Dynamic Feedback Linearizability of Mobile Vehicles with Active Caster Wheels

Xavier Povill Clarós,* Daniel Quiles Pastor, Bernat Ramis Vich, and Lara San Martín Suárez
Universitat Politècnica de Catalunya
(Barcelona Tech)
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In this article, a controller for the non-linear dynamics of a caster wheels vehicle is designed, using the method of dynamic feedback linearization. Numerical simulations of the performance of the controller are included.

Keywords: Actively driven caster wheels, dynamic feedback linearization, differential flatness, non-linear control

The aim of this article is to design a non-linear controller for the dynamical system arising from the equations of motion of a caster wheels vehicle (given the initial position of the vehicle and the desired final position).

I. INTRODUCTION

In this first part, the basics of the dynamic feedback linearization techniques that will be used later will be reviewed.

In linear control theory, there exists a characterization of controllable systems based on the controllability matrix [1]. However, finding out whether a non-linear system is controllable and, if it is so, designing a control, still remains an open problem.

The models considered are expressed in *affine form*:

$$\dot{x} = f(x) + \sum_{k=1}^{m} g_k(x)u_k \tag{1}$$

where $x \in \mathbb{R}^n$ is the vector of state variables and $u_k \in \mathbb{R}$, $k = 1, \dots m$ are the control variables. Consider that $f(x), g_k(x) \in \mathbb{R}^n$ are known vector functions of the state variables. The aim consists in choosing the control functions $u_k(x)$, $k = 1, \dots m$ such that x goes from the starting state x_0 to a certain final state x_f. Notice, however, that in some cases this is not possible.

If the system was single input ($m = 1$) we would wish to find a diffeomorphism $T : \mathbb{R}^n \to \mathbb{R}^n$ for the state variables $y = T(x)$, together with a feedback law $\bar{u} = \alpha(x) + \beta(x)u$, such that, in the new state variables y, the system reads:

$$\dot{y} = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ & \vdots & & \ddots & \\ 0 & \dots & 0 & 0 & 1 \\ 0 & \dots & 0 & 0 & 0 \end{pmatrix} y + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \bar{u} \tag{2}$$

This is called *Brunovsky canonical form.* A system is said to be *static feedback linearizable* (SFL) if there exists such a diffeomorphism T (and a feedback law) that turns the system into (2) [2]. The system (2) is clearly controllable, since the last state variable is accessible via a control, and the variable y_i has access to the variable y_{i-1}, for $i = 2, \dots, n$.

In the multi-input case ($m > 1$), it is desirable to find the diffeomorphism T that decouples the system into m independent subsystems, such that each subsystem looks like (2) and each state variable is not involved in any other subsystem but its own.

Since the state variables belong to a differentiable manifold, let us introduce several tools that come in handy.

A distribution D generated by a set of vector fields $\{X_1, \dots, X_r\}$ is the set of all vector fields over a manifold that can be generated by linear combinatons of $\{X_1, \dots, X_r\}$ using any set of functions $\{F_1, \dots, F_r\}$ over the manifold as coefficients: $D = \{\sum_{i=1}^{r} F_i(x)X_i(x),$ for any set of functions $F_1, \dots, F_r\}$.

Also recall the concept of Lie derivative of functions and vector fields (an introduction to the topic can be found in [3]). Notice, as a particular case, that the temporal derivative of a function of the state variables is equivalent to computing the following Lie derivative: $\dot{h}(x) = \mathcal{L}_f h(x) + \sum_{i=1}^{m} \mathcal{L}_{g_i} h(x)u_i$.

A distribution D generated by the vector fields $\{X_1, \dots, X_r\}$ is said to be involutive if $[X_i, X_j] = 0$ for any pair of generating fields, $i, j = 1, \dots, r$.

All in all, the algorithm we follow to static feedback linearize systems in affine form is:

1. Compute the following distributions:
   - $D_0 =< g_1, g_2, \dots, g_m >$
   - $D_1 =< g_1, g_2, \dots g_m, ad_f g_1, \dots ad_f g_m >$
   - $\dots$
   - $D_k =< g_1, g_2, \dots, ad_f g_1, \dots ad_f g_m, \dots, g_m, ad_f^k g_1, \dots ad_f^k g_m >$

   up to $k \in \mathbb{N}$ such that $D_k = \mathbb{R}^n$. Note that the notation $ad_f^r g_i$ stands for the r-th Lie bracket of the vector fields g_i with respect to f, i.e. $ad_f^r g_i := [f, [\dots^{(r)}, [f, g_i] \dots]$.

* xavier.povill@estudiantat.upc.edu

2. Check that all these distributions are involutive. Otherwise, the system will not linearizable.

3. If d_i is the dimension of D_i, define $r_0 = d_0$, $r_1 = d_1 - d_0, \ldots, r_k = d_k - d_{k-1}$; and let k_j, $j = 1, \ldots, m$ be the number of $r_i, i = 1, \ldots, k$ that are greater or equal to j.

4. Find m functions $h_1, \ldots, h_m$ such that $dh_i \perp D_{k_i - 2}$ and such that $h_1, \ldots, h_m$ are differentially independent. This step consists in solving a system of partial differential equations.

5. The diffeomorphism is given by the $h_i, i = 1, \ldots, m$ functions and their Lie derivatives:

$$y = \begin{pmatrix} h_1(x) \\ \mathcal{L}_f h_1(x) \\ \vdots \\ \mathcal{L}_f^{k_1-1} h_1(x) \\ h_2 \\ \vdots \\ \mathcal{L}_f^{k_2-1} h_2(x) \\ \vdots \\ h_k \\ \vdots \\ \mathcal{L}_f^{k_m-1} h_m(x) \end{pmatrix} \tag{3}$$

Moreover, the feedback control law is:

$$W = \begin{pmatrix} \mathcal{L}_f^{k_1} h_1(x) \\ \mathcal{L}_f^{k_2} h_2(x) \\ \vdots \\ \mathcal{L}_f^{k_m} h_m(x) \end{pmatrix} + \begin{pmatrix} \sum_{i=1}^m \mathcal{L}_{g_i} \mathcal{L}_f^{k_1-1} h_1(x) u_i \\ \sum_{i=1}^m \mathcal{L}_{g_i} \mathcal{L}_f^{k_2-1} h_2(x) u_i \\ \vdots \\ \sum_{i=1}^m \mathcal{L}_{g_i} \mathcal{L}_f^{k_m-1} h_m(x) u_i \end{pmatrix} \tag{4}$$

As remarked in step 2, a system may not be linearizable through this algorithm. Luckily, there exist techniques to transform a system into an equivalent one that is static feedback linearizable. The whole process is called dynamic feedback linearization.

A *prolongation* of a system like (1) consists in setting some controls u_k as state variables as follows:

For a given control variable u_i, add the new state variables $z_1, \ldots, z_r$ to the system:

$$\begin{aligned} z_1 &= u_i \\ \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\cdots \\ \dot{z}_r &= v \end{aligned}$$

where v is the new control. This prolongation can be applied for as many control variables as wished and for any $r \in \mathbb{N}$.

II. KINEMATIC MODELS

The kinematic model of a mobile vehicle with two caster wheels will be presented in order to study the effect of adding an offset in one wheel to the control of the system. In particular, it will be considered that each wheel has two rotating axes along the normal vector and parallel vector to the ground plane. Coordinates (x, y) will refer to the connection of the front caster wheel with the board in the inertial frame, θ to the angular orientation of the board and θ_i to the angular orientation of the front $(i = 1)$ and back $(i = 2)$ caster wheels with respect to the vehicle. The mobile vehicle is assumed to be a rigid body, the caster wheels are assumed to not slip on the ground and the height of the vehicle is assumed to remain constant. The connections of the caster wheels with the board are equally spaced to the center of the board by a distance a. The parameter γ_2 is defined as $l_2 \sin \beta_2$, the distance offset to the back caster wheel, where β_2 corresponds to the angle offset of the back caster wheel with respect to the perpendicular position. Fig. 1 provides a schematic of the vehicle.

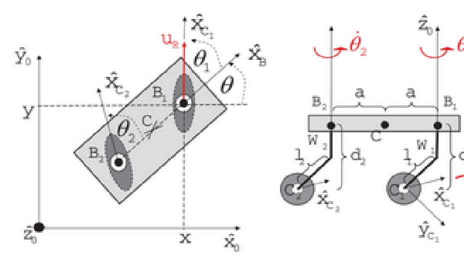

FIG. 1. Schematic of the mobile vehicle with two caster wheels and offsets.

The dynamic equations of the system are presented in (5). Two variations of the problem will be studied:

- Model 1: $\gamma_2 = 0$
- Model 2: $\gamma_2 \neq 0$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cos(\theta + \theta_1) \\ 0 & 0 & \sin(\theta + \theta_1) \\ 0 & \frac{-\gamma_2}{\gamma_2 + 2a\cos\theta_2} & \frac{\sin(\theta_1 - \theta_2)}{\gamma_2 + 2a\cos\theta_2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_R \end{pmatrix} \tag{5}$$

III. CONTROL DESIGN

In both systems it is easy to check that they are not SFL since the distribution $D_0 =< g_1, g_2, g_3 >$ generated by the control functions is not involutive, in both cases because $[g_1, g_3]$ is not generated by D_0. That means that it is necessary to find a prolongation to get an SFL system.

In both models the prolongation is the same. It is defined by taking $z = u_R$ as a new state variable. Let v be the control of this new variable. Then the new system is:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \cos(\theta+\theta_1)z \\ \sin(\theta+\theta_1)z \\ \frac{\sin(\theta_1-\theta_2)}{\gamma_2+2a\cos\theta_2}z \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{-\gamma_2}{\gamma_2+2a\cos\theta_2} & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ v \end{pmatrix} \tag{6}$$

The corresponding distributions that are obtained from that system are (using tangent space notation):

$$\begin{aligned} D_0 =& < \frac{\partial}{\partial\theta_1}, \frac{\gamma_2}{\gamma_2+2a\cdot\cos(\theta_2)}\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\theta_2}, \frac{\partial}{\partial z} > \\ D_1 =& < D_0, \frac{-\gamma_2\sin(\theta+\theta_1)z}{\gamma_2+2a\cdot\cos(\theta_2)}\frac{\partial}{\partial x} + \frac{\gamma_2\cos(\theta+\theta_1)z}{\gamma_2+2a\cdot\cos(\theta_2)}\frac{\partial}{\partial y} + \\ &+ \frac{(\gamma_2\cos(\theta_1-\theta_2)+2a\cdot\cos(\theta_1))z}{(\gamma_2+2a\cdot\cos(\theta_2))^2}\frac{\partial}{\partial\theta}, \sin(\theta+\theta_1)z\frac{\partial}{\partial x} - \\ &- \cos(\theta+\theta_1)z\frac{\partial}{\partial y} - \frac{\cos(\theta_1-\theta_2)z}{\gamma_2+2a\cdot\cos(\theta_2)}\frac{\partial}{\partial\theta}, -\cos(\theta+\theta_1)\frac{\partial}{\partial x} - \\ &- \sin(\theta+\theta_1)\frac{\partial}{\partial y} - \frac{\sin(\theta_1-\theta_2)}{\gamma_2+2a\cdot\cos(\theta_2)}\frac{\partial}{\partial\theta} > \end{aligned} \tag{7}$$

Clearly D_0 is involutive, because no pair of the g_i vectors has a cross dependence between variables. Moreover, the distribution D_0 has constant rank if $\gamma_2+2a\cdot\cos(\theta_2) \neq 0$. D_1 has 6 components, so we only need to check that they are independent (that means that they span $\mathbb{R}^6$). The determinant of the vectors is:

$$\det(D_1) = z^2\frac{2a\cdot\cos(\theta_1)}{(\gamma_2+2a\cdot\cos(\theta_2))^2} \tag{8}$$

According to equation (8), $D_1 = \mathbb{R}^6$ if $z \neq 0$ (meaning the control $u_R \neq 0$), $\theta_1 \neq \pm\pi/2$ (meaning the front wheel is not perpendicular to the orientation of the vehicle) and $\theta_2 \neq \arccos(-\gamma_2/2a)$, which limits the maximum angle of the back wheel and is also necessary for having the vectors of both distributions be well defined. In case of the model 1, with $\gamma_2 = 0$, the last condition simplifies to $\theta_2 \neq \pm\pi/2$

Therefore, under the restrictions listed above, with this prolongation both models are SFL.

The diffeomorphism to obtain the Brunovsky canonical form will now be constructed. According to the theory explained above, one can find that $r_0 = d_0 = 3$, $r_1 = d_1 - d_0 = 6 - 3 = 3$ (d_i = dimension of D_i) $\Rightarrow k_1 = k_2 = k_3 = 2$. So it is necessary to find $dh_1, dh_2, dh_3 \perp D_0$

A. No offset

In this model, the functions h_i are trivially obtained:

$$h_1 = x \qquad h_2 = y \qquad h_3 = \theta$$

Computing the temporal derivatives of the h_i functions we get the diffeomorphism:

$$y = \begin{pmatrix} y_1 \\ \dot{y}_1 \\ y_2 \\ \dot{y}_2 \\ y_3 \\ \dot{y}_3 \end{pmatrix} = \begin{pmatrix} x \\ \cos(\theta+\theta_1)z \\ y \\ \sin(\theta+\theta_1)z \\ \theta \\ \frac{\sin(\theta_1-\theta_2)z}{2a\cdot\cos(\theta_2)} \end{pmatrix} \tag{9}$$

And the regular feedback law:

$$\begin{aligned} W = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} &= \begin{pmatrix} \frac{-\sin(\theta+\theta_1)\sin(\theta_1-\theta_2)z^2}{2a\cdot\cos(\theta_2)} \\ \frac{\cos(\theta+\theta_1)\sin(\theta_1-\theta_2)z^2}{2a\cdot\cos(\theta_2)} \\ 0 \end{pmatrix} + \\ &+ \begin{pmatrix} -\sin(\theta+\theta_1)z & 0 & \cos(\theta+\theta_1) \\ \cos(\theta+\theta_1)z & 0 & \sin(\theta+\theta_1) \\ \frac{\cos(\theta_1-\theta_2)z}{2a\cdot\cos(\theta_2)} & \frac{\cos(\theta_1)z}{2a\cdot\cos^2(\theta_2)} & \frac{\sin(\theta_1-\theta_2)}{2a\cdot\cos(\theta_2)} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ v \end{pmatrix} \end{aligned} \tag{10}$$

Where $w_i = \ddot{y}_i$ is the control in the Brunovsky form. Then, by inverting the feedback law, it is possible to get the original controls from the Brunovsky form ones (w_i):

$$\begin{aligned} v =& \cos(\theta+\theta_1)w_1 + \sin(\theta+\theta_1)w_2 \\ u_1 =& -\frac{\sin(\theta+\theta_1)}{z}w_1 + \frac{\cos(\theta+\theta_1)}{z}w_2 - \frac{\sin(\theta_1-\theta_2)z}{2a\cdot\cos(\theta_2)} \\ u_2 =& \frac{2a\cdot\cos^2(\theta_2)}{z\cdot\cos(\theta_1)}\left(\frac{\cos(\theta_1-\theta_2)z}{2a\cdot\cos(\theta_2)}u_1 + \frac{\sin(\theta_1-\theta_2)}{2a\cdot\cos(\theta_2)}v - w_3\right) \end{aligned} \tag{11}$$

Now, we have all the tools to make a MATLAB program able to solve the initial and final values problem, computing the values of the controls along time.

It is only necessary to decide initial and final values of the state variables in the original system, and then transform these conditions through the diffeomorphism for y. From these conditions, one can build an interpolating polynomial, which will be used to build a trajectory for the y variables in Brunovsky's form. Then, the new control laws w are taken to be the k_i-th derivatives of these polynomials. Finally, as mentioned above, the original controls u_1, u_2, v can be recovered using the equations in (11).

B. Offset in second wheel

In this model, the functions h_i can be taken to be:

$$h_1 = x \qquad h_2 = y$$

$$h_3 = \theta + \frac{\gamma_2}{\sqrt{4a^2-\gamma_2^2}} \log \frac{\sqrt{4a^2-\gamma_2^2} + (2a-\gamma_2)\tan(\theta_2/2)}{\sqrt{4a^2-\gamma_2^2} - (2a-\gamma_2)\tan(\theta_2/2)}$$

Where h_3 is a solution of the partial differential equation:

$$\frac{\partial h_3}{\partial \theta}\left(\frac{-\gamma_2}{\gamma_2 + 2a\cdot\cos(\theta_2)}\right) + \frac{\partial h_3}{\partial \theta_2} = 0$$

It is easy to check that h_3 solves the equation and, taking into account that $2a$ is the distance between axis while γ_2 is the offset distance of the caster wheel, it seems safe to suppose that $2a > \gamma_2$, so all the square roots have real solution.

As before, the new state functions y can be obtained from the h_i and their first derivatives. As h_1 and h_2 are the same as in the previous case, y_1 and y_2 as the same as before. The only changes are in $y_3 = h_3$ (the new one) and in $\dot{y}_3$. It can be checked that the derivative reduces to:

$$\dot{y}_3 = \frac{\sin(\theta_1-\theta_2)z}{\gamma_2 + 2a\cdot\cos(\theta_2)}$$

Computing the second derivative as before, one can obtain W for the Brunovsky form system. Then, inverting the relation, again one can find the original controls in terms of the controls of the Brunovsky form: v has the same form as in the previous model (see 11) and the new controls u_1 and u_2 are:

$$\begin{aligned} u_1 &= -\frac{\sin(\theta+\theta_1)}{z}w_1 + \frac{\cos\theta+\theta_1)}{z}w_2 - \\ &- \left(\frac{\sin(\theta_1-\theta_2)z}{\gamma_2+2a\cdot\cos(\theta_2)} - \frac{\gamma_2}{\gamma_2+2a\cdot\cos(\theta_2)}u_2\right) \\ u_2 &= \tilde{u}_2\cdot\frac{z\cdot\cos(\theta_1)}{2a\cdot\cos^2(\theta_2)}\cdot\frac{(\gamma_2+2a\cdot\cos(\theta_2))^2}{(\gamma_2\cos(\theta_1-\theta_2)+2a\cdot\cos(\theta_1))} \end{aligned} \tag{12}$$

Where $\tilde{u}_2$ is u_2 from model 1 (11). As u_2 only depends on u_1 and the state variables, substituting it in u_1 one can obtain u_1 as a function of the state variables and the Brunovsky form controls w_i. As before, we have now all the tools necessary to solve the problem of initial and final values numerically.

IV. NUMERICAL SIMULATION

The previous controller has been implemented in Matlab. To solve the system of ODE's, the numerical integrator `ode45` is used. This integrator is based on the Dormand-Prince method (a combination of 4th and 5th order Runge-Kutta). The chosen initial and final conditions are

$$\begin{pmatrix} x_0 \\ y_0 \\ \theta_0 \\ \theta_{10} \\ \theta_{20} \\ z_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} x_f \\ y_f \\ \theta_f \\ \theta_{1f} \\ \theta_{2f} \\ z_f \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \\ 0.9 \\ 0.3 \\ 0.5 \\ 1 \end{pmatrix}$$

together with parameters $T = 2$ and $a = 5$. After the numerical simulation, the error of the final value of the state variables in 2-norm is $7.83\cdot10^{-4}$ for the first model ($\gamma_2 = 0$) and $1.35\cdot10^{-3}$ for the second model ($\gamma_2 = 1$). Figure 2 shows the time evolution of both systems, which is quite similar. One can observe that the trajectories are quite smooth and monotonous (except for the angles of the wheels θ_1, θ_2 and the control $z = u_R$). In order to implement the system in practice, the remaining controls could be recovered from equations (11) and (12).

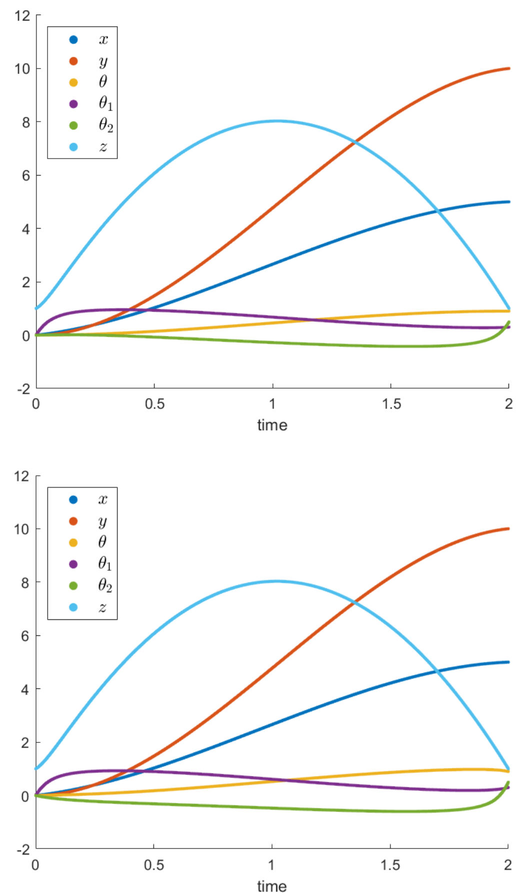

FIG. 2. Time-evolution of the model with $\gamma_2 = 0$ (up) and $\gamma_2 = 1$ (down).

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