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THE CANONICAL TUTTE POLYNOMIAL FOR SIGNED GRAPHS

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Abstract. We construct a new polynomial invariant for signed graphs, the trivariate Tutte polynomial, which contains among its evaluations the number of proper colorings and the number of nowhere-zero flows. In this, it parallels the Tutte polynomial of a graph, which contains the chromatic polynomial and flow polynomial as specializations. While the Tutte polynomial of a graph is equivalently defined as the dichromatic polynomial or Whitney rank polynomial, the dichromatic polynomial of a signed graph (defined more widely for biased graphs by Zaslavsky) does not, by contrast, give the number of nowhere-zero flows as an evaluation in general. The trivariate Tutte polynomial contains Zaslavsky's dichromatic polynomial as a specialization. Furthermore, the trivariate Tutte polynomial gives as an evaluation the number of proper colorings of a signed graph under a more general sense of signed graph coloring in which colors are elements of an arbitrary finite set equipped with an involution.

1. Introduction

The *trivariate Tutte polynomial* of a signed graph is a moderately-sized special case of the "huge polynomial Tutte invariant" of weighted gain graphs [**4**] and, in the sense of [**7**], is the "canonical Tutte polynomial" for switching equivalence classes of signed graphs [**11**, §3]: it has a subset expansion, satisfies a deletion-contraction recurrence and is universal for deletion-contraction invariants, and satisfies duality and convolution formulas. Here we show that the trivariate Tutte polynomial also contains the number of nowhere-zero flows and the number of proper colorings of a signed graph as specializations, in this way resembling the Tutte polynomial of a graph in its guise as the dichromate [**9**]. The trivariate Tutte polynomial contains Zaslavsky's dichromatic polynomial [**12**] as a specialization; in contrast to the Tutte polynomial of a graph, the dichromatic polynomial of a signed graph does not in general give the number of nowhere-zero flows of a signed graph as an

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evaluation. A more expansive treatment of the trivariate Tutte polynomial is given in our preprint [**6**], including its generalization to pairs of matroids on a common ground set and the enumeration of the analogue of graph tensions defined for signed graphs by Chen [**2**]; while all (nowhere-zero) tensions for graphs arise from (proper) vertex colourings, only some (nowhere-zero) tensions for signed graphs arise from (proper) signed graph vertex colourings.

1.1. The Tutte polynomial of a graph

Let $\Gamma = (V, E)$ be a finite graph, $k(\Gamma)$ the number of connected components of Γ, and Γ_A the restriction of Γ to $A \subseteq E$. The Tutte polynomial of Γ has subset expansion

$$T_\Gamma(X,Y) = \sum_{A\subseteq E} (X-1)^{k(\Gamma_A)-k(\Gamma)}(Y-1)^{|A|-|V|+k(\Gamma_A)}. \tag{1}$$

Alternatively, letting $\Gamma\backslash e$ and Γ/e denote the deletion and contraction of Γ by an edge e, the Tutte polynomial is defined by the recurrence

$$T_\Gamma(X,Y) = \begin{cases} T_{\Gamma/e}(X,Y) + T_{\Gamma\backslash e}(X,Y) & \text{if } e \text{ is an ordinary edge of } \Gamma,\\ XT_{\Gamma/e}(X,Y) & \text{if } e \text{ is a bridge of } \Gamma,\\ YT_{\Gamma\backslash e}(X,Y) & \text{if } e \text{ is loop of } \Gamma,\\ 1 & \text{if } \Gamma \text{ has no edges,} \end{cases} \tag{2}$$

where a loop is an edge whose endpoints coincide, and a bridge is an edge whose deletion increases the number of connected components. For a finite additive abelian group G, the number of nowhere-zero G-flows of Γ is

$$(-1)^{|E|-|V|+k(\Gamma)}T_\Gamma(0, 1-|G|), \tag{3}$$

and for a finite set X the number of proper vertex colorings of Γ using colors from X is equal to

$$(-1)^{|V|-k(\Gamma)}|X|^{k(\Gamma)}T_\Gamma(1-|X|, 0). \tag{4}$$

Theorems 3.4 and 3.5 give counterparts to formulas (3) and (4) for signed graphs.

If U is a graph invariant multiplicative over disjoint unions and satisfying

$$U_\Gamma = \begin{cases} \alpha\, U_{\Gamma/e} + \beta U_{\Gamma\backslash e} & \text{if } e \text{ is an ordinary edge of } \Gamma,\\ xU_{\Gamma/e} & \text{if } e \text{ is a bridge of } \Gamma,\\ \gamma y^\ell & \text{if } \Gamma \text{ consists of } \ell \geq 0 \text{ loops,} \end{cases}$$

then

$$U_\Gamma = \alpha^{r(\Gamma)}\beta^{|E|-r(\Gamma)}\gamma^{k(\Gamma)}T_\Gamma\left(x/\alpha, y/\beta\right). \tag{5}$$

The Tutte polynomial is defined more generally for a matroid $M = (E, r)$ with ground set E and rank function r by

$$T_M(X,Y) = \sum_{A\subseteq E}(X-1)^{r(E)-r(A)}(Y-1)^{|A|-r(A)}. \tag{6}$$

2. Signed graphs and their matroids

A *signed graph* is a pair $\Sigma = (\Gamma, \sigma)$, where $\Gamma = (V, E)$ is a finite graph, called the *underlying graph* of Σ, and σ is a function $\sigma : E \to \{-1, 1\}$ that associates a sign to each edge of Γ, called the *signature* of Σ. A cycle $C = (v_1, e_1, v_2, \dots, v_k, e_k, v_1)$ in Γ is called *balanced* in Σ if $\prod_{i=1}^{k} \sigma(e_i) = 1$ and *unbalanced* otherwise. The signed graph $\Sigma = (\Gamma, \sigma)$ is itself called *balanced* if each cycle of Γ is balanced in Σ and *unbalanced* otherwise. Let $k(\Sigma) := k(\Gamma)$, and let $k_b(\Sigma)$ and $k_u(\Sigma)$ denote the number of balanced and unbalanced connected components of Σ, respectively. *Switching* at a vertex v means negating the sign of every edge that is incident with v, while keeping the sign of each loop attached to v. Two signed graphs $\Sigma_1 = (\Gamma_1, \sigma_1)$ and $\Sigma_2 = (\Gamma_2, \sigma_2)$ are *switching equivalent* if the graph Γ_1 is isomorphic to the graph Γ_2, and if, under such an isomorphism, the signature σ_1 can be obtained from σ_2 by a sequence of switchings at vertices.

The *deletion* of an edge e in $\Sigma = (\Gamma, \sigma)$ yields the signed graph $\Sigma \backslash e := (\Gamma \backslash e, \sigma')$, where σ' is the restriction of σ to $E \backslash \{e\}$ and where $\Gamma \backslash e$ is the graph obtained from Γ by deleting e as a graph edge. The *contraction* of a non-loop edge e of Γ that has positive sign in $\Sigma = (\Gamma, \sigma)$ yields the signed graph $\Sigma / e := (\Gamma / e, \sigma')$, where σ' is the restriction of σ to $E \setminus \{e\}$ and where Γ / e is the graph obtained from Γ by contracting e as a graph edge. By switching we can always ensure that the sign of a non-loop edge is positive. When e is a loop with positive sign in Σ we set $\Sigma / e = \Sigma \backslash e$. We need not define contraction of negative edges (which requires the definition of signed graphs to be enlarged to include half-arcs and free loops [**10**]).

2.1. Two matroids associated with a signed graph

The *cycle matroid* $M(\Gamma)$ of the underlying graph $\Gamma = (V, E)$ of $\Sigma = (\Gamma, \sigma)$ is the matroid on ground set E whose circuits are edge sets of a subdivided loop. An edge e is a *bridge in* Γ if $k(\Gamma \backslash e) > k(\Gamma)$ and *ordinary in* Γ if e is neither a bridge nor loop. A subdivision of the graph consisting of two loops on a common vertex is a *tight handcuff*, and a subdivision of the graph consisting of two loops joined by an edge is a *loose handcuff*. A loose handcuff or a tight handcuff in Σ is *contrabalanced* if neither of the cycles it contains is balanced in Σ. The *frame matroid* $F(\Sigma)$ of Σ is the matroid on ground set E whose circuits are the edge sets of subdivisions of a loop that are balanced, or subdivisions of handcuffs that are contrabalanced. A *circuit path edge* of Σ is an edge of a loose handcuff that belongs to neither of its cycles. An edge e is *ordinary in* Σ if it belongs to some circuit of Σ consisting of at least two edges (i.e. e is not a positive loop in Σ) and deleting e does not increase the number of unbalanced connected components.

3. The trivariate Tutte polynomial, flows and colorings

Definition 3.1. The *trivariate Tutte polynomial* of a signed graph $\Sigma = (\Gamma, \sigma)$ with underlying graph $\Gamma = (V, E)$ is defined by

$$T_\Sigma(X, Y, Z) := \sum_{A \subseteq E} (X-1)^{k(\Sigma_A) - k(\Sigma)} (Y-1)^{|A| - |V| + k_b(\Sigma_A)} (Z-1)^{k_u(\Sigma_A)}, \tag{7}$$

in which Σ_A is obtained from Σ by deleting the edges not in A.

The trivariate Tutte polynomial $T_\Sigma(X, Y, Z)$ includes the Tutte polynomial of $M(\Gamma)$ and the Tutte polynomial of $F(\Sigma)$ as specializations:

$$T_{M(\Gamma)}(X,Y) = T_\Sigma(X,Y,Y), \quad \text{and} \tag{8}$$

$$T_{F(\Sigma)}(X,Y) = (X-1)^{k_u(\Sigma)} T_\Sigma\left(X, Y, X/(X-1)\right). \tag{9}$$

Theorem 3.2 ([**6**])**.** *The trivariate Tutte polynomial $T_\Sigma = T_\Sigma(X,Y,Z)$ of a signed graph $\Sigma = (\Gamma, \sigma)$ with underlying graph $\Gamma = (V, E)$ satisfies, for each positive edge $e \in E$,*

$$T_\Sigma = \begin{cases} T_{\Sigma/e} + T_{\Sigma\backslash e} & \text{if } e \text{ is an ordinary edge of } \Gamma, \\ T_{\Sigma/e} + (X-1)T_{\Sigma\backslash e} & \text{if } e \text{ is a bridge of } \Gamma \text{ and a circuit path edge of } \Sigma, \\ XT_{\Sigma/e} & \text{if } e \text{ is a bridge of } \Gamma \text{ not a circuit path edge of } \Sigma, \\ YT_{\Sigma\backslash e} & \text{if } e \text{ is a loop of } \Gamma \text{ positive in } \Sigma, \\ 1+(Z-1)\left[1+Y+\cdots+Y^{\ell-1}\right] & \text{if } \Sigma \text{ is a single vertex with } \ell \geq 1 \text{ negative loops}, \\ 1 & \text{if } \Sigma \text{ has no edges.} \end{cases}$$

Theorem 3.3 (Recipe Theorem [**6**])**.** *Let R be an invariant of signed graphs preserved by switching and multiplicative over disjoint unions. Suppose that there are constants $\alpha, \beta, \gamma, x, y$ and z, with $\gamma \neq 0$, such that, for a signed graph $\Sigma = (\Gamma, \sigma)$ with underlying graph $\Gamma = (V, E)$ and positive edge $e \in E$,*

$$R_\Sigma = \begin{cases} \alpha R_{\Sigma/e} + \beta R_{\Sigma\backslash e} & \text{if } e \text{ is ordinary in } \Gamma \text{ and in } \Sigma, \\ \alpha R_{\Sigma/e} + \gamma R_{\Sigma\backslash e} & \text{if } e \text{ is ordinary in } \Gamma \text{ and } k_u(\Sigma\backslash e) < k_u(\Sigma), \\ \alpha R_{\Sigma/e} + \frac{\beta(x-\alpha)}{\gamma} R_{\Sigma\backslash e} & \text{if } e \text{ bridge in } \Gamma \text{ and a circuit path edge in } \Sigma, \\ x R_{\Sigma/e} & \text{if } e \text{ bridge in } \Gamma, \text{ not a circuit path edge in } \Sigma, \\ y R_{\Sigma\backslash e} & \text{if } e \text{ is a loop in } \Gamma \text{ and in } \Sigma, \\ \beta^{\ell-1}\gamma + (z-\gamma)\sum_{i=0}^{\ell-1} y^{\ell-1-i}\beta^i & \text{if } \Sigma \text{ consists of } \ell \geq 1 \text{ negative loops}, \\ 1 & \text{if } \Sigma \text{ has no edges.} \end{cases}$$

Then R_Σ is a polynomial in α, β, x, y and z over $\mathbb{Z}[\gamma, \gamma^{-1}]$ and

$$R_\Sigma = \alpha^{r_M(E)} \beta^{|E|-r_F(E)} \gamma^{r_F(E)-r_M(E)} T_\Sigma\left(x/\alpha, y/\beta, z/\gamma\right). \tag{10}$$

3.1. Flows

Given a graph $\Gamma = (V, E)$, we call a pair (v, e) with $v \in V$ and $e \in E$ an edge containing v a *half-edge*. (A loop comprises two half-edges.) A *bidirected graph* is a pair (Γ, ω), where $\Gamma = (V, E)$ is a graph in which every half-edge (v, e) receives an orientation $\omega(v, e) \in \{-1, 1\}$. (The two half-edges associated with a loop at a vertex consist of the same vertex-edge pair but receive orientations independently.) The orientation ω is *compatible* with the signature σ of a signed graph $\Sigma = (\Gamma, \sigma)$ if for each edge $e = uv$ we have

$$\sigma(e) = -\omega(u,e)\omega(v,e). \tag{11}$$

Let G be a finite additive abelian group. For $k \in \mathbb{N}$ and $x \in G$, we let $kx = \sum_{i=1}^{k} x$ and $(-k)x = -kx$. Let $2G := \{2x : x \in G\}$.

A *G-flow of a bidirected graph* $(\Gamma = (V, E), \omega)$ [**1**] is a function $f : E \to G$ such that at each vertex v of Γ

$$\sum_{(v,e):v\in e} \omega(v, e)f(e) = 0, \tag{12}$$

where the summation runs over half-edges (v, e) incident with v, so if e is a loop it contributes with two terms to the sum (if the loop is positive these terms cancel each other, while if the loop is negative they have the same sign). A *G-flow of a signed graph* $\Sigma = (\Gamma, \sigma)$ is a function $f : E \to G$ such that f is a G-flow for the bidirected graph (Γ, ω), where ω is an orientation of Γ compatible with σ. A G-flow is *nowhere-zero* if $f(e) \neq 0$ for all $e \in E$.

Theorem 3.4 ([**6**]). *Let G be a finite additive abelian group. Then, for a signed graph $\Sigma = (\Gamma = (V, E), \sigma)$, the number of nowhere-zero G-flows of Σ is equal to*

$$(-1)^{|E|-|V|+k(\Gamma)} T_\Sigma\left(0, 1 - |G|, 1 - |G|/|2G|\right).$$

Proof (sketch). The number of nowhere-zero G-flows satisfies a deletion-contraction recurrence, given in [**3**], with parameters $(x, y, z, \alpha, \beta, \gamma) = (0, |G| - 1, |G|/|2G| - 1, 1, -1, -1)$ in Theorem 3.3. □

When $2G = G$, i.e. G is of odd order, the number of nowhere-zero G-flows of Σ given in Theorem 3.4 is by (9) an evaluation of the Tutte polynomial of the frame matroid $F(\Sigma)$. Theorem 3.4 is equivalent to the special case of [**5**, Theorem 4.6] of flows of a map taking values in an abelian group (a flow of a signed graph can be regarded as a flow of a map with the same underlying graph, the twisted edges of the map corresponding to negative edges of the signed graph [**5**, Remark 4.12]). Theorem 3.4 in the form given by the subset expansion (7) of the trivariate Tutte polynomial was found independently by Qian [**8**, Theorem 4.3].

3.2. Colorings

Let X be a finite set and ι an involution on X. A (X, ι)-*coloring* of a signed graph $\Sigma = (\Gamma, \sigma)$ with underlying graph $\Gamma = (V, E)$ is a function $f : V \to X$ such that, for each edge $e = uv$, $f(u) \neq f(v)$ if $\sigma(e) = +1$ and $\iota(f(u)) \neq f(v)$ if $\sigma(e) = -1$.

Theorem 3.5 ([**6**]). *The number of (X, ι)-colorings of Σ is equal to*

$$(-1)^{|V|-k(\Sigma)}|X|^{k(\Sigma)} T_\Sigma\left(1 - |X|, 0, 1 - t/|X|\right), \quad \textit{where } t = |\{x : \iota(x) = x\}|. \tag{13}$$

Proof (sketch). The number of (X, ι)-colorings satisfies the deletion-contraction of Theorem 3.3 with $(x, y, z, \alpha, \beta, \gamma) = (1 - |X|, 0, 1 - t/|X|, -1, 1, 1)$, except in taking value $|X|^k$ on edgeless signed graphs with k vertices. □

Remark 3.6. When $t = 1$, the evaluation of the trivariate Tutte polynomial in (13) is an evaluation of the Tutte polynomial of $F(\Sigma)$ at $(1 - |X|, 0)$; otherwise, when $t \neq 1$, the number of (X, ι)-colorings of Σ is not given by an evaluation of the Tutte polynomial of $F(\Sigma)$.

When $X = \{0, \pm 1, \ldots, \pm n\}$ (or $X = \{\pm 1, \ldots, \pm n\}$), with ι negation, (X, ι)-colorings coincide with Zaslavsky's notion of *proper (non-zero) n-colorings* of a signed graph [**11**] and Theorem 3.5 yields Zaslavsky's Theorem 2.4 in [**10**]. When X is the set of elements of a finite additive abelian group G and ι is negation, Theorem 3.5 gives that the number of proper colorings of Σ using elements of G as colors is equal to

$$(-1)^{|V|-k(\Sigma)}|G|^{k(\Sigma)}T_\Sigma\left(1-|G|,0,1-1/|2G|\right),$$

as $t=\frac{|G|}{|2G|}$ is the number of self-inverse elements of G. In a way that can be made precise [**6**], this formula is dual to that in Theorem 3.4, giving the number of nowhere-zero G-flows of Σ.

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