Recovering piecewise constant conductances on networks with boundary

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Abstract

The problem of recovering the conductances of a well-connected spider network with boundary from its Dirichlet-to-Robin map is ill-posed for large networks, so despite there is an exact algorithm to solve it [1], the resulting network is very different from the original one. This problem is the discrete analogous to Calderon's Inverse Problem, in which knowing a-priori that the conductivity is piecewise constant with a bounded number of unknown values makes the problem Lipschitz stable [2].

We propose to introduce the hypothesis analogous for the discrete problem that the conductances are constant in each element of a partition of the set of edges with a small number of elements and we formulate the problem as a polynomial optimization one, in which we minimize the difference between the Dirichlet-to-Robin map of the recovered network and the given one plus a term which penalizes the deviation from this hypothesis. We show examples in which we are able to accurately recover the conductances solving this problem.

1 Introduction

The *Inverse Boundary Value Problem* arised for the first time around 1950 due to Alberto Calderón's work. However, it was not until 1980 that he published "On an Inverse Boundary Value Problem" detailing his work on the subject. This problem appeared as a consequence of an engineering problem on geophysical electrical prospection in which the objective is to deduce some internal terrain properties from surface electrical measurements.

These works have motivated several developments in the inverse problem field until nowadays. More recently, this problem has been also considered for medical purposes on *Electrical Impedance Tomography* (EIT), which is a medical imaging technique where an image containing visual information of internal parts of the body is obtained from electrical measurements on the boundary.

The mathematical corresponding problem that Calderón proposed is whether it is possible to determine the conductivity of a body by means of current and voltage measurements at its boundary. This problem of recovering conductances from boundary or surface current and potential measurements is a non–linear inverse problem and it is exponentially ill–posed, since its solution is highly sensitive to changes in the boundary data.

Since its appearance, Calderón's Inverse Problems have been treated in many ways. For instance, Sylvester and Uhlmann treated the uniqueness of solution; Curtis, Ingerman and Morrow have worked on critical circular planar networks conductivity reconstruction; Borcea, Druskin, Guevara and Mamonov have gone into EIT problems in depth and their last works on the subject treat numerical conductivity reconstruction.

Inverse boundary value problems have been considered both over the continuum and the discrete fields. In this work we define a new class of boundary value problems on finite networks associated with Schrödinger operators. The novelty lies on the fact that on a part of the boundary no data is prescribed, whereas in another part of the boundary both the values of the function as of its normal derivative are given. These problems are not self–adjoint, and hence we worry about the study of existence and uniqueness through the adjoint problem.

We show that the overdetermined partial boundary value problem are the key in the framework of inverse boundary value problems on finite networks, since they provide the theoretical foundations of the recovery algorithm. In fact, this type of problems were implicitly considered in some previous works, but only for specific networks and boundary data. We analyze the uniqueness and existence of solution of overdetermined partial boundary value problems through the non–singularity of the partial Dirichlet–to–Neumann maps. These maps allow us to determined the value of the solution in the part of the boundary with no prescribed data. Afterwards, we give explicit formulae for the acquirement of boundary spike conductances on critical planar networks and execute a full conductance recovery for spider networks. This algorithm is an adaptation of the one proposed for the Combinatorial Laplacian and when the corresponding Dirichlet–to–Neumann map is singular.

2 Recovering conductances

Let $\Gamma = (V, c)$ be a finite network; that is, a finite connected graph without loops nor multiple edges, with vertex set V. Let E be the set of edges of the network Γ. Each edge (x, y) has been assigned a *conductance* $c(x, y)$, where $c : V \times V \longrightarrow [0, +\infty)$. Moreover, $c(x, y) = c(y, x)$ and $c(x, y) = 0$ if $(x, y) \notin E$. Then, $x, y \in V$ are *adjacent*, $x \sim y$, iff $c(x, y) > 0$.

The set of functions on a subset $F \subseteq V$, denoted by $\mathcal{C}(F)$, and the set of non–negative functions on F, $\mathcal{C}^+(F)$, are naturally identified with $\mathbb{R}^{|F|}$ and the positive cone of $\mathbb{R}^{|F|}$, respectively. We denote by $\displaystyle\int_F u(x)dx$ or simply by $\displaystyle\int_F u$ the value $\displaystyle\sum_{x \in F} u(x)$. Moreover, if F is a non empty subset of V, its characteristic function is denoted by $\chi_{_F}$. When $F=\{x\}$, its characteristic function will be denoted by ε_x. If $u\in\mathcal{C}(V)$, we define the *support* of u as $\mathsf{supp}(u)=\{x\in V: u(x)\neq 0\}$.

If we consider a proper subset $F\subset V$, then its *boundary* $\delta(F)$ is given by the vertices of $V\setminus F$ that are adjacent to at least one vertex of F. The vertices of $\delta(F)$ are called *boundary vertices* and when a boundary vertex $x\in\delta(F)$ has a unique neighbour we call the edge joining them a *boundary spike*. It is easy to prove that $\bar F=F\cup\delta(F)$ is connected when F is. Any function $\omega\in\mathcal{C}^+(\bar F)$ such that $\mathsf{supp}(\omega)=\bar F$ and $\displaystyle\int_{\bar F}\omega^2=1$ is called *weight* on $\bar F$. The set of weights is denoted by $\Omega(\bar F)$. We denote by k_F the function $k_F(x)=\sum\limits_{x\in F}c(x,y)$.

We define the *normal derivative of* $u\in\mathcal{C}(\bar F)$ *on* F as the function in $\mathcal{C}(\delta(F))$ given by

$$\left(\frac{\partial u}{\partial \mathsf{n}_{_F}}\right)(x)=\int_F c(x,y)\big(u(x)-u(y)\big)\,dy,\quad \text{for any } x\in\delta(F).$$

Any function $K\in\mathcal{C}(F\times F)$ will be called a *kernel on* F. The *integral operator associated with* K is the endomorphism $\mathcal{K}\colon\mathcal{C}(F)\longrightarrow\mathcal{C}(F)$ that assigns to each $f\in\mathcal{C}(F)$, the function $\mathcal{K}(f)(x)=\displaystyle\int_F K(x,y)\,f(y)\,dy$ for all $x\in V$. Conversely, given an endomorphism $\mathcal{K}\colon\mathcal{C}(F)\longrightarrow\mathcal{C}(F)$, the associated kernel is given by $K(x,y)=\mathcal{K}(\varepsilon_y)(x)$. Clearly, kernels and operators can be identified with matrices, after giving a label on the vertex set. In addition, a function $u\in\mathcal{C}(F)$ can be identified with the kernel $K(x,x)=u(x)$ and $K(x,y)=0$ otherwise and hence with a diagonal matrix that will be denoted by D_u.

The *combinatorial Laplacian* of Γ is the linear operator $\mathcal{L}:\mathcal{C}(V)\longrightarrow\mathcal{C}(V)$ that assigns to each $u\in\mathcal{C}(V)$ the function defined for all $x\in V$ as

$$\mathcal{L}(u)(x)=\int_V c(x,y)\big(u(x)-u(y)\big)\,dy.$$

Given $q\in\mathcal{C}(V)$ the *Schrödinger operator* on Γ with *potential* q is the linear operator $\mathcal{L}_q:\mathcal{C}(V)\longrightarrow\mathcal{C}(V)$ that assigns to each $u\in\mathcal{C}(V)$ the function $\mathcal{L}_q(u)=\mathcal{L}(u)+qu$. It is well–known that any Schrödinger operator is self–adjoint. The relation between the values of the Schrödinger operator with potential q on F and the values of the normal derivative at $\delta(F)$ is given by the *First Green Identity*,

$$\int_F v\,\mathcal{L}_q(u)=\frac{1}{2}\int_{\bar F}\int_{\bar F}c_{_F}(x,y)(u(x)-u(y))(v(x)-v(y))\,dxdy+\int_F quv-\int_{\delta(F)}v\,\frac{\partial u}{\partial\mathsf{n}_{\mathsf{F}}},$$

where $u,v\in\mathcal{C}(\bar F)$ and $c_{_F}=c\cdot\chi_{_{(\bar F\times\bar F)\setminus(\delta(F)\times\delta(F))}}$. A direct consequence of the above identity is the so–called *Second Green Identity*

$$\int_F\Big(v\,\mathcal{L}_q(u)-u\,\mathcal{L}_q(v)\Big)=\int_{\delta(F)}\left(u\,\frac{\partial v}{\partial\mathsf{n}_{\mathsf{F}}}-v\,\frac{\partial u}{\partial\mathsf{n}_{\mathsf{F}}}\right),\quad\text{for all}\quad u,v\in\mathcal{C}(\bar F).$$

We define the *energy associated with* F *and* q as the symmetric bilinear form $\mathcal{E}_q^F:\mathcal{C}(\bar F)\times\mathcal{C}(\bar F)\longrightarrow\mathbb{R}$ given for any $u,v\in\mathcal{C}(\bar F)$ by

$$\mathcal{E}_q^F(u,v)=\frac{1}{2}\int_{\bar F}\int_{\bar F}c_{_F}(x,y)\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)\,dx\,dy+\int_{\bar F}q\,u\,v.$$

From the First Green Identity, for any $u, v \in \mathcal{C}(\bar{F})$ we get that

$$\mathcal{E}_q^F(u,v) = \int_F v\mathcal{L}_q(u) + \int_{\delta(F)} v \left[\frac{\partial u}{\partial \mathsf{n}_\mathsf{F}} + qu\right].$$

For any weight $\sigma \in \Omega(\bar{F})$, the so–called *potential associated with* σ is the function in $\mathcal{C}(\bar{F})$ defined as $q_\sigma = -\sigma^{-1}\mathcal{L}(\sigma)$ on F, $q_\sigma = -\sigma^{-1}\dfrac{\partial \sigma}{\partial \mathsf{n}_F}$ on $\delta(F)$. So, through this section, will suppose that the above condition holds. In particular, for any $g \in \mathcal{C}(\delta(F))$ the following Dirichlet problem

$$\mathcal{L}_q(u) = 0 \quad \text{on} \quad F \quad \text{and} \quad u = g \quad \text{on} \quad \delta(F),$$

has a unique solution u_g.

The map $\Lambda_q \colon \mathcal{C}(\delta(F)) \longrightarrow \mathcal{C}(\delta(F))$ that assigns to any function $g \in \mathcal{C}(\delta(F))$ the function $\Lambda_q(g) = \dfrac{\partial u_g}{\partial \mathsf{n}_F} + qg$ is called *Dirichlet–to–Robin map.* Moreover, λ is the lowest eigenvalue of Λ_q and its associated eigenfunctions are multiple of σ. In addition, if $N_q \in \mathcal{C}(\delta(F) \times \delta(F))$ is the kernel of Λ_q, its associated matrix $\mathsf{N_q}$ is an irreducible and symmetric M–matrix. Usually $\mathsf{N_q}$ is called the *response matrix* of the network. Given $A, B \in \delta(F)$ a pair of disjoint subsets, we consider the submatrix of the response matrix $\mathsf{N_q}(A;B) = \big(N_q(x,y)\big)_{(x,y)\in A\times B}$.

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