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REGULARIZATIONS OF SOLUTIONS FOR CHARGED DUST DISTRIBUTIONS IN THE EINSTEIN–MAXWELL THEORY

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Dedicated to the memory of Professor Dubravko Tadić

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Regularization of singular solutions for the static spherically symmetric extremelly charged dust in the Majumdar-Papapetrou system has been investigated. Singularities are of such a type that solutions become physically unacceptable since physically relevant quantities (metric invariants) are singular as well. With a simple redefinition of the charge/energy distributions, these solutions can be regularized. A spectrum of solutions with a number of zero-nodes in the metric tensor is found, and it is shown that their regularization can be accomplished either by using a δ-shell, or a thick shell distribution of matter. The bifurcating behaviour of regular solutions is not present any more, but quantized-like behaviour in the total mass allocated to the solutions is observed.

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1. *Introduction*

It is always interesting to address the problem of the gravitational collapse of classical matter within the Einstein theory, since an exact solution of the Einstein equations with the general form of matter is not known. Various possible scenarios of the collapse have been envisaged and investigated. Recent objections to a rather popular one, i.e. to the black-hole scenario, have been further encouraged by constructing alternative models that could, under certain conditions, prevent the formation of black holes [1, 2, 3, 4, 5, 6]. However, under different initial energy conditions, the collapsing system could also admit timelike or null singularities which (for an observer co-moving with the matter) could be naked, i.e. the singularities are encountered before the boundary enters its Schwarzschild radius.

In this paper we investigate further the regularization of a newly found class of static solutions to Einstein–Maxwell equations for the extremelly charged dust (ECD) model of matter, and discuss the regularization procedures for removing the naked singularity that is present in these solutions. The solutions show remarkable discrete properties with respect to the amount of allocated mass. The singularities (zeros of the metric profile function that we use) are mimicking the behaviour of a quantum system with regard to its higher and higher excitations. It will be shown that such solutions can be regularized by modifications in the distribution of ECD. First we use the δ-shells, which may not be considered to be very physical, but are a useful tool in the approach to difficult nonlinear problems [3, 5, 7, 8]. We also use more illustrative and more general thick shells.

The paper can be outlined as follows: In Sec. 2, we review equations of the structure of static spherically symmetric configurations of ECD in the Majumdar-Papapetrou (MP) formalism [9, 10, 11], and we give the expressions for the total mass and charge of the system. In Sec. 3, we discuss the properties of solutions that one obtains in the dust-free regions of space. The solutions with non-negative rest-energy density of the ECD are considered in Sec. 4. There we review a new class of solutions that we found. These solutions involve a zero crossing in the metric profile function that we are using, $U(R) = (-g_{RR})^{1/2} = (g_{tt})^{-1/2}$, which corresponds to a spacetime singularity, and they do not show the bifurcating behaviour typical of regular solutions in this context [12]. In addition, solutions with more than one zero in the metric profile function, corresponding to higher amounts of allocated mass, are shown to exist. Their behaviour follows the pattern set up by the first one. In Sec. 5, we focus on the singular solutions considered in earlier sections, and we show that in all cases the introduction of a single spherical δ-shell of ECD suffices to remove the spacetime singularity, leaving the structure of the spacetime outside of the shell unchanged. In some cases, depending on the position of the shell, the rest-energy density of the ECD on the shell must be allowed to be negative. Sec. 6 is devoted to the thick-shell regularization. Apart from its more attractive form, the procedure shows that is is possible to accomplish the regularization with a more general dust distribution than the one used before. In Sec. 7 conclusions are given.

2. *The Majumdar–Papapetrou formalism*

In a system involving electrically charged dust (pressureless fluid), the coupled Einstein–Maxwell equations,

$$G^{\mu}{}_{\nu} = 8\pi\, T^{\mu}{}_{\nu} = 8\pi \left(M^{\mu}{}_{\nu} + E^{\mu}{}_{\nu}\right), \tag{1}$$

$$F^{\mu\nu}{}_{;\nu} = 4\pi\, j^{\mu}, \tag{2}$$

must be satisfied. In the Einstein equation (1), $G^{\mu}{}_{\nu}$ is the Einstein tensor, and the energy-momentum tensor $T^{\mu}{}_{\nu}$ consists of

$$M^{\mu}{}_{\nu} = \rho\, u^{\mu} u_{\nu}\,, \tag{3}$$

due to the dust with the rest-energy density ρ and 4-velocity u_{μ}, and

$$E^{\mu}{}_{\nu} = \frac{1}{4\pi}\left(-F^{\mu\sigma}F_{\nu\sigma} + \frac{1}{4}\,\delta^{\mu}_{\nu}\,F^{\rho\sigma}F_{\rho\sigma}\right), \tag{4}$$

due to the electromagnetic field of strength $F_{\mu\nu}$. The non-homogeneous Maxwell equation (2) relates $F_{\mu\nu}$ to the electric charge 4-current j^{μ}, semicolon denoting covariant differentiation. Using σ for the density of the electric charge, and u_{μ} for its 4-velocity, the charge 4-current is

$$j^{\mu} = \sigma\, u^{\mu}\,. \tag{5}$$

The homogeneous Maxwell equation, $F_{[\mu\nu;\sigma]} = 0$, brackets denoting antisymmetrization, is satisfied by expressing $F_{\mu\nu}$ in terms of the 4-potential A_{μ},

$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}\,. \tag{6}$$

In the case of static dust, following the assertion of Majumdar [9] and of Papapetrou [10], one may adopt the harmonic coordinates where the line element has the form

$$\mathrm{d}s^2 = U^{-2}\,\mathrm{d}t^2 - U^2\,(\mathrm{d}X^2 + \mathrm{d}Y^2 + \mathrm{d}Z^2)\,, \tag{7}$$

and the metric profile function U depends on the spacelike coordinates X, Y and Z only. The non-zero components of the Einstein tensor for this metric are

$$G^{0}{}_{0} = (-2U\nabla^2 U + \delta^{kl}\partial_k U \partial_l U)/U^4, \tag{8}$$

and

$$G^{i}{}_{j} = (2\delta^{ik}\partial_k U \partial_j U - \delta^{i}_{j}\delta^{kl}\partial_k U \partial_l U)/U^4. \tag{9}$$

(We use roman indices $i, j, ..$ to loop over spacelike coordinates, while greek indices loop over all four coordinates. $\nabla^2 = \delta^{kl}\partial_k\partial_l$ denotes the three-dimensional Laplacian operating in the harmonic coordinates.) The 4-velocity of static matter, normalized so that $u_{\mu}u^{\mu} = 1$, is

$$u_{\mu} = U^{-1}\delta^{0}_{\mu}\,. \tag{10}$$

The energy-momentum tensor due to static dust (3) is, therefore,

$$M^{\mu}{}_{\nu} = \rho\,\delta^{\mu}_{0}\delta^{0}_{\nu}\,, \tag{11}$$

where we assume that the rest-energy density of the dust, $\rho = \rho(X,Y,Z)$, is nonnegative. The electromagnetic 4-potential has the scalar component only, $A_\mu = \Phi\,\delta^0_\mu$, $\Phi = \Phi(X,Y,Z)$. The nonzero components of the energy-momentum tensor (4) due to electrostatic field are

$$E^0{}_0 = \delta^{kl}\partial_k\Phi\partial_l\Phi/8\pi\,, \tag{12}$$

and

$$E^i{}_j = (2\delta^{ik}\partial_k\Phi\partial_j\Phi - \delta^i_j\delta^{kl}\partial_k\Phi\partial_l\Phi)/8\pi. \tag{13}$$

Consequently, the nontrivial Einstein equations are

$$(-2U\nabla^2U + \delta^{kl}\partial_kU\partial_lU)/U^4 = 8\pi\rho + \delta^{kl}\partial_k\Phi\partial_l\Phi\,, \tag{14}$$

and

$$(2\delta^{ik}\partial_kU\partial_jU - \delta^i_j\delta^{kl}\partial_kU\partial_lU)/U^4 = 2\delta^{ik}\partial_k\Phi\partial_j\Phi - \delta^i_j\delta^{kl}\partial_k\Phi\partial_l\Phi\,. \tag{15}$$

Contracting (15) with δ^j_i gives

$$U^{-4}\delta^{kl}\partial_kU\partial_lU = \delta^{kl}\partial_k\Phi\partial_l\Phi\,, \tag{16}$$

which can be used to eliminate the $\delta^{kl}\partial_k\Phi\partial_l\Phi$ term from (14). The Einstein equations thus reduce to a nonlinear version of the Poisson equation,

$$\nabla^2U = -4\pi\rho U^3. \tag{17}$$

Eq. (16) leads to the simple relation between the metric, U, and the electrostatic potential, Φ, i.e. among the metric and the electrostatic field. Setting $k=l$ in (16) one obtains

$$\frac{\partial_kU}{U^2} = \mp\partial_k\Phi\,, \tag{18}$$

which directly integrates to give

$$U^{-1} = \pm\Phi + const. \tag{19}$$

The $\pm$ sign in the above relations, and throughout the paper, reflects the (global) freedom in the choice of the sign of the electric charge.

The only nontrivial component of the nonhomogeneous Maxwell equations (2) is $F^{0\nu}{}_{;\nu} = 4\pi j^0$. The LHS can be rewritten using the well-known relation $F^{\mu\nu}{}_{;\nu} = |g|^{-1/2}\partial_\nu(|g|^{1/2}F^{\mu\nu})$, $g = \det[g_{\mu\nu}]$, valid for any antisymmetric tensor $F^{\mu\nu}$. Then using (6) and (18) on the LHS, and (5) and (10) on the RHS, it reduces to

$$\nabla^2U = \mp 4\pi\sigma U^3, \tag{20}$$

which has the same structure as (17). However, (17) and (20) must be satisfied simultaneously. This is possible only if $\rho = \pm\sigma$, i.e. if the rest-energy density of the dust ρ equals (in relativistic units) the ($\pm$) density of the electric charge σ. As the rest-energy density of the dust, ρ, is taken to be non-negative, it follows that the density of the electric charge is of the same sign everywhere. Such matter is called extremelly charged, or electrically counterpoised dust (ECD). Static configurations of ECD satisfy the equation

$$\nabla^2 U = -4\pi\rho U^3, \qquad \sigma = \pm\rho\,, \tag{21}$$

and are known as Majumdar–Papapetrou (MP) systems [9, 10]. For a formulation of the MP formalism in $d \geq 4$ dimensional spacetimes, as well as for a recent compilation of references dealing with MP systems, see Ref. [11].

While one of the most remarkable features of the MP systems is that they do not require any spatial symmetry, spherically symmetric and asymptotically flat MP systems have played an important role in obtaining valuable insights into the properties of astrophysically plausible systems, see eg. Ref. [13], and are sometimes called 'Bonnor stars'. Since in this paper we are considering such systems, we use the rest of this section to prepare the relations that are specific for spherical symmetry. The harmonic line element (7) can be written

$$\mathrm{d}s^2 = U(R)^{-2}\,\mathrm{d}t^2 - U(R)^2\,(\mathrm{d}R^2 + R^2\,\mathrm{d}\Omega^2)\,, \tag{22}$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2$, and the MP equation (21) reads

$$R^{-2}\,(R^2 U')' = -4\pi\,\rho\,U^3, \tag{23}$$

the prime denoting differentiation with respect to the harmonic coordinate R. The energy-momentum tensor due to matter $M^\mu{}_\nu$ is still given by (11), while the energy-momentum tensor due to electrostatic field acquires the diagonal form

$$E^\mu{}_\nu = \Phi'^2\,\mathrm{diag}(+,+,-,-)/8\pi\,. \tag{24}$$

Here we comment on the structure of the complete energy-momentum tensor, $T^\mu{}_\nu = M^\mu{}_\nu + E^\mu{}_\nu$, with regard to the fulfillment of the standard energy conditions. The total rest-energy density, $\rho_{\mathrm{tot}} = T^t{}_t = \rho + \Phi'^2/8\pi$, involves the contribution from the rest-energy density of the ECD, ρ, which is non-negative by assumption, and the manifestly non-negative contribution of the electrostatic field. The total rest-energy density ρ_{tot} is therefore non-negative. The pressures are of purely electrostatic origin. The transverse pressures, $p_{\mathrm{tr}} = -T^\vartheta{}_\vartheta = -T^\varphi{}_\varphi = \Phi'^2/8\pi$, are non-negative, while the radial pressure, $p_{\mathrm{rad}} = -T^R{}_R = -\Phi'^2/8\pi$, is of equal strength, but of the opposite sign. Noting that $\rho_{\mathrm{tot}} = p_{\mathrm{tr}} + \rho$, and recalling that by assumption $\rho > 0$, these relations can be summarized as follows:

$$-p_{\mathrm{tr}} = p_{\mathrm{rad}} \leq 0 \leq p_{\mathrm{tr}} \leq \rho_{\mathrm{tot}} = p_{\mathrm{tr}} + \rho\,. \tag{25}$$

The weak [WEC; $\rho_{\rm tot} \geq 0$ and $\rho_{\rm tot} + p_i \geq 0$], null [NEC; $\rho_{\rm tot} + p_i \geq 0$], strong [SEC; $\rho_{\rm tot} + \sum_i p_i \geq 0$ and NEC] and dominant [DEC; $\rho_{\rm tot} \geq 0$ and $\rho_{\rm tot} \geq |p_i|$], energy conditions are all satisfied, as expected for classical matter and fields.

If ECD is localized within some radius, we expect the spacetime to be asymptotically flat as $R \to \infty$. The strength of the electrostatic field is expected to fall off as R^{-2} (see next section), i.e. we expect U and Φ to approach constant values. Without loss of generality, one can require

$$\lim_{R\to\infty} U = 1 \qquad \text{and} \qquad \lim_{\mathrm{R}\to\infty} \Phi = 0\,. \tag{26}$$

This determines the value of the integration constant in (19), and we have

$$U^{-1} = \pm\Phi + 1 \tag{27}$$

everywhere.

The total amount of electric charge in a spherically symmetric MP system that does not involve spacetime singularities is given by the invariant volume integral of the conserved quantity j^0

$$Q = \int j^0 \sqrt{-g}\, \mathrm{d}^3 X\,. \tag{28}$$

Using (5) to eliminate j^0, and then (23) to eliminate ρ, one obtains $Q = \pm \int\limits_0^\infty (R^2 U')' \mathrm{d}R$, which integrates to

$$Q = \mp R^2 U' |_0^\infty\,. \tag{29}$$

The gravitational mass of such system is given by the Tolman–Whittaker [14, 15] expression

$$M = \int (2T^0_0 - T^\mu_\mu)\sqrt{-g}\, \mathrm{d}^3 X\,. \tag{30}$$

Using (11) and (24), the integral is $4\pi \int\limits_0^\infty (\rho + \Phi'^2/4\pi) U^2\, R^2 \mathrm{d}R$, and using (23) and (27) to eliminate ρ and Φ' it integrates to

$$M = -R^2 U'/U |_0^\infty\,. \tag{31}$$

In asymptotically flat nonsingular MP spacetimes, $M = \pm Q$ equals the Arnowitt-Deser-Misner (ADM) mass of the configuration [16]. However, when singularities are present, the integrals (30) and (28) cannot be straightforwardly applied. We deal with these issues in some detail in next sections.

We close this section by giving some relations that are useful when physical interpretation of the solutions is considered. The first deals with the rescaling of solutions. If certain $U(R)$ and $\rho(R)$ solve Eq. (23), one is allowed to rescale the functions according to

$$\begin{array}{rcl} U(R) & \longrightarrow & U^*(R) = U(\alpha R)\,, \\ \rho(R) & \longrightarrow & \rho^*(R) = \alpha^2 \rho(\alpha R)\,, \end{array} \tag{32}$$

where $\alpha > 0$ is the scaling parameter. As a consequence, the mass scales according to

$$M \longrightarrow M^* = M/\alpha\,. \tag{33}$$

Another important issue is the relation among the the harmonic coordinates with the line element (22) and the curvature coordinates with the line element

$$\mathrm{d}s^2 = B(r)\,\mathrm{d}t^2 - A(r)\,\mathrm{d}r^2 - r^2\,\mathrm{d}\Omega^2\,. \tag{34}$$

Using the relations

$$r = UR\,, \qquad B = U^{-2}\,, \qquad A^{-1/2} = 1 + \frac{R}{U}\frac{\mathrm{d}U}{\mathrm{d}R}\,, \tag{35}$$

the metric components can be transformed from one coordinate system to another. Throughout this paper, we will consistently use (uppercase) R for the harmonic coordinate and (lowercase) r for the curvature coordinate.

3. The $\rho = 0$ solution (ERN)

If the density of the ECD is zero throughout the space, the nonlinear differential Eq. (21) reduces to the simple Laplace equation. Its general spherically symmetrical solution is

$$U(R) = k + m/R\,, \tag{36}$$

where k and m are integration constants. To obtain asymptotically flat spacetime for $R \to \infty$, one sets $k = 1$, according to (26). Clearly, setting $m = 0$ yields Minkowski spacetime.

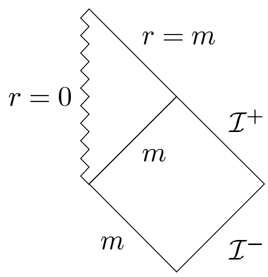
Let us now consider the case when $m > 0$. In the metric (22), at $R = 0$, the component $g_{RR} = -U^2$ diverges, and $g_{tt} = U^{-2}$ has a double zero. The area of the spacelike surface $R = 0$ is $4\pi m^2$, and all metric invariants (e.g. $R_{\mu\nu\kappa\lambda}R^{\mu\nu\kappa\lambda} = 8m^2(m^2 + 6R^2)/(m + R)^8$) are regular. This suggests that at $R = 0$ there is an event horizon beyond which the metric can be analytically extended into the region $R < 0$. At $R = -m$, on the other hand, the metric component g_{tt} diverges, and the area of the spacelike surface $R = -m$ is zero. This indicates a point-like singularity beyond which there is no analytical extension of the metric.

In the curvature coordinates, using the relations (35), the metric is

$$\mathrm{d}s^2 = (1 - m/r)^2\mathrm{d}t^2 - (1 - m/r)^{-2}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2, \tag{37}$$

which is the extremal Reissner-Nordström (ERN) metric, describing a static spherically symmetric black hole of mass $m > 0$ and electric charge $q = \pm m$. The harmonic coordinate R, and the curvature coordinate r, are related by $r = R + m$. At $R = 0$ ($r = m$), there is the event horizon, and at $R = -m$ ($r = 0$), a point-like singularity (Fig. 1).

$m > 0$ $m < 0$



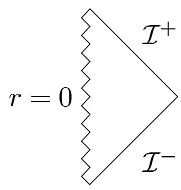


Fig. 1. Penrose–Carter diagrams of the ERN spacetime: the $m > 0$ black-hole case (left) and the $m < 0$ naked singularity case (right). Future and past null-infinities are labeled $\mathcal{I}^+$ and $\mathcal{I}^-$. The $m > 0$ diagram can be continued (tessellated) in the upward (timelike) direction.

The charge integral (28) evaluated in the range $-m < R < \infty$ gives $Q = 0$. This is not surprising since $\rho = 0$ everywhere. The mass integral (30), on the other hand, diverges. Again, since it integrates the energy of the electrostatic field of a point charge (singularity at $R = -m$, $r = 0$), in analogy with classical electrodynamics, this is not an unexpected feature.

It is instructive to consider the solution (36) with negative ADM mass as well. Such solution is generated by setting $m < 0$. Then at $R = |m|$ the metric profile function U has a zero crossing, the metric component g_{RR} has a double zero, and g_{tt} diverges. The area of the spacelike surface $R = |m|$ is zero, and metric invariants (eg. $R_{\mu\nu\kappa\lambda}R^{\mu\nu\kappa\lambda}$) are singular. The metric cannot be analytically extended through this singularity. Since there is no event horizon to shield this singularity from the outer space, this is a naked singularity (Fig. 1). (Note that the metric one obtains with $m < 0$ and $|m| < R < \infty$ is isometric to that with $m > 0$ and $-\infty < R < -m$.) The charge integral (28) evaluated in the range $|m| < R < \infty$ gives $Q = 0$ and the mass integral (30) diverges.

4. *Solutions with non-negative ECD density*

Spherically symmetric Majumdar-Papapetrou systems can be constructed by numerical integration of (23) for the assumed distribution of the ECD specified by the non-negative function $\rho(R)$. However, the structure of the resulting spacetime strongly depends on the shape of $\rho(R)$.

If $\rho(R)$ vanishes as $R \to \infty$ more rapidly than R^{-3}, the solutions are asymptotically flat at infinity and behave like $1 + m_\infty/R$, where the parameter m_∞ is the ADM mass of the configuration. The boundary conditions for such solutions can be formulated as

$$\lim_{R\to\infty} U(R) = 1 \qquad \text{and} \qquad \lim_{\mathrm{R}\to\infty} \mathrm{R}^2\mathrm{U}'(\mathrm{R}) = -\mathrm{m}_\infty \,. \tag{38}$$

Given the ECD distribution $\rho(R)$, Eq. (38) allows integration of the second order ordinary differential equation (23) as an initial value problem. If the resulting function $U(R)$ has a node at some $R > 0$, then there is a naked singularity. This can be seen by following the same argument as in the case of negative mass ERN spacetime (Sec. 3). On the other hand, if there is no nodes, but $U(R)$ diverges as one approaches $R = 0$, then the spacetime involves an event horizon, and the solution to (23) in the range $0 < R < \infty$ covers only the region of the spacetime outside of the event horizon. Only the solutions where $U(R)$ is finite and nonzero for $0 \leq R < \infty$ are regular everywhere. Such solutions can be obtained by imposing an additional boundary condition to control the behaviour of the function $U(R)$ as $R \to 0$. Since for $R \to 0$ one can expect the general solution to behave like $U(R) = m_0/R + \mathit{const.}$, the boundary condition can be written

$$\lim_{R\to 0} R\,U(R) = m_0 \,, \tag{39}$$

where $m_0 = 0$ ensures the regularity at $R = 0$. The spacetime is asymptotically flat and regular throughout if $U(R)$ fulfils the conditions (38) and (39) with $m_0 = 0$, and has no nodes.

To accommodate the additional boundary condition, one can use a factor η multiplying the source term in (23), i.e. one solves

$$R^{-2}\,(R^2U')' = -4\pi\,\eta\,\rho(R)\,U^3(R)\,, \tag{40}$$

and that with the conditions (38) and (39) now presents a boundary value problem. A number of diverse ECD distributions considered in Ref. [12] revealed that regular solutions do not exist for values of η greater than some critical value η_c, which however depends on the particular choice of $\rho(R)$. The solution corresponding to η_c was called critical, and the corresponding ADM mass was labeled m_c. In the range $0 < \eta < \eta_\mathrm{c}$, the critical solution bifurcates into two independent solutions with ADM masses m^- and m^+ that obey $0 < m^- < m_\mathrm{c} < m^+$. As $\eta \to 0$ the 'low mass (m^-) bifurcation branch' produces spacetimes that are asymptotically flat everywhere. The 'high mass (m^+) bifurcation branch', on the other hand, produces spacetimes that asymptotically coincide with the external part of the ERN metric. Such (regular) configurations allow arbitrarily large red-shifts [13, 17], and are known as 'quasi black holes' [12, 18].

Here we used the ECD distribution

$$\rho(R) = \eta\,\rho_0\,\exp(-(R/\tilde{R})^2)\,, \tag{41}$$

where ρ_0 and $\tilde{R}$ scale the solution so that the critical parameters are $m_\mathrm{c} = 1 = \eta_\mathrm{c}$ (for numerical values see Ref. [12]). By numerical integration, we computed the solutions to (40) for the metric profile functions $U(R)$ with zero, one, and two nodes, subject to boundary conditions (39) with $m_0 = 0$, and (38) with a wide range of values for m_∞. The relation among the ADM mass m_∞ and the source strength η in these solutions is shown in Fig. 2). The solutions without nodes generate the regular spacetimes that were discussed above. The upper mass bifurcation branch leads to quasi black-hole spacetimes, while the lower mass bifurcation branch passes through the Minkowski spacetime ($\eta = 0$, $m_\infty = 0$) and enters the regime where both η and m_∞ are negative, which we do not consider any further. The solutions with one or more nodes (outermost node corresponding to the naked singularity) show similar behaviour in the relation among η and m_∞, except that the critical point (bifurcation turning point) is displaced into the negative ADM mass region. Only the upper part of the upper mass bifurcation branch is in the range of positive m_∞ values.

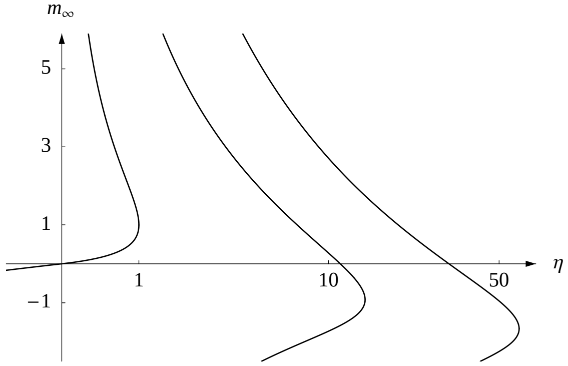


Fig. 2. Relation among the ADM mass m_∞ and the source strength η of the ECD distribution (41) in solutions with zero, one and two nodes (reading from left to right) in the metric profile function $U(R)$ (shown in Fig. 3), satisfying the boundary conditions (38) and (39) with $m_0 = 0$. (Non-linear scaling is used on the η-axis.)

The metric profile functions $U(R)$ for the solutions with zero, one, and two nodes, obtained with $m_\infty = 1$, are shown in Fig. 3. Numerical values of the related parameters are given in Table 1. Since all these solutions obey the same boundary condition as $R \to \infty$, they can not be distinguished by a distant observer. The

TABLE 1. Numerical values of the parameters of the solutions shown in Fig. 3: source strength η, coordinate of the outermost node R_0, first derivative of the metric profile function at the node $U'(R_0)$, coordinate of the outermost maximum R_m, $U(R_\mathrm{m})$, the integrals (28) and (30), Q and M. For solutions with nodes the integrals Q and M are evaluated over the range $R_0 < R < \infty$.

Nodes	η	R_0	$U'(R_0)$	R_m	$U(R_\mathrm{m})$	Q	M
0	1.000	-	-	-	-	1.000	1.000
1	7.291	0.927	2.541	2.795	1.304	3.183	∞
2	19.558	1.416	1.995	3.375	1.263	4.998	∞

metric components in the curvature coordinates for the solution with one node are shown in Fig. 4. If $U(R)$ has its (outermost) maximum at $R_{\rm m}$, then at $r_{\rm m} = R_{\rm m}\,U(R_{\rm m})$ there is the minimum in $B(r)$, and $A(r_{\rm m}) = 1$. In the example shown in Fig. 4, we have $r_{\rm m} > 2m_\infty$, but with increasing the value of m_∞ (going up the bifurcation branch), the ratio $r_{\rm m}/m_\infty$ decreases and it is possible to reach solutions where $r_{\rm m} < m_\infty$. One such example is the $m_\infty = 50$ solution shown if Fig. 5 where it can be seen that by climbing the upper bifurcation branch, the metric is coming arbitrarily close to having a horizon that shields the singularity from the outer space. We therefore conclude that the singular solutions can be used to construct quasi-black holes in the similar way the regular solutions can be used [12].

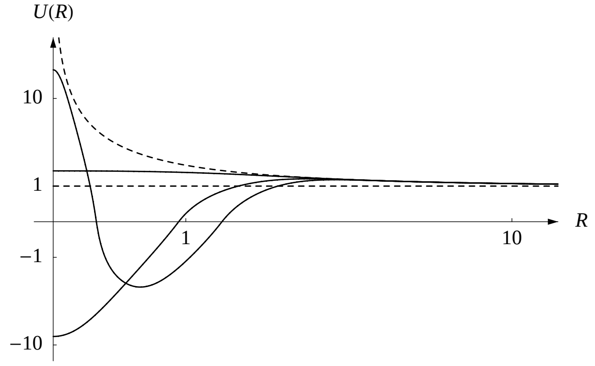


Fig. 3. Metric (22) profile function $U(R)$ for the ECD distribution (41), satisfying boundary conditions (38) with $m_\infty = 1$ and (39) with $m_0 = 0$. Solutions with zero, one and two nodes (solid lines) are shown. Profile function $U = 1 + 1/R$ representing unit mass/charge ERN spacetime and $U = 1$ (dashed lines) is shown for comparison. (Non-linear scaling is used on both axes.)

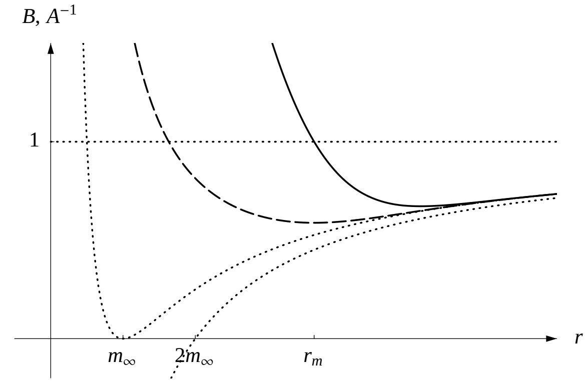


Fig. 4. The metric (34) components $g_{tt} = B(r)$ (dashed line) and $g^{rr} = 1/A(r)$ (solid line) for the solution with one node satisfying boundary conditions $m_0 = 0$ and $m_\infty = 1$. The metric components $B(r) = 1/A(r) = (1 - m_\infty/r)^2$ for the ERN, $B(r) = 1/A(r) = 1 - 2m_\infty/r$ for the Schwarzschild, and $A = B = 1$ for the Minkowski metric are shown for comparison (dotted lines).

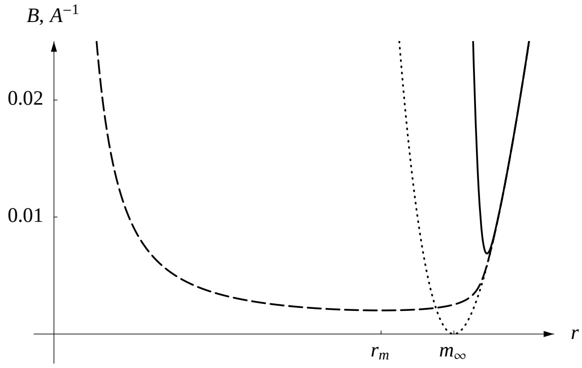


Fig. 5. The metric (34) components $g_{tt} = B(r)$ (dashed line) and $g^{rr} = 1/A(r)$ (solid line) for the solution with one node satisfying boundary conditions $m_0 = 0$ and $m_\infty = 50$. The metric components $B(r) = 1/A(r) = (1 - m_\infty/r)^2$ for the ERN metric are shown for comparison (dotted line).

In the case of the regular solutions, the charge (28) and mass (30) integrals obey the expected relation: $M = \pm Q = m_\infty$. This is not so in the case of the solutions with nodes where the integrals are evaluated over the range $R_0 < R < \infty$, R_0 being the coordinate of the outermost node (naked singularity). The values in the Table 1 show that the value of the charge integral Q is higher than the value the distant observer would expect on the basis of the asymptotic behaviour of the ERN metric. We obtained $Q > m_\infty$ and the mass integral M diverges.

5. Regularization of solutions I: The δ-shell

In all singular MP systems considered in preceding sections, we have shown that the integrals (28) for the charge, Q, and (30) for the mass, M, of the system do not give the values which are in accord with the asymptotic behaviour of the ERN metric. The ERN metric is characterized by the single parameter m, which is the ADM mass, and in relativistic units its value equals the ($\pm$) charge of the system. The relation $M = \pm Q =$ ADMmass) is obeyed by the regular MP systems, but not by those that involve a spacetime singularity.

We now look for the possible modification of the ECD distribution for the singular configurations that can remove the singularity, and thus restore the expected relation among the mass and charge parameters, keeping the external metric and electrostatic field unchanged. It turns out that in all singular configurations considered so far, this can be accomplished by introducing a single spherical shell of ECD. The amount of the ECD on the shell must be adjusted so that in the inside of the shell the metric becomes flat and the electrostatic field vanishes. To simplify the calculations, we will use δ-shells, keeping in mind that similar behaviour is expected for shells with finite thickness (see next section).

Let us first consider the ERN black hole case ($U(R) = 1 + m/R$, $m > 0$, or in curvature coordinates, metric (37)). The δ-shell distribution of ECD located at harmonic radius $R = a$ (corresponding to curvature radius $r = m + a$) can be written

$$\rho_\delta(R) = \eta_\delta\, \delta(R - a)\,. \tag{42}$$

One can now substitute ρ_δ into (23), multiply the equation by R^2 and integrate over the shell, obtaining $R^2 U'(R)|_{a-\epsilon}^{a+\epsilon} = -4\pi a^2 \eta_\delta U^3(a)$. If the metric is to remain unchanged outside of the shell, and the spacetime is to be flat ($U = const$) in the inside, letting $\epsilon \to 0$, the density of the ECD on the shell is

$$\eta_\delta = -\frac{U'_+(a)}{4\pi U_+^3(a)}\,, \tag{43}$$

where $U_+(a)$ and $U'_+(a)$ is the metric as it approaches the shell from the outside, i.e. the original ERN metric. The δ-shell ECD distribution that produces flat space inside, and the ERN metric in the outside, is therefore

$$\rho_\delta(R) = \frac{m\, a}{4\pi\,(m + a)^3}\,\delta(R - a)\,. \tag{44}$$

When the δ-shell is located outside of the region where there originally was a horizon, i.e. at the harmonic radius $R = a > 0$ (corresponding to the curvature radius $r = m + a > m$), the density of the ECD on the shell (44) is positive. Both the event horizon, and the central singularity, are removed from the system. The situation is different if the shell is placed at $R = a$, $-m < a < 0$, i.e. in the black-hole region. Such shell removes the singularity leaving the horizon where it was. The curvature coordinate $r = R + m$ decreases as one approaches the shell from the $R > a$ ERN spacetime, it reaches the finite value $r = a + m > 0$ at the shell, and starts increasing again as one passes the shell and continues moving through the flat $R < a$ region. There is no feature in the spacetime to prevent one from continuing indefinitely. The curvature coordinate r can be interpreted by recalling that the area of the surface $r = const$ is $4\pi r^2$. Therefore the resulting spacetime can be understood as the ERN metric ‘outside the shell on one side’ and Minkowski (flat) space ‘outside the shell on the other side’, as if two external solutions were glued together, one on each side of the shell. An important feature of the ECD shell placed within the ERN black hole is that the energy density of ECD (44) on the shell must be negative. The Penrose-Carter diagrams for the ERN spacetime regularized by the δ-shell placed outside of the event horizon, and by the δ-shell placed inside the black-hole region, is shown in Fig. 6.

Irrespective of the position of the shell, both the charge integral (28),

$$Q = 4\pi \int\limits_{a-\epsilon}^{a+\epsilon} \rho_\delta(R)\, U^3\, R^2\, \mathrm{d}R = m\,, \tag{45}$$

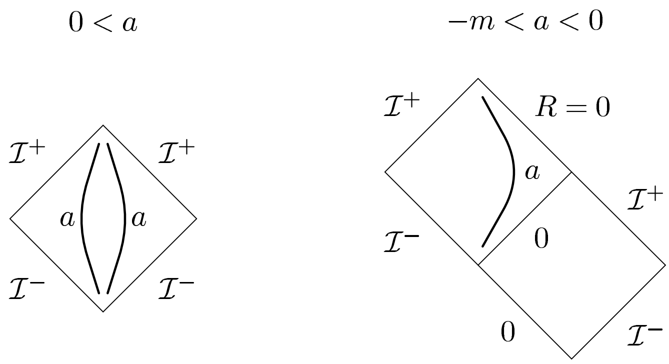


Fig. 6. Penrose–Carter diagrams of $m > 0$ ERN spacetime regularized by a spherical δ-shell of ECD of harmonic radius $R = a$ (curvature radius $r = m + a$): cases $0 < a$ (left) and $-m < a < 0$ (right). Future and past null-infinities are labeled $\mathcal{I}^+$ and $\mathcal{I}^-$, shell is labeled with a. The $-m < a < 0$ diagram can be continued (tessellated) in the upward (timelike) direction.

and the mass integral (30),

$$M = 4\pi \int_{a-\epsilon}^{a+\epsilon} \rho_\delta \, U^2 \, R^2 \, \mathrm{d}R + \int_a^\infty \Phi'^2 \, U^2 \, R^2 \, \mathrm{d}R = m \,, \tag{46}$$

give the expected results: the relation $M = \pm Q = m > 0$ is thus restored. The same procedure can be carried out in the case of ERN spacetime with negative ADM mass (naked singularity). We can shield the singularity with a spherical shell of ECD. The expression (44) giving the ECD density on a δ-shell located at $R = a > |m|$ is valid again, and since $m < 0$, the density of the ECD is negative for all $a > |m|$. Once the shell is introduced into the system, the singularity disappears and $M = \pm Q = m < 0$ holds.

We now proceed to the solutions with the non-zero ECD density and with the naked singularities considered in Sec. 4. Denote $U(R)$ the solution of (23) for the ECD density $\rho(R) \geq 0$, satisfying the boundary conditions (39) with $m_0 = 0$ and (38) with $m_\infty > 0$. Let $U(R)$ have one or more nodes, outermost being at $R = R_0$ (this node corresponds to the singularity). Then $U(R)$ also has a maximum at $R = R_\mathrm{m}$, $R_0 < R_\mathrm{m} < \infty$. We now introduce the δ-shell at the harmonic radius $R = R_0 + a$, $a > 0$, and require that inside the shell the spacetime becomes flat, while outside the shell the metric remains unchanged. The metric which provides regular solutions is

$$U_\delta(R) = \begin{cases} U(R_0 + a) \,, & R < R_0 + a \,, \\ U(R) \,, & R \geq R_0 + a \,. \end{cases} \tag{47}$$

This also means that the ECD must be removed from the inside, while in the outside it must remain unchanged. The ECD density now including the δ-shell can be written as

$$\rho_\delta(R) = \eta_\delta\,\delta(R-(R_0+a)) + \mathcal{H}(R-(R_0+a))\,\rho(R)\,, \tag{48}$$

where $\mathcal{H}(x)$ is the unit step function. Substituting (47) and (48) into (23), and integrating over the shell, one obtains the density of the ECD on the δ-shell,

$$\eta_\delta = -\frac{U'(R_0+a)}{4\pi\,U^3(R_0+a)}\,, \tag{49}$$

which is negative if the shell is between the node and the maximum of $U(R)$ (where $U'>0$), and positive if the shell is outside of the maximum (where $U'<0$).

The charge integral (28) can now be written as the sum of two terms, the first being the contribution of the ECD placed on the shell,

$$4\pi \int\limits_{R_0+a-\epsilon}^{R_0+a+\epsilon} \eta_\delta \delta(R-(R_0+a))\,U_\delta^3(R)\,R^2\,\mathrm{d}R\,, \tag{50}$$

and the second being the contribution of ECD distributed outside of the shell,

$$4\pi \int\limits_{R_0+a}^{\infty} \rho(R)\,U_\delta^3(R)\,R^2\,\mathrm{d}R\,. \tag{51}$$

Numerical calculations with arbitrarily positioned shells ($a>0$) revealed that the sum of the two terms precisely matches the value of the ADM mass of the configuration. In the limit $a\to 0$, i.e. the shell approaching the singularity, the sum of the two terms is

$$Q = -R_0^2 U'(R_0) + 4\pi \int\limits_{R_0}^{\infty} \rho(R)\,U^3(R)\,R^2\,\mathrm{d}R\,. \tag{52}$$

Analysis of the numerical values of the parameters of the solutions with nodes given in Table 1 reveals that the relation $\pm Q = m_\infty$ is restored. In a similar way we tested the behaviour of the mass integral (30), where the relation $M = m_\infty$ was recovered as well. However, the contribution of the shell to the mass integral diverges as the shell approaches the singularity, so the relation analogous to (52) cannot be given.

6. *Regularization of the solutions II: The thick shell*

The spacetime that we have obtained by placing the δ-shell into the ERN black-hole region deserves some more comments with regard to its two not so ordinary features: negative ECD density on the shell and the increasing curvature radius r

on both sides of the shell. Another interesting situation that we encountered is the need for the negative ECD density in the δ-shell regularizing the solution with non-negative ECD density and the naked singularity. The δ-shell that we have used in the preceding section hides its internal structure from us. To visualize the situation better, we now use thick shells to regularize the solutions.

In the analytic construction of the thick shell in Ref. [19], the metric profile function $U(R)$ within the shell is a parabola determined by the requirements that at its outer surface (harmonic radius R_o) it smoothly matches the external ERN metric, and that at its inner surface (harmonic radius R_i), it smoothly matches the flat metric, smoothly meaning that $U(R)$ is of the differentiability class C^1. These conditions could be formulated as $U'_\mathrm{sh}(R_\mathrm{i}) = 0$, $U_\mathrm{sh}(R_\mathrm{o}) = 1 + m/R_\mathrm{o}$, $U'_\mathrm{sh}(R_\mathrm{o}) = -m/R_\mathrm{o}^2$. The ECD density resulting from such thick shell is, generally, discontinuous (nonzero) at the surfaces.

We wish to construct the thick shell that will regularize the the ERN black-hole spacetime, as well as the solutions with non-negative ECD density and the metric profile function $U(R)$ with nodes (naked singularities), described in Sec. 4, analogously to the way this was done in the preceding section with the δ-shell. We also require that the shell does not induce jump discontinuities in the ECD density, which can be accomplished by requiring that the metric is C^2 continuous across the surfaces. This leads to five conditions: $U_\mathrm{sh}^{(1,2)}(R_\mathrm{i}) = 0$, and $U_\mathrm{sh}^{(0,1,2)}(R_\mathrm{o}) = U^{(0,1,2)}(R_\mathrm{o})$, where $U(R)$ is the original metric profile function to be regularized. Therefore, sticking to the polynomial model, its degree must be raised to four. The general analytical expressions for the metric profile function and the ECD density within such thick shell can be straightforwardly derived, but are quite complicated, and are therefore not shown.

We start with the ERN black-hole spacetime by placing the thick shell inside the black-hole region (in harmonic radii, this is the range $-m < R_\mathrm{i} < R_\mathrm{o} < 0$, and in curvature radii this is $0 < r_\mathrm{i} < r_\mathrm{o} < m$). In Fig. 7, we show (among other things) the dependence of the curvature radius r on the harmonic radius R which is in general given by $r = U(R)\,R$, Eq. (35). Outside the thick shell on the right side of the diagram is the ERN spacetime, and $r = R + m$, while on the left side the spacetime is flat, $U(R) = U_0 < 0$, and $r = U_0 R$. The minimum value of r (throat) occurs within the shell. The rest-energy density of the ECD, ρ, is negative within the shell, while the rest-energy density of the electrostatic field (which is equal to the transverse pressure, and the negative of the radial pressure), is positive. The total rest-energy density, which is the sum of the two, $\rho_\mathrm{tot} = \rho + p_\mathrm{tr}$, is negative at least close to the inner edge of the shell where all p_tr falls off more rapidly than ρ, so none of the standard energy conditions (Sec. 2) are satisfied in this solution.

In Fig. 8 we show the example where the spacetime with the non-negative ECD density and with the naked singularity is regularized by the thick shell located in the region where the acceleration due to gravity is oriented outward. As it can be seen, the rest-energy density of the ECD in the shell is (mostly) negative, and as one approaches the internal surface of the shell, it vanishes less rapidly than the transverse pressure. It follows that none of the standard energy conditions are satisfied. The need for the negative ECD density within the shell makes this regularized configuration similar to the one that we obtained when the thick shell was placed in the ERN black hole. However, the most important difference in this case, besides of not involving the event horizon, is that the curvature radius decreases in the region that is understood as internal flat spacetime.

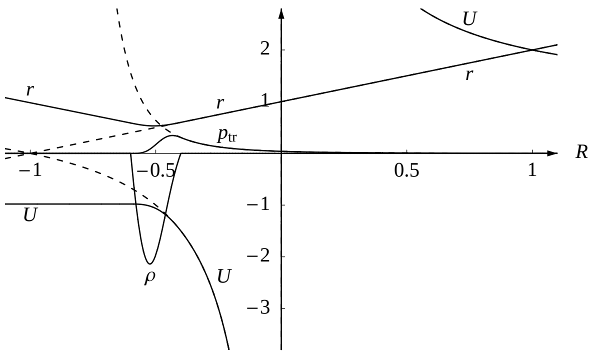


Fig. 7. The $m = 1$ ERN black-hole spacetime (dashed lines) regularized by the thick shell of ECD [19] located in the black-hole region (solid lines): metric (22) profile function $U(R)$, ECD density ρ, transverse pressure $p_{\mathrm{tr.}} = \Phi'^2/8\pi$, curvature radius r. Harmonic radii of the inner and outer surface of the shell are $R_{\mathrm{i}} = -0.6$ and $R_{\mathrm{o}} = -0.4$.

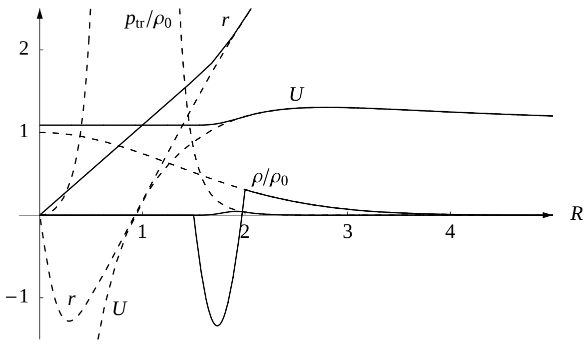


Fig. 8. Solution for the density (41) with the metric profile function $U(R)$ with one node satisfying boundary conditions (38) with $m_\infty = 1$ and (39) with $m_0 = 0$ (dashed lines) regularized with a thick shell of ECD (solid lines): metric profile function $U(R)$, ECD density ρ, transverse pressure $p_{\mathrm{tr.}} = \Phi'^2/8\pi$, curvature radius r. Harmonic radii of the inner and outer thick shell surfaces are $R_{\mathrm{i}} = 1.5$ and $R_{\mathrm{o}} = 2.0$ are less than the harmonic radius of the maximum of $U(R)$, see also Table 1.

7. *Discussion and conclusions*

In this paper we have shown that in the Majumdar–Papapetrou system (a subset of the Einstein–Maxwell system covering static configurations of extremally charged dust, ECD), starting from the prescribed spherically symmetric distribution of ECD with non-negative rest-energy density, one can obtain solutions with the naked point singularity and positive ADM mass. In such solutions the radial component of the metric tensor in harmonic coordinates, $g_{RR} = U^2(R)$, has at least one double zero, i.e. $U(R)$ has a zero crossing. The outermost zero corresponds to the naked singularity. The regular solutions considered in Ref. [12] show bifurcating behaviour of the ADM mass with regard to the amount of allocated matter (source strength). The solutions considered here stem from the same system, but they have only one branch that in the low source strength limit leads to quasi-black-hole configurations. We have shown, in two ways, that the naked singularity can be removed by a modification of the ECD distribution: by introducing either a spherical δ-shell, or a thick shell of ECD. The external metric and ECD distribution remains unchanged, while inside the shell the spacetime is flat. We have used before the δ-shells to obtain basic insight into the structure of the regularized system, and in the similar context δ-shells were used to regularize the multiple black-hole ERN spacetime in Ref. [8]. Here however, to investigate the behaviour of the pressures within the shell, we have used thick shells. If the shell is sufficiently close to the centre of the system, i.e. in the region where the acceleration of the gravitational force is directed outward, and the electrical field reverses its direction relative to outer space, the rest-energy density of the ECD within the shell must be be negative. This is an expected feature since it has been known for a long time that the requirement that the charged structure of sufficiently small size has finite mass, within the classical (Einstein–Maxwell) theory, leads to negative energy density in its inside (see e.g. Refs. [20, 21]). Roughly speaking, this can be understood by observing that the energy of the electrostatic field, which contributes to the mass of the configuration, diverges as the the object shrinks in size. Therefore, if the total mass is to remain constant, the diverging contribution of the positive electrostatic component must be compensated by the appropriate amount of negative energy density allocated within the object. Our solutions can, once regularized by the shell of sufficiently small radius, be understood as a cloud of ECD with non-negative energy density surrounding the core object of negative energy density and electrical charge of the opposite sign. Standard energy conditions are violated within the shell. We have also shown that the rest-energy density of the regularizing shell placed behind the event horizon of the ERN black-hole spacetime must be negative.

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REGULARIZACIJA RJEŠENJA ZA RASPODJELE NABIJENE PRAŠINE U EINSTEIN–MAXWELLOVOJ TEORIJI

Istražujemo regularizaciju rješenja za statičku sferno simetričnu potpuno nabijenu prašinu u Majumdar-Papapetrouovom sistemu. Rješenja u njihovim koordinatama pokazuju singularnosti. Te su singularnosti takve naravi da su rješenja fizikalno neprihvatljiva jer su važne fizičke veličine (metričke invarijante) također singularne. Ta se rješenja mogu regularizirati jednostavnom promjenom definicije raspodjela naboja/energije. Našli smo spektar rješenja s nekoliko nul-čvorova u metričkom tenzoru, i pokazali da se može postići njihova regularizacija bilo s δ sferom, ili s debelom sfernom raspodjelom tvari. Nestaje grananje regularnih rješenja, ali se za ukupnu masu pridruženu rješenjima opaža ponašanje slično kvantizaciji.