Synchronization of complex-valued chaotic systems

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Abstract: Synchronization is a collective behaviour in a complex dynamical network by which the trajectories of all its agents converge to a common state due to the inherent coupling between them plus, in some cases, the existence of external inputs. Due to obvious difficulty reasons, networks of chaotic systems have often been taken as benchmark examples where to apply synchronization techniques. In turn, complex-valued dynamical networks are gaining research interest because of the variety of physical magnitudes that allow a complex-variable representation. In this article, synchronization in complex-valued chaotic systems is firstly induced by feedback and adaptive control techniques, and then, a complex-valued sliding mode control strategy is tested for the same purpose. Numerical validations carried out using Matlab show that the complex sliding mode controller outperforms both the feedback and the adaptive controllers.

I. INTRODUCTION

Networks of coupled dynamical systems have arisen general interest by their ability to represent large-scale and complex physical systems. In these systems the nodes denote the individual dynamical system and the edges denote the several interaction between them. An interesting feature of these systems is the emergence of collective behaviours, such as synchronization. This is particularly relevant in chaotic systems, as they defy synchronization.

In order to induce this outcome many control schemes can be used, such as feedback control, adaptive control, intermittent control, pinning control...

On the other hand, many physical problems require the implementation of complex variables such as Lorenz systems or rotating fluids. Therefore, many differences and aspects about stability and dynamical analysis should be taken into account.

Driven by the above discussions, this paper considers the synchronization of network-coupled complex-variable chaotic systems with complex couplings. Based on the Lyapunov stability theory, sufficient conditions for synchronization of the network via feedback control are derived in [1]. Furthermore, adaptive technique and sliding mode control (also with hysteresis and boundary layer) are designed for more practical applications.

This paper is organized as follows. In Section II a brief description of the target system and the control techniques is provided. Section III gathers the numerical simulations. A comparison of the performance of the different control approaches is carried out in Section IV. Finally, conclusions are drawn in Section V.

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II. MATHEMATICAL ASPECTS

The complex-variable dynamical network with complex coupling can be described by the following equations:

$$\dot{y}_k(t) = g\left(y_k(t), z_k(t)\right) + \varepsilon \sum_{l=1}^{N} c_{kl}\Gamma_1 y_l(t) \tag{1a}$$

$$\dot{z}_k(t) = h\left(y_k(t), z_k(t)\right) + \varepsilon \sum_{l=1}^{N} d_{kl}\Gamma_2 z_l(t), \tag{1b}$$

where $\varepsilon > 0$ is the coupling strength, $\Gamma_1 = \mathrm{diag}(\gamma_1^1, ..., \gamma_1^m)$ and $\Gamma_2 = \mathrm{diag}(\gamma_2^1, ..., \gamma_2^n)$ are the inner coupling matrices, and Cc_{kl} and Dd_{kl} are the zero-row-sum outer coupling matrices.

Let $x_k(t) = (y_k^T(t), z_k^T(t))^T$ and $f(x_k(t)) = (g^T(y_k, z_k), h^T(y_k, z_k))^T$, then the controlled network can be written as

$$\begin{aligned}\dot{x}_k(t) =& f\left(x_k(t)\right) + \varepsilon \sum_{l=1}^{N} c_{kl}\tilde{\Gamma}_1 x_l(t) + \\ &+ \varepsilon \sum_{l=1}^{N} d_{kl}\tilde{\Gamma}_2 x_l(t) + u_k(t),\end{aligned} \tag{2}$$

where $\tilde{\Gamma}1 = \mathrm{diag}(\underbrace{\gamma_1^1, \dots, \gamma_1^m}_{m}, \underbrace{0, \dots, 0}_{n})$ and $\tilde{\Gamma}2 = \mathrm{diag}(\underbrace{0, \dots, 0}_{m}, \underbrace{\gamma_2^1, \dots, \gamma_2^n}_{n})$ are now the inner coupling matrices.

Let the error be: $e_k(t) = x_k(t) - s_k(t)$, with s(t) being a solution of an isolated node satisfying $\dot{s}(t) = f(s(t))$. Then error dynamics is:

$$\begin{aligned}\dot{e}_k(t) =& f\left(x_k(t)\right) - f\left(s_k(t)\right) + \varepsilon \sum_{l=1}^{N} c_{kl}\tilde{\Gamma}_1 e_l(t) + \\ &+ \varepsilon \sum_{l=1}^{N} d_{kl}\tilde{\Gamma}_2 e_l(t) + u_k(t).\end{aligned} \tag{3}$$

Therefore, synchronization is achieved if $\lim_{t\to\infty} ||e_k(t)|| = 0$. The following controllers will be tested:

1. A linear feedback controller [1]:

$$u_k(t) = -\varepsilon\theta_k\tilde{\Gamma}e_k(t), \qquad k = 1, 2, \dots, N, \tag{4}$$

where $\tilde{\Gamma} = \tilde{\Gamma}_1 + \tilde{\Gamma}_2$, $\theta_k > 0$ are constants.

2. An adaptive controller [1]:

$$\begin{aligned} &u_k(t) = -\varepsilon\theta_k\tilde{\Gamma}e_k(t), \\ &\dot{\theta}_k(t) = \eta_k e_k^T(t)\tilde{\Gamma}^s\overline{\mathrm{e_k(t)}}, \qquad k = 1, 2, \dots, N, \end{aligned} \tag{5}$$

where $\eta_k > 0$ are the adaptive gains and $\tilde{\Gamma}^s = \tilde{\Gamma}^T + \overline{\tilde{\Gamma}}$.

3. A sliding mode controller [2]:

$$u_k(t) = -\kappa sign(e_k(t)) = -\kappa\frac{e_k(t)}{||e_k(t)||}, \qquad \kappa \in \mathbb{C}. \tag{6}$$

Sufficient conditions to guarantee synchronization are derived in [1] for the linear and adaptive controllers, and follow straightforwardly from [2] for the sliding mode controller.

III. NUMERICAL SIMULATIONS

Simulations are conducted on the complex-variable Chen system of the form (1):

$$\begin{cases} \dot{y}_{k_1} = \mu\left(y_{k_2} - y_{k_1}\right) \\ \dot{y}_{k_2} = (\omega - \mu)y_{k_1} - y_{k_1}z_k + \omega y_{k_2} \\ \dot{z}_k = \left(\bar{y}_{k_1}y_{k_2} + y_{k_1}\bar{y}_{k_2}\right)/2 - \nu z_k, \end{cases} \tag{7}$$

where y_{k_1} and y_{k_2} are complex variables and z_k is real. In order to show its chaotic behavior one chooses $\mu = 27$, $\nu = 1$, $\omega = 23$, and $y_{k1}(0) = 2 + j$, $y_{k2}(0) = 2 + j$ and $z_k(0) = 2 + j$ as initial conditions.

The chaotic attractor behaviour of the network is seen in Fig. 1, while orbits are portrayed in Fig. 2.

Once the individual dynamics is shown, different control methods are used in the interest of achieving synchronization of a network with five coupled complex-variable Chen systems (7).

The complex inner coupling matrices in (2),(3) are $\tilde{\Gamma}_1$ =diag$(1 + j, 1 - j, 0)$ and $\tilde{\Gamma}_2$ =diag$(0, 0, 1)$, and the outer couplings, also complex, are the following:

$$C = \begin{bmatrix} -4-5j & 3+3j & 0 & 1+2j & 0 \\ 1-3j & -2-j & 1+4j & 0 & 0 \\ 0 & 3+4j & -3+j & -2j & -3j \\ 2j & 0 & 2-2j & -6-3j & 4+3j \\ 0 & 0 & 1+3j & 1-j & -2-2j \end{bmatrix}$$

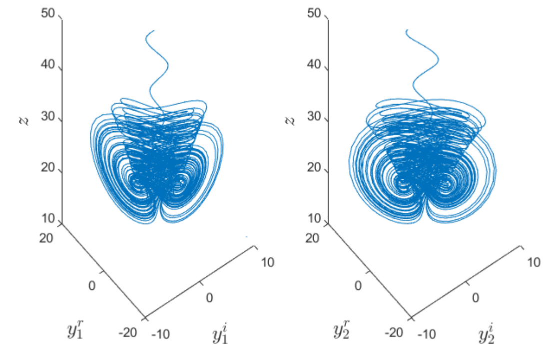

FIG. 1. Chaotic attractor of the complex-variable Chen system.

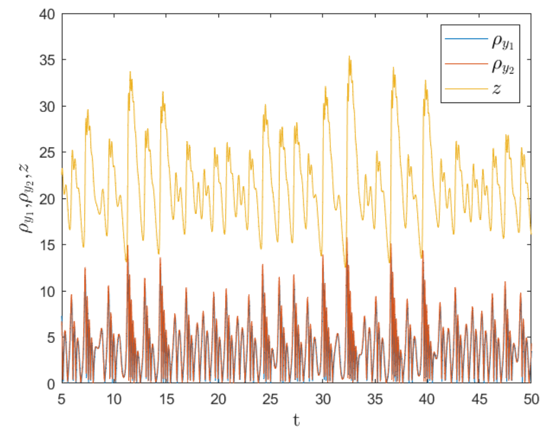

FIG. 2. Orbits of $z(t)$ and modules $\rho_{y_j}(t)$, j=1,2.

$$D = \begin{bmatrix} -5 & 3 & 0 & 2 & 0 \\ -3 & -1 & 4 & 0 & 0 \\ 0 & 4 & 1 & -2 & -3 \\ 2 & 0 & -2 & -3 & 3 \\ 0 & 0 & 3 & -1 & -2 \end{bmatrix}.$$

Moreover, the coupling strength is chosen as $\epsilon = 2$, and the initial conditions as $x_k(0) = (k + kj, k - kj, k)^T$, $k = 1, \dots, 5$.

The performance of the linear feedback controller (4) with $\theta_k = 11$ is illustrated in Fig. 3, while that of the adaptive controller (5) with $\nu_k = 0.01$ and $\theta_k(0) = 11$ is portrayed in Fig. 4. In both cases the errors tend to 0 and the controllers are stabilized.

In turn, it can be clearly seen in Fig. 5 that synchronization is also achieved with the sliding mode controller (6) with gain $\kappa = 398e^{\frac{i\pi}{20}}$.

It is well known that practical implementations of sliding mode controllers do not allow an infinite switching frequency of the control signal. Instead, regularization techniques have to be applied in a neighborhood of the switching surface to make it feasible. In this paper we

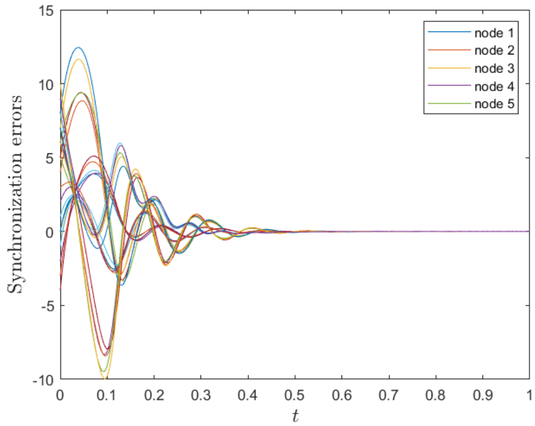

FIG. 3. Error dynamics using linear feedback control.

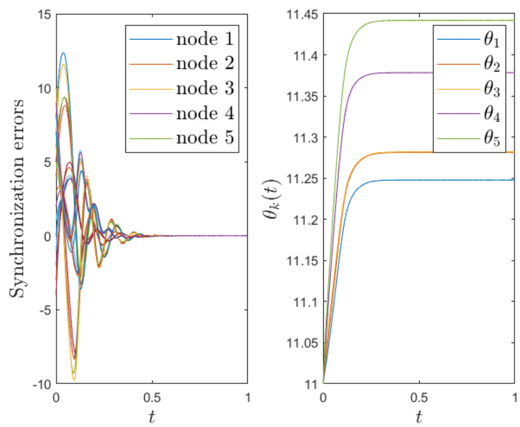

FIG. 4. Error dynamics using adaptive controllers.

select, and compare, hysteric and boundary layer implementations [2]. In both cases, which also use the control gain of the ideal implementation, the chattering phenomenon is observed.

Sliding mode control-based hysteric controllers are implemented as:

$$u = \begin{cases} -\kappa\frac{e}{|\epsilon|} & \text{if } |e| > \epsilon_h \\ u_{k-1} & \text{if } |e| \leq \epsilon_h, \end{cases} \tag{8}$$

where $\epsilon_h = 0.1$ has been chosen. The results are in Fig. 6.

The amplitude of the chattering depends on ϵ_h and on $|\kappa|$ if the integration step is smaller than ϵ_h, which is the case.

Instead, boundary layer implementations use:

$$u = \begin{cases} -\kappa\frac{e}{|e|} & \text{if } |e| > \epsilon_b \\ -\kappa\frac{e}{\epsilon_b} & \text{if } |e| \leq \epsilon_b, \end{cases} \tag{9}$$

with $\epsilon_h = 0.1$. The results are now in Fig. 7. Again, the amplitude of the chattering in this case depends on

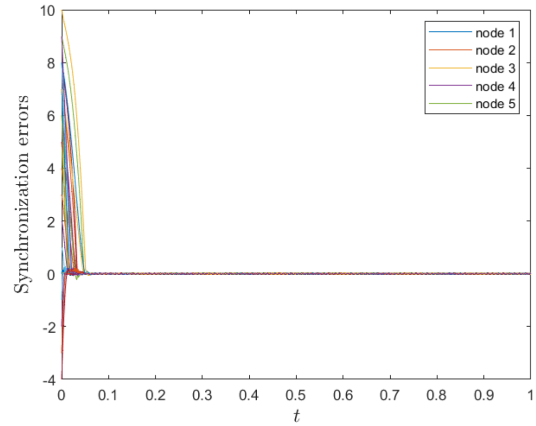

FIG. 5. Error dynamics using sliding mode control.

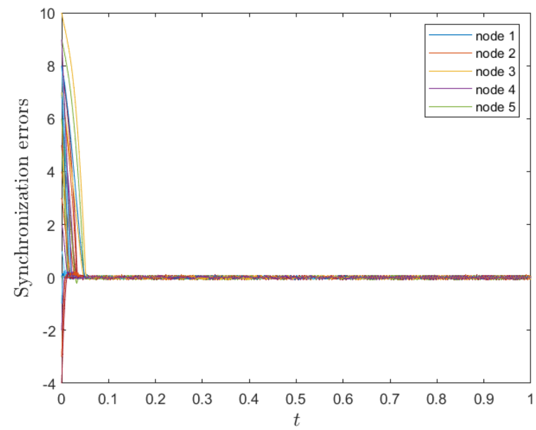

FIG. 6. Error dynamics using sliding mode control with a hysteretic implementation.

ϵ_b and on $|\kappa|$ if the integration step is smaller than ϵ_b, which is the case.

IV. COMPARISON OF THE PERFORMANCE

In order to conduct a fair comparison between the different control systems, they have to be tested on a level playing field. The feature that has been equalised in this case is the ∞-norm of the control input ($||u||_\infty(t) = max_i|u_i(t)|$) around 398, as it can be seen in Fig. 8. In order to achieve this condition, several parameters such as gains and boundaries have been adjusted.

Under these conditions, the average error of each network has been computed and the results are seen in Fig. 9. One can easily see how linear and adaptive controllers have a similar behaviour achieving synchronization in 5 tenths of a second. In contrast, the controllers related to sliding mode control achieve it much earlier, in less than one tenth of a second.

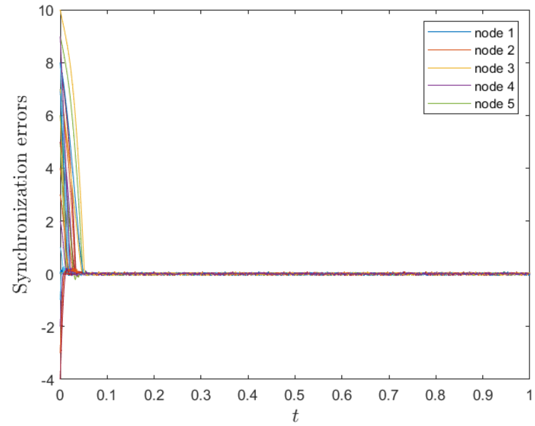

FIG. 7. Error dynamics using sliding mode control with a boundary layer implementation.

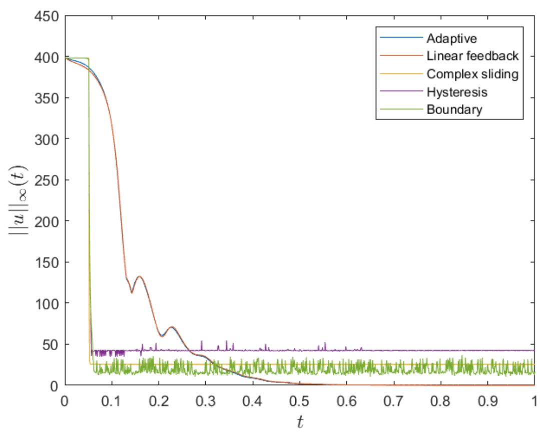

FIG. 8. Control strength of the different controllers.

These results could be foreseen, as the figures shown in the numerical simulations section have been carried with the parameters we have adjusted to equalize $||u||_\infty$.

V. CONCLUSION

Different control approaches have been discussed in order to achieve synchronization in complex valued chaotic system. It has been showed that through all of them synchronization can be achieved, however, there are better approaches than others.

By imposing equal initial control strength, it has been concluded that the best system, in terms of fastest settling time, is the complex sliding control. This control technique, in terms of implementation, may be achieved by means of hysteresis or boundary layer control, which offer a similar response.

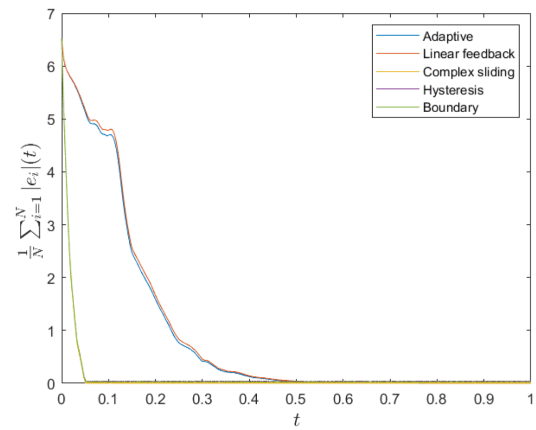

FIG. 9. Average error using the same initial control strength.

VI. REFERENCES

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