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CONTROLLABILITY PROPERTIES FOR MULTI-AGENT LINEAR SYSTEMS. A GEOMETRIC APPROACH

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Abstract

This work addresses the controlability of a class of multi-agent linear systems that they are interconnected via communication channels. Multiagent systems have attracted much attention because they have great applicability in multiple areas, such as power grids, bioinformatics, sensor networks, vehicles, robotics and neuroscience, for example. Consequently, they have been widely studied by scientists in different fields specially in the field of control theory. Recently has taken interest to analyze the control properties as consensus controllability of multi-agent dynamical systems motivated by the fact that the architecture of communication network in engineering multi-agent systems is usually adjustable. In this paper, the control condition is analyzed under geometrical point of view. in the case of multiagent linear systems that can be described by k agents with dynamics $\dot{x}^i = A_i x^i + B_i u^i$, $i = 1, \dots, k$.

Key words

Controllability, invariant subspaces.

1 Introduction

Controllability is a basic concept of control system theory, especially if we want to do whatever with the given dynamic system under control input, necessarily the system must be controllable. It is particularly important for practical implementations ([Chen, 1970], [Chen, 2017], [Garcia-Planas, Tarragona, 2016], [Heniche, Kamwa, 2002], [Kundur, 1994], [Liu, Slotine, Barabási, 2011], [Wang et Al. 2016]).

In recent years has grown the interest in the study of control multi-agent systems, as well as the increasing interest in distributed control and coordination of networks consisting of multiple autonomous agents. It is due to that they appear in different areas, and there are an amount of bibliography as [Javier, Campos-Cantdn, 2013], [Saber, Murray, 2004], [Sun el Al., 2017], [Trumpf and Trentelman, 2018], [Wang, Cheng, Hu, 2008], [Xie, Wang, 2006]. In particular multisystems can help to model brain Neural Networks for best understanding brain function, where the concept of brain cognitive control defined by neuroscientists is related to the mathematical concept of control defined by physicists, mathematicians, and engineers, where the state of a multisystem can be adjusted by a particular input.

Also, network control theory can be a powerful tool for understanding and manipulating biomedical networks such as intracellular molecular interaction networks, [Czeizler et Al., 2018].

A basic tool in the structure theory of linear dynamical systems is the concept of an invariant subspaces. It is well known that One of the most a control input can always be chosen in a special form, namely constant state feedback $u(t) = Fx(t)$ that can be expressed in algebraic terms by means (A, B)-invariant subspaces addressing the input side of a control system,

In this work the controllability character of multiagent systems consisting of k agents having identical linear dynamic mode, with dynamics

$$\dot{x}^i = Ax^i + Bu^i \quad i = 1, \dots, k \tag{1}$$

are analyzed under geometrical point of view.

The presence of invariant subspaces has been taken into account by other authors such as [Xue, Roy, 2019] where they study the structural controllability of linear dynamic networks formed by interconnected homogeneous subsystems, showing that structural controllability at the subsystem and network level is necessary but not sufficient for the structural controllability of the complete model. They also show that the presence of certain high multiplicity structural modes, which they call invariant structural network modes, are barriers to structural controllability.

In what follows, we denote by $M_{n\times m}(\mathbb{K})$ the set of matrices with n rows and m columns to coefficients in the field $\mathbb{K}$ that can be the field of real $\mathbb{R}$ or complex $\mathbb{C}$ numbers, and we denote by $Gl(n;\mathbb{R})$ the group of n-order invertibe square matrices invertible to coefficients the field $\mathbb{R}$.

2 Preliminaries

The communication topology among agents of the system is defined by means an indirect graph. It should be noted that graph models are commonly used in network representations.

In this particular setup, we consider a graph $\mathcal{G} = (\mathcal{V},\mathcal{E})$ of order k with the set of vertices $\mathcal{V} = \{1,\dots,k\}$ and the set of edges $\mathcal{E} = \{(i,j) \mid i,j \in \mathcal{V}\} \subset \mathcal{V}\times\mathcal{V}$.

Given an edge (i,j) i is called the parent node and j is called the child node and j is in the neighbor of i, concretely we define the neighbor of i and we denote it by $\mathcal{N}_i$ to the set $\mathcal{N}_i = \{j \in \mathcal{V} \mid (i,j) \in \mathcal{E}\}$.

The graph is called undirected if verifies that $(i,j) \in \mathcal{E}$ if and only if $(j,i) \in \mathcal{E}$. The graph is called connected if there exists a path between any two vertices, otherwise is called disconnected.

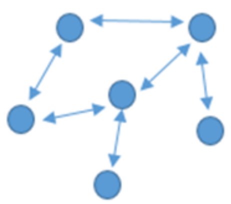

Figure 1. Undirected connected graph

Associated to the graph it can be consider the Laplacian matrix of the graph defined in the following manner

$$\mathcal{L} = (l_{ij}) = \begin{cases} |\mathcal{N}_i| & \text{if } i=j \\ -1 & \text{if } j \in \mathcal{N}_i \\ 0 & \text{otherwise} \end{cases} \qquad (2)$$

Remark 2.1. *The following properties are verified.*

i) If the graph is undirected then the matrix $\mathcal{L}$ is symmetric, then there exist an orthogonal matrix P such that $P\mathcal{L}P^t = \mathcal{D}$.

ii) If the graph is undirected then 0 is an eigenvalue of $\mathcal{L}$ and $(1,\dots,1)^t$ is the associated eigenvector.

iii) If the graph is undirected and connected the eigenvalue 0 is simple.

For more information on graph theory, see [West, 2007].

About matrices, we need to remember Kronecker product of matrices because it will be useful in our study.

Given a couple o matrices $A = (a_{ij}) \in M_{n\times m}(\mathbb{C})$ and $B = (b_{ij}) \in M_{p\times q}(\mathbb{C})$, remember that the Kronecker product is defined as follows.

Definition 2.1. *Let $A = (a^i_j) \in M_{n\times m}(\mathbb{C})$ and $B \in M_{p\times q}(\mathbb{C})$ be two matrices, the Kronecker product of A and B, write $A\otimes B$, is the matrix*

$$A\otimes B = (a^i_j B) \in M_{np\times mq}(\mathbb{C})$$

Kronecker product verifies the following properties

1) $(A+B)\otimes C = (A\otimes C) + (B\otimes C)$
2) $A\otimes(B+C) = (A\otimes B) + (A\otimes C)$
3) $(A\otimes B)\otimes C = A\otimes(B\otimes C)$
4) $(A\otimes B)^t = A^t\otimes B^t$
5) If $A \in Gl(n;\mathbb{C})$ and $B \in Gl(p;\mathbb{C})$), then $A\otimes B \in Gl(np;\mathbb{C})$) and $(A\otimes B)^{-1} = A^{-1}\otimes B^{-1}$
6) If the products AC and BD are possible, then $(A\otimes B)(C\otimes D) = (AC)\otimes(BD)$

Corollary 2.1. *The vector $\mathbf{1}_k\otimes v$ is an eigenvector corresponding to the zero eignevalue of $\mathcal{L}\otimes I_n$.*

Proof.

$$(\mathcal{L}\otimes I_n)(\mathbf{1}_k\otimes v) = \mathcal{L}\mathbf{1}_k\otimes v = 0\otimes v = 0$$

Consequently, if $\{e_1,\dots,e_n\}$ is a basis for $\mathbb{C}^n$, then $\mathbf{1}_k\otimes e_i$ is a basis for the nullspace of $\mathcal{L}\otimes I_n$.

Associated to the Kronecker product, can be defined the vectorizing operator that transforms any matrix A into a column vector, by placing the columns in the matrix one after another,

Definition 2.2. *Let $X = (x^i_j) \in M_{n\times m}(\mathbb{C})$ be a matrix, and we denote $x_i = (x^1_i,\dots,x^n_i)^t$ for $1 \le i \le m$ the i-th column of the matrix X. We define the vectorizing operator vec, as*

$$\begin{aligned} vec : M_{n\times m}(\mathbb{C}) &\longrightarrow M_{nm\times 1}(\mathbb{C}) \\ X &\longrightarrow \begin{pmatrix} x_1 \; x_2 \; \dots \; x_m \end{pmatrix}^t \end{aligned}$$

Obviously, vec is an isomorphism.

See [Lancaster,Tismenetsky, 1985] for more information and properties.

3 Control Properties

If we want to do whatever with the given dynamic system under control input, necessarily the system must be "controllable". Below we recall the well known concepts of controllability and stability for a better understanding of the work.

Definition 3.1. *The dynamical system $\dot{x} = Ax + Bu$ is said to be controllable if for every initial condition $x(0)$ and every vector $x_1 \in \mathbb{R}^n$, there exist a finite time t_1 and control $u(t) \in \mathbb{R}^m$, $t \in [0,t_1]$, such that $x(t_1) = x_1$.*

This definition requires only that any initial state $x(0)$ can be steered to any final state x_1 at time t_1. However, the trajectory of the dynamical system between 0 and t_1 is not specified. Furthermore, there is no constraints posed on the control vector $u(t)$ and the state vector $x(t)$.

For simplicity and if confusion is not possible, we will write (A, B) for dynamical system $\dot{x} = Ax + Bu$.

It is easier to compute the controllability using the following matrix

$$\mathcal{C} = \begin{pmatrix} B \; AB \; A^2B \; \dots \; A^{n-1}B \end{pmatrix}. \tag{3}$$

called controllability matrix, thanks to the following well-known result.

Theorem 3.1. *The dynamical system $\dot{x} = Ax + Bu$ is controllable if and only if* $\operatorname{rank} \mathcal{C} = n$.

As we says, controllability of the dynamical system $\dot{x} = Ax + Bu$ implies that each initial state can be steered to 0 on a finite time-interval. If only is required that this to happen asymptotically for $t \to \infty$, we have the following concept.

Definition 3.2. *The system $\dot{x} = Ax + Bu$ is called stabilizable if for each initial state $x(0) \in \mathbb{R}^n$ there exists a (piece-wise continuous) control input $u : [0, \infty) \longrightarrow \mathbb{R}^m$ such that the state-response with $x(0)$ verifies*

$$\lim_{t\to\infty} x(t) = 0.$$

Remark 3.1. *i) All controllable systems are stabilizable but the converse is false.*
ii) If the matrix A in the system $\dot{x} = Ax + Bu$ is Hurwitz then, the system is stabilizable.

It is important the following result

Theorem 3.2. *The system $\dot{x} = Ax + Bu$ is stabilizable if and only if there exists some feedback F such that $\dot{x} = (A - BF)x$ is asymptotically stable.*

The controllability and stabilizable characters are preserved under feedback

Definition 3.3. *Two systems (A_1, B_1) and (A_2, B_2) are feedback equivalent, if and only if, there exist $P \in Gl(n, \mathbb{R})$, $Q \in Gl(m, \mathbb{R})$ and $F \in M_{m\times n}(\mathbb{R})$ such that*

$$(A_2, B_2) = (P^{-1}A_1P + P^{-1}B_1F, P^{-1}B_1Q)$$

Proposition 3.1. *Let (A_1, B_1) and (A_2, B_2) feedback equivalent systems, then*

i) (A_1, B_1) is controllable if and only if (A_2, B_2) is
i) (A_1, B_1) is stabilizable if and only if (A_2, B_2) is

4 Invariant (A, B)-subspaces

In this section we recall the definition of invariant subspace under (A, B)-map.

Definition 4.1. *A subspace $G \subset \mathbb{C}^n$ is invariant under (A, B) if and only if*

$$AG \subset G + \operatorname{Im} B \tag{4}$$

In other words, a subspace G of the state space is called (A, B)-invariant invariant, if for every starting point in G there exists a control input such that the corresponding trajectory stays in G in the future.

Notice that if $B = 0$, this definition coincides with the definition of A-invariant subspace.

We can construct invariant subspaces in the following manner. Let $H \subset \mathbb{C}^n$ be a subspace, we define

$$G_{k+1} = H \cap \{x \in \mathbb{C}^n \mid Ax \in G_k + \operatorname{Im} B\},\; G_0 = H,$$

limit of recursion exists and we will denote by $G(H)$. This subspace is the supremal (A, B)-invariant subspace contained in H. Taking $H = \mathbb{C}^n$, we will write it as G^*.

Example 4.1. *Let (A, B) be the pair $A = \left(\begin{smallmatrix} 0&0&0\\0&1&0\\0&1&1 \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} 1\\0\\0 \end{smallmatrix}\right)$ and $H = \{(x, y, z) \mid z = 0\}$,*
Computation of G_1:

$$\begin{pmatrix} 0&0&0\\0&1&0\\0&1&1 \end{pmatrix}\begin{pmatrix} x\\y\\z \end{pmatrix} = \begin{pmatrix} 0\\y\\y+z \end{pmatrix} = \begin{pmatrix} \mu\\\nu\\0 \end{pmatrix} + \begin{pmatrix} \lambda\\0\\0 \end{pmatrix}$$

$[(x, y, -y)] \cap H = [(x, 0, 0)] = G_1$.
Computation of G_2:

$$\begin{pmatrix} 0&0&0\\0&1&0\\0&1&1 \end{pmatrix}\begin{pmatrix} x\\y\\z \end{pmatrix} = \begin{pmatrix} \mu\\0\\0 \end{pmatrix} + \begin{pmatrix} \lambda\\0\\0 \end{pmatrix}$$

$[(x, 0, 0)] \cap H = [(1, 0, 0)] = G_2 = G_1$. *Then $G = G_1$.*
Obviously $AG \subset G + Im\, B$.

Proposition 4.1. *Let (A, B) be a pair of matrices. A subspace $G \subset \mathbb{C}^n$ is invariant under (A, B) if and only if is invariant under $(A + BF, B)$ for all feedback $F \in M_{m\times n}(\mathbb{C})$.*

Proof. Suppose that $AG \subset G + \operatorname{Im} B$, then for all $x \in G$, there exists $y \in G$, $v = Bw \in \operatorname{Im} B$ such that $Ax = y + Bw$ so, for any $F \in M_{m\times n}(\mathbb{C})$, we have

$$\begin{gathered} Ax + BFx - BFx = y + Bw \\ (A + BF)x = y + B(Fx + w). \end{gathered}$$

Consequently, for all $x \in G$, $(A+BF)G \subset G+\operatorname{Im} B$.

Reciprocally, suppose that $(A + BF)G \subset G + \operatorname{Im} B$, then for all $x \in G$, there exists $y \in G$, $v = Bw \in \operatorname{Im} B$ such that $(A+BF)x = y+Bw$ so, $Ax = y-BFxBw$ and $Ax = y + B(-Fx + w)$. Then, for all $x \in G$ we have $AG \subset G + \operatorname{Im} B$.

Proposition 4.2. *Let (A_1, B_1) and (A_2, B_2) be two equivalent pairs under equivalence defined in 3.3. Then $G \subset \mathbb{C}^n$ is an invariant subspace under (A_1, B_1) if and only if $P^{-1}G$ is invariant under (A_2, B_2).*

Proof. Suppose that $A_1G \subset G + \operatorname{Im} B$. Then $A_2P^{-1}G = (P^{-1}A_1P + P^{-1}B_1F)P^{-1}G = P^{-1}(A_1G + B_1FP^{-1}G) \subset P^{-1}(G + \operatorname{Im} B_1) = P^{-1}G + \operatorname{Im} PB_2Q^{-1}) = P^{-1}G + P\operatorname{Im} B_2Q^{-1}) = P^{-1}G + \operatorname{Im}B_2R^{-1}) \subset (P^{-1}G + \operatorname{Im} B_2$

5 Controllability subspaces

In this section we are going to study a particular case of invariant subspaces. First of all we observe the following result.

Proposition 5.1. *Let (A, B) be a pair of matrices. Then*

$$G = [B, AB, \dots, A^{n-1}B]$$

is a (A, B)-invariant subspace.

Proof.

$$AG = A[B, AB, \dots, A^{n-1}B] = [AB, A^2B, \dots, A^nB]$$

Now, it suffices to apply the Cayley-Hamilton theorem.

Theorem 5.1. *Let*

$$C_r = \begin{pmatrix} I & & & & B & & & \\ A & I & & & & B & & \\ & \ddots & \ddots & & & & \ddots & \\ & & & I & & & & B \\ & & & A & & & & & B \end{pmatrix} \in M_{nr\times(n(r-1)+mr)}(\mathbb{C})$$

be the r-controllability matrix. Suppose r being the least such that $\operatorname{rank} C_r < (n(r-1)+mr)$*, and let* $\begin{pmatrix} v_1 \dots v_r\ w_1 \dots w_{r+1}\end{pmatrix} \in \operatorname{Ker} C_r$ *(v_i are vectors in $\mathbb{C}^n$ and w_i vectors in $\mathbb{C}^m$). Then $G = [v_1, \dots, v_r]$ is a (A, B)-invariant subspace.*

Proof. We consider $v = \lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_{r-1} v_{r-1} + \lambda_r v_r$, $Av = \lambda_1 Av_1 + \lambda_2 Av_2 + \dots + \lambda_{r-1} Av_{r-1} + \lambda_r Av_r = \lambda_1(-v_2 - Bw_2) + \lambda_2(-v_3 - Bw_3) + \dots + \lambda_{r-1}(-v_r - Bw_r) - \lambda_r Bw_{r+1} = (\lambda_1 v_2 - \lambda_2 v_3 - \dots - \lambda_{r-1} v_r) + B(-\lambda_1 w_2 - \lambda_2 w_3 - \dots - \lambda_{r-1} w_r - \lambda_r w_{r+1}) \in G + \operatorname{Im} B$.

Definition 5.1. *The space sum of all spaces G in theorem before is a invariant subspace that we will call controllability subspace and we will denote it by $\mathcal{C}(A, B)$.*

Notice that $\mathcal{C}(A, B)$ is the set of states in which the system is controllable.

Corollary 5.1. *Let (A, B) be a pair of matrices. In this case the invariant subspace G obtained in the above theorem, coincides with the controllability (A, B)-invariant subspaces $[B, AB, \dots, A_{r-1}B]$.*

Proof. Making block-row elemental transformations to the matrix C_r we obtain the equivalent matrix

$$\begin{pmatrix} I_n & & & & B & & & \\ 0 & I_n & & & -AB & B & & \\ & \ddots & \ddots & & & & \ddots & \\ & & & I_n & (-1)^{r-2}A_{r-2}B & & -AB & B \\ & & & 0 & (-1)^{r-1}A^{r-1}B & & & -AB & B \end{pmatrix}.$$

5.1 Controllability subspaces of multiagent systems

Writing

$$\mathcal{X}(t) = \begin{pmatrix} x^1(t) \\ \vdots \\ x^k(t) \end{pmatrix}, \quad \dot{\mathcal{X}}(t) = \begin{pmatrix} \dot{x}^1(t) \\ \vdots \\ \dot{x}^k(t) \end{pmatrix},$$

$$\mathcal{U}(t) = \begin{pmatrix} u^1(t) \\ \vdots \\ u^k(t) \end{pmatrix},$$

$$\mathcal{A} = \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_k \end{pmatrix}, \ \mathcal{B} = \begin{pmatrix} B_1 & & \\ & \ddots & \\ & & B_k \end{pmatrix},$$

Following this notation we can describe the multisystem as a system:

$$\dot{\mathcal{X}}(t) = \mathcal{A}\mathcal{X}(t) + \mathcal{B}\mathcal{U}(t)$$

Clearly, this system is controllable if and only if each subsystem is controllable, and, in this case, there exist a feedback

$$\mathcal{F} \begin{pmatrix} F_1 & & \\ & \ddots & \\ & & F_k \end{pmatrix}$$

in which we obtain the desired solution.

We consider the vector space $\mathbb{R}^n \times \overset{k}{\dots} \times \mathbb{R}^n$ and a subspace $\mathcal{H} = H_1 \times \dots \times H_k$ a subspace.

(Observe that the decomposition of $\mathcal{H}$ in product of subspaces H_i in each factor $\mathbb{R}^n$ is unique).

Proposition 5.2. *The subspace $\mathcal{H}$ is $(\mathcal{A}, \mathcal{B})$-invariant, if and only if each H_i is (A_i, B_i)-invariant.*

In the particular case where $A_1 = \dots = A_k$, we generate subspaces $(\mathcal{A}, \mathcal{B})$- invariants by making the product of k subspaces (A, B)- invariants equal or not.

6 Consensus

We are interested in take the output of the system to a reference value and keep it there, we can ensure that if the system is controllable.

Roughly speaking, we can define the consensus as a collection of processes such that each process starts with an initial value, where each one is supposed to output the same value and there is a validity condition that relates outputs to inputs. More concretely, the consensus problem is a canonical problem that appears in the coordination of multi-agent systems. The objective is that Given initial values (scalar or vector) of agents, establish conditions under which through local interactions and computations, agents asymptotically agree upon a common value, that is to say: to reach a consensus.

Definition 6.1. *Consider the system 1. We say that the consensus is achieved using local information if there is a state feedback* $u^i = K\sum_{j\in\mathcal{N}_i}(x^i - x^j)$ *such that*

$$\lim_{t\to\infty} \|x^i - x^j\| = 0,\ 1 \le i,j \le k.$$

The closed-loop system obtained under this feedback is as follows

$$\left.\dot{\mathcal{X}} = \mathcal{A}\mathcal{X} + \mathcal{B}\mathcal{K}\mathcal{Z},\right\}$$

where $\mathcal{X}$, $\dot{\mathcal{X}}$, $\mathcal{A}$, $\mathcal{B}$ are as before and

$$\mathcal{K} = \begin{pmatrix} K & & \\ & \ddots & \\ & & K \end{pmatrix},\ \mathcal{Z} = \begin{pmatrix} \sum_{j\in\mathcal{N}_1} x^1 - x^j \\ \vdots \\ \sum_{j\in\mathcal{N}_k} x^k - x^j \end{pmatrix}.$$

Following this notation we can conclude the following.

Proposition 6.1. *The closed-loop system can be described as*

$$\dot{\mathcal{X}} = ((I_k \otimes A) + (I_k \otimes BK)(\mathcal{L}\otimes I_n))\mathcal{X} \tag{5}$$

Calling $\mathcal{A}_1 = ((I_k \otimes A) + (I_k \otimes BK)(\mathcal{L}\otimes I_n))$ the system is written as $\dot{\mathcal{X}} = \mathcal{A}_1\mathcal{X}$.

Assuming $\mathcal{X}(0) = 0$, the equation 5 can be solved as

$$\begin{array}{l}\mathcal{X}(t) = \\ \int_0^t e^{((I_k\otimes A)+(I_k\otimes BK)(\mathcal{L}\otimes I_n))(t-s)}\mathcal{X}(s)dsds.\end{array} \tag{6}$$

In our particular setup, we have that there exists an orthogonal matrix $P \in Gl(k,\mathbb{R})$ such that $P\mathcal{L}P^t = \mathcal{D} =$ diag $(\lambda_1,\dots,\lambda_k)$, $(\lambda_1 \ge \dots \ge \lambda_k)$.

Corollary 6.1. *The closed-loop system can be described in terms of the matrices* A, B, *the feedback* K *and the eigenvalues of* $\mathcal{L}$.

Proof. Following properties of Kronecker product we have that

$$\begin{array}{l}(P\otimes I_n)(I_k\otimes A)(P^t\otimes I_n) = (I_k\otimes A)\\ (P\otimes I_n)(I_k\otimes BK)(P^t\otimes I_n) = \\ (I_k\otimes BK)\\ (P\otimes I_n)(\mathcal{L}\otimes I_n)(P^t\otimes I_n) = (\mathcal{D}\otimes I_n)\end{array}$$

and calling $\widehat{\mathcal{X}} = (P\otimes I_n)\mathcal{X}$, we have

$$\dot{\widehat{\mathcal{X}}} = ((I_k\otimes A) + (I_k\otimes BK)(\mathcal{D}\otimes I_n))\widehat{\mathcal{X}}.$$

Equivalently,

$$\dot{\widehat{\mathcal{X}}} = \begin{pmatrix} A+\lambda_1 BK & & \\ & \ddots & \\ & & A+\lambda_k BK \end{pmatrix}\widehat{X}. \tag{7}$$

Calling $\mathcal{A}_2 = \begin{pmatrix} A+\lambda_1 BK & & \\ & \ddots & \\ & & A+\lambda_k BK \end{pmatrix}$, the system is written as $\dot{\widehat{\mathcal{X}}} = \mathcal{A}_2\widehat{\mathcal{X}}$

Now let $\mathcal{H}$ be a $\mathcal{A}_1$-invariant subspace, i.e. $\mathcal{A}_1\mathcal{H}\subset\mathcal{H}$, we have the following proposition

Proposition 6.2. *The subspace* $\mathcal{H}$ *is* $\mathcal{A}_1$*-invariant if and only if* $(P\otimes I_n)\mathcal{H}$ *is* $\mathcal{A}_2$ *invariant.*

Proof. $\mathcal{A}_1\mathcal{H} = ((I_k\otimes A)+(I_k\otimes BK)(\mathcal{L}\otimes I_n))\mathcal{H}\subset\mathcal{H}$.

Equivalently $(P\otimes I_n)((I_k\otimes A) + (I_k\otimes BK)(\mathcal{L}\otimes I_n))(P^t\otimes I_n)(P\otimes I_n)\mathcal{H} \subset (P\otimes I_n)\mathcal{H}$

That is to say $\mathcal{A}_2(P\otimes I_n)\mathcal{H}\subset(P\otimes I_n)\mathcal{H}$.

Taking into account proposition 4.1, we have the following corollary.

Corollary 6.2. *Let* $\mathcal{H}$ *be a* $\mathcal{A}_1$*-invariant subspace and* $\bar{H}_1\times\dots\times\bar{H}_k$ *the unique decomposition of* $(P\otimes I_n)\mathcal{H}$ *in* $\mathbb{R}^n \times \overset{k}{\dots} \times \mathbb{R}^n$. *Then,* $\bar{H}_i$ *is* (A_i,B_i)*-invariant, for each* $i = 1,\dots,k$.

Other properties.

Corollary 6.3. *The system 1 is consensus stabilizable if and only if the systems* $A+\lambda_i BK$ *are stable by means the same* K.

7 Conclusion

In this paper, we have analyzed the controllability subspaces of multiagent linear systems and consensus controllability subspaces in the case of multiagent linear systems having all agents the same dynamics.

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