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LETTER TO THE EDITOR

ON BIANCHI-I COSMIC STRINGS

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Einstein field equations with the cosmological term are solved for an axially symmetric Bianchi-I Letelier string coupled with a magnetic field. It is shown that in the presence of Λ, the initial singularity occurs faster. A formula to obtain numerical value of $\varphi^2 = \frac{A^2}{B^2}$ is given.

The gravitational effects of the gauge cosmic strings have been extensively studied[1-5]. A model of a cloud formed by massive strings was used as a source by Letelier[6] for Bianchi-I and Kantowski-Sachs space-times. The strings forming the cloud were the generalization of the relativistic string model of Takabayashi[7] — the p-strings. The total energy-momentum tensor for a cloud of massive strings is

$$T_\mu^\nu = \varrho V_\mu V^\nu - \lambda x_\mu x^\nu \tag{1}$$

where ϱ is the rest energy density for a cloud of strings with particles attached along the extension. Thus,

$$\varrho = \varrho_p + \lambda \tag{2}$$

where ϱ_p is the particle energy density, λ the tension density of the string, V^μ the 4-vector representing the velocity of the cloud of particles and x^μ, the 4-vector, representing the direction of anisotropy. In this work the Einstein field equations with the cosmological term are solved for an axially symmetric Bianchi-I Letelier string coupled with a magnetic field*, and thereby achieved the generalization of the work of Banerjee et al.[8] (cited henceforth as Ref. 8).

*We adopt the signature $(+, -, -, -)$ of the space-time and use units so that $c = \hbar = 1$. We have $V_\mu V^\mu = 1$, $x^\mu x_\mu = -1$, $V_\mu x^\mu = 0$.

An axially symmetric Bianchi-I metric is

$$ds^2 = dt^2 - e^{2\alpha}dx^2 - e^{2\beta}(dy^2 + dz^2) \tag{3}$$

where $\alpha = \alpha(t)$, $\beta = \beta(t)$.

Consider a system of string-dust with a magnetic field along the x-direction as the source for the metric. The energy-momentum tensor for such a system would be given by

$$T_\mu^\nu = T_{\mu\,string}^\nu + E_{\mu\,mag}^\nu,$$

where $T_{\mu\,string}^\nu$ is given by (1) and

$$E_{\mu\,mag}^\nu = \frac{1}{4\pi}\left[-(F_\mu^\alpha F_\mu^\nu) + \frac{1}{4}(F_{\alpha\beta}F^{\alpha\beta})\,\delta_\mu^\nu\right]. \tag{4}$$

In the co-moving coordinate system, we have

$$T_{0\,string}^0 = \varrho, \qquad T_{1\,string}^1 = \lambda, \qquad T_{\mu\,string}^\nu = 0 \tag{5}$$

for $\mu, \nu = 2, 3$ and also for $\mu \neq \nu$.

As the magnetic field is along the x-direction alone, F_{23} is the only non-zero component of the Maxwell tensor $F_{\mu\nu}$. From Maxwell equations $F_{[\mu\nu,\sigma]} = 0$ and $[F^{\mu\nu}(\sqrt{-g})]_{,\mu} = 0$, we find $F_{23} =$ constant A. Then

$$E_0^0 = E_1^1 = -E_2^2 = -E_3^3 = -\frac{A^2}{8\pi}\,e^{-4\beta} \tag{6}$$

(For details, one may refer to Ref. 8).

Now, the Einstein field equations (with the cosmical constant Λ)

$$R_\mu^\nu - \frac{1}{2}Rg_\mu^\nu + \Lambda g_\mu^\nu = -T_\mu^\nu \tag{7}$$

in the conventional notations, assume the following form

$$G_0^0 = 2\dot{\alpha}\dot{\beta} + \dot{\beta}^2 = \varrho + \frac{A^2}{8\pi}\,e^{-4\beta} + \Lambda \tag{8}$$

$$G_1^1 = 2\ddot{\beta} + 3\dot{\beta}^2 = \lambda + \frac{A^2}{8\pi}\,e^{-4\beta} + \Lambda \tag{9}$$

$$G_2^2 = G_3^3 + \ddot{\alpha} + \dot{\alpha}^2 + \ddot{\beta} + \dot{\beta}^2 + \dot{\alpha}\dot{\beta} = -\frac{A^2}{8\pi}\,e^{-4\beta} + \Lambda. \tag{10}$$

Since the number of unknown parameters α, β, ϱ, λ exceeds the number of equations, we close the system by assuming a relation between the metric coefficients given by

$$\alpha = a\beta \tag{11}$$

as has been done in Ref. 8, where a is a constant.

Then (10) reduces to

$$(a+1)\,\ddot{\beta} + (a^2 + a + 1)\,\dot{\beta}^2 = -\frac{A^2}{8\pi}\,e^{-4\beta} + \Lambda. \tag{12}$$

For $a \neq -1$, Eq. (12) can be written as an integral equation

$$\int \mathrm{d}\left[\dot{\beta}^2\, e^{2\beta\left(\frac{a^2+a+1}{a+1}\right)}\right] = -\frac{A^2}{4\pi\,(a+1)}\int\left[e^{2\beta\left(\frac{a^2-a-1}{a+1}\right)} + \right.$$

$$\left. + \frac{2\Lambda}{(a+1)}\, e^{2\left(\frac{a^2+a+1}{a+1}\right)}\right]\mathrm{d}\beta + B,$$

where B is a constant of integration.

Hence,

$$\int \frac{e^{2\beta}\mathrm{d}\beta}{\left[Be^{-2\beta\left(\frac{a^2-a-1}{a+1}\right)} - \frac{A^2}{8\pi\,(a^2-a-1)} - \frac{A^2\Lambda}{4\pi\,(a^2+a+1)\,(a+1)}\,e^{4\beta}\right]^{1/2}} =$$

$$= \pm\,(t - t_0) \tag{13}$$

where t_0 is another constant of integration. In Ref. 8, (13) has been solved for two different cases: $a^2 - a - 1/(a+1) = -1$ and 2. In our case the first condition gives $a = 0$ i. e., $\alpha = 0$ and reduces (12) to

$$\ddot{\beta} + \dot{\beta}^2 = -\frac{A^2}{8\pi}\,e^{-4\beta} + \Lambda \tag{14}$$

For $y = \dot{\beta}^2$, (14) admits the solution

$$y = Be^{-2\beta} + \left[\frac{A^2}{8\pi}\,e^{-4\beta} + \Lambda\right]. \tag{15}$$

Now (13) becomes

$$\int \frac{e^{2\beta}\mathrm{d}\beta}{\left[Be^{2\beta} + \frac{A^2}{8\pi} + \Lambda e^{4\beta}\right]^{1/2}} = \pm\,(t - t_0)$$

which on integration yields

$$e^{2\beta} = \frac{B}{2\Lambda}(\eta \cosh \tau - 1) \tag{16}$$

where

$$\eta = \left(1 - \frac{A^2}{B^2}\frac{\Lambda}{2\pi}\right)^{1/2}, \quad \tau = 2\sqrt{\Lambda}\,(t - t_0).$$

Approximating (16), we obtain

$$e^{2\beta} = p + \Lambda q \tag{17}$$

to first order in Λ, where

$$p = B(t - t_0)^2 - \frac{A^2}{B}\frac{1}{8\pi}$$

$$q = \frac{B}{3}(t - t_0)^4 - \frac{A^2}{B}\frac{1}{4\pi}(t - t_0)^2 - \frac{A^4}{B^3}\frac{1}{64\pi^2}.$$

Eq. (17) shows

$$e^{2\beta}_{ours} = e^{2\beta}_{B} + \Lambda q,$$

where the suffix 'ours' indicates our result and the suffix 'B' indicates that obtained in Ref. 8. In the limit $\Lambda \to 0$, we get the result of Ref. 8.

The proper volume R^3, expansion scalar Θ and shear scalar σ^2 for the metric (4) are, respectively

$$R^3_{ours} = e^{\alpha + 2\beta} = e^{2\beta} = \frac{B}{2\Lambda}(\eta \cosh \tau - 1) = R^2_B + \Lambda q \tag{18}$$

$$\Theta_{ours} = V^a{}_{;a} = \dot{\alpha} + 2\dot{\beta} = \frac{1}{R^3_{ours}}\left[\frac{B}{\sqrt{\Lambda}}\,\eta \sin h\tau\right] =$$

$$= \frac{2B}{p}(t - t_0) + \Lambda\left\{\frac{4}{3}\frac{B}{p}(t - t_0)^3 - 2B(t - t_0)\frac{q}{p^2} - \frac{A^2}{B}(t - t_0)\frac{1}{2\pi p}\right\},$$

hence

$$\Theta_{ours} = \Theta_B + \Lambda X \tag{19}$$

where X stands for the quantity in the curly bracket above, and

$$\sigma^2_{ours} = \sigma_{\mu\nu}\sigma^{\mu\nu} = \dot{\alpha}^2 + 2\dot{\beta}^2 - \frac{1}{3}\Theta^2 = \frac{1}{6}\left[\frac{1}{R^3_{ours}}\left(\frac{B}{2\Lambda}\,\eta \sinh \tau\, 2\sqrt{\Lambda}\right)\right]^2,$$

hence

$$\sigma^2_{ours} = \sigma^2_B + \frac{2}{6}\sigma_B \Lambda \left\{ \frac{4}{3}\frac{B}{P}(t-t_0)^3 - \frac{A^2}{B}\frac{1}{2\pi p}(t-t_0) - \frac{2B}{p^2}(t-t_0)\,q \right\} \quad (20)$$

to first order in Λ.

Similarly straightforward calculations to first order in Λ yield

$$\varrho_{ours} = \varrho_B + \Lambda\,(Y) \quad (21)$$

and

$$\lambda_{ours} = \lambda_B + \Lambda\,(Z) \quad (22)$$

where ϱ_B and λ_B are given in Ref. 8 and

$$Y = \frac{4}{3}B^2(t-t_0)^4\frac{1}{p^2} - B^2\frac{2q}{p^3}(t-t_0)^3 - \frac{A^2}{8\pi}\frac{1}{p^2}(t-t_0)^2 - 1$$

$$Z = \frac{2Bq}{p^2} + \frac{4B}{p}(t-t_0)^2 - \frac{A^2}{B}\frac{1}{2\pi p} - \frac{A^2}{2\pi p^2}(t-t_0)^2 - \frac{2B}{p^3}(t-t_0)^2 -$$

$$- \frac{3}{4}\frac{A^2}{\pi}\frac{q}{p^3} - 1.$$

In our case, the condition for singularity in the initial epoch i. e. $R^3 \to 0$ gives

$$T^2_{ours} = (t-t_0)^2 = \frac{A^2}{8\pi B^2} - \frac{\Lambda}{24\pi^2}\frac{A^4}{B^4} \quad (23)$$

to first order in Λ. Here we find

$$T_{ours} - T_B = -ve.$$

Hence the introduction of Λ in the string-filled universe speeds up the occurrence of singularity.

The tension density λ of the string vanishes at the instant specified by the cubic equation

$$\left(\frac{2}{3}\Lambda B^3 + 4B^3\Lambda - \frac{A^2}{2\pi}B^2\Lambda - B^3\Lambda\right)T^3 +$$

$$+\left(B^3 - \frac{2}{3}\Lambda B^2 C - 8B^2 C\Lambda - \frac{A^2}{2\pi}\Lambda B + \frac{A^2}{\pi}BC\Lambda - \frac{A^2}{4\pi}B\Lambda\right)T^2 +$$

$$+\left(4BC^2\Lambda-\frac{3A^2}{8\pi}B-CB^2-4B^2C\Lambda-5C^2B\Lambda+\frac{A^2}{\pi}AC-\right.$$

$$\left.-\frac{A^2}{2\pi}C^2\Lambda-2B^2\Lambda\right)T+\left(\frac{3A^2C}{8\pi}+\right.$$

$$\left.+4BC^2\Lambda-\frac{A^2}{B}\frac{\Lambda}{2\pi}C^2+\frac{3A^2C\Lambda}{2\pi}+C^3\Lambda\right)=0. \tag{24}$$

One of the roots of (24), in the limit $\Lambda \to 0$, reduces to the condition

$$T^2=(t-t_0)^2=\frac{3A^2}{8\pi B^2},$$

which is the corresponding value obtained in Ref. 8.

Another result of significance is regarding η, which we require to be real on physical grounds. Hence,

$$\eta^2=\frac{B^2}{\Lambda}-\frac{A^2}{2\pi}\geq 0$$

i. e.

$$\Lambda\leq 2\pi\varphi^2,\quad \text{where}\quad \varphi^2=\frac{A^2}{B^2}.$$

Therefore

$$\varphi\geq\pm\sqrt{\frac{\Lambda}{2\pi}} \tag{25}$$

which gives the numerical estimation of the ratio φ. In absence of the magnetic field, φ becomes physically meaningless.

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O BIANCHI-I KOZMIČKIM STRUNAMA

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Riješena je Einsteinova jednadžba polja s kozmološkim članom za aksijalno-simetričnu Bianchi-I Letelierovu strunu vezanu za magnetsko polje. Pokazano je da se u prisustvu Λ početni singularitet brže pojavljuje.