cap S_{i-1}(w) = S_{i-1}(v) = S_{i-1}(w)$. So assume $S_{i-1,i}(v) \cap S_{i-1}(w) \neq \emptyset$. In particular we have $S_{i-1,i}(v) \neq \emptyset$ and hence, from Definition 1, we get that $S_{i-1,i}(v) = S_{i-1}(v)$. So our assumption implies that $S_{i-1}(v) \cap S_{i-1}(w) \neq \emptyset$. Now, by applying statement (1) of Lemma 2, we have $S_{i-1}(v) \subseteq S_{i-1}(w)$ and $S_{i-1}(w) \subseteq S_{i-1}(v)$. Hence $S_{i-1,i}(v) \cap S_{i-1}(w) = S_{i-1}(v) \cap S_{i-1}(w) = S_{i-1}(v) = S_{i-1}(w)$.

By Definition 1 and Lemma 1 we know that $|S_{i-1}(w)| = d^{i-1}$ and that, either $S_{k,i}(v) = \emptyset$ or $|S_{k,i}(v)| = |S_k(v)| = d^k$. Therefore, from (21) we conclude that

$$
|S_i^\star(v) \cap S_{i-1}^\star(w)| = |S_{i-1}(w)| - \sum_{k=0}^{i-2} |S_{k,i}(v)| - |S'| \geqslant d^{i-1} - \sum_{k=0}^{i-2} d^k - |S'|.
$$

Thus, since

$$
d^{i-1} - \sum_{k=0}^{i-2} d^k = d^{i-1} - \frac{d^{i-1}-1}{d-1} = \frac{(d-2)d^{i-1}+1}{d-1} > 0,
$$

we have $|S_i^\star(v) \cap S_{i-1}^\star(w)| = 0$ if and only if $|S'| \neq 0$. Therefore $S_i^\star(v) \cap S_{i-1}^\star(w) = \emptyset$ if and only if $S_{i-1,i}(v) \cap S_{i-1}(w) = S_{i-1}(v) = S_{i-1}(w)$.

By statement (1) of Lemma 2, we have $S_{i-1}(v) = S_{i-1}(w)$ if and only if $v_{[i,D]} = w_{[i,D]}$. Thus, since $w \in S_1(v)$, we conclude that $S_{i-1}(v) = S_{i-1}(w)$ if and only if $v_i = v_{i+1} = \cdots = v_D = w_D$; if and only if $G = B(d,D)$ and $v_i = v_{i+1} = \cdots = v_D = w_D$.

Therefore the proof of (a) will be completed by showing that if $S_{i-1}(v) = S_{i-1}(w)$, then $S_{i-1,i}(v) \cap S_{i-1}(w) = S_{i-1}(v) = S_{i-1}(w)$. Let us prove this. Assume $S_{i-1}(v) = S_{i-1}(w)$. Then $S_{i-1}(v) = S_{i-1}(w) \subseteq S_i(v)$, because $w \in S_1(v)$. Therefore, by Definition 1, we have $S_{i-1,i}(v) = S_{i-1}(v)$. So $S_{i-1,i}(v) \cap S_{i-1}(w) = S_{i-1}(v) \cap S_{i-1}(w) = S_{i-1}(v) = S_{i-1}(w)$, as we wanted to prove.

Now let us demonstrate (b); that is, we have to prove that if $S_i^\star(v) \cap S_{i-1}^\star(w) = \emptyset$, then $S_i^\star(v) \cap S_i^\star(w) \neq \emptyset$.

So let us assume $S_i^\star(v) \cap S_{i-1}^\star(w) = \emptyset$ and thus, by (a), we have $G = B(d,D)$ and $v_i = v_{i+1} = \cdots = v_D = w_D$.

If $i = D$ and $d \geqslant 3$ there is nothing to prove, because in this case we have $S_D^\star(v) \cap S_D^\star(w) \neq \emptyset$ by the case $i = j = D$ of statement (3) of Proposition 3; whereas if $i = D$ and $d = 2$, then, since $v_D = w_D$, we also have $S_D^\star(v) \cap S_D^\star(w) \neq \emptyset$ by the case $i = j = D$ of statement (3) of Propositions 4.

Hence assume $i < D$. Since $v_i = v_{i+1} = \cdots = v_D = w_D$, we have $v_{[i+1,D-1]} = v_{[i+2,D]}$ and $v_D = w_D$, which implies $v_{[i+1,D]} = w_{[i+1,D]}$, because $w \in S_1(v)$. Thus we conclude from Lemma 2 that $S_i(v) = S_i(w)$ and so $S_i(v) \cap S_i(w) \neq \emptyset$. Therefore if $d \geqslant 3$, then it follows from statement (1) of Proposition 3 that $S_i^\star(v) \cap S_i^\star(w) \neq \emptyset$. Whereas if $d = 2$, since $i < D$, we also have $S_i^\star(v) \cap S_i^\star(w) \neq \emptyset$, because of condition (a) of statement (1) of Proposition 4. This concludes the proof of (b).

Case $i \leqslant j < D$ of statement (3) of Propositions 3 and 4

We have to prove the following two statements:

(a) If $d \geqslant 3$, then there exists a unique integer j, $i \leqslant j \leqslant D$, such that the intersection $S_i^\star(v) \cap S_j^\star(w)$ is nonempty.

(b) If $d = 2$, then the intersection $S_i^\star(v) \cap S_j^\star(w)$ is empty for all integer j, $i \leqslant j \leqslant D$, if and only if $G = B(d, D)$ and $v_i = v_{i+1} = \cdots = v_D \neq w_D$.

Before proving (a) and (b) let us demonstrate the following claim: for any $v \in V$ and $w \in S_1(v)$, either $S_i(v) \cap S_{D-1}(w) \neq \emptyset$ or $S_i(v) \cap S_D(w) \neq \emptyset$.

Indeed, the claim clearly holds whenever $G = B(d, D)$, because in this case we have $S_D(w) = V$. If G is the Kautz digraph $K(d, D)$, we conclude from statements (1) and (2) of Lemma 2 that if $v_{i+1} = w_D$, then $S_i(v) \cap S_{D-1}(w) \neq \emptyset$; while if $v_{i+1} \neq w_D$, then $S_i(v) \cap S_D(w) \neq \emptyset$. This finishes the proof of our claim.

The above claim guarantees that the set of integers $\{\ell : i \leqslant \ell \leqslant D \text{ and } S_i(v) \cap S_\ell(w) \neq \emptyset\}$ is nonempty. Set $\ell_0 = \min\{\ell : i \leqslant \ell \leqslant D \text{ and } S_i(v) \cap S_\ell(w) \neq \emptyset\}$. So ℓ_0 is an integer such that $i \leqslant \ell_0 \leqslant D$, $S_i(v) \cap S_{\ell_0}(w) \neq \emptyset$, and $S_i(v) \cap S_k(w) = \emptyset$ for $i \leqslant k < \ell_0$. In particular, for $i \leqslant k < \ell_0$, we have $S_i(v) \not\subseteq S_k(w)$, and hence $S_i(v) \not\subseteq S_{k,\ell_0}(w)$ for $i \leqslant k < \ell_0$.

Let us prove (a).

So we assume $d \geqslant 3$ and we have to demonstrate that there exists a unique integer j, $i \leqslant j \leqslant D$, such that the intersection $S_i^\star(v) \cap S_j^\star(w)$ is nonempty. By Proposition 2, it is enough to prove that there exists an integer j_0, $i \leqslant j_0 \leqslant D$, such that the $S_i^\star(v) \cap S_{j_0}^\star(w) \neq \emptyset$.

If $i = D$ the result holds by taking $j_0 = D$, because in this case, by the case $i = j = D$ of statement (3) of Proposition 3, we have $S_D^\star(v) \cap S_D^\star(w) \neq \emptyset$. Whereas if $i < D$ the result holds by taking $j_0 = \ell_0$. Indeed, since $S_i(v) \not\subseteq S_{k,\ell_0}(w)$ for $i \leqslant k < \ell_0$, by applying statement (1) of Proposition 3, we have $S_i^\star(v) \cap S_{\ell_0}^\star(w) \neq \emptyset$.

This concludes the proof of (a).

Now let us prove (b).

Assume $d = 2$. First, let us prove that if for all integer j, $i \leqslant j \leqslant D$, the intersection $S_i^\star(v) \cap S_j^\star(w)$ is empty, then $G = B(d, D)$ and $v_i = v_{i+1} = \cdots = v_D \neq w_D$.

Observe that if $i = D$, then the above implication is a direct consequence of the case $i = j = D$ of statement (3) of Proposition 4. Thus we only must prove the implication in the case $i < D$.

Hence, assume $i < D$. Let us consider the integer ℓ_0 defined above. By assumption, $S_i^\star(v) \cap S_{\ell_0}^\star(w) = \emptyset$. Therefore, by statement (1) of Proposition 4, we conclude that $\ell_0 = D$, $v_{[i,D-1]} = v_{[i+1,D]}$, and $S_{i-1,\ell_0}(w) \neq \emptyset$. Since $i < D$ we have $v_{[i,D-1]} = v_{[i+1,D]}$ if and only if $G = B(d, D)$ and $v_i = v_{i+1} = \cdots = v_D$. To conclude it only remains to show that $v_D \neq w_D$.

Since $\ell_0 = D$ and $S_{i-1,\ell_0}(w) \neq \emptyset$, by Remark 2 we have $S_{i-1}(w) \cap S_k(w) = \emptyset$ for all $i - 1 < k < D$. By statement (1) of Lemma 2, we have $S_{i-1}(w) \cap S_k(w) = \emptyset$ for all $i - 1 < k < D$ if and only if $w_{[i,D-(k-i)-1)]} \neq w_{[k+1,D]}$ for all $i-1 < k < D$; if and only if $v_{[i+1,D-(k-i)-1)]} \neq v_{[k+2,D]}$ for $v_{D-(k-i)} \neq w_D$ for all $i-1 < k < D$. Since $v_i = v_{i+1} = \cdots = v_D$ we have $v_{[i+1,D-(k-i)-1)]} = v_{[k+2,D]}$ for all $i-1 < k < D-1$. Therefore we have $S_{i-1}(w) \cap S_k(w) = \emptyset$ for all $i - 1 < k < D$ if and only if $v_{D-(k-i)} \neq w_D$ for all $i - 1 < k < D$; if and only if $v_D \neq w_D$.

Reciprocally, let us demonstrate that if $G = B(d, D)$ and $v_i = v_{i+1} = \cdots = v_D \neq w_D$, then the intersection $S_i^\star(v) \cap S_j^\star(w)$ is empty for all integer j, $i \leqslant j \leqslant D$. If $v_i = v_{i+1} = \cdots = v_D \neq w_D$, then $v_{[i+1,D-(j-i)]} \neq w_{[j+1,D]}$ for all j, $i \leqslant j < D$. Therefore, by statement(1) of Lemma 2, we have $S_i(v) \cap S_j(v) = \emptyset$ for all j, $i \leqslant j < D$, and hence $S_i^\star(v) \cap S_j^\star(w) = \emptyset$ for all j, $i \leqslant j < D$. It remains to be proved that we also have $S_i^\star(v) \cap S_D^\star(w) = \emptyset$. Indeed, since $S_i^\star(v) = S_i(v) \setminus \left(\bigcup_{k=0}^{i-1} S_k(v)\right)$ and $S_D^\star(w) = V \setminus \left(\bigcup_{k=0}^{D-1} S_k(w)\right)$, we conclude that

$$S_i^\star(v) \cap S_D^\star(w) = S_i(v) \setminus \bigcup_{k=0}^{i-1} (S_k(v) \cup S_k(w)) = \emptyset,$$

because, since $d = 2$ and $v_i = v_{i+1} = \cdots = v_D \neq w_D$, we have $S_{i-1}(v) \cup S_{i-1}(w) = S_i(v)$. (This last equality can be easily checked by using the sequence representation of the vertices.)