On polynomials associated to Voronoi diagrams of point sets and crossing numbers

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Abstract

Three polynomials are defined for sets S of n points in general position in the plane: The Voronoi polynomial with coefficients the numbers of vertices of the order-k Voronoi diagrams of S, the circle polynomial with coefficients the numbers of circles through three points of S enclosing k points, and the $E_{\leq k}$ polynomial with coefficients the numbers of (at most k)-edges of S. We present several formulas for the rectilinear crossing number of S in terms of these polynomials and their roots. We also prove that the roots of the Voronoi polynomial lie on the unit circle if and only if S is in convex position. Further, we present bounds on the location of the roots of these polynomials.

1 Introduction

Let S be a set of $n \geq 4$ points in general position in the plane, meaning that no three points of S are collinear and no four points of S are cocircular. The Voronoi diagram of order k of S, $V_k(S)$, is a subdivision of the plane into cells such that points in the same cell have the same k nearest points of S. Voronoi diagrams have found many applications in a wide range of disciplines, see e.g. [7, 27]. We define the *Voronoi polynomial* $p_V(z) = \sum_{k=1}^{n-1} v_k z^{k-1}$, where v_k is the number of vertices of $V_k(S)$. Proximity information among the points of S is also encoded by the *circle polynomial* of S, which we define as $p_C(z) = \sum_{k=0}^{n-3} c_k z^k$, where c_k is the number of circles passing through three points of S that enclose exactly k other points of S. The numbers v_k and c_k are related via the well-known relation

$$v_k = c_{k-1} + c_{k-2} \tag{1}$$

where $c_{-1} = 0$ and $c_{n-2} = 0$, see e.g. [21]. The two polynomials $p_V(z)$ and $p_C(z)$ are especially interesting due to their connection to the rectilinear crossing number problem.

The rectilinear crossing number of a point set S, $\overline{cr}(S)$, is the number of pairwise edge crossings of the complete graph K_n when drawn with straight-line segments on S, i.e. the vertices of K_n are the points of S. Equivalently, $\overline{cr}(S)$ is the number of convex quadrilaterals with vertices in S. We denote $\overline{cr}(S)$ as $\alpha\binom{n}{4}$, with $0 \leq \alpha \leq 1$. Note that for S in convex position, $\alpha = 1$. The rectilinear crossing number problem consists in, for each n, finding the minimum value of $\overline{cr}(S)$ among all sets S of n points, no three of them collinear, commonly denoted as $\overline{cr}(K_n)$. The limit of $\overline{cr}(K_n)/\binom{n}{4}$, when n tends towards infinity, is the so-called *rectilinear crossing number constant* α^*. This problem is solved only for $n \leq 27$ and $n = 30$, and the current best bound for the rectilinear crossing number constant is $\alpha^* > 0,37997$, see

39th European Workshop on Computational Geometry, Barcelona, Spain, March 29–31, 2023.
This is an extended abstract of a presentation given at EuroCG'23. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear eventually in more final form at a conference with formal proceedings and/or in a journal.

the survey [3] and the web page [5]. A fruitful approach to the rectilinear crossing number problem is proving bounds on the numbers of j-edges and of $(\leq k)$-edges of S [1, 2, 6, 11, 20]. An (oriented) j-edge of S is a directed straight line ℓ passing through two points of S such that the open half-plane bounded by ℓ and on the right of ℓ contains exactly j points of S. The number of j-edges of S is denoted by e_j, and $E_{\leq k} = \sum_{j=0}^{k} e_j$ is the number of $(\leq k)$-edges. We then also consider the $E_{\leq k}$ *polynomial* $p_E(z) = \sum_{k=0}^{n-3} E_{\leq k} z^k$, which also encodes information on higher order Voronoi diagrams, since the number of j-edges e_j is the number of unbounded cells of the order-$(j+1)$ Voronoi diagram of S (see, e.g., Proposition 30 in [13]). Note that $p_E(z)$ has no term $E_{\leq n-2}$. For an illustration of the defined polynomials for a particular point set, see Figure 1.

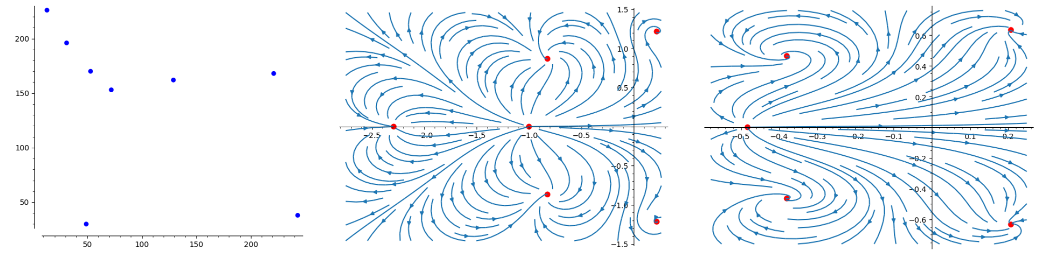

Figure 1 Left: 1591-th entry of the order type database for 8 points [4]. With complex stream plots of its Voronoi polynomial (center): $p_V(z) = 10 + 23z + 27z^2 + 24z^3 + 17z^4 + 9z^5 + 2z^6$, and its $E_{\leq k}$ polynomial (right): $p_E(z) = 4 + 13z + 22z^2 + 34z^3 + 43z^4 + 52z^5$; roots are red points.

For a point set S, we show that $\overline{cr}(S)$ appears in the first derivatives of these three polynomials when evaluated at $z = 1$ and, in addition, we obtain appealing formulas for $\overline{cr}(S)$ in terms of the roots of the polynomials. Motivated by this, we study the location of such roots, showing several bounds on their modulus. As a particular result, we also prove that the roots of the Voronoi polynomial lie on the unit circle if and only if S is in convex position. Furthermore, the circle polynomial comes into play when considering the random variable X that counts the number of points of S enclosed by the circle defined by three points chosen uniformly at random from S. The probability generating function of X is $p_C(z)/\binom{n}{3}$. In [24] a central limit theorem for random variables with values in $\{0, \ldots, n\}$ was shown, under the condition that the variance is large enough and that no root of the probability generating function is too close to $1 \in \mathbb{C}$. We show that the random variable X does not approximate a normal distribution, and use the result from [24] to derive that $p_C(z)$ has a root close to $1 \in \mathbb{C}$.

Throughout this work, points (a, b) in the plane are identified with complex numbers $z = a + ib$. To avoid cumbersome notation we omit indicating the point set S where it is clear from context; for example, each polynomial considered depends on a point set S but we write $p_C(z)$ instead of $p_C^S(z)$.

2 Known relations

A main source is the work by Lee [19], from where several of the following formulas can be obtained.

- For any point set S, and $0 \leq k \leq n-3$, it holds that, see [8, 12, 13, 19, 21],

$$c_k + c_{n-k-3} = 2(k+1)(n-k-2). \tag{2}$$

- From [15] we get the following two equations.

$$\sum_{k=0}^{n-3} k \cdot c_k = \binom{n}{4} + \overline{cr}(S) = (1+\alpha)\binom{n}{4}. \tag{3}$$

This was essentially also obtained in [29], though not stated in terms of $\overline{cr}(S)$.

$$\sum_{k=0}^{n-3} k^2 \cdot c_k = \binom{n}{5} + \binom{n}{4} + (n-3)\overline{cr}(S). \tag{4}$$

- For $k \leq \frac{n-3}{2}$ it holds that, see Lemma 3.1 in [14],

$$c_k \geq (k+1)(n-k-2), \text{ and } c_{n-k-3} \leq (k+1)(n-k-2). \tag{5}$$

Next Equations (6), (7) and (8) hold for a point set S in convex position.

$$2c_k = c_{k-1} + c_{k+1} + 2. \tag{6}$$

Then, the number of vertices of $V_k(S)$ fulfills, see e.g. Proposition 34, Equation (4) in [13],

$$v_k = c_{k-1} + c_{k-2} = (2k-1)n - 2k^2 \ . \tag{7}$$

This implies that

$$v_k = v_{n-k}. \tag{8}$$

- For every set S of n points in general position, the relation between $E_{\leq k}$ and c_k is, see e.g. Proposition 33 in [13],

$$c_k + E_{\leq k} = (k+1)(2n-k-2). \tag{9}$$

3 Properties of the Voronoi, circle and $E_{\leq_k}$ polynomials

For every set S of n points in general position:

▶ **Proposition 1.** *Polynomials $p_C(z) = \sum_{k=0}^{n-3} c_k z^k$, and $p_V(z) = \sum_{k=1}^{n-1} v_k z^{k-1}$ satisfy*

$$p_V(z) = (1+z)p_C(z). \tag{10}$$

▶ **Proposition 2.** *The circle polynomial $p_C(z) = \sum_{k=0}^{n-3} c_k z^k$ satisfies*

1. $p_C(1) = \binom{n}{3}$.
2. $p'_C(1) = \binom{n}{4} + \overline{cr}(S)$.
3. $p''_C(1) = \binom{n}{5} + (n-4)\overline{cr}(S)$.
4. $p_C(-1) = \frac{n-1}{2}$ *for n odd.*

▶ **Proposition 3.** *The Voronoi polynomial $p_V(z) = \sum_{k=1}^{n-1} v_k z^{k-1}$ satisfies*

1. $p_V(1) = 2\binom{n}{3}$.
2. $p'_V(1) = \binom{n}{3} + 2\binom{n}{4} + 2\overline{cr}(S)$.
3. $p''_V(1) = 2\binom{n}{4} + 2\binom{n}{5} + 2(n-3)\overline{cr}(S)$.
4. $p_V(-1) = 0$.
5. $p'_V(-1) = \frac{n-1}{2}$ *for n odd.*

▶ **Proposition 4.** *The $E_{\leq_k}$ polynomial $p_E(z) = \sum_{k=0}^{n-3} E_{\leq_k} z^k$ satisfies:*

1. $p_E(1) = 3\binom{n}{3}$.

2. $p'_E(1) = \sum_{k=0}^{n-3} kE_{\leq k} = 9\binom{n}{4} - \overline{cr}(S)$.
3. $p''_E(1) = \sum_{k=0}^{n-3} k(k-1)E_{\leq k} = 35\binom{n}{5} - (n-4)\overline{cr}(S)$.
4. $p_E(-1) = \frac{n(n-1)}{2}$ *for n odd.*

Using Lemma 1 of Aziz and Mohammad in [9], we get an intriguing family of formulas for the rectilinear crossing number.

▶ **Proposition 5.** *The coefficients of the polynomials $p_V(z)$, $p_C(z)$ and $p_E(z)$ satisfy*

$$\textbf{1)}\ \overline{cr}(S) = \frac{4}{3(n-3)} \sum_{k=1}^{n-3} \sum_{j=0}^{n-3} c_j \frac{z_k^{j+1}}{(z_k-1)^2} = \sum_{j=0}^{n-3} c_j \left(\frac{4}{3(n-3)} \sum_{k=1}^{n-3} \frac{z_k^{j+1}}{(z_k-1)^2} \right), \tag{11}$$

where the z_k are the $(n-3)$-th roots of -3.

$$\textbf{2)}\ \overline{cr}(S) = \frac{2}{3(n-1)} \sum_{k=1}^{n-1} \sum_{j=1}^{n-1} v_j \frac{z_k^{j}}{(z_k-1)^2} = \sum_{j=1}^{n-1} v_j \left(\frac{2}{3(n-1)} \sum_{k=1}^{n-1} \frac{z_k^{j}}{(z_k-1)^2} \right), \tag{12}$$

where the z_k are the $(n-1)$-th roots of -3.

$$\textbf{3)}\ \overline{cr}(S) = -\frac{4}{n-3} \sum_{k=1}^{n-3} \sum_{j=0}^{n-3} E_{\leq j} \frac{z_k^{j+1}}{(z_k-1)^2} = \sum_{j=0}^{n-3} E_{\leq j} \left(\frac{-4}{n-3} \sum_{k=1}^{n-3} \frac{z_k^{j+1}}{(z_k-1)^2} \right), \tag{13}$$

where the z_k are now the $(n-3)$-th roots of $-\frac{1}{3}$.

4 On the roots of the Voronoi, circle and $E_{\leq k}$ polynomials

In this section, we study properties for the roots of these polynomials. By Proposition 1, $p_V(z)$ has the same roots as $p_C(z)$ plus the root $z=-1$. A direct relation between roots of polynomials and the rectilinear crossing number can be derived from the well-known relation

$$\frac{P'(z)}{P(z)} = \sum_{i=1}^{n} \frac{1}{z-a_i}, \tag{14}$$

where $P(z)$ is a polynomial of degree n with roots $a_1, \ldots, a_n$, and z is any complex number such that $P(z) \neq 0$. For the circle polynomial $p_C(z)$ and $z=1$, using Proposition 2 we get

$$\frac{\binom{n}{4} + \overline{cr}(S)}{\binom{n}{3}} = \sum_{i=1}^{n-3} \frac{1}{1-a_i}, \tag{15}$$

where the a_i are the roots of $p_C(z) = \sum_{k=0}^{n-3} c_k z^k$. Note that 1 is never a root of a polynomial whose coefficients are all positive, as is the case in the polynomials introduced in this work. Using Equation (14) and the reciprocal polynomial $p^*_C(z) = \sum_{k=0}^{n-3} c_k z^{n-k-3}$ we obtain

▶ **Proposition 6.**

$$\sum_{k=0}^{n-3} (n-k-3)c_k = 3\binom{n}{4} - \overline{cr}(S). \tag{16}$$

▶ **Proposition 7.**

$$\frac{-2\binom{n}{4} + 2\overline{cr}(S)}{\binom{n}{3}} = \sum_{i=1}^{n-3} \frac{1+a_i}{1-a_i}, \tag{17}$$

where the a_i are the roots of $p_C(z) = \sum_{k=0}^{n-3} c_k z^k$ (also works for the roots of $p_V(z)$).

Of particular interest is $p_V(z) = \sum_{k=1}^{n-1} v_k z^{k-1}$ for a set of n points in convex position. By Equation (7), $v_k = (2k-1)n - 2k^2$. By Equation (8), $p_V(z)$ is a palindromic polynomial, so it has roots a_i and $1/a_i$. Then, for sets S of n points in convex position,

$$\sum_{i=1}^{n-2} \frac{1}{1-a_i} = \frac{n-2}{2}, \tag{18}$$

where the a_i are the roots of $p_V(z) = \sum_{k=1}^{n-1} v_k z^{k-1}$. For S in convex position, from Proposition 7 we also have,

$$\sum_{i=1}^{n-2} \frac{1+a_i}{1-a_i} = 0. \tag{19}$$

For our next result we use a theorem due to Malik [23], also see [28], Corollary 14.4.2.

▶ **Theorem 4.1.** *Let S be a set of points in general position. Then S is in convex position if and only if all the roots of the Voronoi polynomial of S, $p_V(z) = \sum_{k=1}^{n-1} v_k z^{k-1}$, lie on the unit circle.*

In order to find a lower bound on the largest modulus of the roots of $p_C(z)$ with S not in convex position, we use two theorems. The first one is due to Laguerre [18], Theorem 1, see also [25], and [28], Theorem 3.2.1b. The second theorem is due to Obrechkoff [26], also see [10] and [22], Chapter IX, 41, Exercise 5.

▶ **Theorem 4.2.** *For every set S of $n > 3$ points in general position with rectilinear crossing number $\overline{cr}(S) = \alpha \cdot \binom{n}{4}$, the Voronoi polynomial $p_V(z) = \sum_{k=1}^{n-1} v_k z^{k-1}$ has a root of modulus at least $1 + \frac{(1-\alpha)\pi^2}{16(n-3)^2} + O\left(\frac{1}{n^4}\right)$.*

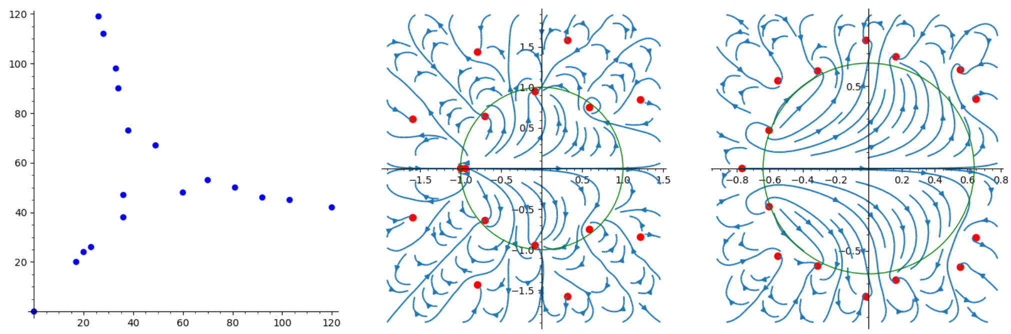

Figure 2 Left: Point set S minimizing the rectilinear crossing number for $n = 18$ [5]. With complex stream plots of its Voronoi (center), and $E_{\leq k}$ (right) polynomials; with roots as red points and circles illustrating, respectively, the bounds of Theorems 4.2 and 4.5.

For an illustration of Theorem 4.2 see Figure 2, center. We further show that the Voronoi polynomial $p_V(z)$ has a root close to point 1 in the complex plane. Thereto, we apply Theorem 1.2 from Michelen and Sahasrabudhe [24].

▶ **Theorem 4.3.** *Let α be a constant from $(0, 1]$ and let S be a set of n points in general position with $\frac{\overline{cr}(S)}{\binom{n}{4}} = \alpha$. Then the Voronoi polynomial of S, $p_V(z) = \sum_{k=1}^{n-1} v_k z^{k-1}$, has a root ζ such that $|1-\zeta| \in o\left(\frac{log(n)}{n}\right)$.*

In the following, we study the location of the roots of the $E_{\leq k}$ polynomial $p_E(z) = \sum_{k=0}^{n-3} E_{\leq k} z^k$ of a point set S. Note that its coefficients $E_{\leq k}$ form an increasing sequence of positive numbers. The well-known Eneström-Kakeya theorem [17] tells us that all the roots of $p_E(z)$ are contained in the unit disk, and more precisely, that they are contained in an annulus: The absolute values of the roots of $p_E(z)$ lie between the greatest and the least of

$$\frac{E_{\leq n-4}}{E_{\leq n-3}}, \frac{E_{\leq n-5}}{E_{\leq n-4}}, \ldots, \frac{E_{\leq 1}}{E_{\leq 2}}, \frac{E_{\leq 0}}{E_{\leq 1}}.$$

We give a lower bound on the largest modulus of the roots of the $E_{\leq k}$ polynomial.

▶ **Theorem 4.4.** *Let S be a set of $n > 3$ points in general position, with rectilinear crossing number $\overline{cr}(S) = \alpha \cdot \binom{n}{4}$. Then the $E_{\leq k}$ polynomial of S, $p_E(z) = \sum_{k=0}^{n-3} E_{\leq k} z^k$, has a root of modulus at least $\frac{3+\alpha}{9-\alpha}$.*

Next, we show a better lower bound on the largest modulus of $p_E(z)$ when n is large enough. Thereto, we apply a theorem of Titchmarsh ([31], p. 171), also see [16], Theorem A.

▶ **Theorem 4.5.** *Let S be a set of n points with h of them on the boundary of the convex hull of S, in general position. Then $p_E(z) = \sum_{k=0}^{n-3} E_{\leq k} z^k$ has a root of modulus at least*

$$\left(\frac{3\binom{n}{3}}{h} \right)^{-\frac{1}{n-3}}.$$

For an illustration of Theorem 4.5, see Figure 2, right.

5 Discussion

We have introduced three polynomials $p_V(z)$, $p_C(z)$, $p_E(z)$ for sets S of n points in general position in the plane, showing their connection to $\overline{cr}(S)$ and several bounds on the location of their roots. The obvious open problem is using bounds on such roots to improve upon the current best bound on the rectilinear crossing number problem. To the best of our knowledge, this approach has not been explored so far. Besides, we think that the presented polynomials are interesting objects of study on their own, given the many applications of Voronoi diagrams. For some of the formulas presented for one of the polynomials, like Equation (15), there are analogous statements for the other polynomials considered.

Further, several other polynomials on point sets can be considered. The reader interested in crossing numbers has probably in mind the j-edge polynomial $p_e(z) = \sum_{j=0}^{n-2} e_j z^j$ of a point set S. For this, the known formula for the rectilinear crossing number $\overline{cr}(S)$ in terms of the numbers of j-edges e_j of S, see [20], Lemma 5, translates into

$$2\overline{cr}(S) - 6\binom{n}{4} = p_e''(1) - (n-3)p_e'(1). \tag{20}$$

As is the case for the Voronoi polynomial and the circle polynomial, for sets S of n points in convex position, the j-edge polynomial $p_e(z)$ has all its roots on the unit circle. This is readily seen since $p_e(z)$ is then n times the all-ones polynomial, $p_e(z) = n \sum_{j=0}^{n-2} z^j$, as $e_j = n$ for all j, if S is in convex position. Its roots are the $(n-1)$-th roots of unity, except $z = 1$.

We finally propose to study the presented polynomials for random point sets. The expected rectilinear crossing number is known for sets of n points chosen uniformly at random from several convex shapes K, see e.g. [30], Section 1.4.5. pp. 63–64, and [3].

Acknowledgments. M. C., A. d.-P., C. H., and D. O. were supported by project PID2019-104129GB-I00/MCIN/AEI/10.13039/501100011033. C.H. was supported by project Gen. Cat. DGR 2021-SGR-00266. D. F.-P. was supported by grants PAPIIT IN120520 and PAPIIT IN115923 (UNAM, México).

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