Primitive ideals in rational, nilpotent Iwasawa algebras

DOI:
10.1016/j.aim.2022.108371

Advances in Mathematics 403 (2022) 108371

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Primitive ideals in rational, nilpotent Iwasawa algebras

Adam Jones

ARTICLE INFO

Article history:
Received 9 June 2021
Received in revised form 15 March 2022
Accepted 19 March 2022
Available online 1 April 2022
Communicated by Roman Bezrukavnikov

MSC:
primary 14G22, 16D25, 16W60, 17B10, 22E35
secondary 16S34, 22E60

Keywords:
Non-commutative algebra
Representation theory
Compact p-adic Lie groups
Iwasawa algebras
Rigid geometry
Primitive ideals

ABSTRACT

Given a p-adic field K and a nilpotent uniform pro-p group G, we prove that all primitive ideals in the K-rational Iwasawa algebra KG are maximal, and can be reduced to a particular standard form. Setting $\mathcal{L}$ as the associated $\mathbb{Z}_p$-Lie algebra of G, our approach is to study the action of KG on a Dixmier module $\widehat{D(\lambda)}$ over the affinoid envelope $\widehat{U(\mathcal{L})}_K$, and to prove that all primitive ideals can be reduced to annihilators of modules of this form.

Contents

E-mail address: adam.jones-5@manchester.ac.uk.

https://doi.org/10.1016/j.aim.2022.108371

1. Introduction

Fix $p > 2$ a prime, and let $K/\mathbb{Q}_p$ be a finite extension with ring of integers $\mathcal{O}$, uniformiser π, residue field k.

1.1. Background

Let G be a compact p-adic Lie group, and recall that we define the *completed group algebra* of G over $\mathcal{O}$ as:

$$\mathcal{O}G := \varprojlim \mathcal{O}[G/N] \tag{1}$$

where the limit is taken over all open normal subgroups N of G. Continuous, $\mathcal{O}$-linear representations of G are closely related to $\mathcal{O}G$-modules.

This paper is part of an ongoing project to classify the prime ideal structure of $\mathcal{O}G$, towards which much progress has been made in [7], [2], [6], [16] and [15]. In the same vein as those works, we aim to prove that all prime ideals in $\mathcal{O}G$ can be reduced to a particular standard form. Specifically, recall the following definition [15, Definition 1.1]:

Definition 1.1. We say that a prime ideal P of $\mathcal{O}G$ is *standard* if there exists a closed, normal subgroup H of G such that:

- $G_0 := \frac{G}{H}$ is torsionfree.

- $H = (P+1) \cap G$.
- The image of P in $\mathcal{O}G_0$ is centrally generated.

We say that P is *virtually standard* if $P \cap \mathcal{O}U$ is a finite intersection of standard prime ideals of $\mathcal{O}U$ for some open normal subgroup U of G.

The essence of this definition is that P is standard when it can be constructed using only *augmentation ideals* of the form $(H-1)\mathcal{O}G$, for H a closed subgroup of G, and centrally generated ideals, i.e. the *obvious* prime ideals.

In our case, we will assume further that G is a *uniform pro-p group* in the sense of [12, Definition 4.1]. This is a safe reduction since all compact p-adic Lie groups have an open, uniform normal subgroup.

Let us recall the main conjecture within the study of two-sided ideals in non-commutative Iwasawa algebras, first proposed in [4, Question N], and stated in [15, Conjecture 1.1]:

Conjecture 1.2. *Let G be a solvable, uniform pro-p group, and let P be a prime ideal in $\mathcal{O}G$. Then P is virtually standard, and moreover if $p \in P$ then P is standard.*

Note: There is a version of this conjecture for non-solvable groups, which requires us to exclude the case where $\mathcal{O}G/P$ is a finitely generated $\mathcal{O}$-module, but this will not concern us here.

When the prime ideal P contains p, we can reduce to studying the *mod-p Iwasawa algebra*:

$$kG := \tfrac{\mathcal{O}G}{(\pi)} = \varprojlim k[G/N].$$

We know that Conjecture 1.2 holds for all prime ideals P of kG whenever G is nilpotent by [2, Theorem A], and also when G is *abelian-by-procyclic* by [16, Theorem 1.4].

In the case where the prime ideal P does not contain p, however, the picture is very different. Define the *rational Iwasawa algebra* or *Iwasawa algebra of continuous distributions* as

$$KG := \mathcal{O}G \otimes_{\mathcal{O}} K.$$

This is a Noetherian, topological K-algebra, and the prime ideals of $\mathcal{O}G$ not containing p are in bijection with prime ideals in KG, via the map $P \mapsto P \otimes_{\mathcal{O}} K$. We aim to prove the analogue of Conjecture 1.2 for prime ideals in KG.

Note: 1. This conjecture is trivially true for G abelian.

2. Definition 1.1 still makes sense if we replace $\mathcal{O}$ with k or K at each occurrence, so we may also talk about prime ideals in kG or KG being standard or virtually standard.

Indeed, a prime ideal in $\mathcal{O}G$ containing (resp. not containing p) is (virtually) standard if and only if the corresponding prime ideal in kG (resp. KG) is (virtually) standard.

3. The requirement that prime ideals in KG are only virtually standard is necessary, since they are *not* all standard. For example, if $G = \mathbb{Z}_p$, and K contains a p'th root of unity ζ, then there is a continuous group homomorphism $G \to K^\times, r \mapsto \zeta^r$, which extends to a ring homomorphism $KG \to K$. If we let P be the kernel of this map, then P is a prime ideal of KG and P is not standard, since $(P+1) \cap G = G^p$ and G/G^p is not torsionfree.

In this paper, we prove a version of Conjecture 1.2 for KG, in the case where G is nilpotent.

1.2. Alternative formulation

There is an alternative way of describing standard prime ideals in $\mathcal{O}G$ and KG, and thus formulating Conjecture 1.2, which will be of more practical use:

Firstly, for any two-sided ideal I of $\mathcal{O}G$, recall from [2, Definition 5.2] that we define

$$I^\dagger := \{g \in G : g - 1 \in I\} = (P+1) \cap G,$$

a closed, normal subgroup of G, and we say that I is *faithful* if $I^\dagger = 1$, i.e. if the natural map $G \to \left(\frac{\mathcal{O}G}{I}\right)^\times, g \mapsto g + I$ is injective.

Setting $G_I := \frac{G}{I^\dagger}$, the kernel of the natural surjection $\mathcal{O}G \to \mathcal{O}G_I$ is the augmentation ideal $(I^\dagger - 1)\mathcal{O}G$, and the image of I under this surjection is a faithful ideal of $\mathcal{O}G_I$.

Note: If I is prime and $p \in I$, it follows from [2, Lemma 5.2] that G_I is torsionfree, but this need not be true if $p \notin I$. Roughly speaking, this is why we can only generally assert that prime ideals not containing p are *virtually standard*, and not standard.

If P is a faithful, prime ideal of $\mathcal{O}G$, then to prove that P is standard, we see using Definition 1.1 that it is only required to prove that P is centrally generated. Using [1, Corollary A], we know that $Z(\mathcal{O}G) = \mathcal{O}Z(G)$, so P is centrally generated precisely when $P = (P \cap \mathcal{O}Z(G))\mathcal{O}G$.

More generally, if I is a right ideal of $\mathcal{O}G$ and H is a closed subgroup of G, we say that H *controls* I if $I = (I \cap \mathcal{O}H)\mathcal{O}G$, i.e. I is generated as a right ideal by a subset of $\mathcal{O}H$. Define the *controller subgroup* of I by $I^\chi := \bigcap\{U \leq_o G : U \text{ controls } I\}$, and it follows from [3, Theorem A] that a closed subgroup H of G controls an ideal $I \trianglelefteq \mathcal{O}G$ if and only if $I^\chi \subseteq H$, so in particular I^χ controls I.

If I is a two-sided ideal, then I^χ is a closed, normal subgroup of G by [2, Lemma 5.3(a)], and to prove that I is centrally generated, all that is required is to prove that I is controlled by $Z(G)$, i.e. $I^\chi \subseteq Z(G)$.

So, to summarise, given a prime ideal P of $\mathcal{O}G$, to prove that P is standard, we need only to prove that the quotient $G_P = \frac{G}{P^\dagger}$ is torsionfree, and that the image of P under the surjection $\mathcal{O}G \to \mathcal{O}G_P$ is controlled by $Z(G_P)$. Therefore, we deduce the following alternative formulation for Conjecture 1.2:

Alternative Formulation: Let G be a solvable, uniform pro-p group. We conjecture that every faithful prime ideal of $\mathcal{O}G$ is controlled by $Z(G)$.

Note that we could replace $\mathcal{O}$ with K at any point in this subsection without affecting the sensibility of any definitions or conclusions.

1.3. Main results

When studying Iwasawa algebras, rather than studying general prime ideals, we may be interested specifically in classifying *primitive ideals*, i.e. the annihilators of simple $\mathcal{O}G$-modules.

However, since G is a pro-p group, the Iwasawa algebra $\mathcal{O}G$ has a unique maximal left ideal $\mathfrak{m} = (G - 1, \pi)$, which is in fact two-sided. Therefore $\mathfrak{m}$ is the only primitive ideal in $\mathcal{O}G$. The rational Iwasawa algebra KG, on the other hand, has many simple modules and primitive ideals.

Theorem A. *Let G be a nilpotent, uniform pro-p group. Then every primitive ideal of KG is maximal and virtually standard. Moreover, every faithful, primitive ideal of KG is standard.*

As explained above, we see that to prove Theorem A, it suffices to show that all faithful, primitive ideals in KG are controlled by $Z(G)$.

Now, recall from [19, III 2.1.2] the definition of a *p-valuation* $\omega : G \to \mathbb{R} \cup \{\infty\}$. We recap the key properties of p-valuations in section 2, but for now, just recall that if G is uniform, then G carries a complete p-valuation given by $\omega(g) := \sup\{n \in \mathbb{N} : g \in G^{p^{n+1}}\}$, so this concept gives rise to a larger class of torsionfree compact p-adic Lie groups which, in particular, contains the class of all closed subgroups of uniform groups.

If we assume that (G, ω) is a complete, nilpotent p-valued group of finite rank, then it follows from [2, Theorem A] that all faithful prime ideals in the mod-p Iwasawa algebra kG are controlled by $Z(G)$. One might think that these techniques could be generalised to the characteristic 0 case to prove the same result. Unfortunately, the author showed in [15] that these techniques fail in characteristic 0, and they can only be used to establish a much weaker control theorem for primitive ideals ([15, Theorem 1.2]).

However, in this paper, we adapt the argument given in [15] with some new techniques, and prove the following much stronger control theorem for general prime ideals:

Theorem B. *Let G be a nilpotent, complete p-valued group of finite rank. Then there exists an abelian normal subgroup A of G such that A controls every faithful prime ideal in KG.*

Of course, if we could show that this subgroup A is central, then Theorem A would follow immediately, and would remain true for prime ideals as opposed to just primitive ideals. But unfortunately, this need not always be the case.

For example, if $G = H \rtimes \mathbb{Z}_p$ for H abelian and $(G, H) \nsubseteq Z(G)$, then the subgroup A given by Theorem B is H, which is not central. We prove Theorem B in section 3.

Theorem B is the strongest result we have obtained to date concerning general prime ideals in KG, but all subsequent results require the additional assumption that our prime ideals are *primitive*.

The key idea is that we want to define a class of KG-representations M whose annihilator ideals completely describe the primitive ideal structure of KG. Using [25, Theorem 5.2], we have a dense, faithfully flat embedding of KG into the *locally analytic distribution algebra* $D(G, K)$ as defined in [26, Definition 2.1, Proposition 2.3], so it makes sense to restrict to the class of coadmissible $D(G, K)$-modules, which naturally have the structure of KG-modules. However, since $D(G, K)$ is non-noetherian, this may present difficulties, so instead we restrict our attention to larger, Noetherian completions of KG:

Returning to the case where G is uniform, let $\mathcal{L}_G = \log(G)$ be the $\mathbb{Z}_p$-Lie algebra of G as defined in [12, Theorem 4.30], and set $\mathcal{L} := \frac{1}{p}\mathcal{L}_G$. Recall from [17, Definition 1.2] that we define the *affinoid enveloping algebra* of $\mathcal{L}$ with coefficients in K to be:

$$\widehat{U(\mathcal{L})}_K := \left(\varprojlim_{n \in \mathbb{N}} U(\mathcal{L})/\pi^n U(\mathcal{L}) \right) \otimes_{\mathcal{O}} K \tag{2}$$

This is a Noetherian, Banach K-algebra, and recall from [5, Theorem 10.4] that there exists a continuous, dense embedding of K-algebras:

$$KG \hookrightarrow \widehat{U(\mathcal{L})}_K, g \mapsto \exp(\log(g)). \tag{3}$$

Unlike the embedding $KG \to D(G, K)$, this map is not faithfully flat, but we can still use it to study the representation theory of KG via the representation theory of $\mathcal{L}$.

In section 2, we recall from a previous work [17] how we define the *Dixmier module* $\widehat{D(\lambda)}$ of $\widehat{U(\mathcal{L})}_K$, corresponding to a linear form $\lambda \in \operatorname{Hom}_{\mathbb{Z}_p}(\mathcal{L}, \mathcal{O})$. It follows from [17, Theorem A] that using the annihilators of these modules, we can completely describe the primitive ideal structure of $\widehat{U(\mathcal{L})}_K$.

So now, we are interested in the restricted action of KG on $\widehat{D(\lambda)}$, and the key result we need in the proof of Theorem A is the following:

Theorem C. *Let G be a nilpotent, uniform pro-p group such that $\mathcal{L}$ is powerful, let F/K be a finite extension, and let $\lambda \in Hom_{\mathbb{Z}_p}(\mathcal{L}, \mathcal{O}_F)$ such that $\lambda|_{Z(\mathcal{L})}$ is injective. Then $P := \operatorname{Ann}_{KG}\widehat{D(\lambda)}_F$ is controlled by $Z(G)$.*

Note: To say that $\mathcal{L}$ is powerful just means that $[\mathcal{L}, \mathcal{L}] \subseteq p\mathcal{L}$.

We prove this result in section 4. The key idea is that we know that the annihilator $P := \text{Ann}_{KG}\widehat{D(\lambda)}$ is controlled by an abelian normal subgroup A of G by Theorem B, so we consider the action of KA on $\widehat{D(\lambda)}$, and prove that the kernel of this action is controlled by $Z(G)$.

In section 5, we apply [17, Theorem A], to prove that it suffices to know that Dixmier annihilators are controlled by $Z(G)$ to establish the same result for all primitive ideals, and Theorem A follows immediately from Theorem C.

Acknowledgments

I am very grateful to Konstantin Ardakov for many helpful comments, and also to the anonymous referees at Advances in Mathematics. I would also like to thank EPSRC and the Heilbronn Institute for Mathematical Research for funding and supporting me in my research.

2. Preliminaries

Notation: For $g, h \in G$, we denote the group commutator by $(g, h) := ghg^{-1}h^{-1}$. Moreover, we write $H \trianglelefteq_c^i G$ to mean that H is a closed, *isolated* normal subgroup of G, i.e. $\frac{G}{H}$ is torsionfree.

2.1. Non-commutative valuations

Let us first recap some basic notions of ring filtrations and valuations. Throughout, let R be any ring.

Definition 2.1. A *filtration* on R is a map $w : R \to \mathbb{Z} \cup \{\infty\}$ such that $w(0) = \infty$ and for all $r, s \in R$:

- $w(r + s) \geq \min\{w(r), w(s)\}$.
- $w(rs) \geq w(r) + w(s)$

We say that w is *separated* if $w(r) = \infty$ if and only if $r = 0$, and w is a *valuation* if $w(rs) = w(r) + w(s)$ for all $r, s \in R$.

For each $n \in \mathbb{Z}$, we define $F_nR := \{r \in R : w(r) \geq n\}$, and define the *associated graded ring* $\text{gr}_w R$ to be $\underset{n\in\mathbb{Z}}{\oplus} \frac{F_nR}{F_{n+1}R}$ with multiplication $(r+F_{n+1}R)(s+F_{m+1}R) = rs+F_{n+m+1}R$.

Note that w is a valuation if and only if $\text{gr}_w R$ is a domain.

If $r \in R$ and $w(r) = n$ then we denote $\text{gr}(r) := r + F_{n+1}R \in \text{gr } R$.

Recall from [21, Ch. II Definition 2.2.1] that a filtration $w : R \to \mathbb{Z} \cup \{\infty\}$ is *Zariskian* if the *Rees ring* $\tilde{R} := \underset{n\in\mathbb{Z}}{\oplus} F_nR$ is Noetherian, and $F_1R \subseteq J(F_0R)$. We will not use this definition very often, but we will usually always assume that our filtrations are Zariskian.

Note that if w is Zariskian, then it is separated and both R and $\mathrm{gr}_w\ R$ are Noetherian, since they arise as quotients of the Rees ring.

Example. 1. If R carries a filtration w, then the matrix ring $M_n(R)$ carries a filtration $w_n(A) = \min\{w(a_{i,j}) : i, j = 1, \cdots, n\}$ – the *standard matrix filtration*.

2. If I is a two-sided ideal of R and R carries a filtration w, then the quotient ring $\frac{R}{I}$ carries the *quotient filtration* given by $\overline{w}(r + I) = \sup\{w(r + y) : y \in I\}$. Note that $\mathrm{gr}_{\overline{w}}\ \frac{R}{I} = \frac{\mathrm{gr}_w\ R}{\mathrm{gr}\ I}$, and if w is Zariskian then $\overline{w}$ is Zariskian.

Now, recall the following definition ([16, Definition 3.1])

Definition 2.2. Let Q be a simple artinian ring, and let $v : Q \to \mathbb{Z} \cup \{\infty\}$ be a filtration. We say that v is a *non-commutative valuation* if the completion $\widehat{Q}$ of Q with respect to v is isomorphic to a matrix ring $M_n(Q(D))$, where:

- $Q(D)$ is the ring of quotients of some non-commutative DVR D with uniformiser ν,
- the extension of v to $\widehat{Q}$ is given by the standard matrix filtration corresponding to the ν-adic filtration on $Q(D)$.

Note that if v is a non-commutative valuation on Q, then for all $z \in Z(Q)$, $q \in Q$, $v(qz) = v(q) + v(z)$, a property which will be very useful to us in section 3.

The following construction allows us to define a non-commutative valuation on the artinian ring of quotients $Q(R)$ of a Zariskian filtered ring R. This construction was derived in [2, Section 3], and we state it fully since we will need it for some proofs in section 3.

Construction 2.3. *Let R be a prime, Noetherian ring with a Zariskian filtration w such that $gr_w\ R$ is commutative and the graded ideal $(gr_w\ R)_{\geq 0}$ is non-nilpotent. Then for each minimal prime ideal $\mathfrak{q}$ of $gr_w\ R$, we can construct a non-commutative valuation on $Q(R)$ using the following data:*

- $S := \{r \in R\colon \mathrm{gr}(r) \notin \mathfrak{q}\}$ *– an Ore set in R such that $S^{-1}R = Q(R)$*
- w' *– a Zariskian filtration on $Q(R)$ such that $w'(r) \geq w(r)$ for all $r \in R$, and $w'(s^{-1}r) = w'(r) - w(s)$ for all $s \in S$. The associated graded $gr_{w'}\ Q(R)$ is the homogeneous localisation of $gr_w\ R$ at $\mathfrak{q}$.*
- Q' *– the completion of $Q(R)$ with respect to w', an artinian ring.*
- U *– the positive part of Q', a Noetherian ring.*
- z *– a regular, normal element of $J(U)$ such that $z^nU = F_{nw'(z)}Q'$ for all $n \in \mathbb{Z}$.*
- $v_{z,U}$ *– the z-adic filtration on Q', topologically equivalent to w'.*
- $\widehat{Q}$ *– a simple quotient of Q'.*
- V *– the image of U in $\widehat{Q}$.*
- $\overline{z}$ *– the image of z in V.*

- $v_{\overline{z},V}$ – *the* $\overline{z}$*-adic filtration on* $\widehat{Q}$.
- $\mathcal{B}$ – *a maximal order in* $\widehat{Q}$, *equivalent to* V, *satisfying* $\mathcal{B} \subseteq \overline{z}^{-r}V$ *for some* $r \geq 0$, *isomorphic to* $M_n(D)$ *for some non-commutative DVR* D.
- $v_{\overline{z},\mathcal{B}}$ – *the* $\overline{z}$*-adic filtration on* $\mathcal{B}$.
- $v_{\mathfrak{q}}$ – *the* $J(\mathcal{B})$*-adic filtration on* $\widehat{Q}$, *topologically equivalent to* $v_{\overline{z},\mathcal{B}}$.

Then $v = v_{\mathfrak{q}}$ *defines a non-commutative valuation on* $Q(R)$, *whose completion is* $\widehat{Q}$, *and the natural map* $R \to Q(R)$ *is continuous. Moreover, if* $w(x) \geq 0$ *then* $v(x) \geq 0$.

2.2. Crossed products

Given a ring R and a group H, recall from [23] that a *crossed product* of R with H, denoted $R * H$, is a ring extension $R \subseteq S$, free as a left R-module with basis $\{\overline{h} : h \in H\} \subseteq S^{\times}$ in bijection with H such that for each $g, h \in H$:

- $\overline{g}R = R\overline{g}$ and
- $\overline{g}R\overline{h}R = \overline{gh}R$.

Furthermore, given a sequence $H_1, \cdots, H_r$ of groups, we denote an *iterated crossed product* $R * H_1 * H_2 * \cdots * H_r$ inductively to mean a crossed product of $R * H_1 * \cdots * H_{r-1}$ with H_r.

Let us recap some properties of crossed products that we will use throughout.

Lemma 2.4. *Let* R *be a Noetherian* $\mathbb{Q}$*-algebra,* F *a finite group. Then if* P *is a prime ideal of a crossed product* $S = R * F$, *then:*

i. $P \cap R$ *is semiprime in* R.

ii. $J := (P \cap R) \cdot S$ *is semiprime in* S, *and* P *is a minimal prime above* J.

iii. $S/J = (P/P \cap R) * F$.

Proof. We will prove that $P \cap R$ is an F-prime ideal, i.e. it is F-invariant and for any F-invariant ideals A, B of R, if $AB \subseteq P \cap R$ then $A \subseteq P \cap R$ or $B \subseteq P \cap R$.

Having established this, part i follows from the fact that all minimal primes above $P \cap R$ form a single F-orbit by [23, Lemma 14.2(ii)], part iii is obvious since $J = \bigoplus_{g \in F} (P \cap R)\overline{g}$, and part ii is part iii together with [22, Proposition 10.5.8] and [23, Theorem 4.4].

So, suppose $A, B \trianglelefteq R$ are F-invariant, i.e. for all $g \in F$, $\bar{g}A = A\bar{g}$ and $\bar{g}B = B\bar{g}$, and suppose that $AB \subseteq P \cap R$. Then AS, BS are two-sided ideals of S, and $(AS)(BS) \subseteq P$. So since P is prime, we can assume without loss of generality that $AS \subseteq P$.

So since $AS = \bigoplus_{g \in F} A\bar{g}$, it follows that $A \subseteq P \cap R$, and hence $P \cap R$ is F-prime as required. □

Lemma 2.5. *Let R be a Noetherian ring, F a finite group. Then if P is a primitive ideal of a crossed product $R * F$, then $P \cap R$ is semiprimitive.*

Proof. Let $S = R * F$, then since P is primitive, $P = \text{Ann}_S M$ for some irreducible S-module M. Since F is finite, M is finitely generated over R, so since R is Noetherian, we can choose a maximal R-submodule U of M.

For each $g \in F$, $\bar{g} \cdot U$ is a maximal R-submodule of M, so set $M_g := M/\bar{g} \cdot U$, an irreducible R-module, and let $Q_g := Ann_R M_g$, a primitive ideal of R. Clearly if $r \in P \cap R = Ann_R M$ then $rN_g = 0$ for all $g \in F$, so $P \cap R \subseteq \underset{g \in F}{\cap} Q_g$.

Also, $\underset{g \in F}{\cap} \bar{g} \cdot U$ is an S-submodule, so by simplicity of M, $\underset{g \in F}{\cap} \bar{g} \cdot U = 0$. So if $r \in \underset{g \in F}{\cap} Q_g$ then $rM_g = 0$ for all g, so $rM \subseteq \bar{g} \cdot U$ for all g, i.e. $rM \subseteq \underset{g \in F}{\cap} \bar{g} \cdot U = 0$ and hence $r \in Ann_R M = P \cap R$. Hence:

$$P \cap R = \underset{g \in F}{\cap} Q_g$$

Hence $P \cap R$ is semiprimitive as required. □

2.3. p-valued groups

Let G be a group. Recall from [19, III 2.1.2] that we define a *p-valuation* on G to be a map $\omega : G \to \mathbb{R} \cup \{\infty\}$ such that for all $g, h \in G$:

- $\omega(g) = \infty$ if and only if $g = 1$.
- $\omega(g^{-1}h) \geq \min\{\omega(g), \omega(h)\}$.
- $\omega((g, h)) \geq \omega(g) + \omega(h)$.
- $\omega(g^p) = \omega(g) + 1$.
- $\omega(g) > \frac{1}{p-1}$.

Note that if (G, ω) is a p-valued group then G is torsionfree, and carries a topology defined by the metric $d(g, h) := c^{-\omega(g^{-1}h)}$ for some $c > 1$. We will always assume that G is complete with respect to this topology, in which case we can define *p-adic exponentiation* in G, i.e. for all $g \in G$, $\alpha \in \mathbb{Z}_p$, if $\alpha = \lim\limits_{n \to \infty} \alpha_i$ for $\alpha_i \in \mathbb{Z}$, we define $g^\alpha := \lim\limits_{n \to \infty} g^{\alpha_i} \in G$.

Given $d \in \mathbb{N}$, we say that G has *finite rank* d if there exists a subset $\underline{g} := \{g_1, \cdots, g_d\} \subseteq G$ such that for every $g \in G$, there exists a unique $\alpha \in \mathbb{Z}_p^d$ such that $g = \underline{g}^\alpha := g_1^{\alpha_1} \cdots g_d^{\alpha_d}$, and $\omega(g) := \min\{v_p(\alpha_i) + \omega(g_i) : i = 1, \cdots, d\}$. We call such a subset $\underline{g}$ an *ordered basis* for (G, ω).

Remark. We say that a topological group G is *p-valuable* if there exists a p-valuation ω on G such that (G, ω) is a complete p-valued group of finite rank.

Example. If G is uniform, and $\omega(g) := \sup\{n \in \mathbb{N} : g \in G^{p^{n+1}}\}$, then (G,ω) is a complete p-valued group, and any minimal topological generating set for G is an ordered basis for (G,ω).

Definition 2.6. We say that a p-valuation $\omega : G \to \mathbb{R} \cup \{\infty\}$ is *abelian* if:

- There exists $n \in \mathbb{N}$ such that $\omega(G) \subseteq \frac{1}{n}\mathbb{Z}$.
- For all $g, h \in G$, $\omega((g,h)) > \omega(g) + \omega(h)$.

Using [27, Lemma 26.13], if (G,ω) is any integer valued p-valued group of finite rank (e.g. a uniform group), then we can choose $c > 0$ such that $\omega_c(g) := \omega(g) - c$ is an abelian p-valuation on G.

Now, suppose that (G,ω) is a complete p-valued group of rank d, with ordered basis $\underline{g} = \{g_1, \cdots, g_d\}$ then the Iwasawa algebra $\mathcal{O}G$ is isomorphic to the power series ring $\mathcal{O}[[b_1, \cdots, b_d]]$ as an $\mathcal{O}$-module (and as a ring if G is abelian), where each variable b_i corresponds with $g_i - 1$.

Moreover, if ω is an abelian p-valuation, taking values in $\frac{1}{n}\mathbb{Z}$ for some $n \in \mathbb{N}$, then recall from [15, Section 2.2] that we can define a filtration w on $\mathcal{O}G$ via:

$$w(\sum_{\alpha \in \mathbb{N}^d} \lambda_\alpha b_1^{\alpha_1} \cdots b_d^{\alpha_d}) = \inf\{v_\pi(\lambda_\alpha) + \sum_{i \le d} en\alpha_i\omega(g_i) : \alpha \in \mathbb{N}^d\},$$

where e is the ramification index of $K/\mathbb{Q}_p$.

We call w the *Lazard filtration* on $\mathcal{O}G$. Using [25, Theorem 4.5], we see that $\mathrm{gr}_w\ \mathcal{O}G \cong k[t, t_1, \cdots, t_d]$, where k is the residue field of K, $t = \mathrm{gr}(\pi)$ and $t_i = \mathrm{gr}(b_i)$, and hence is commutative. Note that for any $g \in G$, $w(g-1) \ge en\omega(g)$, with equality if $g = g_i$ for some i.

Furthermore, since $\mathcal{O}G$ is complete with respect to w and $\mathrm{gr}_w\ \mathcal{O}G$ is Noetherian, it follows from [21, Ch. II Theorem 2.1.2] that w is a Zariskian filtration. Hence for any two-sided ideal I of $\mathcal{O}G$, the quotient filtration $\overline{w}$ on $\mathcal{O}G/I$ is Zariskian.

In particular, if I is a prime ideal then we can use Construction 2.3 to define a non-commutative valuation v on the Goldie ring of quotients $Q(\mathcal{O}G/I)$ such that the natural map $\tau : (\mathcal{O}G, w) \to (Q(\mathcal{O}G/I), v)$ is continuous.

2.4. Prime ideals in KG

Fixing (G,ω) a complete p-valued group of finite rank, we will now examine some basic properties of prime ideals in KG. First of all, the following lemma allows us to simplify the statement of Theorem C to remove reference to the finite extension F/K:

Lemma 2.7. *Let F/K be a finite extension, and let I' a right ideal of FG. Setting $I := I' \cap KG$, we have that if I' is controlled by $U \le_c G$ then I is controlled by U.*

Proof. We will first suppose that U is open in G. Then given $r \in I$, choose a complete set of coset representatives $\{g_1, \cdots, g_r\}$ for U in G, then $r = \sum_{1 \leq i \leq r} r_i g_i$ for some $r_i \in KU \subseteq FU$.

So since $I = I' \cap KG$ and I' is controlled by U, it follows that $r_i \in I' \cap FU \cap KG = I' \cap KU = I$ for each i, and hence I is controlled by U.

So, let I^χ be the controller subgroup of I, i.e. the intersection of all open subgroups of G controlling I. So since this includes all open subgroups of G controlling I', we have that $I^\chi \subseteq I'^\chi$, hence any closed subgroup controlling I' also controls I. □

Now, recall that a two-sided ideal P of a ring R is *completely prime* if the quotient $\frac{R}{P}$ is a domain. The following result is the characteristic 0 analogue of [2, Theorem 8.6], and it uses a similar argument.

Theorem 2.8. *Let P be a prime ideal of $KZ(G)$. Then PKG is a completely prime ideal of KG, and if P is faithful then PKG is faithful.*

Proof. Let $Z := Z(G)$. Then Z is a closed, isolated subgroup of G by [2, Lemma 8.4(a)], and hence G/Z is a p-valuable group by [19, IV.3.4.2]. We will prove that if P is a prime ideal of $\mathcal{O}Z$ with $p \notin P$ then $P\mathcal{O}G$ is completely prime, and it is faithful if P is faithful. The result for the rational Iwasawa algebra follows immediately.

Let Q be the field of fractions of $\mathcal{O}Z/P$. If we let w be the Lazard filtration on $\mathcal{O}Z$, then since w is a Zariskian filtration and the associated graded is a commutative, infinite dimensional k-algebra, it follows from Construction 2.3 that there exists a valuation v' on Q such that the natural map $\tau : \mathcal{O}Z \to Q$ is continuous, and if $w(x) \geq 0$ then $v'(\tau(x)) \geq 0$.

Furthermore, if $v'(\tau(z-1)) = 0$ for some $z \in Z$ then $v'(\tau(z-1)^n) = 0$ for all n since v' is a valuation, which is a contradiction since $(z-1)^n$ converges to zero in $\mathcal{O}G$, and hence in Q by continuity of τ. Therefore $v'(\tau(z-1)) > 0$ for all $z \in Z(G)$, and after choosing an ordered basis $\{z_1, \cdots, z_n\}$ for Z and an integer M such that $Mv'(\tau(z_i - 1)) \geq w(z_i - 1)$ for all i, then we obtain an equivalent valuation $v := Mv'$ on Q such that $v(\tau(x)) \geq w(x)$ for all $x \in \mathcal{O}Z$.

Recall that if we fix an ordered basis $\{g_1, \cdots, g_e\}$ for $\frac{G}{Z}$, then every element of $\mathcal{O}G$ has the form $\sum_{\alpha \in \mathbb{N}^e} \mu_\alpha \underline{c}^\alpha$ for some $\mu_\alpha \in \mathcal{O}Z$ where $c_i = g_i - 1$. Define a map $u : \mathcal{O}G \to \mathbb{Z} \cup \{\infty\}$ via:

$$u : \mathcal{O}G \to \mathbb{Z} \cup \{\infty\}, \sum_{\alpha \in \mathbb{N}^e} \mu_\alpha \underline{c}^\alpha \mapsto \inf\{v(\tau(\mu_\alpha)) + w(\underline{c}^\alpha) : \alpha \in \mathbb{N}^e\}. \tag{4}$$

Since v is a separated valuation, it is clear that $u(\sum_{\alpha \in \mathbb{N}^e} \mu_\alpha \underline{c}^\alpha) = \infty$ if and only if $\mu_\alpha \in P$ for all α, i.e. if and only if $\sum_{\alpha \in \mathbb{N}^e} \mu_\alpha \underline{c}^\alpha \in P\mathcal{O}G$. Therefore $u^{-1}(\infty) = P\mathcal{O}G$. So following the proof of [2, Theorem 8.6], we will prove that u is a valuation on $\mathcal{O}G$, from which it will follow that $P\mathcal{O}G = u^{-1}(\infty)$ is a completely prime ideal.

Firstly, it is clear from the definition that $u(r+s) \geq \min\{u(r), u(s)\}$, $u(\mu) = v(\tau(\mu))$ and $u(\mu r) = u(\mu) + u(r)$ for all $r, s \in \mathcal{O}G$, $\mu \in \mathcal{O}Z$. It is also clear that if $r_1, r_2, \cdots \in \mathcal{O}G$ with $r_i \to 0$ as $i \to \infty$ then $u(r_1 + r_2 + \cdots) \geq \inf\{u(r_i) : i \geq 1\}$, therefore to prove that u is a filtration it remains to prove that $u(\underline{c}^\alpha \underline{c}^\beta) \geq u(\underline{c}^\alpha) + u((\underline{c})^\beta)$ for all $\alpha, \beta \in \mathbb{N}^r$.

Write $\underline{c}^\alpha \underline{c}^\beta = \sum_{\gamma \in \mathbb{N}^e} \lambda_\gamma^{\alpha,\beta} \underline{c}^\gamma$, then by the definition of the Lazard filtration, $w(\sum_{\gamma \in \mathbb{N}^e} \lambda_\gamma^{\alpha,\beta} \underline{c}^\gamma)$ $= \inf\{w(\lambda_\gamma^{\alpha,\beta}) + w(\underline{c}^\gamma) : \gamma \in \mathbb{N}^d\}$. So since $u(x) \geq w(\tau(x))$ for all $x \in \mathcal{O}Z$, we have:

$$u(\underline{c}^\alpha \underline{c}^\beta) = \inf\{v(\tau(\lambda_\gamma^{\alpha,\beta})) + w(\underline{c}^\gamma) : \gamma \in \mathbb{N}^e\} \geq \inf\{w(\lambda_\gamma^{\alpha,\beta}) + w(\underline{c}^\gamma) : \gamma \in \mathbb{N}^e\} = w(\underline{c}^\alpha) + w(\underline{c}^\beta) = u(\underline{c}^\alpha) + u(\underline{c}^\beta).$$

So u is a filtration on $\mathcal{O}G$, and to verify that it is a valuation, we will show that the associated graded $\mathrm{gr}_u\ \mathcal{O}G$ is a domain. First note that the definition of u gives rise to a natural inclusion of graded rings $\mathrm{gr}_v\ \mathcal{O}Z/P \to \mathrm{gr}_u\ \mathcal{O}G$, and this gives rise to an isomorphism of graded rings $\mathrm{gr}_v\ (\mathcal{O}Z/P)[Y_1, \cdots, Y_e] \to \mathrm{gr}_u\ \mathcal{O}G$ where Y_i is sent to $\mathrm{gr}(c_i)$. Therefore $\mathrm{gr}_u\ \mathcal{O}G$ is a domain and u is a valuation as required.

Finally, if P is faithful, then suppose $g \in G$ and $g - 1 \in P\mathcal{O}G$. Then write $g = zg_1^{\alpha_1} \cdots g_e^{\alpha_e}$ for some $z \in Z$, $\alpha_i \in \mathbb{Z}_p$, and it follows that:

$$h - 1 = (z-1) + (z-1) \sum_{0 \neq \gamma \in \mathbb{N}^e} \binom{\alpha}{\gamma} \underline{c}^\alpha + \sum_{0 \neq \gamma \in \mathbb{N}^e} \binom{\alpha}{\gamma} \underline{c}^\alpha.$$

Therefore, we see that $z - 1 \in P$ and hence $z = 1$ since P is faithful. It also follows that for each $0 \neq \gamma \in \mathbb{N}^e$, $\binom{\alpha}{\gamma} \in P$, and hence $\binom{\alpha}{\gamma} = 0$ since $P \cap \mathcal{O} = 0$. This is only possible if $\alpha = (\alpha_1, \cdots, \alpha_e) = 0$, and hence $h = zg_1^{\alpha_1} \cdots g_e^{\alpha_e} = 1$ and $P\mathcal{O}G$ is faithful as we require. □

In particular, it follows from this result that standard prime ideals in KG are completely prime.

2.5. Completions of KG

For the rest of this section, fix G a uniform pro-p group, let $\mathcal{L} := \frac{1}{p}\log(G)$ be its $\mathbb{Z}_p$-Lie algebra of G, and let $\mathfrak{g} = \mathcal{L} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

To reiterate, we aim to study the action of KG on certain $\widehat{U(\mathcal{L})}_K$-modules using the dense embedding $KG \to \widehat{U(\mathcal{L})}_K$. However, this embedding is not faithfully flat, so representation theoretic information is lost when passing from KG to $\widehat{U(\mathcal{L})}_K$.

Perhaps a better choice for a completion of KG would be the *distribution algebra* $D(G, K)$ of G with coefficients in K in the sense of [25]. In this case, the natural dense embedding $KG \to D(G, K)$ is faithfully flat by [25, Theorem 4.11], but unfortunately $D(G, K)$ is not Noetherian, so it would be difficult in practice to extract general ring-theoretic information from $D(G, K)$.

However, for each $n \geq 0$, consider the crossed products

$$D_{p^n} = D_{p^n}(G) := \widehat{U(p^n\mathcal{L})}_K * \tfrac{G}{G^{p^n}}$$

as defined in [5, Proposition 10.6], which arise as Banach completions of KG with respect to the extension of the dense embedding $KG^{p^n} \to \widehat{U(p^n\mathcal{L})}_K$ to $KG = KG^{p^n} * \frac{G}{G^{p^n}}$. These algebras give rise to an inverse system:

$$KG \to D(G,K) \to \cdots D_{p^3} \to D_{p^2} \to D_p \to D_0 = \widehat{U(\mathcal{L})}_K.$$

i.e. $D(G,K) = \varprojlim_{n\to\infty} D_{p^n}$, so since $D(G,K)$ is faithfully flat over KG, we want to approximate $D(G,K)$ using the Noetherian Banach algebras D_{p^n}, and thus limit how much information we lose. Indeed, using [5, Proposition 10.6(e), Corollary 10.11], we see that for all KG-modules M, $D_{p^n} \otimes_{KG} M \neq 0$ for all sufficiently high n.

Lemma 2.9. *Let A be a free abelian pro-p group of rank d, $\mathcal{A} := \frac{1}{p}\log(A)$. Then $\frac{A}{A^p} = C_1 \times \cdots \times C_d$ where each $C_i = \langle c_i \rangle = \langle g_i A^p \rangle$ is a cyclic group of order p, and $D_p = D_p(A)$ is an iterated crossed product:*

$$D_p = \widehat{U(p\mathcal{A})}_K * C_1 * \cdots * C_d.$$

where for each $i = 1, \cdots, d$ $\overline{c_i}^r = \overline{c_i^r}$ for $0 \le r < p$, and $\overline{c_i}^p = g_i^p$.

Proof. Firstly, it is clear that since $A = \mathbb{Z}_p^d$ that $\frac{A}{A^p} = \frac{\mathbb{Z}_p^d}{(p\mathbb{Z}_p)^d} = (\frac{\mathbb{Z}_p}{p\mathbb{Z}_p})^d = C_1 \times \cdots \times C_d$ as required.

For the second statement, it suffices to prove that $KA = KA^p * C_1 * \cdots * C_d$, and that this decomposition satisfies the same properties, since it will be preserved after passing to the completion.

Choose a $\mathbb{Z}_p$-basis $\{g_1, \cdots, g_d\}$ for A, and we may assume that $C_i = \langle c_i \rangle$ where $c_i = g_i A^p$. Then every element $r \in KA$ has the form $\sum_{\alpha \in [p-1]^d} r_\alpha g_1^{\alpha_1} \cdots g_d^{\alpha_d}$ for some $r_\alpha \in KA^p$, and r is sent to $\sum_{\alpha \in [p-1]^d} r_\alpha \overline{c_1^{\alpha_1} \cdots c_d^{\alpha_d}}$ under the isomorphism $KA \to KA^p * \frac{A}{A^p}$.

So, since g_i, g_j and $g_i g_j$ are sent to $\overline{c_i}$, $\overline{c_j}$ and $\overline{c_i c_j}$ respectively, it follows that $\overline{c_i c_j} = \overline{c_i} \cdot \overline{c_j}$ for each i, j. Hence $KA = KA^p * C_1 * \cdots * C_d$.

Finally, for $0 \le r < p$, g_i^r is sent to $\overline{c_i^r}$, and hence $\overline{c_i}^r = \overline{c_i^r}$, and $g_i^p \in KA^p$ is sent to g_i^p, so $\overline{c_i}^p = g_i^p$ as required. □

2.6. Dixmier modules

Recall from [17, Definition 2.3] the following definition:

Definition 2.10. Let $\lambda : \mathfrak{g} \to K$ be a $\mathbb{Q}_p$-linear form such that $\lambda(\mathcal{L}) \subseteq \mathcal{O}$ (i.e. $\lambda \in Hom_{\mathbb{Z}_p}(\mathcal{L}, \mathcal{O})$).

- A *polarisation* of $\mathfrak{g}_K = \mathfrak{g} \otimes_{\mathbb{Q}_p} K$ at λ is a solvable subalgebra $\mathfrak{b}$ of $\mathfrak{g}_K$ such that for any subspace $\mathfrak{b} \subseteq V \subseteq \mathfrak{g}_K$, $\lambda([V,V]) = 0$ if and only if $V = \mathfrak{b}$.
- Given a polarisation $\mathfrak{b}$ of $\mathfrak{g}_K$ at λ, let $\mathcal{B} := \mathfrak{b} \cap (\mathcal{L} \otimes_{\mathbb{Z}_p} \mathcal{O})$, and let K_λ be the one-dimensional $\mathfrak{b}$-module induced by λ. Define the *affinoid Dixmier module* of $\widehat{U(\mathcal{L})}_K$ induced by λ to be $\widehat{D(\lambda)} = \widehat{D(\lambda)}_{\mathfrak{b}} := \widehat{U(\mathcal{L})}_K \otimes_{\widehat{U(\mathcal{B})}_K} K_\lambda$

Note: If it is unclear what the base field K is, we may sometimes write $\widehat{D(\lambda)}_K$.

So, fixing $\lambda \in \mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{L}, \mathcal{O})$, let $\mathfrak{b}$ be a polarisarion of $\mathfrak{g}$ at λ, and we see that KG acts on $\widehat{D(\lambda)}_{\mathfrak{b}}$ via the embedding $KG \to \widehat{U(\mathcal{L})}_K$. Set $P := \mathrm{Ann}_{KG}\widehat{D(\lambda)}$, and using [17, Theorem 4.4], we see that this does not depend on the choice of polarisation.

Definition 2.11. Define the λ-*scalar ideal* of $\mathfrak{g}$ to be $\mathfrak{a}_\lambda$, the largest ideal $\mathfrak{a}$ of $\mathfrak{g}$ such that $\lambda(\mathfrak{a}) = 0$. Also, set $\mathcal{A}_\lambda := \mathfrak{a}_\lambda \cap \mathcal{L}$, and define the λ-scalar subgroup of G, $A_\lambda := \exp(p\mathcal{A}_\lambda) \trianglelefteq_c^i G$.

Note: For any choice of polarisation $\mathfrak{b}$ of $\mathfrak{g} \otimes_{\mathbb{Q}_p} K$ at λ, it follows from [17, Lemma 2.3] that $\mathfrak{a}_\lambda \subseteq \mathfrak{b}$.

Lemma 2.12. *Let A_λ be the λ-scalar subgroup of G. If $P = Ann_{KG}\widehat{D(\lambda)}$, then $A_\lambda = P^\dagger = \{g \in G : g - 1 \in P\}$. In particular, P is faithful if and only if the restriction of λ to $Z(\mathfrak{g})$ is injective.*

Proof. Firstly, since $\mathfrak{a}_\lambda \subseteq \mathfrak{b}$, we see that $\mathfrak{a}_\lambda\widehat{D(\lambda)} = \mathfrak{a}_\lambda\widehat{U(\mathcal{L})}_K \otimes_{\widehat{U(\mathcal{B})}_K} K_\lambda = \widehat{U(\mathcal{L})}_K\mathfrak{a}_\lambda \otimes_{\widehat{U(\mathcal{B})}_K} K_\lambda = 0$.

So since $A_\lambda - 1 \subseteq \mathfrak{a}_\lambda\widehat{U(\mathcal{L})}_K$, it is clear that $A_\lambda - 1 \subseteq P$, i.e. $A_\lambda \subseteq P^\dagger$.

Now, since $T = P^\dagger$ is a closed, normal subgroup of G, $\mathcal{T} := \frac{1}{p}\log(T)$ is an ideal of $\mathcal{L}$, and it contains $\frac{1}{p}\log(A_\lambda) = \mathcal{A}_\lambda$. Also, since $(T-1)\widehat{D(\lambda)} = 0$, it follows that $\mathcal{T}\widehat{D(\lambda)} = 0$. This is only possible if $\mathcal{T} \subseteq \mathcal{B}$ and $\lambda(\mathcal{T}) = 0$.

Setting $\mathfrak{t} := \mathcal{T} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, $\mathfrak{t}$ is an ideal of $\mathfrak{g}$, $\mathfrak{a}_\lambda \subseteq \mathfrak{t}$ and $\lambda(\mathfrak{t}) = 0$. So by the definition of $\mathfrak{a}_\lambda$, this means that $\mathfrak{a}_\lambda = \mathfrak{t}$.

So, for any $u \in \mathcal{T}$, there exists $i \in \mathbb{N}$ such that $\pi^i u \in \mathcal{A}_\lambda = \mathfrak{a}_\lambda \cap \mathcal{L}$, and this means that $u \in \mathcal{A}_\lambda$. So $\mathcal{A}_\lambda = \mathcal{T}$, and it follows immediately that $A_\lambda = T$.

Finally, since G is nilpotent, $\mathcal{L}$ is nilpotent, and thus if $\mathcal{A}_\lambda \neq 0$, then it must have non-trivial intersection with $Z(\mathfrak{g})$. So since P is faithful if and only if $A_\lambda = 1$ (i.e. if and only if $\mathfrak{a}_\lambda = 0$), and any subspace of $Z(\mathfrak{g})$ is an ideal of $\mathfrak{g}$, it follows that P is faithful precisely when nothing in $Z(\mathfrak{g})$ is sent to zero under λ, i.e. $\lambda|_{Z(\mathfrak{g})}$ is injective. □

This lemma is useful to know, because it implies that for any Dixmier annihilator P, $P^\dagger$ is a closed, isolated normal subgroup of G, and hence we can replace G by $G_P = \frac{G}{P^\dagger}$, which is still a nilpotent, uniform group, and $P_0 = \frac{P}{(P^\dagger - 1)KG}$ becomes a faithful Dixmier annihilator.

Note that this lemma explains why we need the assumption that $\lambda|_{Z(\mathfrak{g})}$ is injective in the statement of Theorem C, since it is generally untrue that non-faithful prime ideals in KG are controlled by $Z(G)$.

3. The logarithm of automorphisms

In this section, we will study general prime ideals within the rational Iwasawa algebra KG of a p-valuable group G. This is of course equivalent to studying prime ideals in $\mathcal{O}G$ that do not contain p.

The methods we use are inspired by those used in [2] to prove that faithful prime ideals in the mod-p Iwasawa algebra kG are standard, namely the study of *Mahler expansions of G-automorphisms*. Unfortunately, these methods do not work in characteristic 0, as demonstrated in [15, Section 3.3], and the best result they can be used to obtain is a weak control theorem for faithful *primitive* ideals ([15, Theorem 1.2]).

The methods we consider in this section involve the *logarithm* of a G-automorphism, which in many ways is the characteristic 0 version of the Mahler expansion. While these methods are not yet sufficient to prove standardness in full generality, we can employ them together with techniques from [2, Section 7] to adapt the argument used in [15, Section 3.4] and ultimately reprove the weak control theorem [15, Theorem 1.2] for all faithful prime ideals, rather than just primitive ideals. This culminates in the proof of Theorem B, given at the end of the section.

3.1. Bounded ring automorphisms

Let R be a ring carrying a complete Zariskian filtration w. Recall from [2] that a function $f : R \to R$ is *bounded* if $\inf\{w(f(r)) - w(r) : r \in R\} > -\infty$, in which case we define the *degree* of f to be the number $\deg_w(f) := \inf\{w(f(r)) - w(r) : r \in R\}$.

If we set $B(R)$ as the space of bounded, additive maps $f : R \to R$, then $B(R)$ is a ring with pointwise addition and composition as multiplication, and $\deg_w$ defines a complete separated filtration on $B(R)$.

Lemma 3.1. *If $f : R \to R$ is an additive map such that $\deg_w(f) > 0$, and I is a two-sided ideal of R such that $f(I) \subseteq I$. Then if $\overline{w}$ is the quotient filtration on R/I, and $\overline{f} : R/I \to R/I$ is the map induced from f, then $\deg_{\overline{w}}(\overline{f}) > 0$.*

Proof. Let $\mu := \deg_w(f) > 0$. Then given $r \in R$, $w(f(r)) - w(r) \geq \mu$, we want to prove that $\overline{w}(\overline{f}(r + I)) - \overline{w}(r + I) > 0$:

By definition, $\overline{w}(r + I) = \sup\{w(r + y) : y \in I\}$, so let us suppose for contradiction that $\overline{w}(r+I) \geq \overline{w}(\overline{f}(r+I))$, and hence there exists $y \in I$ such that $w(r+y) \geq w(f(r)+u)$

for all $u \in I$. In particular, since $f(I) \subseteq I$, $w(r+y) \geq w(f(r)+f(y)) = w(f(r+y)) \geq w(r+y)+\mu > w(r+y)$ – contradiction.

Therefore $\overline{w}(\overline{f}(r+I)) > \overline{w}(r+I)$ for all $r \in R$, so since $\overline{w}$ is integer valued, it follows that $\deg_{\overline{w}}(\overline{f}) \geq 1 > 0$. □

Now, suppose that R is a $\mathbb{Z}_p$-algebra, with $p \neq 0$ and $w(p) > 0$. The following lemma will be useful to us several times in this section:

Lemma 3.2. *Given $m \in \mathbb{N}$, $a, b \in R$ such that $w(a) = 0$, $w(b) \geq 1$, and a and b commute:*

- $v_p\binom{p^m}{k} = m - v_p(k)$ *for all* $0 < k < p^m$.
- $\min\{m - v_p(k) + (p^m - k)w(b) : 0 < k < p^m\} \to \infty$ *as* $m \to \infty$
- $w((a+b)^{p^m} - a^{p^m}) \geq \min\{p^m w(b), m - v_p(k) + (p^m - k)w(b) : 0 < k < p^m\}$, *and thus* $(a+b)^{p^m} - a^{p^m} \to 0$ *as* $m \to \infty$.

Proof. Firstly, if $k = a_0 + a_1 p + \cdots + a_t p^t$ for some $0 \leq a_i < p$, we define $s(k) := a_0 + a_1 + \cdots + a_t$. Then using [19, III 1.1.2.5] we see that $v_p(k!) = \frac{k - s(k)}{p-1}$. Therefore, $v_p\binom{p^m}{k} = v_p\left(\frac{p^m!}{k!(p^m-k)!}\right) = \frac{p^m - s(p^m) - k + s(k) - (p^m - k) + s(p^m - k)}{p-1} = \frac{s(k) + s(p^m - k) + 1}{p-1}$.

But since $k < p^m$, we may assume that $t = m-1$, i.e. $k = a_0 + a_1 p + \cdots + a_{m-1}p^{m-1}$. And since $k \neq 0$, let i be maximal such that $a_{m-i} \neq 0$, so $k = a_{m-i}p^{m-i} + \cdots + a_{m-1}p^{m-1}$ and hence $v_p(k) = m - i$.

Now, $p^m = (p-1)p^{m-i} + (p-1)p^{m-i+1} + \cdots + (p-1)p^{m-1} + p^{m-i}$, and thus $p^m - k = (p - a_{m-i})p^{m-i} + (p - a_{m-i+1} - 1)p^{m-i+1} + \cdots + (p - a_{m-1} - 1)p^{m-1}$ and we deduce that $s(p^m - k) = ip - s(k) - (i-1) = i(p-1) - s(k) + 1$.

Therefore, $v_p\binom{p^m}{k} = \frac{s(k) + s(p^m - k) - 1}{p-1} = \frac{i(p-1)}{p-1} = i = m - v_p(k)$ as required.

To prove the second statement, we just need to prove that for any k, $m - v_p(k) + (p^m - k)w(b) \to \infty$ as $m \to \infty$:

If $v_p(k) \leq \frac{m}{2}$ then $m - v_p(k) + (p^m - k)w(b) \geq \frac{m}{2} \to \infty$.

If $v_p(k) > \frac{m}{2}$ then $k = p^{v_p(k)}y$ with $v_p(y) = 0$, so $m - v_p(k) + (p^m - k)w(b) \geq p^{v_p(k)}(p^{m - v_p(k)} - y)w(b)$

$\geq p^{\frac{m}{2}} \to \infty$ as required.

Finally, $(a+b)^{p^m} - a^{p^m} = \sum_{0 \leq k < p^m} \binom{p^m}{k} a^k b^{p^m - k}$, so $w((a+b)^{p^m} - a^{p^m}) \geq \min\{p^m w(b), w(\binom{p^m}{k} a^k b^{p^m - k}) : 0 < k < p^m\}$, and $w(\binom{p^m}{k} a^k b^{p^m - k}) \geq v_p\binom{p^m}{k} w(p) + kw(a) + (p^m - k)w(b) \geq m - v_p(k) + (p^m - k)w(b)$ as required. □

Now, fix a ring automorphism $\varphi : R \to R$ such that $\deg_w(\varphi - 1) > 0$.

Lemma 3.3. *The sequence $\varphi^n - 1$ converges to 0 in $B(R)$ as $v_p(n) \to \infty$.*

Proof. We just need to prove that $\deg_w(\varphi^n - 1) \to \infty$ as $v_p(n) \to \infty$. Since $\deg_w(\varphi^n - 1) > 0$ for all n, it suffices to prove that $\deg_w(\varphi^{p^m} - 1) \to \infty$ as $m \to \infty$.

Now, $\varphi^{p^m}-1=((\varphi-1)+1)^{p^m}-1=\sum_{0\leq k<p^m}\binom{p^m}{k}(\varphi-1)^{p^m-k}$. But since $\deg_w$ defines a ring filtration on $B(R)$ and $\deg_w(\varphi-1)\geq 1$, it follows from Lemma 3.2 that if $k>0$ then

$\deg_w(\binom{p^m}{k}(\varphi-1)^{p^m-k})\geq w\binom{p^m}{k}+(p^m-k)\deg_w(\varphi-1)\geq m-v_p(k)+(p^m-k)\deg_w(\varphi-1)\to\infty$ as $m\to\infty$, and clearly if $k=0$ then $\deg_w(\binom{p^m}{k}(\varphi-1)^{p^m-k})\geq p^m\deg_w(\varphi-1)\to\infty$ as required. □

Now, let us suppose that R is a prime, Noetherian ring, $\mathrm{gr}_w\ R$ is commutative, and that the positively graded ideal $(\mathrm{gr}_w\ R)_{\geq 0}$ is not nilpotent. Then the simple, artinian ring $Q(R)$ carries a non-commutative valuation v, which we can describe using Construction 2.3. Clearly any automorphism φ of R extends to $Q(R)$.

The following theorem allows us to pass from (R,w) to $(Q(R),v)$ without difficulty. The proof is similar to that of an analogous result in a characteristic p setting, namely the proof of [16, Proposition 3.5].

Theorem 3.4. *If $\varphi\in Aut(R)$ and $\deg_w(\varphi-1)>0$, then there exists $n\in\mathbb{N}$ with $\deg_v(\varphi^n-1)>0$.*

Proof. Using Construction 2.3, we see that we have a sequence of filtrations $w', v_{z,U}, v_{\underline{z},V}, v_{\underline{z},\mathcal{B}}, v$ on $Q(R)$, and our strategy is to prove that φ^n-1 has positive degree for some n with respect to each of these in turn. Let us first consider w'.

Since $\deg_w(\varphi-1)>0$, i.e. $w(\varphi(r)-r)>w(r)$ for all $r\in R$, it follows that the induced graded automorphism $\overline{\varphi}$: $\mathrm{gr}_w\ R\to\mathrm{gr}_w\ R, r+F_nR\mapsto\varphi(r)+F_nR$ is just the identity. Therefore, since $\mathrm{gr}_{w'}\ Q(R)=(\mathrm{gr}\ R)_{\mathfrak{q}}$, it follows that the induced morphism $\overline{\varphi}$: $\mathrm{gr}_{w'}\ Q(R)\to\mathrm{gr}_{w'}\ Q(R)$ is also the identity, and hence $\deg_{w'}(\varphi-1)\geq 1$.

So, using Lemma 3.3, we can choose $n_1\in\mathbb{N}$ such that $\deg_{w'}(\varphi^{n_1}-1)\geq w'(z)$. But since $z^nU=F_{nw'(z)}Q(R)$, it follows that if $r\in z^nU$ then $w'((\varphi^{n_1}-1)(r))\geq w'(r)+w'(z)\geq nw'(z)+w'(z)=(n+1)w'(z)$, and hence $(\varphi^{n_1}-1)(z^nU)\subseteq F_{(n+1)w'(z)}Q(R)=z^{n+1}U$, and hence $\deg_{v_{z,U}}(\varphi^{n_1}-1)\geq 1$.

Now, recall that $\widehat{Q}$ is a simple quotient of the completion Q' of $Q(R)$ with respect to $v_{z,U}$, i.e. a quotient of Q' by a maximal ideal $\mathfrak{m}$. But since Q' is artinian, all maximal ideals are minimal prime ideals, and hence there are only finitely many of them. Since φ^{n_1} is continuous, $\varphi^{n_1}(\mathfrak{m})$ is also a minimal prime ideal of Q', i.e. φ^{n_1} permutes the set of minimal prime ideals of Q'. Since this set is finite, all permutations have finite order, so there exists m such that $\varphi^{mn_1}(\mathfrak{m})=\mathfrak{m}$.

Therefore, there exists n_2 such that φ^{n_2} induces an automorphism $\overline{\varphi^{n_2}}:\widehat{Q}\to\widehat{Q}$, and since $Q(R)$ is simple, the composition $Q(R)\hookrightarrow Q'\twoheadrightarrow\widehat{Q}$ must be injective. Therefore, we can think of $\overline{\varphi^{n_2}}$ as an extension of φ^{n_2} to $\widehat{Q}$, so sometimes we may just call it φ^{n_2} for convenience.

Now, if $r\in\overline{z}^nV$, then r is the image of z^ns in $\widehat{Q}$, for some $s\in U$. So since $\deg_{v_{z,U}}(\varphi^{n_2}-1)\geq 1$, it follows that $(\varphi^{n_2}-1)(z^ns)\in z^{n+1}U$, and hence $(\overline{\varphi^{n_2}}-1)(r)\in\overline{z}^{n+1}V$. Therefore $\deg_{v_{\overline{z},V}}(\overline{\varphi^{n_2}}-1)\geq 1$.

Now, $\mathcal{B}$ is a maximal order in $\widehat{Q}$, equivalent to V, and $\mathcal{B} \subseteq \overline{z}^{-r}V$. So let $I := \{x \in V : Bx \subseteq V\}$, then I is a two-sided ideal of V with $z^rV \subseteq I$. Since $\deg_{v_{\overline{z},V}}(\varphi^{n_2} - 1) > 0$, it follows from Lemma 3.3 that we can choose $m \in \mathbb{N}$ such that $(\varphi^{mn_2} - 1)(V) \subseteq z^rV$, and hence there exists $n_3 \in \mathbb{N}$ such that $(\varphi^{n_3} - 1)(V) \subseteq I$.

In particular, $\varphi^{n_3}(I) \subseteq I$, and it follows from Noetherianity of V that $\varphi^{n_3}(I) = I$, and hence $I = \varphi^{-n_3}(I)$.

Therefore, given $x \in \varphi^{n_3}(\mathcal{B})$, $x = \varphi^{n_3}(b)$ for some $b \in \mathcal{B}$, and given $c \in I$, $xc = \varphi^{n_3}(b)c = \varphi^{n_3}(b\varphi^{-n_3}(c))$, and since $\varphi^{-n_3}(c) \in I$, it follows that $b\varphi^{-n_3}(c) \in V$, and hence $xc \in V$.

So setting $\mathcal{O}(I) = \{b \in B : bI \subseteq V\}$, it follows that $\mathcal{O}(I)$ contains $\mathcal{B}$ and $\varphi^{n_3}(\mathcal{B})$. But $\mathcal{O}(I)$ is an order in $\widehat{Q}$, equivalent to V, by [22, Lemma 5.1.12], so since $\mathcal{B}$ and $\varphi^{n_3}(\mathcal{B})$ are maximal orders, this means that $\mathcal{O}(I) = \mathcal{B} = \varphi^{n_3}(\mathcal{B})$, and hence φ^{n_3} is an automorphism of $\mathcal{B}$.

Also, we can choose n_3 such that $\deg_{v_{\overline{z},V}}(\varphi^{n_3} - 1) \geq r + 1$, and hence $(\varphi^{n_3} - 1)(\overline{z}^n\mathcal{B}) \subseteq (\varphi^{n_3} - 1)(z^{n-r}V) \subseteq z^{n+1}v \subseteq z^{n+1}\mathcal{B}$. Therefore $\deg_{\overline{z},\mathcal{B}}(\varphi^{n_3} - 1) \geq 1$.

Finally, $\mathcal{B} = M_n(D)$ for some non-commutative DVR D, so let ν be a uniformiser in D, and it follows that all two-sided ideals of $\mathcal{B}$ have the form $\nu^n\mathcal{B}$. Since v is the ν-adic filtration on $\widehat{Q}$, and v is topologically equivalent to $v_{\overline{z},\mathcal{B}}$, we know that there exists $k \in \mathbb{N}$ such that $z^r\mathcal{B} \subseteq \nu^2\mathcal{B}$. By Lemma 3.3, we can choose $n \geq n_3$ such that $\deg_{v_{\overline{z},\mathcal{B}}}(\varphi^n - 1) \geq k$, i.e. $(\varphi^n - 1)(\mathcal{B}) \subseteq \overline{z}^k\mathcal{B} \subseteq \mu^2\mathcal{B}$.

So, if we assume for induction that $(\varphi^n - 1)(\nu^i\mathcal{B}) \subseteq \nu^{i+1}\mathcal{B}$ for all $i < m$, then $(\varphi^n - 1)(\nu^m) = \varphi^n(\nu^m) - \nu^m = (\varphi^n - 1)(\nu^{m-1})\varphi^n(\nu) + \nu^{m-1}(\varphi^n - 1)(\nu)$. But $(\varphi^n - 1)(\nu) \in \nu^2\mathcal{B}$, $(\varphi^n - 1)(\nu^{m-1}) \in \nu^m\mathcal{B}$ and $\varphi^n(\nu) \in \nu\mathcal{B}$, therefore $(\varphi^n - 1)(\nu^m) \in \nu^{m+1}\mathcal{B}$.

It follows that $(\varphi^n - 1)(\nu^m\mathcal{B}) \subseteq \nu^{m+1}\mathcal{B}$ for all m, and hence $\deg_v(\varphi^n - 1) \geq 1$ as required. □

3.2. Bounded group automorphisms

Now, let (G, ω) be a complete, p-valued group of finite rank. We may assume that ω is an *abelian p-valuation* as defined in Definition 2.6, and we let w be the corresponding Lazard filtration on $\mathcal{O}G$ – a complete Zariskian filtration.

Fix an ordered basis $\underline{g} = \{g_1, \cdots, g_d\}$ for (G, ω), so that $\mathcal{O}G$ is isomorphic to the space of power series $\mathcal{O}[[b_1, \cdots, b_d]]$ as an $\mathcal{O}$-module, where $b_i = g_i - 1$. Recall that for any $g \in G$, $w(g - 1) \geq en\omega(g)$, with equality if $g = g_i$ for some i.

Now, recall the following definition ([2, Definition 4.5]):

Definition 3.5. An automorphism $\varphi \in \mathrm{Aut}(G)$ is *bounded* if $\inf\{\omega(\varphi(g)g^{-1}) - \omega(g) : g \in G\} > \frac{1}{p-1}$, and we define the *degree* of φ to be the number $\deg_\omega(\varphi) = \inf\{\omega(\varphi(g)g^{-1}) - \omega(g) : g \in G\}$.

Let $\mathrm{Aut}^\omega(G)$ be the group of bounded automorphisms of G.

Lemma 3.6. *If $\varphi \in Aut^{\omega}(G)$ then φ extends to a continuous, $\mathcal{O}$-linear automorphism of $\mathcal{O}G$ such that* $\deg_w(\varphi - 1) > 0$.

Proof. Clearly φ extends to an $\mathcal{O}$-linear automorphism of $\mathcal{O}[G]$, so we need only prove that $\deg_w(\varphi - 1) > 0$, and it will follow that φ is continuous and extends to $\mathcal{O}G$.

Firstly, for each $i = 1, \cdots, d$, $(\varphi - 1)(g_i - 1) = \varphi(g_i) - g_i = (\varphi(g_i)g_i^{-1} - 1)g_i$. But since $\varphi \in Aut^{\omega}(G)$, we know that $\omega(\varphi(g_i)g_i^{-1}) > \omega(g) + \frac{1}{p-1}$, so $w(\varphi(g_i)g_i^{-1} - 1) \geq en\omega(\varphi(g_i)g_i^{-1}) \geq en\omega(g_i) = w(g_i - 1)$. Hence $w((\varphi - 1)(g_i - 1)) - w(g_i - 1) > 0$.

Now, given $a, b \in \mathcal{O}[G]$:

$$(\varphi-1)(ab) = \varphi(a)\varphi(b) - ab = (\varphi(a) - a)\varphi(b) + a(\varphi(b) - b) = (\varphi-1)(a)\varphi(b) + a(\varphi-1)(b).$$

If we assume that $w((\varphi - 1)(a)) - w(a) > 0$ and $w((\varphi - 1)(b)) - w(b) > 0$, then it follows that $w(\varphi(b)) = w(b)$ and $w((\varphi(a) - a)\varphi(b) + a(\varphi(b) - b)) > \min\{w(a) + w(\varphi(b), w(a) + w(b)\} = w(a) + w(b) = w(ab)$, therefore $w((\varphi - 1)(ab)) - w(ab) > 0$.

So since $w((\varphi-1)(g_i-1)) - w(g_i-1) > 0$ for each $i = 1, \cdots, d$, it follows that for every $\alpha \in \mathbb{N}^d$, $\lambda \in \mathcal{O}$, $w(\lambda(\varphi-1)((g_1-1)^{\alpha_1} \cdots (g_d-1)^{\alpha_d})) - w(\lambda(g_1-1)^{\alpha_1} \cdots (g_d-1)^{\alpha_d}) > 0$. In particular, since $\mathcal{O}[G]$ is generated as an additive group by the monomials $\lambda b_1^{\alpha_1} \cdots b_d^{\alpha_d}$, it follows that $w(\varphi(s)) = w(s)$ for all $s \in \mathcal{O}[G]$.

Now, given $r \in \mathcal{O}[G]$ with $w(r) = t$, it follows from the definition of w that $r = \sum\limits_{\alpha \in A} \lambda_\alpha b_1^{\alpha_1} \cdots b_d^{\alpha_d} + s$, where $A := \{\alpha \in \mathbb{N}^d : \sum\limits_{1 \leq i \leq d} \alpha_i en\omega(g_i) = t\}$ and $w(s) > t$. Since $w(\varphi(s)) = w(s)$, it is clear that $w((\varphi-1)(s)) \geq w(s) > t$, so to prove that $w((\varphi-1)(r)) > t$, it remains to show that $w((\varphi - 1)(\sum\limits_{\alpha \in A} \lambda_\alpha b_1^{\alpha_1} \cdots b_d^{\alpha_d})) > t$.

But $w(\lambda_\alpha b_1^{\alpha_1} \cdots b_d^{\alpha_d}) = t$ for all $\alpha \in A$, so we have seen that $w((\varphi-1)(\lambda_\alpha b_1^{\alpha_1} \cdots b_d^{\alpha_d})) > t$ for each α, and it follows immediately that $w((\varphi - 1)(\sum\limits_{\alpha \in A} \lambda_\alpha b_1^{\alpha_1} \cdots b_d^{\alpha_d})) > t$ as required. $\square$

Now, fix a prime ideal P of $\mathcal{O}G$ such that $p \notin P$, and fix an automorphism $\varphi \in Aut^{\omega}(G)$ such that P is invariant under the extension of φ to $\mathcal{O}G$, i.e. $\varphi(P) = P$. Hence φ induces an automorphism $\overline{\varphi}$ of $\frac{\mathcal{O}G}{P}$. We will also assume that $P \neq (G - 1)\mathcal{O}G$.

Proposition 3.7. *If G is nilpotent, $p \notin P$ and $P \neq (G - 1)\mathcal{O}G$, then the quotient ring $\mathcal{O}G/P$ is infinitely generated over $\mathcal{O}$.*

Proof. Let $Q := K \otimes_{\mathcal{O}} P \trianglelefteq KG$, and since $p \notin P$, Q is a proper ideal of KG, and $KG/Q \cong K \otimes_{\mathcal{O}} \frac{\mathcal{O}G}{P}$, so Q is prime. If $\mathcal{O}G/P$ is finitely generated over $\mathcal{O}$ then KG/Q is a finite dimensional prime K-algebra, and thus it is simple, meaning that $Q = \mathrm{Ann}_{KG}M$ for some finite dimensional simple KG-module M.

If $M = KGm$ for some $m \in M$ then M carries a natural filtration given by $v(rm) = \sup\{w_K(r+y) : y \in KG \text{ and } ym = 0\}$, where w_K is the extension of the Lazard filtration on $\mathcal{O}G$ to KG. Then since M is finite dimensional over K, it is complete with respect to v, and hence M has the structure of a simple module over the completion of KG

with respect to w_K, which is isomorphic to the affinoid enveloping algebra $\widehat{U(\mathcal{L})}_K$. So set $Q' := \mathrm{Ann}_{\widehat{U(\mathcal{L})}_K} M$, a primitive ideal in $\widehat{U(\mathcal{L})}_K$ of finite codimension.

But since $\mathcal{L}$ is nilpotent, we see using [17, Theorem A] that $Q' \cap \widehat{U(p^n\mathcal{L})}_K = \mathrm{Ann}_{\widehat{U(p^n\mathcal{L})}_K} \widehat{D(\lambda)}$ for some $n \in \mathbb{N}$ and some linear form $\lambda \in (p^n\mathcal{L})^*$. Thus $Q' \cap U(\mathfrak{g}) = \mathrm{Ann}_{U(\mathfrak{g})} D(\lambda)$ has finite codimension in $U(\mathfrak{g})$, meaning that $\lambda = 0$ by [11], and hence $\mathcal{L} \subseteq \mathfrak{g} \subseteq Q'$.

But since $G = \exp(\mathcal{L})$, this means that $G - 1 \subseteq Q' \cap \mathcal{O}G = P$, and hence $\mathcal{O}G/P$ is a quotient of $\mathcal{O} = \mathcal{O}G/(G-1)$, meaning that either $p \in P$ or $P = (G-1)\mathcal{O}G$, contradicting our assumptions. Therefore $\mathcal{O}G/P$ is infinitely $\mathcal{O}$-generated as required. □

In light of this proposition, we will also assume from now on that $\mathcal{O}G/P$ is infinitely generated over $\mathcal{O}$.

Theorem 3.8. *There exists a non-commutative valuation v on $Q(\mathcal{O}G/P)$ such that the natural map $\tau : (\mathcal{O}G, w) \to (Q(\mathcal{O}G/P), v)$ is continuous, and there exists $n \in \mathbb{N}$ such that* $\deg_v(\overline{\varphi}^n - 1) \geq v(p)$.

Proof. Let $\overline{w}$ be the quotient filtration on $\frac{\mathcal{O}G}{P}$, so that $\mathrm{gr}_{\overline{w}} \frac{\mathcal{O}G}{P} \cong \frac{\mathrm{gr}_w \mathcal{O}G}{\mathrm{gr}_w P}$. So since w is Zariskian, it follows that $\mathrm{gr}_{\overline{w}} \frac{\mathcal{O}G}{P}$ is commutative and Noetherian, and since $\frac{\mathcal{O}G}{P}$ is complete with respect to $\overline{w}$, it follows from [21, Theorem 2.1.2] that $\overline{w}$ is Zariskian.

Moreover, since $\mathrm{gr}_w \mathcal{O}G \cong k[t, t_1, \cdots, t_d]$ as a graded ring, the quotients $\mathrm{gr}_{\overline{w}} \frac{\mathcal{O}G}{P}/(\mathrm{gr}_{\overline{w}} \frac{\mathcal{O}G}{P})^i_{\geq 0}$ are finite dimensional over k for each $i \geq 1$. So if $(\mathrm{gr}_{\overline{w}} \frac{\mathcal{O}G}{P})_{\geq 0}$ is nilpotent in $\mathrm{gr}_{\overline{w}} \frac{\mathcal{O}G}{P}$ then $\mathrm{gr}_{\overline{w}} \frac{\mathcal{O}G}{P}$ is also finite dimensional over k, and hence $\mathcal{O}G/P$ is finitely generated over $\mathcal{O}$ – a contradicting our assumption. Hence $(\mathrm{gr}_{\overline{w}} \frac{\mathcal{O}G}{P})_{\geq 0}$ is non-nilpotent.

Therefore, using Construction 2.3, we can define a non-commutative valuation v on $Q(\mathcal{O}G/P)$ such that the inclusion $(\mathcal{O}G/P, \overline{w}) \to (Q(\mathcal{O}G/P), v)$ is continuous. Since the surjection $(\mathcal{O}G, w) \to (\mathcal{O}G/P, \overline{w})$ is continuous, it follows that the composition τ is also continuous.

Now, since $\deg_w(\varphi - 1) > 0$ by Lemma 3.6, and $(\varphi - 1)(P) \subseteq P$, it follows from Lemma 3.1 that $\deg_{\overline{w}}(\varphi - 1) > 0$. Using Theorem 3.4, it follows that there exists $n \in \mathbb{N}$ such that $\deg_v(\varphi^n - 1) > 0$ as required. □

3.3. The logarithm

Given a complete p-valued group (G, ω) of finite rank, and a faithful prime ideal P of $\mathcal{O}G$ such that $p \notin P$ and $\mathcal{O}G/P$ is infinitely generated over $\mathcal{O}$, we now want to take steps towards proving a control theorem for P. Again, assume that there is an automorphism $\varphi \in \mathrm{Aut}^\omega(G)$ with $\varphi \neq 1$ such that $\varphi(P) = P$, and let $\overline{\varphi}$ be the automorphism of $\mathcal{O}G/P$ induced from φ.

We will now assume further that there is a closed, central subgroup A of G such that:

- $\varphi(g)g^{-1} \in A$ for all $g \in G$.

- For all $a \in A$, $\varphi(a) = a$.

It is straightforward to show that these properties are satisfied for φ^n for all $n \in \mathbb{N}$.

Now, using Theorem 3.8, we fix a non-commutative valuation v on $Q = Q(\mathcal{O}G/P)$ such that the natural map $\tau : (\mathcal{O}G, w) \to (Q, v)$ is continuous, and a natural number $n \in \mathbb{N}$ such that $\deg_v(\overline{\varphi}^n - 1) \geq v(p)$. After replacing φ by φ^n if necessary, we may assume that $n = 1$.

Let $\widehat{Q}$ be the completion of Q with respect to v, and clearly $\overline{\varphi}$ extends continuously to $\widehat{Q}$.

Also, since A is central in G, $P \cap \mathcal{O}A$ is a prime ideal of $\mathcal{O}A$. Therefore, the field of fractions of $\frac{\mathcal{O}A}{P \cap \mathcal{O}A}$ is contained in $\widehat{Q}$. So, let F be the closure of this field of fractions in $\widehat{Q}$, then F is a central subfield of $\widehat{Q}$ carrying a complete valuation $v_F = v|_F$.

Definition 3.9. Define the *logarithm* of $\overline{\varphi}$ to be the derivation of $\widehat{Q}$ defined by the logarithm series at $\overline{\varphi}$. Specifically:

$$\log(\overline{\varphi}) = \sum_{k \geq 1} \frac{(-1)^{k+1}}{k} (\overline{\varphi} - 1)^k \tag{5}$$

Note: 1. This definition never makes sense if $p \in P$, because in this case if $p \mid k$ then $k = 0$ in $\widehat{Q}$.

2. This definition makes sense when $p \notin P$ because $\deg_v(\overline{\varphi} - 1) \geq v(p)$ so $\deg_v(\overline{\varphi} - 1)^k \geq kv(p)$ for all k and $\deg_v(\frac{(-1)^{k+1}}{k}(\overline{\varphi} - 1)^k) \geq kv(p) - v(k) = kv(p) - v_p(k)v(p) \to 0$ as $k \to \infty$. So since $B(\widehat{Q})$ is complete with respect to $\deg_v$ by [2, Lemma 2.4], it follows that the logarithm series must converge to an element of $B(\widehat{Q})$, and since $\overline{\varphi}$ is an automorphism, this must be a derivation.

3. See the proof of [24, Theorem 4] for details of why $\log(\overline{\varphi})$ is a derivation of $\widehat{Q}$.

Proposition 3.10. *Fix $g \in G$, then:*

i. $\deg(\log(\overline{\varphi})) \geq 1$.

ii. The series $\log(\varphi(g)g^{-1}) := \sum_{k \geq 1} \frac{(-1)^{k+1}}{k} \tau(\varphi(g)g^{-1} - 1)^k$ *converges in* F.

iii. $\log(\overline{\varphi})(g) = \log(\varphi(g)g^{-1})\tau(g)$.

iv. $\log(\overline{\varphi})(F) = 0$

v. $\log(\overline{\varphi})$ *is a continuous, F-linear derivation.*

Proof. *i.* Since $\deg_v(\overline{\varphi}) \geq v(p)$, it follows that $\deg(\frac{(-1)^{k+1}}{k}(\overline{\varphi} - 1)^k) \geq kv(p) - v(k) = (k - v_p(k))v(p) \geq 1$, and hence $\deg_v(\log(\overline{\varphi})) \geq 1$.

ii − iii. For any $g \in G$, $(\overline{\varphi} - 1)(g) = \tau(\varphi(g) - g) = \tau(\varphi(g)g^{-1} - 1)\tau(g)$, and if we suppose for induction that $(\overline{\varphi} - 1)^k(g) = \tau(\varphi(g)g^{-1} - 1)^k\tau(g)$, then $(\overline{\varphi} - 1)^{k+1}(g) = (\overline{\varphi} - 1)(\tau(\varphi(g)g^{-1} - 1)^k\tau(g)) = \overline{\varphi}(\tau(\varphi(g)g^{-1} - 1)^k)\overline{\varphi}(\tau(g)) - \tau(\varphi(g)g^{-1} - 1)^k\tau(g)$.

But since $\varphi(g)g^{-1} \in A$, it follows from our assumption that $\overline{\varphi}(\tau(\varphi(g)g^{-1}-1)^k) = \tau(\varphi(g)g^{-1}-1)^k$, and hence $(\overline{\varphi}-1)^{k+1}(g) = \tau(\varphi(g)g^{-1}-1)^k\overline{\varphi}(\tau(g)) - \tau(\varphi(g)g^{-1}-1)^k\tau(g) = \tau(\varphi(g)g^{-1}-1)^{k+1}\tau(g)$.

Therefore, $\log(\overline{\varphi})(g) = \sum_{k\geq 1}\frac{(-1)^{k+1}}{k}\tau(\varphi(g)g^{-1}-1)^k\tau(g)$, so multiplying on the right by $\tau(g)^{-1}$ gives that $\log(\varphi(g)g^{-1}) = \sum_{k\geq 1}\frac{(-1)^{k+1}}{k}\tau(\varphi(g)g^{-1}-1)^k$ converges to $\log(\overline{\varphi})(g)\tau(g)^{-1}$, and since $\tau(\varphi(g)g^{-1}-1) \in \tau(\mathcal{O}A) \subseteq F$, it follows that $\log(\varphi(g)g^{-1}) \in F$.

iv. Again, since $\varphi(a) = a$ for all $a \in A$, it follows that $(\overline{\varphi}-1)(\tau(a)) = 0$, and hence $\log(\overline{\varphi})(a) = 0$. So since $\log(\varphi)$ is $\mathcal{O}$-linear and continuous, it follows that $\log(\overline{\varphi})(s) = 0$ for all $s \in \tau(\mathcal{O}A) = \mathcal{O}A/\mathcal{O}A \cap P$.

Moreover, since F is the completion of the field of fractions of $\tau(\mathcal{O}A)$, and $\log(\overline{\varphi})$ is a derivation, it follows that $\log(\overline{\varphi})(s) = 0$ for all $s \in F$ as required.

v. We know that $\log(\overline{\varphi})$ is a continuous derivation, and since $\log(\overline{\varphi})(F) = 0$ it follows that it is F-linear. □

Remark. Without our assumptions that $\varphi(g)g^{-1} \in A$ for all g, and φ is trivial when restricted to A, this proposition fails.

Fix $\lambda := \inf\{v(\log(\overline{\varphi})(\varphi(g)g^{-1})) : g \in G\}$, and let $U := \{g \in G : v(\log(\varphi(g)g^{-1})) > \lambda\}$. The following lemma depends on the assumption that P is faithful:

Lemma 3.11. $1 \leq \lambda < \infty$*, and* U *is a proper, open subgroup of* G *containing* G^p *and* (G,G)*.*

Proof. For convenience, set $\psi(g) := \varphi(g)g^{-1}$. Then since $\varphi(g)g^{-1} \in Z(G)$ for all $g \in G$, it follows that $\psi : G \to A$ is a group homomorphism.

Using Proposition 3.10, we see that $\deg(\overline{\varphi}) \geq 1$, and that $\log(\overline{\varphi})(g) = \log(\psi(g))g$ for all $g \in G$. So since $v(g) = 0$, it follows that $v(\log(\psi(g))) = v(\log(\overline{\varphi})(g)) \geq \deg(\overline{\varphi}) \geq 1$ for all $g \in G$, and hence $\lambda \geq 1$.

If $\lambda = \infty$ then $v(\log(\psi(g))) = \infty$ and hence $\log(\psi(g)) = 0$ for all $g \in G$. But the function on $\widehat{Q}$ defined by the logarithm series is injective by [12, Corollary 6.25(*ii*)], and hence $\tau(\psi(g)-1) = 0$ for all $g \in G$, i.e. $\psi(g)-1 \in P$. But P is faithful, so this means that $\psi(g) = \varphi(g)g^{-1} = 1$ for all $g \in G$, and hence φ is trivial – contradicting our assumption.

Therefore $1 \leq \lambda < \infty$, and clearly λ is an integer, so there exists $g \in G$ such that $v(\log(\psi(g))) = \lambda$, and hence $U \subsetneq G$.

Furthermore, given $g \in G$, $v(\log(\psi(g^p))) = v(\log(\psi(g)^p)) = v(p\log(\psi(g)) = v(\log(\psi(g))) + v(p) \geq \lambda + v(p) > \lambda$, and hence $G^p \subseteq U$.

Also, if $g, h \in U$ then

$$v(\log(\psi(gh))) = v(\log(\psi(g)\psi(h))) = v(\log(\psi(g)) + \log(\psi(h))) \geq \min\{v(\log(\psi(g))), v(\log(\psi(h)))\} > \lambda,$$

so since $\frac{G}{G^p}$ is a finite group and $\frac{U}{G^p}$ is closed under the group operation, it follows that U is an open subgroup of G.

Finally, if $g, h \in G$, $v(\log(\psi(g,h))) = v(\log((\psi(g), \psi(h)))) = v(\log(1)) = \infty > \lambda$, so $(G, G) \subseteq U$ as required. $\square$

So, using [2, Lemma 4.2], choose an ordered basis $\{g_1, \cdots, g_d\}$ for (G, ω) such that $\{g_1^p, \cdots, g_r^p, g_{r+1}, \cdots, g_d\}$ is an ordered basis for U for some $1 \leq r \leq d$. Thus $v(\log(\varphi(g_i)g_i^{-1})) = \lambda$ for $i = 1, \cdots, r$ and $v(\log(\varphi(g_i)g_i^{-1})) > \lambda$ for all $i > r$.

Now, set $a := \log(\varphi(g_1)g_1^{-1})$. Then $a \in F$ by Proposition 3.10, and $v(a) = \lambda < \infty$ so $a \neq 0$, thus a is a unit in F. So for each $i = 1, \cdots, d$, set $z_i := a^{-1}\log(\varphi(g_i)g_i^{-1}) \in F$.

Since F is central, it follows from the definition of a non-commutative valuation that $v(z_i) = v(\log(\varphi(g_i)g_i^{-1})) - v(a)$ for each i, so $v(z_1) = \cdots = v(z_r) = 0$, and $v(z_i) > 0$ if $i > r$.

From now on, set $\delta := a^{-1}\log(\overline{\varphi}) : \widehat{Q} \to \widehat{Q}$, which is an F-linear derivation of $\widehat{Q}$, and using Proposition 3.10 we see that for all $g \in G$, $\delta(g) = a^{-1}\log(\overline{\varphi})(g) = a^{-1}\log(\varphi(g)g^{-1})g$. But if $g = \underline{g}^{\alpha} := g_1^{\alpha_1} \cdots g_d^{\alpha_d}$ for some $\alpha \in \mathbb{Z}_p^d$ then

$$\begin{gathered}\log(\varphi(g)g^{-1}) = \log((\varphi(g_1)g_1^{-1})^{\alpha_1} \cdots (\varphi(g_d)g_d^{-1})^{\alpha_d})) = \\ \alpha_1 \log(\varphi(g_1)g_1^{-1}) + \cdots + \alpha_d \log(\varphi(g_d)g_d^{-1}).\end{gathered}$$

Therefore, $\delta(\underline{g}^{\alpha}) = (\alpha_1 z_1 + \cdots + \alpha_d z_d)\underline{g}^{\alpha}$ for all $\alpha \in \mathbb{Z}_p^d$.

Furthermore, since δ is F-linear, it follows that $\delta^n(\underline{g}^{\alpha}) = (\alpha_1 z_1 + \cdots \alpha_d z_d)^n \underline{g}^{\alpha}$ for all $n \in \mathbb{N}$.

3.4. A convergence argument

We will now show how we can study convergence of δ^{p^m} to prove a control theorem for P.

Notation: For any $\alpha \in \mathbb{Z}_p$, denote by α' the unique integer in $\{0, \cdots, p-1\}$ such that $\alpha \equiv \alpha' \pmod{p}$. Also, let $\mathcal{V} = \{z \in F : v(q) \geq 0\}$, $\mathcal{V}^+ = \{z \in F : v(q) > 0\}$, so that $\frac{\mathcal{V}}{\mathcal{V}^+}$ is a field extension of $\mathbb{F}_p$.

Lemma 3.12. *For all $m \in \mathbb{N}$, $z_1^{p^m}, \cdots, z_r^{p^m}$ are $\mathbb{F}_p$-linearly independent modulo $\mathcal{V}^+$.*

Proof. Let us suppose that there exist integers $\alpha_1, \cdots, \alpha_r \in \{0, 1, \cdots, p-1\}$ such that $\alpha_1 z_1^{p^m} + \cdots + \alpha_r z_r^{p^m} \in \mathcal{V}^+$, i.e. $v(\alpha_1 z_1^{p^m} + \cdots + \alpha_r z_r^{p^m}) > 0$.

Firstly, note that $\alpha_i^{p^m} \equiv \alpha_i \pmod{p}$ by Fermat's Little theorem, so $0 \equiv \alpha_1 z_1^{p^m} + \cdots + \alpha_r z_r^{p^m} \equiv (\alpha_1 z_1 + \cdots + \alpha_r z_r)^{p^m} \pmod{\mathcal{V}^+}$, which implies that $\alpha_1 z_1 + \cdots + \alpha_r z_r \in \mathcal{V}^+$.

But $z_i = a^{-1}\log(\varphi(g_i)g_i^{-1})$, so $\alpha_i z_i = a^{-1}\log((\varphi(g_i)g_i^{-1})^{\alpha_i})$. But since $g \mapsto \varphi(g)g^{-1}$ defines a group homomorphism, it follows that $\alpha_i z_i = a^{-1}\log(\varphi(g_i^{\alpha_i})g_i^{-\alpha_i})$, and hence $\alpha_1 z_1 + \cdots + \alpha_r z_r = a^{-1}\log(\varphi(g_1^{\alpha_1})g_1^{-\alpha_1} \cdots \varphi(g_r^{\alpha_r})g_r^{-\alpha_r}) = a^{-1}\log(\varphi(\underline{g}^{\alpha})g^{-\alpha}) \in \mathcal{V}^+$.

Therefore, it follows that $v(\log(\varphi(\underline{g}^\alpha)\underline{g}^{-\alpha})) > v(a) = \lambda$, and hence $\underline{g}^\alpha \in U$ by the definition of U. But we know that $\{g_1^p, \cdots, g_r^p, g_{r+1}, \cdots, g_d\}$ is an ordered basis for U, which means that $p \mid \alpha_i$, i.e. $\alpha_i = 0$, for all i as required. □

Lemma 3.13. *For any $m \in \mathbb{N}$, $\alpha \in \mathbb{Z}_p^d$, $\delta^{p^m}(\underline{g}^\alpha) = (\alpha_1' z_1^{p^m} + \cdots + \alpha_r' z_r^{p^m})\underline{g}^\alpha + z^{(m,\alpha)}\underline{g}^\alpha$ for some $z^{(m,\alpha)} \in \mathcal{V}^+$.*

Proof. Firstly, we know that $\delta^{p^m}(\underline{g}^\alpha) = (\alpha_1 z_1 + \cdots + \alpha_d z_d)^{p^m}\underline{g}^\alpha$, and we know that $v(\alpha_{r+1}z_{r+1} + \cdots + \alpha_d z_d) > 0$. Also, using Lemma 3.12, we see that $v(\alpha_1 z_1 + \cdots + \alpha_r z_r) = 0$ if $p \nmid \alpha_i$ for some i. Therefore:

$$(\alpha_1 z_1 + \cdots + \alpha_d z_d)^{p^m} \equiv (\alpha_1 z_1 + \cdots + \alpha_r z_r)^{p^m} \equiv \alpha_1^{p^m} z_1^{p^m} + \cdots + \alpha_r^{p^m} z_r^{p^m} \pmod{\mathcal{V}^+}.$$

Since $\alpha_i^{p^m} \equiv \alpha_i \equiv \alpha_i' \pmod p$ for all m, it follows that $\alpha_1^{p^m} z_1^{p^m} + \cdots + \alpha_r^{p^m} z_r^{p^m} \equiv \alpha_1' z_1^{p^m} + \cdots + \alpha_r' z_r^{p^m} \pmod{\mathcal{V}^+}$.

Therefore, $(\alpha_1 z_1 + \cdots + \alpha_d z_d)^{p^m} = \alpha_1' z_1^{p^m} + \cdots + \alpha_r' z_r^{p^m} + z^{(m,\alpha)}$ for some $z^{(m,\alpha)} \in \mathcal{V}^+$, and hence $\delta^{p^m}(\underline{g}^\alpha) = (\alpha_1' z_1^{p^m} + \cdots + \alpha_r' z_r^{p^m})\underline{g}^\alpha + z^{(m,\alpha)}\underline{g}^\alpha$ as required. □

Now, define $T := \begin{pmatrix} z_1 & z_2 & \cdots & z_r \\ z_1^p & z_2^p & \cdots & z_r^p \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ z_1^{p^{r-1}} & z_2^{p^{r-1}} & \cdots & z_r^{p^{r-1}} \end{pmatrix}$. Then using Lemma 3.13, we see that:

$$\begin{pmatrix} \delta(\underline{g}^\alpha) \\ . \\ . \\ . \\ \delta^{p^{r-1}}(\underline{g}^\alpha) \end{pmatrix} = T \begin{pmatrix} \alpha_1'\underline{g}^\alpha \\ . \\ . \\ . \\ \alpha_r'\underline{g}^\alpha \end{pmatrix} + \begin{pmatrix} z^{(0,\alpha)}\underline{g}^\alpha \\ . \\ . \\ . \\ z^{(r-1,\alpha)}\underline{g}^\alpha \end{pmatrix}. \tag{6}$$

But T is a matrix of Vandermonde type, in the sense of [7, Section 1.1], and the entries of T all lie in $\mathcal{V}$ and are $\mathbb{F}_p$-linearly independent modulo $\mathcal{V}^+$ by Lemma 3.12, so it follows from [7, Lemma 1.1] that $\det(T)$ has value zero. Therefore T is invertible, and its inverse has value 0, i.e. the minimum value over all the entries of T^{-1} is zero.

Define $u : \widehat{Q}^r \to \widehat{Q}, (s_1, \cdots, s_r)^T \mapsto s_1 + \cdots + s_r$, and define:

$$\overline{\iota} : \widehat{Q} \to \widehat{Q}, s \mapsto uT^{-1} \begin{pmatrix} \delta(\tau(s)) \\ . \\ . \\ . \\ \delta^{p^{r-1}}(\tau(s)) \end{pmatrix} \tag{7}$$

Note that $\bar{\iota}$ is continuous and F-linear. Also for any $\alpha \in \mathbb{Z}_p^d$, using (6) we see that:

$$\bar{\iota}(\underline{g}^\alpha) = uT^{-1}\begin{pmatrix} \delta(\underline{g}^\alpha) \\ . \\ . \\ . \\ \delta^{p^{r-1}}(\underline{g}^\alpha) \end{pmatrix} = u\begin{pmatrix} \alpha_1'\underline{g}^\alpha \\ . \\ . \\ . \\ \alpha_r'\underline{g}^\alpha \end{pmatrix} + uT^{-1}\begin{pmatrix} z^{(0,\alpha)}\underline{g}^\alpha \\ . \\ . \\ . \\ z^{(r-1,\alpha)}\underline{g}^\alpha \end{pmatrix}.$$

But the entries of T^{-1} all have value at least 0, so we deduce that $\bar{\iota}(\underline{g}^\alpha) = (\alpha_1' + \cdots + \alpha_r')\underline{g}^\alpha + z^{(\alpha)}\underline{g}^\alpha$ for some $z^{(\alpha)} \in \mathcal{V}^+$.

For each $m \in \mathbb{N}$, define $\iota_m : \widehat{Q} \to \widehat{Q}, q \mapsto \bar{\iota}^{p^m(p-1)}(q)$, which is also continuous and F-linear. Also, for each $k \geq 0$, define $\widehat{Q}_k := \{q \in Q : v(q) \geq k\}$.

Proposition 3.14. *Given $m \in \mathbb{N}$:*

- *The composition $\iota_m\tau : \mathcal{O}G \to \widehat{Q}$ is continuous and $\mathcal{O}$-linear.*
- *There exists $k_m \in \mathbb{N}$ such that $k_m \to \infty$ as $m \to \infty$, and for all $\alpha \in \mathbb{Z}_p^d$,*

$$\iota_m(\underline{g}^\alpha) \equiv \begin{cases} \underline{g}^\alpha & \text{if } v_p(\alpha_1 + \cdots + \alpha_r) = 0 \\ 0 & \text{if } v_p(\alpha_1 + \cdots + \alpha_r) > 0 \end{cases} (\text{mod } \widehat{Q}_{k_m}).$$

Moreover, choose a sequence of integers $m_1 < m_2 < \cdots$ such that $k_{m_1} < k_{m_2} < \cdots$, then for every $s \in \mathcal{O}G$, $i \in \mathbb{N}$, $(\iota_{m_i} - \iota_{m_{i+1}})(\tau(s)) \in \widehat{Q}_{k_{m_i}}$.

Proof. The first statement is obvious, since τ and ι_m are both continuous and $\mathcal{O}$-linear.

Now, we know that $\bar{\iota}$ is F-linear, and for any $\alpha \in \mathbb{Z}_p^d$, $\bar{\iota}(\underline{g}^\alpha) = (\alpha_1' + \cdots + \alpha_r' + z^{(\alpha)})\underline{g}^\alpha$ for some $z^{(\alpha)} \in F$ with $v(z^{(\alpha)}) \geq 1$. Therefore, $\iota_m(\underline{g}^\alpha) = \bar{\iota}^{p^m(p-1)}(\underline{g}^\alpha) = (\alpha_1' + \cdots + \alpha_r' + z^{(\alpha)})^{p^m(p-1)}\underline{g}^\alpha$.

But if $v_p(\alpha_1' + \cdots + \alpha_r') = 0$ then $(\alpha_1' + \cdots + \alpha_r')^{p-1} \equiv 1 \pmod p$, so $(\alpha_1' + \cdots + \alpha_r' + z^{(\alpha)})^{p-1} = 1 + y^{(\alpha)}$ for some $y^{(\alpha)} \in F$ with $v(y^{(\alpha)}) \geq 1$, and $\iota_m(\underline{g}^\alpha) = (1 + y^{(\alpha)})^{p^m}\underline{g}^\alpha$.

On the other hand, if $v_p(\alpha_1' + \cdots + \alpha_r') > 0$ then $v(\iota_m(\underline{g}^\alpha)) \geq p^m(p-1)$.

Let $\gamma := \inf\{v(y^{(\alpha)}) : \alpha \in \mathbb{Z}_p^d, v(\alpha_1 + \cdots + \alpha_r) = 0\} \geq 1$, and for each $m \in \mathbb{N}$, define $t_m := \min\{p^m\gamma, m - v_p(k) + (p^m - k)\gamma : 0 < k < p^m\}$. Then since $\gamma = v(y^{(\alpha)})$ for some $\alpha \in \mathbb{Z}_p^d$, it follows from Lemma 3.2 that $t_m \to \infty$ as $m \to \infty$.

Furthermore, also using Lemma 3.2, we see that:

$v((1+y^{(\alpha)})^{p^m} - 1) \geq \min\{p^m v(y^{(\alpha)}), m - v_p(k) + (p^m - k)v(y^{(\alpha)}) : 0 < k < p^m\} \geq t_m$.

Hence $v(\iota_m(\underline{g}^\alpha) - \underline{g}^\alpha) \geq t_m$ for all m whenever $v(\alpha_1 + \cdots + \alpha_r) = 0$.

So, let $k_m := \min\{p^m(p-1), t_m\} \to \infty$ as $m \to \infty$, then $v(\iota_m(\underline{g}^\alpha)) \geq k_m$ if $v(\alpha_1 + \cdots + \alpha_r) > 0$, and $v(\iota_m(\underline{g}^\alpha) - \underline{g}^\alpha) \geq k_m$ if $v_p(\alpha_1 + \cdots + \alpha_r) = 0$ as required.

In particular, if $k_{m_1} < k_{m_2} < \cdots$, then for every $g \in G$, $\iota_{m_i}(g) \equiv \iota_{m_{i+1}}(g) \pmod{\widehat{Q}_{k_{m_i}}}$ for all i, i.e. $(\iota_{m_i} - \iota_{m_{i+1}})(g) \in \widehat{Q}_{k_{m_i}}$. So since ι_m is $\mathcal{O}$-linear for every m, it follows that

$(\iota_{m_i} - \iota_{m_{i+1}})(\tau(s)) \in \widehat{Q}_{k_{m_i}}$ for every $s \in \mathcal{O}[G]$, and since $(\iota_{m_i} - \iota_{m_{i+1}})\tau$ is continuous, this means that $(\iota_{m_i} - \iota_{m_{i+1}})(\tau(s)) \in \widehat{Q}_{k_{m_i}}$ for every $s \in \mathcal{O}G$ as required. □

Using this proposition, it follows that there exists a continuous, $\mathcal{O}$-linear map $\iota : \mathcal{O}G \to \widehat{Q}$ such that $\iota(P) = 0$, $\iota(s) \equiv \iota_m(\tau(s)) \pmod{\widehat{Q}_{k_{m_i}}}$ for every i, and for every $\alpha \in \mathbb{Z}_p^d$:

$$\iota(\underline{g}^\alpha) = \begin{cases} \underline{g}^\alpha & \text{if } v_p(\alpha_1 + \cdots + \alpha_r) = 0 \\ 0 & \text{if } v_p(\alpha_1 + \cdots + \alpha_r) > 0 \end{cases} \tag{8}$$

Finally, since U contains (G,G) and G^p by Lemma 3.11, the quotient $\frac{G}{U}$ has the structure of an $\mathbb{F}_p$-vector space, with basis $\{g_1U, \cdots, g_rU\}$. Therefore, the map $\chi : \frac{G}{U} \to \frac{\mathbb{Z}}{p\mathbb{Z}}, \underline{g}^\alpha U \mapsto \alpha_1 + \cdots + \alpha_r + p\mathbb{Z}$ is a non-zero $\mathbb{F}_p$-linear map, so $\ker(\chi) = \frac{V}{U}$ for some proper open subgroup V of G, and it follows that for all $g \in G$:

$$\iota(g) = \begin{cases} g & \text{if } g \notin V \\ 0 & \text{if } g \in V \end{cases} \tag{9}$$

Now we are ready to prove a control theorem. Firstly, let $C^\infty(G, \mathcal{O})$ be the space of locally constant functions $f : G \to \mathcal{O}$, and recall from [3, Proposition 2.5 and Lemma 2.9] that there is a natural action $\rho : C^\infty(G, \mathcal{O}) \to \mathrm{End}_{\mathcal{O}}\mathcal{O}G$ such that for any open subgroup U of G, if $f \in C^\infty(G, \mathcal{O})$ is constant on the cosets of U then the action of f on $\mathcal{O}G$ can be described explicitly:

If $x \in \mathcal{O}G$ and $x = \sum\limits_{g \in C} x_g g$, where C is a set of coset representatives for U in G and $x_c \in \mathcal{O}U$, then

$$\rho(f)(x) = \sum_{g \in C} f(g) x_c g$$

In particular, define $f : G \to \mathcal{O}, g \mapsto \begin{cases} 1 & \text{if } g \notin V \\ 0 & \text{if } g \in V \end{cases}$. Then clearly $f \in C^\infty(G, \mathcal{O})$ is constant on the cosets of V, so $\rho(f)(g) = \begin{cases} g & \text{if } g \notin V \\ 0 & \text{if } g \in V \end{cases}$.

Therefore, it follows from (9) that $\iota(g) = \tau\rho(f)(g)$ for all $g \in G$, so since ι, τ and $\rho(g)$ are continuous and $\mathcal{O}$-linear, it follows that $\iota = \tau\rho(f)$. Therefore, since $\iota(P) = 0$, it follows that $\rho(f)(P) \subseteq P$.

Proposition 3.15. *P is controlled by V.*

Proof. This is now identical to the proof of [15, Theorem 1.4]. Firstly, suppose that $C = \{x_1, \cdots, x_t\}$ is a complete set of coset representatives for V in G, then for all $r \in \mathcal{O}G$, $r = \sum\limits_{i \le t} r_i x_i$ for some $r_i \in \mathcal{O}V$.

Suppose we can choose C such that if $r \in P$ then $r_1 \in P \cap \mathcal{O}V$. Then since $rx_1^{-1}x_i \in P$ for all $i = 1 \cdots, t$ and $rx_1^{-1}x_i$ has x_1 component r_i, it follows that $r_i \in P \cap \mathcal{O}V$ for each i, and hence P is controlled by V.

It remains to prove that we can choose such a set C of coset representatives such that if $\sum_{i\leq t} r_i x_i \in P$, then at least one of the r_i lies in $P \cap \mathcal{O}V$.

Since $G^p \subseteq V$, it follows that V has ordered basis $\{g_1^p, \cdots, g_s^p, g_{s+1}, \cdots, g_d\}$ and thus $C = \{g_1^{b_1} \cdots g_s^{b_r} : 0 \leq b_i < p\}$ is a complete set of coset representatives for V in G.

So for each $\underline{b} \in [p-1]^s$, let $g_{\underline{b}} = g_1^{b_1} \cdots g_r^{b_r}$ (here $[p-1] = \{0, 1, \cdots, p-1\}$).

Then if $r = \sum_{\underline{b}\in[p-1]^s} r_{\underline{b}} g_{\underline{b}} \in P$, then $\rho(f)(r) = \sum_{\underline{b}\in[p-1]^s} f(g_{\underline{b}}) r_{\underline{b}} g_{\underline{b}}$, and since $\rho(f)(P) \subseteq P$ this also lies in P. But $f(g_{\underline{b}}) = 1$ if $\underline{b} \neq 0$, and $f(g_{\underline{0}}) = 0$ hence $\rho(f)(r) = \sum_{\underline{b}\in[p-1]^s\setminus\{\underline{0}\}} r_{\underline{b}} g_{\underline{b}} \in P$.

Therefore, $r_{\underline{0}} g_{\underline{0}} = r - \rho(f)(r) \in P$, and thus $r_{\underline{0}} \in P \cap \mathcal{O}U$ as required. □

So, since prime ideals in $\mathcal{O}G$ not containing P correspond bijectively with prime ideals in KG, altogether we have now prove the following theorem:

Theorem 3.16. *Let (G, ω) be a complete, p-valued group of finite rank, and let P be a faithful, prime ideal of KG such that KG/P is infinite dimensional over K. Also, let $\varphi \in Aut^\omega(G)$ be an automorphism of G, and let A be a closed, central subgroup of G such that:*

- $\varphi \neq 1$.
- $\varphi(P) = P$.
- $\varphi(g)g^{-1} \in A$ *for all* $g \in G$.
- $\varphi(a) = a$ *for all* $a \in A$.

Then P is controlled by a proper, open subgroup of G.

This result is, in essence, the characteristic 0 version of [2, Theorem B]. This result was sufficient to fully prove Conjecture 1.2 in characteristic p for G nilpotent in [2], but unfortunately our additional assumption that $\varphi(g)g^{-1}$ is fixed by φ for all $g \in G$ restricts the usefulness of this result, which is why, as we will see, we cannot assume that the subgroup A described in the statement of Theorem B is central.

3.5. Control theorem for prime ideals

Now we are ready to prove Theorem B, and the remainder of the argument is similar to the proof of [15, Theorem 1.2], as given in [15, Section 3.5].

Firstly, recall that a prime ideal P of KG is *non-splitting* if for any closed subgroup H of G that controls P, $P \cap KH$ is a prime ideal of KH. Furthermore, a right ideal I of

KG is *virtually non-splitting* if $I = PKG$ for some non-splitting prime ideal P of KU, where U is some open subgroup of G.

The following theorem, analogous to [2, Theorem 5.8] and partially proved in [15, Theorem 4.8], essentially proves that establishing a control theorem for virtually non-splitting ideals is sufficient to establish it for all primes.

Theorem 3.17. *Let (G, ω) be a complete p-valued group of finite rank, let A be a closed subgroup of G, and suppose that all faithful, virtually non-splitting right ideals of KG are controlled by A. Then all faithful, prime ideals of KG are controlled by A.*

Proof. Let P be a faithful, prime ideal of KG, and let $P = I_1 \cap \cdots \cap I_m$ be an essential decomposition for P in the sense of [22, Definition 5.6], with each I_j virtually prime, and $I_1, \cdots, I_m$ forming a single G-orbit.

Setting $m = 1$, $I_1 = P$, it is clear that such a decomposition exists, so we will assume that m is maximal such that a decomposition of this form exists. We know that m is finite because KG/P has finite uniform dimension in the sense of [22]. So, by [15, Proposition 4.4], each I_j is a virtually non-splitting right ideal of KG. Furthermore, since P is faithful, it follows from [15, Lemma 4.2] that each I_j is faithful.

Therefore, by assumption, I_j is controlled by A, so $I_j = (I_j \cap KA)KG$ for each j. So since $P = I_1 \cap \cdots \cap I_r$, we have that

$$(P \cap KA)KG = ((I_1 \cap KA) \cap \cdots \cap (I_r \cap KA))KG$$
$$= (I_1 \cap KA)KG \cap \cdots \cap (I_r \cap KA)KG = I_1 \cap \cdots \cap I_r = P \text{ by [15, Lemma 4.1}(i)\text{]}.$$

Thus P is controlled by A as required. □

Now, in [15], we defined the closed, normal subgroup $C_G(Z_2(G))$ of G to be

$$C_G(Z_2(G)) := \{g \in G\text{: if } (h, G) \subseteq Z(G) \text{ then } (g, h) = 1\}.$$

Note that $C_G(Z_2(G))$ is isolated in G because if $g \in G$ and $g^{p^n} \in C_G(Z_2(G))$ for some $n \in \mathbb{N}$ then $(g^{p^n}, h) = 1$ whenever $(h, G) \subseteq Z(G)$, and hence $\omega((g^{p^n}, h)) = \infty$. But $\omega((g^{p^n}, h)) = \omega((g, h)) + n$ by [27, Proposition 25.1], and hence $\omega((g, h)) = \infty$, so $(g, h) = 1$ and $g \in C_G(Z_2(G))$.

The main result in that paper ([15, Theorem 1.2]) was that all faithful, primitive ideals in KG are controlled by $C_G(Z_2(G))$. With the results proved in this section, we can now generalise this result to all prime ideals.

Theorem 3.18. *Let (G, ω) be a nilpotent, complete p-valued group of finite rank, and let P be a faithful prime ideal of KG. Then P is controlled by $C_G(Z_2(G))$.*

Proof. First, suppose that P is non-splitting, and let $H := P^\chi$ be the controller subgroup of P:

Then $Q := P \cap KH$ is a faithful, prime ideal of KH by the definition of non-splitting, and since H is nilpotent KH/Q is infinite dimensional over K by Proposition 3.7. Since H is the smallest subgroup of G controlling P by [3, Theorem A], Q is not controlled by any proper subgroup of H. Also, note that H is a normal subgroup of G by the proof of [2, Lemma 5.2], so for any $g \in G$, $(g, H) \subseteq H$.

Now, choose $g \in G$ such that $(g, G) \subseteq Z(G)$, and let $A := Z(G) \cap H$. Let φ be the automorphism of H induced by conjugation by g. Then clearly $\varphi(Q) = Q$, and for all $h \in H$, $\varphi(h)h^{-1} = (g, h) \in Z(G) \cap H = A$. Moreover, since A is central in G, it follows that $\varphi(a) = a$ for all $a \in A$. So applying Theorem 3.16 gives that if $\varphi \neq 1$, then Q is controlled by a proper subgroup of H – contradiction.

Therefore $\varphi = 1$, i.e. g centralises H.

Therefore, if $g \in G$ and $(g, G) \subseteq Z(G)$, then $(g, H) = 1$, and hence H is contained in $C_G(Z_2(G))$ as required. Thus P is controlled by $C_G(Z_2(G))$.

Now suppose that $I \trianglelefteq_r KG$ is a faithful and virtually non-splitting right ideal of KG. Then $I = PKG$ for some open subgroup U of G, and some faithful, non-splitting prime P of KU.

We have proved that P is controlled by $C_U(Z_2(U))$, and $C_U(Z_2(U)) = C_G(Z_2(G)) \cap U$ by [15, Lemma 4.9], and hence I is controlled by $C_G(Z_2(G))$.

So, using Theorem 3.17, it follows that every faithful, prime ideal of KG is controlled by $C_G(Z_2(G))$ as required. □

Now we can finally complete the proof of Theorem B. So suppose that (G, ω) is a nilpotent, p-valued group of finite rank.

Proof of Theorem B. Consider the following sequence of subgroups, $A_0 = G$ and for each $i \geq 0$, $A_{i+1} = C_{A_i}(Z_2(A_i))$. Since G is nilpotent, each A_i must also be nilpotent.

Therefore, if A_i is non-abelian then there must exist $g \in A_i$ with $g \notin Z(A_i)$ such that $(g, A_i) \subseteq Z(A_i)$ (i.e. $g \in Z_2(A_i)$). But by definition, if $h \in A_{i+1} = C_{A_i}(Z_2(A_i))$ then $(g, h) = 1$, so since g is not central, this means that $A_{i+1} \neq A_i$.

Moreover, if A_i is abelian, then clearly $A_{i+1} = A_i$, and hence $A_j = A_i$ for all $j > i$. But since A_{i+1} is a closed, isolated normal subgroup of A_i, the chain $G = A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots$ must terminate, i.e. there exists $i \geq 0$ such that $A_j = A_i$ for all $j \geq i$, and hence A_i is abelian. Let $A := A_i$, and we will prove that all faithful, prime ideals in KG are controlled by A.

Let P be a faithful, non-splitting prime ideal of KG, and let us suppose for contradiction that P is not controlled by A. But trivially, P is controlled by $G = A_0$, so let $0 \leq j < i$ be maximal such that P is controlled by A_j.

Since P is non-splitting and A_j is a closed subgroup of G, $Q := P \cap KA_j$ is a faithful prime ideal of KA_j, so it follows from Theorem 3.18 that Q is controlled by $C_{A_j}(Z_2(A_j)) = A_{j+1}$, and hence $P = QKG = (Q \cap KA_{j+1})KA_jKG = (P \cap KA_{j+1})KG$ is controlled by A_{j+1} – contradiction.

Therefore, every faithful, non-splitting prime ideal of KG is controlled by A.

Now suppose that $I \trianglelefteq_r KG$ is a faithful and virtually non-splitting right ideal of KG. Then $I = PKG$ for some faithful, non-splitting prime P of KU, where U is some open subgroup of G.

Again, let $B_0 = U$ and for $j \geq 0$ let $B_{j+1} := C_{B_j}(Z_2(B_j))$. Then using [15, Lemma 4.9], $B_j \subseteq A_j$ for each j, and hence B_i is abelian. So since P is a faithful, non-splitting prime ideal of KU, it follows that P is controlled by $B_i \subseteq A$, and hence $I = PKG = (P \cap KB_i)KUKG \subseteq (I \cap KA)KG$ is controlled by A. So using Theorem 3.17, it follows that every faithful, prime ideal of KG is controlled by A. □

4. Dixmier annihilators

In this section, we will study the action of the rational Iwasawa algebra KG on the affinoid Dixmier module $\widehat{D(\lambda)}$, and ultimately prove Theorem C.

Throughout, fix G a nilpotent, uniform pro-p group, $\mathcal{L} = \frac{1}{p}\log(G)$, and $\mathfrak{g} = \mathcal{L} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. We will assume further that $\mathcal{L}$ is *powerful*, i.e. $[\mathcal{L}, \mathcal{L}] \subseteq p\mathcal{L}$.

4.1. Faithful dixmier annihilators

Let $\lambda : \mathfrak{g} \to K$ be a $\mathbb{Q}_p$-linear map such that $\lambda(\mathcal{L}) \subseteq \mathcal{O}$ and $\lambda|_{Z(\mathfrak{g})}$ is injective. Let $\mathfrak{b}$ be a polarisation of $\mathfrak{g}_K := \mathfrak{g} \otimes_{\mathbb{Q}_p} K$ at λ, and let $\mathcal{B} := \mathfrak{b} \cap \mathcal{L}_K$, where $\mathcal{L}_K := \mathcal{L} \otimes_{\mathbb{Z}_p} \mathcal{O}$.

Fix $P := \mathrm{Ann}_{KG}\widehat{D(\lambda)}$, which does not depend on the choice of polarisation by [17, Theorem 4.5].

Lemma 4.1. *P is a faithful, completely prime ideal of KG.*

Proof. Since $\lambda|_{Z(\mathfrak{g})}$ is injective, it follows from Lemma 2.12 that P is a faithful ideal of KG.

Furthermore, since $\mathcal{L}_K$ is powerful, it follows from [17, Corollary 3.4] that if $I := \mathrm{Ann}_{\widehat{U(\mathcal{L})}_K}\widehat{D(\lambda)}$ then $\frac{\widehat{U(\mathcal{L})}_K}{I}$ is a domain. So since $P = I \cap KG$, this means that $\frac{KG}{P}$ is a domain, and hence P is completely prime ideal as required. □

Therefore, using Theorem B, it follows that P is controlled by an abelian subgroup A of G. Let $\mathcal{A} := \frac{1}{p}\log(A)$, and let $\mathfrak{a} := \mathcal{A} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Then $\mathfrak{a}$ is an abelian ideal of $\mathfrak{g}$, so since P is independent of the choice of polarisation, we may assume that $\mathfrak{a} \subseteq \mathfrak{b}$. In other words, we may assume that the subgroup A acts by scalars on the submodule K_λ of $\widehat{D(\lambda)} = \widehat{U(\mathcal{L})}_K \otimes_{\widehat{U(\mathcal{B})}_K} K_\lambda$.

Our approach will be to study the action of KA on $\widehat{D(\lambda)}$, and prove that the kernel of this action is centrally generated.

Now, following [8], we define the *Tate algebra* over a complete valued field Ω in d variables $t_1, \cdots, t_d$ to be the subring of the power series ring $\Omega[[t_1, \cdots, t_d]]$ defined by:

$$\Omega\langle t_1,\cdots,t_d\rangle := \left\{\sum_{\alpha\in\mathbb{N}^d}\lambda_\alpha t_1^{\alpha_1}\cdots t_d^{\alpha_d}\in\Omega[[t_1,\cdots,t_d]] : \lambda_\alpha\to 0 \text{ as } \alpha\to\infty\right\}$$

Let $\{x_1,\cdots,x_r\}$ be an $\mathcal{O}$-basis for $\mathcal{L}_K/\mathcal{B}$, then using [17, Lemma 3.3], we see that $\widehat{D(\lambda)}$ is isomorphic as a K-vector space to $K\langle x_1,\cdots,x_r\rangle$.

Lemma 4.2. *There exists an $\mathcal{O}$-basis $\{x_1,\cdots,x_r\}$ for $\mathcal{L}_K/\mathcal{B}$ such that if we let $\partial_i := \frac{d}{dx_i}\in End_K\widehat{D(\lambda)}$, then each $u\in\mathfrak{a}$ acts on $\widehat{D(\lambda)}$ by a polynomial $f_u\in K[\partial_1,\cdots,\partial_s]$ for some $s\leq r$ where $f_u(0)=\lambda(u)$. Moreover, if $u\in\mathcal{A}$ then $f_u\in\mathcal{O}[\partial_1,\cdots,\partial_s]$.*

Furthermore, $s=0$ if and only if $\mathcal{A}$ is central, and for each $i=1,\cdots,s$, ∂_i lies in the image of $U(\mathfrak{a})$ under the action.

Proof. Firstly, let $\mathfrak{a}^\perp = \{u\in\mathfrak{g}_K : \lambda([u,\mathfrak{a}])=0\}$, and let $s:=\dim_K\frac{\mathfrak{g}}{\mathfrak{a}^\perp}$. Then fix a basis $\{u_1,\cdots,u_r\}$ for $\mathcal{L}_K/\mathcal{B}$ such that $\{u_{s+1},\cdots,u_r\}$ is a basis for $(\mathfrak{a}^\perp\cap\mathcal{L}_K)/\mathcal{B}$. Then it follows from [17, Proposition 3.5] that each $u\in\mathfrak{a}$ acts on $\widehat{D(\lambda)}$ by a polynomial $f_u\in K[\partial_1,\cdots,\partial_s]$, and that $\partial_1,\cdots,\partial_s$ lie in the image of $U(\mathfrak{a})$ under this action.

Furthermore, we see using [17, Proposition 3.3] that

$$f_u = \sum_{\alpha\in\mathbb{N}^s}\frac{1}{\alpha_1!}\cdots\frac{1}{\alpha_s!}\lambda(\mathrm{ad}(u_s)^{\alpha_r}\cdots\mathrm{ad}(u_1)^{\alpha_1}(u))\partial_1^{\alpha_1}\cdots\partial_s^{\alpha_s},$$

so clearly the constant term is $\lambda(u)$. Moreover, if $u\in\mathcal{A}$ then since $\mathcal{L}$ is powerful,

$\mathrm{ad}(u_r)^{\alpha_r}\cdots\mathrm{ad}(u_1)^{\alpha_1}(u)\in p^{\alpha_1+\cdots+\alpha_r}\mathcal{L}$, and hence $\lambda(\mathrm{ad}(u_r)^{\alpha_r}\cdots\mathrm{ad}(u_1)^{\alpha_1}(u))\in p^{\alpha_1+\cdots+\alpha_r}\mathcal{O}$ for all $\alpha\in\mathbb{N}^r$.

So since $v_p(\alpha_i!)\leq\alpha_i$ for each i, it follows that $\frac{1}{\alpha_1!}\cdots\frac{1}{\alpha_r!}\lambda(\mathrm{ad}(u_r)^{\alpha_r}\cdots\mathrm{ad}(u_1)^{\alpha_1}(u))\in\mathcal{O}$ as required.

Finally, if $s=0$ then $\mathfrak{a}$ acts by scalars on $\widehat{D(\lambda)}$, and hence $[\mathfrak{a},\mathfrak{g}]\subseteq P$ and $\lambda([\mathfrak{a},\mathfrak{g}])=0$. So since P is faithful, it follows from Lemma 2.12 that $[\mathfrak{g},\mathfrak{a}]=0$, and hence A is central. Conversely, if $\mathfrak{a}$ is central then clearly $\widehat{U(\mathcal{A})}_K$ acts by scalars on $\widehat{D(\lambda)}$, so since $\partial_1,\cdots,\partial_s$ lie in the image of this action, and do not act by scalars, it follows that $s=0$. $\square$

Note: It follows from this lemma that the image of $\widehat{U(\mathcal{A})}_K$ in $\mathrm{End}_K\widehat{D(\lambda)}$ is contained in $K\langle\partial_1\cdots,\partial_s\rangle$.

4.2. Results from rigid geometry

We will now prove some technical results using techniques from rigid geometry. A detailed introduction to the theory of rigid geometry and its applications can be found in [8] and [9], whose results we will often use in this section.

First, recall from [8, Definition 3.1.1] that an *affinoid algebra* over K is a quotient of the Tate algebra $K\langle t_1,\cdots,t_r\rangle$ for some $r\in\mathbb{N}$. It follows from [8, Proposition 3.1.5] that any affinoid algebra R carries a complete, separated filtration w_R.

Lemma 4.3. *Let $\phi : K\langle u_1, \cdots, u_d\rangle \to R$ be a map of affinoid algebras, and let $a_1, \cdots, a_r \in R$ lie in the image of ϕ. Then there exists $m \in \mathbb{N}$ such that the image of ϕ inside R contains the affinoid K-subalgebra topologically generated by $\pi^m a_1, \cdots, \pi^m a_r$.*

Proof. For each i, we know that $a_i = \phi(r_i)$ for some $r_i \in K\langle u_1, \cdots, u_d\rangle$, so choose $m \in \mathbb{N}$ such that $w_{\inf}(\pi^m r_i) \geq 0$ and $w_R(\pi^m a_i) \geq 0$ for all i.

Then there exist K-algebra maps $\Theta_1 : K\langle X_1, \cdots, X_r\rangle \to K\langle u_1, \cdots, u_d\rangle$ and $\Theta_2 : K\langle X_1, \cdots, X_r\rangle \to R$ sending X_i to $\pi^m r_i$ and $\pi^m a_i$ respectively, and it is clear that $\Theta_2 = \phi\Theta_1$. Therefore, the image of ϕ must contain the image of Θ_2, which is precisely the affinoid K-algebra topologically generated by $\pi^m a_1, \cdots, \pi^m a_r$ as required. □

Now, let $\overline{K}$ be the algebraic closure of K. Recall that for each $\epsilon \in \mathbb{R}$, we define the *d-dimensional disc of radius* ϵ to be the space

$$\mathbb{D}_\epsilon^d := \{\underline{\alpha} \in \overline{K}^d : v_\pi(\alpha_i) \geq \epsilon \text{ for each } i\}$$

This is an affinoid space, in the sense of [8, Definition 3.1.1], isomorphic to Sp $K\langle u_1, \cdots, u_d\rangle$. Thus all discs are isomorphic, regardless of the radius.

Moreover, the Tate algebra $K\langle u_1, \cdots, u_d\rangle$ can be realised as the space of analytic functions on $\mathbb{D}_0^d$, i.e. the set of all power series in $u_1, \cdots, u_d$ converging on the unit disc, while for each $n \in \mathbb{N}$, the subalgebra $K\langle \pi^n u_1, \cdots, \pi^n u_d\rangle$ is precisely those functions which converge on $\mathbb{D}_{-n}^d$.

Following [20, 5.1.2], for each non-constant polynomial $g(t) := b_0 + b_1 t + \cdots + b_n t^n \in K[t]$ with $b_0 \in \mathcal{O}$, define

$$\chi(g) := \max_{1 \leq j \leq n} -\tfrac{v_\pi(b_j)}{j}.$$

Lemma 4.4. *Let $g(t) \in K[t]$ be a polynomial with $g(0) \in \mathcal{O}$, and let $\beta \in K$ with $v_\pi(\beta) > 0$. Then $\chi(\beta g) < \chi(g)$.*

It follows that if $f_1(t), \cdots, f_d(t) \in K[t]$ are polynomials with $f_i(0) \in \mathcal{O}$ for each i, and $v(\beta) > 0$ then setting $\mu_i := \max_{1 \leq j \leq d} \chi(\beta^i f_j)$ for each $i \geq 0$, we have that $\mu_0 > \mu_1 > \mu_2 > \cdots$.

Proof. Suppose $g(t) = b_0 + b_1 t + \cdots + b_n t^n$, with $b_0 \in \mathcal{O}$, $b_n \neq 0$. Then by definition;

$$\chi(\beta g) = \max_{1 \leq j \leq n} -\tfrac{v_\pi(\beta b_j)}{j} = \max_{1 \leq j \leq n} -\tfrac{v_\pi(b_j)}{j} - \tfrac{v_\pi(\beta)}{j}$$

So since $v_\pi(\beta) > 0$, this maximum is strictly less than $\max_{1 \leq j \leq n} -\frac{v_\pi(b_j)}{j} = \chi(g)$.

To prove the second statement, it suffices to prove that $\mu_0 > \mu_1$ and apply induction. So suppose $\mu_1 = \chi(\beta f_i)$ and $\mu_0 = \chi(f_j)$, then we have that $\mu_1 = \chi(\beta f_i) < \chi(f_i) \leq \chi(f_j) = \mu_0$. □

Recall from [20, Theorem 5.1.2] that if we assume $b_0 \neq 0$, then the set

$$X(g) := \{\alpha \in \overline{K} : v_\pi(g(\alpha)) \geq 0\}$$

is an affinoid subdomain of $\mathbb{A}_K^{1,an} := \overline{K}$, whose G-connected component about 0 is the disc $\mathbb{D}^1_{\chi(g)} = \{\alpha \in \overline{K} : v_\pi(\alpha) \geq \chi(g)\}$.

Furthermore, if $b_0 = 0$, it is clear that $\chi(g) = \chi(1+g)$, and that $X(g) = \{\alpha \in \overline{K} : v_\pi(g(\alpha)) \geq 0\} = \{\alpha \in \overline{K} : v_\pi(1 + g(\alpha)) \geq 0\} = X(1+g)$, so we reach the same conclusion.

Lemma 4.5. *Suppose that K contains a $(p-1)$'st root of p. Then given polynomials $f_1, \cdots, f_r \in \mathcal{O}[t]$, there exist $\alpha \in \overline{K}$, $k \in \{1, \cdots, r\}$ such that $v_p(f_i(\alpha)) \geq -1$ for all i and $v_p(f_k(\alpha)) < \frac{1}{p-1} - 1$.*

Proof. If $\omega \in K$ and $\omega^{p-1} = p$ then $v_p(\omega) = \frac{1}{p-1}$. So for each $j \geq 0$ let

$$Y_j := \{\alpha \in \overline{K} : v_\pi(\omega^j f_i(\alpha)) \geq 0 \text{ for all } i\},$$

and set $\mu_j := \max_{i=1,\cdots,d} \chi(\omega^j f_i)$. Then using [20, Theorem 5.1.2] we see that Y_j is an affinoid subdomain of $\mathbb{A}_1^{an}$ and the G-connected component of Y_j about zero is the closed disc $\mathbb{D}^1_{\mu_j}$ is the.

We want to find $\alpha \in \overline{K}$ such that $v_p(f_i(\alpha)) \geq -1$ for all i, i.e. $v_p(pf_i(\alpha)) \geq 0$, and since $\omega^{p-1} = p$, this just means that $\alpha \in Y_{p-1}$. So it remains to find an element α in the connected component $\mathbb{D}^1_{\mu_{p-1}}$ of Y_{p-1} such that $v_p(f_k(\alpha)) < \frac{1}{p-1} - 1$ for some k, i.e. $v_\pi(f_k(\alpha)) < v_\pi(p)(\frac{1}{p-1} - 1)$.

For each $j \geq 0$, fix $i_j = 1, \cdots, d$ such that $\chi(\omega^j f_{i_j}) = \mu_j$. Using Lemma 4.4 we see that $\mu_0 > \mu_1 > \mu_2 > \cdots$, and we know that for each j, the G-connected component of $X(\omega^{j-1} f_{i_{j-1}})$ about zero is $\mathbb{D}^1_{\mu_{j-1}}$.

In particular, since $\mu_{j-1} > \mu_j$ we have that $\mathbb{D}^1_{\mu_{j-1}} \subsetneq \mathbb{D}^1_{\mu_j}$, so since $\mathbb{D}^1_{\mu_j}$ is G-connected, this means that $\mathbb{D}^1_{\mu_j} \nsubseteq X(\omega^{j-1} f_{i_{j-1}})$. So for each j, we may choose $\alpha_j \in \mathbb{D}^1_{\mu_j} \backslash X(\omega^{j-1} f_{i_{j-1}})$.

But $X(\omega^{j-1} f_{i_{j-1}}) = \{\alpha \in \overline{K} : v_\pi(f_{i_{j-1}}(\alpha)) \geq -(j-1)v_\pi(\omega)\}$, so this means that $v_\pi(f_{i_{j-1}}(\alpha_j)) < -v_\pi(\omega)(j-1)$. But $v_p(\omega) = \frac{1}{p-1}$ so $v_\pi(\omega) = \frac{v_\pi(p)}{p-1}$, thus $v_\pi(f_{i_{j-1}}(\alpha_j)) < -v_p(\pi)\frac{j-1}{p-1} = v_p(\pi)(\frac{1}{p-1} - \frac{j}{p-1})$

So, finally, choose $j = p-1$, and let $k := i_{j-1}$. Then $\alpha_j \in \mathbb{D}^1_{\mu_{p-1}} \subseteq Y_{p-1}$ and $v_\pi(f_k(\alpha_j)) < v_p(\pi)(\frac{1}{p-1} - 1)$ as required. □

Corollary 4.6. *Suppose that K contains a $(p-1)$'st root of p. Then given polynomials $f_1, \cdots, f_r \in \mathcal{O}[t_1, \cdots, t_m]$, there exists $\alpha \in \overline{K}^m$, $k \in \{1, \cdots, r\}$ such that $v_p(f_i(\alpha)) \geq -1$ for all i and $v_p(f_k(\alpha)) < \frac{1}{p-1} - 1$.*

Proof. If $m = 1$, this is precisely Lemma 4.5, so assume that $m > 1$ and choose $(\alpha_1, \cdots, \alpha_{m-1}) \in \mathcal{O}^{m-1}$. For each $i = 1, \cdots, r$, let $g_i(t) := f_i(\alpha_1, \cdots, \alpha_{m-1}, t) \in \mathcal{O}[t]$.

Then using Lemma 4.5, there exists $\alpha_m \in \overline{K}$, $k \in \{1, \cdots, r\}$ such that $-1 \leq v_p(g_i(\alpha_m))$ for all i and $v_p(g_k(\alpha_m)) \leq \frac{1}{p-1} - 1$. So let $\alpha = (\alpha_1, \cdots, \alpha_m) \in \overline{K}^m$, and it follows that $-1 \leq v_p(f_i(\alpha))$ for all i, and $v_p(f_k(\alpha)) \leq \frac{1}{p-1} - 1$. □

4.3. Almost-polynomial maps

Now we will further explore the action of the abelian lattice $\mathcal{A}$ on the Dixmier module $\widehat{D(\lambda)} \cong K\langle x_1 \cdots, x_s\rangle$.

Definition 4.7. A morphism $\phi : K\langle u_1, \cdots, u_d\rangle \to K\langle t_1, \cdots, t_r\rangle$ of K-algebras is called an *almost-polynomial map* if

- $\phi(u_i) \in \mathcal{O}[t_1, \cdots, t_r]$ for each i,
- $t_1, \cdots, t_r$ are contained in the image of ϕ.

Using Lemma 4.3, we see that if ϕ is an almost-polynomial map then there exist $m \in \mathbb{N}$ such that $\mathrm{im}(\phi)$ contains $K\langle \pi^m t_1, \cdots, \pi^m t_r\rangle$.

Example. 1. Since $\mathcal{A}$ is abelian, we know that $\widehat{U(\mathcal{A})}_K \cong K\langle u_1, \cdots, u_d\rangle$ by [17, Lemma 2.1], and using Lemma 4.2 we see that the action $\rho : \widehat{U(\mathcal{A})}_K \to \mathrm{End}_K \widehat{D(\lambda)}$ has image contained in $K\langle \partial_1, \cdots, \partial_s\rangle$, and the map $K\langle u_1, \cdots, u_d\rangle \to K\langle \partial_1, \cdots, \partial_s\rangle$ is an almost-polynomial map.

2. If $\phi : K\langle u_1, \cdots, u_d\rangle \to K\langle t_1, \cdots, t_r\rangle$ is an almost-polynomial map then so is the restriction $K\langle pu_1, \cdots, pu_d\rangle \to K\langle t_1, \cdots, t_r\rangle$. Moreover, if F/K is a finite extension, then the scalar extension $\phi_F : F\langle u_1, \cdots, u_d\rangle \to F\langle t_1, \cdots, t_r\rangle$ is also an almost polynomial map.

Lemma 4.8. *Let $\phi : K\langle u_1, \cdots, u_d\rangle \to K\langle t_1, \cdots, t_r\rangle$ be an almost-polynomial map, and let $f_i := \phi(u_i) \in K[t_1, \cdots, t_r]$ for each i. Then setting $Y := \{\alpha \in \overline{K}^r : v_\pi(f_i(\alpha)) \geq 0$ for each $i\}$, we have that:*

i. Y is an affinoid subdomain of $\mathbb{A}_K^{r,an}$.

ii. The image of ϕ is contained in the set of all functions in $K\langle t_1, \cdots, t_r\rangle$ converging on Y.

Proof. Set $A := \mathrm{im}(\phi)$, then $t_1, \cdots, t_r \in A$ by Definition 4.7. Since $K\langle t_1, \cdots, t_r\rangle$ is affinoid, it follows from Lemma 4.3 that there exists $m \in \mathbb{N}$ such that A contains $T = K\langle \pi^m t_1, \cdots, \pi^m t_r\rangle$, we may of course choose m to be arbitrarily large.

If we set $B := T\langle \zeta_1, \cdots, \zeta_d\rangle/(\zeta_i - f_i(t_1, \cdots, t_r) : i = 1, \cdots, d)$, then there is a natural surjection from B to A, identical on T, which sends t_i to $f_i(t)$. This gives rise to a closed embedding of affinoid varieties $\mathrm{Sp}\ A \hookrightarrow \mathrm{Sp}\ B$.

i. Since each f_i is a polynomial, it is clear that there exists $N > 0$ such that if $\alpha \in \overline{K}^r$ and $v_\pi(\alpha) < -N$ then $v_\pi(f_i(\alpha)) < 0$ for all i. So by choosing $m > N$ we may assume that

$$Y = \{\alpha \in \mathbb{D}^r_{-m} : v_\pi(f_i(\alpha)) \geq 0\}.$$

Hence using [8, Lemma 3.3.10(i)] and the proof of [8, Proposition 3.3.11], we see that $Y = \text{Sp } B$ and that Y is an affinoid subdomain of $\mathbb{A}^{r,an}_K$.

ii. Notice that $K\langle u_1, \cdots, u_d\rangle$ is precisely the set of functions converging on the open unit disc $\mathbb{D}^d_0$, so it follows that the image of $K\langle u_1, \cdots, u_d\rangle$ under ϕ is contained in the set of functions converging on $\{\alpha \in \overline{K}^r : (f_1(\alpha), \cdots, f_d(\alpha)) \in \mathbb{D}^d_0\} = \{\alpha \in \overline{K}^r : v_\pi(f_i(\alpha)) \geq 0 \text{ for all } i\} = Y$ as required. □

The following result will be essential later when proving a control theorem.

Proposition 4.9. *Let $\phi : K\langle u_1, \cdots, u_d\rangle \to K\langle t_1, \cdots, t_r\rangle$ be an almost-polynomial map, and let $f_i := \phi(u_i) \in K[t_1, \cdots, t_r]$ for each i. Then there exists $k \in \{1, \cdots, d\}$ such that $\exp(pf_k(t))$ does not lie in $\phi(K\langle pu_1, \cdots, pu_d\rangle)$.*

Proof. We may assume that K contains a $p-1$'st root of p. If we prove the result in this case, then it follows generally, since if $K' := K(\sqrt[p-1]{p})$ and we can find k such that $\exp(pf_k)$ does not lie in the image of $K'\langle pu_1, \cdots, pu_d\rangle$ under the scalar extension of ϕ, then it will also not lie in the image of $K\langle pu_1, \cdots, pu_d\rangle$ under ϕ.

Let $Y := \{\alpha \in \overline{K}^r : v_\pi(pf_i(\alpha)) \geq 0 \text{ for all } i\}$. Then using Lemma 4.8 we see that Y is an affinoid subdomain of $\mathbb{A}^{1,an}_K$, and that $\phi(K\langle pu_1, \cdots, pu_d\rangle)$ is contained in the set of all functions in $K\langle t_1, \cdots, t_r\rangle$ converging on Y. So it remains to prove that for some k, $\exp(pf_k)$ does not converge on Y, and thus cannot lie in the image of $K\langle pu_1, \cdots, pu_d\rangle$.

Using [18, Example 0.4.1], the disc of convergence for exp is $\{\lambda \in \overline{K} : v_p(\lambda) > \frac{1}{p-1}\}$, so it remains only to find $\alpha \in Y$ such that $v_p(pf_k(\alpha)) \leq \frac{1}{p-1}$ for some k, i.e. $v_p(f_k(\alpha)) \leq \frac{1}{p-1} - 1$.

But $f_1, \cdots, f_d \in \mathcal{O}[t_1, \cdots, t_r]$, so using Corollary 4.6 we know that there exists $\alpha \in \overline{K}^r$ such that $v_p(f_i(\alpha)) \geq -1$ for all i and $v_p(f_k(\alpha)) < \frac{1}{p-1} - 1$ for some k. Hence $v_\pi(pf_i(\alpha)) \geq 0$ for all i, and hence $\alpha \in Y$, and $v_p(pf_k(\alpha)) \leq \frac{1}{p-1}$. □

Now we will explore more closely the image of the Tate algebra under an almost polynomial map.

Theorem 4.10. *Suppose that $\phi : K\langle u_1, \cdots, u_d\rangle \to K\langle t_1, \cdots, t_r\rangle$ is an almost-polynomial map, and let $S := \phi(K\langle u_1, \cdots, u_d\rangle)$. Then S is an integrally closed domain of Krull dimension r.*

Proof. First, we will prove that S has Krull dimension r, and this is very similar to the proof of [17, Proposition 7.5]:

Since $t_1, \cdots, t_r \in S$, it follows from Lemma 4.3 that S contains $K\langle \pi^m t_1, \cdots, \pi^m t_r\rangle$ for some $m \geq 0$. Therefore we have inclusions of commutative affinoid algebras, $K\langle \pi^m t_1, \cdots, \pi^m t_r\rangle \hookrightarrow S \hookrightarrow K\langle t_1, \cdots, t_r\rangle$, which gives rise to a chain of open embeddings of the associated affinoid spectra: Sp $K\langle t_1, \cdots, t_r\rangle \hookrightarrow$ Sp $S \hookrightarrow$ Sp $K\langle \pi^m t_1, \cdots, \pi^m t_r\rangle$.

The notion of the analytic dimension dim X of a rigid variety X is defined in [14], where it is proved to be equal to the supremum of the Krull dimensions of every affinoid algebra R such that Sp R is an affinoid subdomain of X. In particular, if Sp B is an affinoid subdomain of Sp A in the sense of [8, Definition 3.3.9], then K.dim$(B) \leq$ K.dim(A).

Therefore, since the Tate algebras $K\langle t_1, \cdots, t_s\rangle$ and $K\langle \pi^m t_1, \cdots, \pi^m t_r\rangle$ both have dimension r, it remains to prove that the embeddings Sp $S \to$ Sp $K\langle \pi^m t_1, \cdots, \pi^m t_r\rangle$ and Sp $K\langle t_1, \cdots, t_r\rangle \to$ Sp S define affinoid subdomains.

For convenience, set $D :=$ Sp $K\langle t_1, \cdots, t_r\rangle$ and $D_1 :=$ Sp $K\langle \pi^m t_1, \cdots, \pi^m t_r\rangle$. Then D can be realised as the unit disc in r-dimensional rigid K-space, while D_1 is a deformed disc containing D, so fixing coordinates, $D = \{(x_1, \cdots, x_r) \in D_1 : v_\pi(x_i) \geq 0 \text{ for all } i\}$.

But since Sp S contains D, we could instead write $D = \{(x_1, \cdots, x_r) \in \text{Sp } S : v_\pi(x_i) \geq 0 \text{ for all } i\}$, and this is a Weierstrass subdomain of Sp S in the sense of [8, Definition 3.3.7], and hence D is an affinoid subdomain of Sp S by [8, Proposition 3.3.11]. Therefore K.dim$(S) \geq \dim(D) = r$.

Now, set $T := K\langle \pi^m t_1, \cdots, \pi^m t_r\rangle$, and let $f_i(t_1, \cdots, t_r) = \phi(u_i) \in T$ for each i. Define

$$B := T\langle \zeta_1, \cdots, \zeta_d\rangle / (\zeta_i - f_i(t_1, \cdots, t_r) : i = 1, \cdots, d)$$

then B is an affinoid algebra which naturally surjects onto $S = \phi(K\langle u_1, \cdots, u_d\rangle)$, where each $a \in T$ is sent to a, and each ζ_i is sent to $\phi(u_i)$. Therefore, K.dim$(S) \leq$ K.dim(B).

But clearly there is a map $T \to B$, inducing a morphism of affinoid varieties Sp $B \to$ Sp T, and the proof of [8, Proposition 3.3.11] shows that this corresponds to the embedding of the Weierstrass subdomain $Y = \{x \in \text{Sp } T : v_\pi(f_i(x)) \geq 0 \text{ for all } i\}$ into Sp T, and hence Sp B is an affinoid subdomain of Sp $T = D_1$ by [8, Proposition 3.3.11], and hence K.dim$(B) \leq \dim(D_1) = r$.

Therefore $r \leq$ K.dim$(S) \leq$ K.dim$(B) \leq r$, forcing equality, so K.dim$(S) = r$ as required. Moreover, this implies that S and B have the same Krull dimension, and hence S is a quotient of B by a minimal prime ideal.

To prove that S is integrally closed, we will prove that the affinoid variety $Y =$ Sp B is *normal*, i.e. at every point $p \in Y$, the ring of germs of affinoid functions $\mathcal{O}_{Y,p}$ (as defined in [8, Definition 4.1]) is reduced and integrally closed. Using this, it will follow from [9, Proposition 7.3.8] that the localisation $B_{\mathfrak{q}}$ of B at every prime ideal $\mathfrak{q}$ of B is reduced and integrally closed.

Since S is a minimal prime quotient of B, it will follow that S is an integrally closed domain as required.

Using [10, Theorem 5.1.3], we see that since the affine variety $\mathbb{A}_K^r$ is smooth, and hence normal, its analytification $\mathbb{A}_K^{r,an}$ is also normal. So since $Y = \{x \in \mathbb{A}_K^{r,an} : v_\pi(f_i(x)) \geq 0$ for all $i\}$ is an affinoid subdomain of $\mathbb{A}_K^{r,an}$ by Lemma 4.8(ii), it follows that $\mathcal{O}_{\mathbb{A}_K^{r,an},p} = \mathcal{O}_{Y,p}$ for every point $p \in Y$. Hence $\mathcal{O}_{Y,p}$ is reduced and integrally closed as required. □

4.4. Using the crossed product

In this subsection, we will prove a control theorem for kernels of almost-polynomial maps. Throughout, we will assume that K contains a p'th root of unity ζ.

Fix A a free abelian pro-p group of rank d, let $\mathcal{A} := \frac{1}{p}\mathcal{L}_A$ be the associated $\mathbb{Z}_p$-Lie algebra of A, and let $\phi : \widehat{U(\mathcal{A})}_K \to K\langle t_1, \cdots, t_s\rangle$ be an almost-polynomial map.

Consider the crossed product $D_p = D_p(A) = \widehat{U(p\mathcal{A})}_K * \frac{A}{A^p}$ defined in Section 2.5. This is a Banach completion of KA with respect to the extension of the dense embedding $\iota : KA^p \to \widehat{U(p\mathcal{A})}_K$ to KA, and there is a natural map $\tau : D_p \to \widehat{U(\mathcal{A})}_K$. Define $\phi' : D_p \to K\langle t_1, \cdots, t_s\rangle$ and $\phi_A : KA \to K\langle t_1, \cdots, t_s\rangle$ making the following diagram commute:

$$\begin{array}{ccc} D_p & \xrightarrow{\tau} & \widehat{U(\mathcal{A})}_K \\ {\scriptstyle\iota}\uparrow & \searrow^{\phi'} & \downarrow{\scriptstyle\phi} \\ KA & \xrightarrow{\phi_A} & K\langle t_1, \cdots, t_s\rangle \end{array}$$

From now on, set $I = \ker(\phi')$, and let $Q := \ker(\phi_A) = I \cap KA$, and define:

$$U := \{a \in A : \phi(a) \in \phi(\widehat{U(p\mathcal{A})}_K)\}.$$

Proposition 4.11. *U is a proper open subgroup of A containing A^p.*

Proof. Since ϕ is a ring homomorphism, it is clear that for all $a, b \in U$, $ab \in U$, and since KA^p is a subalgebra of $\widehat{U(p\mathcal{A})}_K$, it is clear that $A^p \subseteq U$. Therefore, since $\frac{A}{A^p}$ is a finite group, and $\frac{U}{A^p}$ is closed under multiplication, it follows that U is a subgroup of A containing A^p, and hence it is open.

Finally, since ϕ is an almost polynomial map, it follows from Proposition 4.9 that there exists $u \in \mathcal{A}$ such that $\exp(p\phi(u)) = \phi(\exp(pu))$ does not lie in the image of $\widehat{U(p\mathcal{A})}_K$ under ϕ. But $a := \exp(pu) \in A$ and hence $a \notin U$. Therefore U is a proper subgroup of G. □

Using this proposition, we can fix a $\mathbb{Z}_p$-basis $\{a_1, \cdots, a_d\}$ for A such that $\{a_1, \cdots, a_r, a_{r+1}^p, \cdots, a_d^p\}$ is a $\mathbb{Z}_p$-basis for U, so $a_1, \cdots, a_r \in U$ and $a_{r+1}, \cdots, a_d \notin U$.

Since A is a free abelian pro-p group, we have that $\frac{A}{A^p}$ is a direct product of d copies of the cyclic group of order p, where the i'th copy is generated by the image of a_i in $\frac{A}{A^p}$. Setting $c_i := a_iA^p$, it follows from Lemma 2.9 that:

$$D_p = \widehat{U(p\mathcal{A})}_K * \langle c_1 \rangle * \cdots * \langle c_d \rangle \tag{10}$$

where $\overline{c_i}^t = \overline{c_i^t}$ for $0 \leq t < p$ and $\overline{c_i}^p = a_i^p$.

From now on, let $S := \phi(\widehat{U(p\mathcal{A})}_K) \subseteq K\langle t_1, \cdots, t_s \rangle$, and let $B := \widehat{U(p\mathcal{A})}_K * \langle c_1 \rangle * \cdots * \langle c_r \rangle \leq D_p$. Then since $a_1, \cdots, a_r$ lie in U, the image of B under ϕ is S by the definition of U. Furthermore, since $KU = KA^p * \frac{U}{A^p} = KA^p * \langle c_1 \rangle * \cdots * \langle c_r \rangle$, it is clear that $KU \subseteq B$.

Let $J := I \cap B \trianglelefteq B$ be the kernel of the restriction of ϕ' to B, and let $I' := JD_p$ – an ideal of D_p contained in I.

Lemma 4.12. *I is a prime ideal of D_p, minimal prime above I'.*

Proof. Since $D_p/I \cong \mathrm{im}(\phi') \leq K\langle t_1, \cdots, t_s \rangle$, it is clear that I is a prime ideal of D_p.

Since D_p is a crossed product of B with a finite group, it follows from Lemma 2.4(*ii*) that I is minimal prime above $I' = (I \cap B)D_p$. $\square$

Now, we will need the following small result from Galois theory [28]:

Lemma 4.13. *Let F be a field of characteristic 0, containing a p'th root of unity ζ. Let $r \in F$, and suppose that r has no p'th root in F. Choose a p'th root $\alpha \in \overline{F}$ of r, and let $F' := F(\alpha)$. Then if $\beta \in F'$ and $\beta^p \in F$ then $\beta = c\alpha^m$ for some $c \in F$, $0 \leq m < p$.*

Proof. Since F' is the splitting field for the polynomial $x^p - r$ over F, it is clear that F' is a Galois extension of F. So since $[F' : F] = p$ this means that $Gal(F'/F)$ has order p.

In fact, if we consider the element $\sigma \in Gal(F'/F)$ sending α to $\zeta\alpha$, then $Gal(F'/F)$ is cyclic of order p, generated by σ.

The result is clear if $\beta \in F$, so assume $\beta \notin F$ and $\beta^p \in F$. Then β is a root of the polynomial $x^p - \beta^p \in F[x]$, and hence $\sigma(\beta)$ is also a root. Therefore $\sigma(\beta) = \zeta^m\beta$ for some $0 \leq m < p$, so $\sigma(\alpha^{-m}\beta) = \zeta^{-m}\alpha^{-m}\zeta^m\beta = \alpha^{-m}\beta$.

But since σ generates $Gal(F'/F)$, it follows that $\alpha^{-m}\beta$ is fixed by the Galois group, so since F'/F is a Galois extension, this means that $c := \alpha^{-m}\beta \in F$, and hence $\beta = c\alpha^m$ as required. $\square$

For clarity, we will introduce/recall the following data:

- $I = \ker(\phi') \trianglelefteq D_p$.
- $Q = I \cap KA \trianglelefteq KA$.
- $U = \{a \in A : \phi(a) \in \phi(\widehat{U(p\mathcal{A})}_K)\} = \langle a_1, \cdots, a_r, a_{r+1}^p, \cdots, a_d^p \rangle$.
- $B = \widehat{U(p\mathcal{A})}_K * \langle c_1 \rangle * \cdots * \langle c_r \rangle \leq D_p$.
- $S = \phi(\widehat{U(p\mathcal{A})}_K) = \phi'(B) \leq K\langle t_1, \cdots, t_s \rangle$.
- $J = I \cap B \trianglelefteq B$.
- $I' = JD_p \trianglelefteq D_p$.
- $R := D_p/I'$

Proposition 4.14. *R is a domain.*

Proof. Since $D_p = B * \langle c_{r+1}\rangle * \cdots * \langle c_d\rangle$ and $I' = JD_p$, it follows from Lemma 2.4(*iii*) that $R \cong S * \langle c_{r+1}\rangle * \cdots * \langle c_d\rangle$, where $\overline{c}_i^p = \phi(a_i^p)$ for each i. So using [23, Theorem 4.4] we see that R is reduced. Therefore, we may consider the usual semisimple artinian ring of quotients $Q(R)$ of R, which has the form:

$$Q(R) = Q(S) * \langle c_{r+1}\rangle * \cdots * \langle c_d\rangle,$$

where $Q(S)$ is the field of fractions of S. Note that since $S = \phi(\widehat{U(p\mathcal{H})_K})$ is the image of a Tate algebra under an almost-polynomial map, it follows from Theorem 4.10 that S is an integrally closed domain. It remains to prove that $Q(R)$ is a field.

Let $T_0 := Q(S)$, and for each $i = 1, \cdots, d-r$, define $T_i := T_{i-1} * \langle c_{r+i}\rangle$, so that $T_{d-r} = Q(R)$.

Clearly T_0 is a field, so we will use induction to show that T_i is a field for each i, so in particular, $Q(R)$ is a field. So assume that for some $j > 0$, $T_0, \cdots, T_{j-1}$ are all fields:

Then since $T_j = T_{j-1} * \langle c_{r+j}\rangle$ where $\overline{c}_{r+j}^p = \phi(a_{r+j}^p) \in S$, it follows that

$$T_j = T_{j-1}[x]/(x^p - \phi(a_{r+j}^p))$$

So we only need to show that the polynomial $x^p - \phi(a_{r+j}^p) \in T_{j-1}[x]$ is irreducible over the field T_{j-1}.

Since K contains a p'th root of unity, we see using standard Galois theory that this just means we need to show that this polynomial has no root in T_{j-1}, i.e. that there is no $b \in T_{j-1}$ such that $b^p = \phi(a_{r+j}^p)$.

Let us suppose for contradiction that $b_1^p = \phi(a_{r+j}^p)$ for some $b_1 \in T_{j-1} = T_{j-2} * \langle c_{r+j-1}\rangle$. Then since $\phi(a_{r+j}^p) \in S \subseteq T_{j-2}$ and T_{j-2} is a field containing K, it follows from Lemma 4.13 that $b_1 = b_2\overline{c}_{r+j-1}^{k_1}$ for some $b_2 \in T_{j-2}$, $0 \le k_1 < p$.

Therefore, $b_2^p = \phi((a_{r+j}a_{r+j-1}^{-k_1})^p) \in S$, so applying a second induction, for each $i > 0$, we can find integers $0 \le k_1, \cdots, k_{i-1} < p$ and $b_i \in T_{j-i}$ such that $b_i^p = \phi((a_{r+j}a_{r+j-1}^{-k_1}a_{r+j-2}^{-k_2}\cdots a_{r+j-i+1}^{-k_{i-1}})^p) \in S$.

Taking $i = j$ we have that $b_j \in T_0 = Q(S)$ and $b_j^p \in S$. So since S is integrally closed, it follows that $b_j \in S \subseteq K\langle t_1, \cdots, t_s\rangle$.

Now, $(b_j\phi(a_{r+j}^{-1}a_{r+j-1}^{k_1}\cdots a_{r+1}^{k_{j-1}}))^p = 1$, so it follows that there is a p'th root of unity $\zeta \in K$ such that:

$$\zeta b_j = \phi(a_{r+j}a_{r+j-1}^{-k_1}a_{r+j-1}^{-k_2}\cdots a_{r+1}^{-k_{j-1}}).$$

Therefore, since $b_j \in S$, this means that $\phi(a_{r+j}a_{r+j-1}^{-k_1}a_{r+j-1}^{-k_2}\cdots a_{r+1}^{-k_{j-1}}) \in S = \phi(\widehat{U(p\mathcal{A})_K})$, or in other words $a_{r+j}a_{r+j-1}^{-k_1}a_{r+j-1}^{-k_2}\cdots a_{r+1}^{-k_{j-1}} \in U$ by the definition of U.

This is the required contradiction since $\{a_1, \cdots, a_r, a_{r+1}^p, \cdots, a_d^p\}$ is a $\mathbb{Z}_p$-basis for U, and each k_i is less than p. $\square$

Theorem 4.15. *Let $\phi : \widehat{U(\mathcal{A})}_K \to K\langle t_1, \cdots, t_s\rangle$ be an almost-polynomial map. Then the kernel Q of the restriction of this map to KA is controlled by U.*

Proof. If $I = \ker(\phi') \trianglelefteq D_p$, then using Proposition 4.14 we see that $R = D_p/(I \cap B)D_p$ is a domain. But we know that I is minimal prime above $(I \cap B)D_p$ by Lemma 4.12, so it follows that $I = (I \cap B)D_p$.

So, if $r \in Q = I \cap KA$ then since $KA = KU * \frac{A}{U}$, $r = \sum_{a \in A//U} s_a a$ for some $s_a \in KU \subseteq B$. So since $r \in I = (I \cap B)D_p$ it follows that $s_a \in I \cap B \cap KU = Q \cap KU$ for each a, and hence $r \in (Q \cap KU)KA$. Since our choice of r was arbitrary, this means that $Q = (Q \cap KU)KA$, i.e. Q is controlled by U. □

4.5. Control theorem for dixmier annihilators

Now we can finally conclude our proof of Theorem C. Again, G is a nilpotent, uniform pro-p group, whose $\mathbb{Z}_p$-Lie algebra $\mathcal{L} = \frac{1}{p}\log(G)$ is powerful.

Fix a linear form $\lambda : \mathcal{L} \to \mathcal{O}$ such that the restriction of λ to $Z(\mathfrak{g})$ is injective, then $P := \text{Ann}_{KG}\widehat{D(\lambda)}$ is a faithful prime ideal of KG by Lemma 4.1, and $\frac{KG}{P}$ is a domain.

Proof of Theorem C. Firstly, for any finite extension F/K, if we let $I = \text{Ann}_{FG}\widehat{D(\lambda_F)}$, where λ_F is the scalar extension of λ, then clearly $I \cap KG = P$. So if we prove that I is controlled by $Z(G)$, then it will follow from Lemma 2.7 that P is controlled by $Z(G)$. Therefore, we may pass to field extensions of K without issue. In particular we may assume that $F = K$ contains a p'th root of unity.

Since P is a faithful, prime ideal of KG, it follows from Theorem B that P is controlled by an abelian subgroup of G. So let $A = P^\chi$ be the controller subgroup of P, then A is an abelian normal subgroup of G, so if we let $\mathcal{A} := \frac{1}{p}\log(A)$ then $\mathcal{A}$ is an abelian ideal of $\mathcal{L}$. We want to prove that A is central in G, or equivalently that $\mathcal{A}$ is central in $\mathcal{L}$.

Using Lemma 4.2, we see that the image of $\widehat{U(\mathcal{A})}_K$ in $\text{End}_K\widehat{D(\lambda)}$ is contained in a Tate algebra $K\langle\partial_1, \cdots, \partial_s\rangle$ such that if $s > 0$ then the morphism $\widehat{U(\mathcal{A})}_K \to K\langle\partial_1, \cdots, \partial_s\rangle$ is an almost polynomial map. Moreover, $s = 0$ if and only if $\mathcal{A}$ is central, so let us assume that $s > 0$.

Then using Proposition 4.11 and Theorem 4.15, we can find a proper, open subgroup U of G such that the kernel Q of the restriction of ϕ to KA is controlled by U. But clearly $Q = P \cap KA$, so this is a contradiction since A is the controller subgroup of P. □

5. Primitive ideals

The aim of this section is to prove our main result Theorem A. The essence of our argument is to compare general primitive ideals in KG to Dixmier annihilators.

5.1. Weakly rational ideals

Fix G a uniform, nilpotent pro-p group, and as usual let $\mathcal{L} := \frac{1}{p}\mathcal{L}_G$, and $\mathfrak{g} := \mathcal{L}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$.

Definition 5.1. Given a prime ideal P of KG, we say that P is *weakly rational* if $Z(KG/P)$ is a finite field extension of K.

It follows from [13, Theorem 1.1(1)] that any primitive ideal of KG is weakly rational.

Lemma 5.2. *Let $P \subseteq Q$ be weakly rational ideals of KG, and suppose that P is faithful. Then Q is faithful.*

Proof. Let $F_1 = Z(KG/P)$, $F_2 = Z(KG/Q)$, then F_1, F_2 are finite field extensions of K, and clearly the natural surjection $KG/P \twoheadrightarrow KG/Q$ reduces to a field extension $F_1 \hookrightarrow F_2$.

Since $Q^\dagger = \{g \in G : g-1 \in Q\}$ is a normal subgroup of G, using nilpotence of G we see that if $Q^\dagger \neq 1$, then there exists $z \in Q^\dagger \cap Z(G)$ with $z \neq 1$. Thus $z+P, 1+P \in F_1 \subseteq F_2$, and $z + Q = 1 + Q$, which implies that $z + P = 1 + P$ and hence $z - 1 \in P$. So since P is faithful, $z = 1$ – contradiction.

Therefore, $Q^\dagger = 1$, and hence Q is faithful. □

Now, recall from Section 2.5 the definition of the Banach completions $D_{p^n} = \widehat{U(p^n\mathcal{L})}_K * \frac{G}{G^{p^n}}$ of KG for each $n \in \mathbb{N}$.

Proposition 5.3. *Let P be a primitive ideal of KG, then for all sufficiently high $n \in \mathbb{N}$, there exists a primitive ideal Q_n of $D_{p^n} = \widehat{U(p^n\mathcal{L})}_K * \frac{G}{G^{p^n}}$ such that $Q_n \cap KG = P$.*

Proof. Since P is primitive, $P = \text{Ann}_{KG}M$ for some irreducible KG-module M. Using [5, Proposition 10.6(e), Corollary 10.11], we see that for n sufficiently high, $\widehat{M} := D_{p^n} \otimes_{KG} M \neq 0$.

Since M is irreducible and $\widehat{M} \neq 0$, the natural map $M \to \widehat{M}, m \mapsto 1 \otimes m$ is injective. And since D_{p^n} is a Banach completion of KG with respect to some filtration w, it follows that $\widehat{M}$ is a completion of $M = KGm$ with respect to the filtration $v(rm) = \sup\{w_n(r + y) : y \in KG \text{ and } ym = 0\}$.

Therefore, if $r \in P$, i.e. $rM = 0$, then taking limits shows that $r\widehat{M} = 0$, so $P \subseteq \text{Ann}_{KG}\widehat{M} = (\text{Ann}_{D_{p^n}}\widehat{M}) \cap KG$.

Now, since D_{p^n} is Noetherian and $\widehat{M}$ is a finitely generated D_{p^n}-module, we can choose a maximal submodule $U \leq \widehat{M}$, and let $M' := \widehat{M}/U$ – an irreducible D_{p^n}-module.

Since M is irreducible, the composition $M \hookrightarrow \widehat{M} \twoheadrightarrow M'$ is either injective or zero. If it is zero then $M \subseteq U$, and hence $\widehat{M} \subseteq U$ and $M' = 0$. This contradiction implies that the composition is injective.

Finally, let $Q_n = \text{Ann}_{D_{p^n}}M'$, then Q_n is a primitive ideal of D_{p^n}, and $P \subseteq \text{Ann}_{KG}\widehat{M} \subseteq \text{Ann}_{KG}M' = Q_n \cap KG$. Also, if $r \in Q_n \cap KG$ then $rM' = 0$, so since $M \subseteq M'$, $rM = 0$ and $r \in P$. Thus $P = Q_n \cap KG$ as required. □

5.2. Reduction from KG to KG^{p^n}

Now we begin to explore how we can relate primitive ideals in Iwasawa algebras to Dixmier annihilators:

Proposition 5.4. *Given a primitive ideal P of KG, there exists $m \in \mathbb{N}$ with $m \geq 1$, finite extensions $F_1, \cdots, F_r/K$ and $\mathbb{Q}_p$-linear maps $\lambda_i : \mathfrak{g} \to F_i$ with $\lambda_i(p^m\mathcal{L}) \subseteq \mathcal{O}_{F_i}$ for each $i = 1, \cdots, r$, such that:*

$$P \cap KG^{p^m} = \text{Ann}_{KG^{p^m}} \widehat{D(\lambda_1)}_{F_1} \cap \cdots \cap \text{Ann}_{KG^{p^m}} \widehat{D(\lambda_r)}_{F_r}$$

Proof. Using Proposition 5.3, if P is primitive, then for any sufficiently high $n \geq 1$, there is a primitive ideal Q of $D_{p^n} = \widehat{U(p^n\mathcal{L})}_K * \frac{G}{G^{p^n}}$ such that $Q \cap KG = P$, and hence $Q \cap KG^{p^n} = P \cap KG^{p^n}$.

Let $I = Q \cap \widehat{U(p^n\mathcal{L})}_K$, then using Lemma 2.5 we see that I is a semiprimitive ideal of $\widehat{U(p^n\mathcal{L})}_K$, so choose primitive ideals $J_1, J_2, \cdots, J_r$ of $\widehat{U(p^n\mathcal{L})}_K$ such that $I = J_1 \cap J_2 \cap \cdots \cap J_r$.

Since each J_i is primitive, it follows from [17, Theorem A] that there exists $m \geq n$ such that for each i, $J_i \cap \widehat{U(p^m\mathcal{L})}_K = \text{Ann}_{\widehat{U(p^m\mathcal{L})}_K} \widehat{D(\lambda_i)}_{F_i}$ for F_i/K a finite extension, $\lambda_i : \mathfrak{g} \to F_i$ $\mathbb{Q}_p$-linear with $\lambda_i(p^m\mathcal{L}) \subseteq \mathcal{O}_{F_i}$. Thus:

$P \cap KG^{p^m} = Q \cap KG^{p^m} = I \cap KG^{p^m} = (J_1 \cap \widehat{U(p^m\mathcal{L})}_K) \cap \cdots \cap (J_r \cap \widehat{U(p^m\mathcal{L})}_K) \cap KG^{p^m}$ is an intersection of Dixmier annihilators as required. □

Now, we want to show that all faithful, primitive ideals of KG are centrally generated, which we know is true for Dixmier annihilators by Theorem C. Proposition 5.4 allows us to compare general primitive ideals to Dixmier annihilators, and the following result uses this to prove a reduced version of Theorem A:

Theorem 5.5. *Let G be a nilpotent, uniform pro-p group with centre Z, and let P be a faithful, primitive ideal of KG. Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $P \cap KG^{p^n}$ is controlled by Z^{p^n}.*

Proof. Using Proposition 5.4, we see that for some $m \geq 1$, there are finite extensions $F_1, \cdots, F_r$ and $\mathbb{Q}_p$-linear maps $\lambda_i : \mathfrak{g} \to F_i$ with $\lambda(p^m\mathcal{L}) \subseteq \mathcal{O}_{F_i}$ such that $P \cap KG^{p^m} = \text{Ann}_{KG^{p^m}} \widehat{D(\lambda_1)}_{F_1} \cap \cdots \cap \text{Ann}_{KG^{p^m}} \widehat{D(\lambda_r)}_{F_r}$.

For each $i = 1, \cdots, r$, set $J_i := \text{Ann}_{KG^{p^m}} \widehat{D(\lambda_i)}_{F_i}$ for convenience, clearly these are prime ideals of KG^{p^m}, thus $P \cap KG^{p^m}$ is semiprime and $J_1, \cdots, J_r$ are the minimal primes above $P \cap KG^{p^m}$, hence they are all G-conjugate by the proof of [2, Lemma 5.4(b)]. Note that for all $n \geq m$, $J_i \cap KG^{p^n} = \text{Ann}_{KG^{p^n}} \widehat{D(\lambda)}_{F_i}$ for each i.

Also, since P is faithful, $P \cap KG^{p^m}$ is faithful, so $J_1^\dagger \cap \cdots \cap J_r^\dagger = P^\dagger = 1$. But since $J_1^\dagger, \cdots, J_r^\dagger$ are G-conjugate and G is orbitally sound by [2, Proposition 5.9], this means that the subgroup 1 must have finite index in $J_i^\dagger$ for each i, which means that they are finite. But G is torsionfree, thus $J_i^\dagger = 1$ for all i, i.e. $J_1, \cdots, J_r$ are faithful.

So since $J_i = \mathrm{Ann}_{KG^{p^m}} \widehat{D(\lambda_i)}_{F_i}$ is faithful, it follows from Lemma 2.12 that λ_i is injective when restricted to $Z(\mathfrak{g})$.

Now, since $m \geq 1$, note that for all $n \geq m$, $\frac{1}{p}\log(G^{p^n}) = p^{n-1}\log(G)$ is a powerful Lie lattice. Therefore, using Theorem C, we see that $J_i \cap KG^{p^n} = \mathrm{Ann}_{KG^{p^n}} \widehat{D(\lambda_i)}_{F_i}$ is controlled by $Z(G^{p^n})$ for each i, and using [2, Lemma 8.4(a)], $Z(G^{p^n}) = Z(G) \cap G^{p^n} = Z^{p^n}$.

Therefore, setting $B_{i,n} := J_i \cap KG^{p^n} = \mathrm{Ann}_{KG^{p^n}} \widehat{D(\lambda_i)}$, $B_{i,n} = (B_{i,n} \cap KZ^{p^n})KG^{p^n}$ for each i, so using [15, Lemma 4.1(a)]:

$$P \cap KG^{p^n} = B_{1,n} \cap \cdots \cap B_{r,n} = (B_{1,n} \cap KZ^{p^n})KG^{p^n} \cap \cdots \cap (B_{r,n} \cap KZ^{p^n})KG^{p^n}$$
$$= (B_{1,n} \cap \cdots \cap B_{r,n} \cap KZ^{p^n})KG^{p^n} = (P \cap KZ^{p^n})KG^{p^n}$$

Hence $P \cap KG^{p^n}$ is controlled by Z^{p^n} as required. □

5.3. Extension from KG^{p^n} to KG

The results of the previous subsection show that we can establish Theorem A after passing to G^{p^n} for sufficiently high n. We now just need to extend to KG.

Lemma 5.6. *Let P be a weakly rational ideal of KG. Then $P \cap KZ(G)$ is a maximal ideal of $KZ(G)$.*

Proof. Since P is prime in KG, $Q := P \cap KZ(G)$ is prime in $KZ(G)$. So setting $F := Z(KG/P)$, it is clear that $KZ(G)/Q \hookrightarrow F$. So since $KZ(G)/Q$ is a domain containing K, and F is a finite extension of K, it follows that $KZ(G)/Q$ is a field, and hence Q is maximal. □

Proposition 5.7. *Let G be a nilpotent, uniform pro-p group, and let $P_1 \subseteq P_2$ be faithful, primitive ideals of KG. Then there exists $n \in \mathbb{N}$ such that $P_1 \cap KG^{p^n} = P_2 \cap KG^{p^n}$. It follows that if P is a faithful, primitive ideal of KG then P is maximal.*

Proof. Using Theorem 5.5, we see that there exist $N_1, N_2 \in \mathbb{N}$ such that for all $n_1 \geq N_1$, $n_2 \geq N_2$, $P_i \cap KG^{p^{n_i}}$ is controlled by $Z(G)^{p^{n_i}}$ for each i. So choose $n \geq \max\{N_1, N_2\}$ and we have that $P_1 \cap KG^{p^n}, P_2 \cap KG^{p^n}$ are controlled by $Z(G)^{p^n}$.

Since P_1 is primitive, it is weakly rational, so using Lemma 5.6 we see that $P_1 \cap KZ(G)$ is a maximal ideal of $KZ(G)$. So since $P_1 \cap KZ(G) \subseteq P_2 \cap KZ(G)$, we have that $P_1 \cap KZ(G) = P_2 \cap KZ(G)$, and hence $P_1 \cap KZ(G)^{p^n} = P_2 \cap KZ(G)^{p^n}$. Therefore:

$$P_1 \cap KG^{p^n} = (P_1 \cap KZ(G)^{p^n})KG^{p^n} = (P_2 \cap KZ(G)^{p^n})KG^{p^n} = P_2 \cap KG^{p^n}.$$

Finally, given a faithful, primitive ideal P of KG, let Q be a maximal ideal of KG containing P. Since P and Q are primitive, they are weakly rational, so since P is faithful, Q is faithful by Lemma 5.2. Thus, by the above, there exists $n \in \mathbb{N}$ such that $P \cap KG^{p^n} = Q \cap KG^{p^n}$ is controlled by $Z(G)^{p^n}$.

But $P \cap KZ(G)$ is prime in $KZ(G)$, so $P \cap KG^{p^n} = (P \cap KZ(G)^{p^n})KG^{p^n}$ is prime in KG^{p^n} by Theorem 2.8. So since $P \cap KG^{p^n} = Q \cap KG^{p^n}$, it follows from [23, Theorem 16.6(iii)] that $P = Q$, and hence P is maximal. □

Theorem 5.8. *Let G be a nilpotent, uniform pro-p group. Then all faithful, primitive ideals of KG are controlled by $Z(G)$.*

Proof. Let P be a faithful, primitive ideal of KG, and let $Z = Z(G)$. We want to prove that P is controlled by Z.

Using Theorem 5.5, we know that there exists $n \in \mathbb{N}$ such that $P \cap KG^{p^n}$ is controlled by Z^{p^n}, and hence is prime in KG^{p^n} by Theorem 2.8. So let $J := (P \cap KG^{p^n})KG$, then using Lemma 2.4 we see that J is a semiprime ideal of KG, and P is minimal prime above J.

Let $Q := P \cap KZ$, then Q is prime in KZ, so QKG is prime in KG by Theorem 2.8. And since $P \cap KG^{p^n} = (P \cap KZ^{p^n})KG^{p^n}$, we have that:

$$J = (P \cap KG^{p^n})KG = (P \cap KZ^{p^n})KG \subseteq QKG.$$

But clearly $QKG \subseteq P$, so since QKG is prime and P is minimal prime above J, it follows that $P = QKG = (P \cap KZ)KG$, and hence P is controlled by Z as required. □

Now we can finally prove our main result. First, we just need a small Lemma:

Lemma 5.9. *Let G be a uniform pro-p group, let N be a closed, normal subgroup of G. Then there exists an open, uniform normal subgroup U of G such that $N \cap U$ is a closed, isolated normal subgroup of U.*

Proof. Recall from [29, Definition 1.6] the definition of the *isolater* $i_G(N)$ of N in G, and recall from [29, Proposition 1.7, Lemma 1.8] that $i_G(N)$ is a closed, isolated normal subgroup of G, and N is open in $i_G(N)$.

Therefore, there exists $n \in \mathbb{N}$ such that if $g \in i_G(N)$ then $g^{p^n} \in N$. So if $g = h^{p^n} \in U := G^{p^n}$ and $g^p = h^{p^{n+1}} \in N \subseteq i_G(N)$, then $h \in i_G(N)$, so $g = h^{p^n} \in N$. Hence $N \cap U$ is isolated in U as required. □

Proof of Theorem A. Let P be a primitive ideal of KG, and we want to prove that P is virtually standard, i.e. that $P \cap KU$ is a finite intersection of standard ideals for some open, normal subgroup U of G.

Firstly, if P is faithful, then it follows from Theorem 5.8 that P is controlled by $Z(G)$, and hence is standard, and using Proposition 5.7 we see that P is maximal. So we can assume that P is not faithful.

Let $N := P^\dagger = \{g \in G : g - 1 \in P\}$. Then N is a closed, normal subgroup of G, so by Lemma 5.9, there exists an open, uniform normal subgroup U of G such that $N \cap U$ is isolated in U. Let $Q := P \cap KU$, then Q is a semiprimitive ideal in KU by Lemma 2.5, and $Q^\dagger = N \cap U$ is a closed, isolated normal subgroup of U.

Let $U_1 := \frac{U}{Q^\dagger}$, and let $Q_1 := \frac{Q}{(Q^\dagger - 1)KU}$. Then U_1 is a nilpotent, uniform pro-p group and Q_1 is a faithful semiprimitive ideal of KU_1. Therefore, it follows that Q is a finite intersection of faithful, primitive ideals in KU_1. Since all faithful primitives in KU_1 are maximal and standard, this means that Q_1 is a finite intersection of maximal standard ideals.

Therefore, since Q_1 is a homomorphic image of Q, this means that Q is a finite intersection of maximal, standard ideals, and it follows from Definition 1.1 that P is a virtually standard prime ideal of KG. Therefore, it remains to show that P is maximal.

Using Lemma 2.4(ii), we see that P is minimal prime above the semiprime ideal $(P \cap KU)KG$. So since $P \cap KU$ is semimaximal in KU, it follows from [23, Theorem 16.6(iii)] that P is maximal in KG as required. □

References

[1] K. Ardakov, The centre of completed group algebras of pro-p groups, Doc. Math. 9 (2004) 599–606.
[2] K. Ardakov, Prime ideals in nilpotent Iwasawa algebras, Invent. Math. 190 (2) (2012) 439–503.
[3] K. Ardakov, The controller subgroup of one-sided ideals in completed group rings, Contemp. Math. 562 (2012).
[4] K. Ardakov, K.A. Brown, Ring-theoretic properties of Iwasawa algebras: a survey, Doc. Math. Extra Volume Coates (2006) 7–33.
[5] K. Ardakov, S. Wadsley, On irreducible representations of compact p-adic analytic groups, Ann. Math. 178 (2013) 453–557.
[6] K. Ardakov, S. Wadsley, Verma modules for Iwasawa algebras are faithful, Münster J. Math. 7 (2014) 5–26.
[7] K. Ardakov, F. Wei, J.J. Zhang, Nonexistence of reflexive ideals in Iwasawa algebras of Chevalley type, J. Algebra 320 (1) (2008) 259–275.
[8] S. Bosch, Lectures on Formal and Rigid Analytic Geometry, Lecture Notes in Mathematics, vol. 2105, Springer, Cham, 2014.
[9] S. Bosch, U. Güntzer, R. Remmert, Non-archimedean Analysis, Grundlehren der Mathematischen Wissenschaften, vol. 261, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1984.
[10] B. Conrad, Irreducible components of rigid spaces, Ann. Inst. Fourier 49 (2) (1999).
[11] J. Dixmier, Enveloping Algebras, Graduate Studies in Mathematics, vol. 11, American Mathematical Society, 1996.
[12] J.D. Dixon, M.P.F. du Sautoy, A. Mann, D. Segal, Analytic Pro-p Groups, second edition, Cambridge Studies in Advanced Mathematics, vol. 61, Cambridge University Press, Cambridge, 1999.
[13] G. Dopinescu, B. Schraen, Endomorphism algebras of p-adic representations of p-adic Lie groups, arXiv:1106.2442.
[14] A. Ducros, Espaces Analytiques p-adiques au sens de Berkovich, in: Séminaire BOURBAKI, 2006.
[15] A. Jones, A control theorem for primitive ideals in Iwasawa algebras, J. Algebra 560 (2019) 700–724.
[16] A. Jones, Completed group algebras of Abelian-by-procyclic groups, Adv. Math. 354 (2019).
[17] A. Jones, Affinoid dixmier modules, Algebr. Represent. Theory 42 (2021).
[18] K. Kedlaya, p-Adic Differential Equations, Cambridge Studies in Advanced Mathematics, vol. 125, Cambridge University Press, 2010.
[19] M. Lazard, Groupes analytiques p-adiques, Publ. Math. IHÉS 26 (1965) 389–603.
[20] B. Lewis, Primitive factor rings of p-adic completions of enveloping algebras as arithmetic differential operators, Doctoral Thesis, Queen Mary University of London, 2015.
[21] H. Li, F. Van Oystaeyen, Zariskian Filtrations, Kluwer Academic, Dordrecht, 1996.

[22] J.C. McConnell, J.C. Robson, Noncommutative Noetherian Rings, Graduate Studies in Mathematics, vol. 30, American Mathematical Society, Providence, 2001.
[23] D.S. Passman, Infinite Crossed Products, Pure and Applied Mathematics, vol. 135, Academic Press, Boston, 1989.
[24] C. Praagman, Iterations and logarithms of formal automorphisms, Aequ. Math. 30 (2–3) (1986) 151–160.
[25] P. Schneider, J. Teitelbaum, Algebras of p-adic distributions and admissable representations, Invent. Math. 153 (2003) 145–196.
[26] P. Schneider, J. Teitelbaum, Locally analytic distributions and p-adic representation theory, with applications to GL_2, J. Am. Math. Soc. 15 (2002) 443–468.
[27] P. Schneider, p-adic Lie Groups, Grundlehren der Mathematischen Wissenschaften, vol. 344, Springer, Heidelberg, 2011.
[28] User reuns, https://math.stackexchange.com/questions/3241556/pth-roots-in-an-extension-of-prime-degree-p.
[29] W. Woods, On the structure of virtually nilpotent compact p-adic analytic groups, arXiv:1608.03137, 2016.