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SIMILARITY REDUCTIONS OF A LINEARLY COUPLED KORTEWEG-de VRIES SYSTEM

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Using the direct method of Clarkson and Kruskal we obtain the similarity reductions of a system of two linearly coupled Korteweg-de Vries equations, which was recently studied by Grimshaw and Malomed in relation to the existence of a new type of gap solitons. Particular solutions of this system, both rational and logarithmic, are also determined.

1. *Introduction*

In the study of nonlinear partial differential equations (PDEs) of physical interest, the determination of similarity reductions capable of reducing the number of independent variables by one, plays an important role [1–3]. In the particular case of evolution equations in (1+1) dimensions, such reductions transform the PDEs into ordinary differential equations which are easier to solve, and often permit obtaining explicit particular solutions (similarity solutions) of the given PDEs. The determination of these particular solutions is important as they themselves may be interesting, they may reflect the asymptotic behaviour of more complicated solutions [2], or they may be used as exact references to test the accuracy of programs devised to obtain numerical solutions.

There is a large number of PDEs whose similarity reductions have been determined [3–18]. Most of them are single PDEs, as the number of systems of PDEs with known similarity reductions and similarity solutions is relatively small. One of the systems of PDEs of physical interest, for which no exact similarity reductions had been previously determined, is the following:

$$u_t - uu_x + u_{xxx} = -\lambda v_x \tag{1a}$$

and

$$v_t - \delta v_x - vv_x - \alpha v_{xxx} = -\beta\lambda u_x, \tag{1b}$$

where $u(x,t)$ and $v(x,t)$ are two wave fields, and α, β, δ and λ are real constants. This system of linearly coupled Korteweg-de Vries (KdV) equations was recently studied by Grimshaw and Malomed (GM) [19] in connection with the existence of a novel type of gap solitons, and the need for further numerical simulations with this system was pointed out by these authors.

In the present paper, we determine all the similarity reductions of system (1) which are obtainable by means of the direct method devised by Clarkson and Kruskal (CK) [3]. Particular solutions of system (1), both rational and logarithmic, are also determined. In Appendix A we prove that all similarity reductions of the system (1), which can be obtained by means of the classical Lie's method of infinitesimal transformations, are just particular cases of the similarity reductions obtained by the CK method. In Appendix B, we prove that the system (1) does not possess the Painlevé property for PDEs, as defined by Weiss, Tabor and Carnevale [20], thus clarifying why logarithmic solutions of the system (1) can exist.

2. *Similarity reductions*

To determine the similarity reductions of the system (1) by the CK method [3], we begin by substituting the following expressions into (1):

$$u = U(x,t,P(z(x,t))), \tag{2a}$$

$$v = V(x,t,Q(z(x,t))). \tag{2b}$$

Demanding that the resulting equations be a system of ordinary differential equations (ODEs) for $P(z)$ and $Q(z)$, it is not difficult to prove that U and V must have the following form:

$$U = a(x,t) + b(x,t)P(z) \tag{3a}$$

$$V = c(x,t) + d(x,t)Q(z). \tag{3b}$$

Substituting these expressions into (1), we arrive at the following system of equations for P and Q:

$$P'''(bz_x^3) + P''(3b_xz_x^2 + 3bz_xz_{xx}) + PP'(-b^2z_x)$$

$$+P'(bz_t - abz_x + 3b_{xx}z_x + 3b_x z_{xx} + bz_{xxx}) + Q'(\lambda d z_x)$$

$$+P(b_t - ab_x - ba_x + b_{xxx}) + P^2(-bb_x) + Q(\lambda d_x)$$

$$+a_t - aa_x + a_{xxx} + \lambda c_x = 0 \tag{4a}$$

and

$$Q'''(-\alpha d z_x^3) + Q''(-3\alpha d_x z_x^2 - 3\alpha d z_x z_{xx}) + QQ'(-d^2 z_x)$$

$$+Q'(dz_t - \delta d z_x - cdz_x - 3\alpha d_{xx} z_x - 3\alpha d_x z_{xx} - \alpha d z_{xxx}) + P'(\beta\lambda b z_x)$$

$$+Q(d_t - \delta d_x - cd_x - dc_x - \alpha d_{xxx}) + Q^2(-dd_x) + P(\beta\lambda b_x)$$

$$+c_t - \delta c_x - cc_x - \alpha c_{xxx} + \beta\lambda a_x = 0, \tag{4b}$$

where the primes indicate z derivatives. If we require that these equations be a system of ODEs for $P(z)$ and $Q(z)$, all the coefficients, and also the independent terms appearing in these equations must be functions of z only, multiplied at most by a common factor –the "normalizing coefficient"– which can be cancelled out.

Taking the coefficients of P''' and Q''', respectively, as the normalizing coefficients for Eqs. (4a) and (4b), and considering that $z_x \neq 0$, we found that the necessary condition for these equations to be a system of ODEs for $P(z)$ and $Q(z)$, is that the following equations hold:

$$z(x,t) = kx + \Omega t^2 - \omega_0 t + z_0, \tag{5}$$

$$a(t) = \frac{2\Omega}{k}t - \frac{\omega_1}{k}, \tag{6a}$$

$$c(t) = \frac{2\Omega}{k}t - \frac{\omega_2}{k}, \tag{6b}$$

$$b = \alpha\lambda, \tag{6c}$$

$$d = \beta\lambda/\alpha, \tag{6d}$$

where k, Ω, ω_0, ω_1, ω_2 and z_0 are independent arbitrary constants (with the only condition that $k \neq 0$).

Defining:

$$p(z) \equiv bP(z) = \alpha\lambda P(z), \tag{7a}$$

$$q(z) \equiv dQ(z) = \frac{\beta\lambda}{\alpha}Q(z), \tag{7b}$$

and taking into account Eqs. (5) and (6), Eqs. (3) and (4) transform into:

$$u(x,t) = \frac{2\Omega}{k}t - \frac{\omega_1}{k} + p(z), \tag{8a}$$

$$v(x,t) = \frac{2\Omega}{k}t - \frac{\omega_2}{k} + q(z), \tag{8b}$$

$$p''' - \frac{1}{k^2}pp' + \left[\frac{\omega_1 - \omega_0}{k^3}\right]p' + \frac{\lambda}{k^2}q' + \frac{2\Omega}{k^4} = 0, \tag{8c}$$

$$q''' + \frac{1}{\alpha k^2}qq' + \left[\frac{\delta}{\alpha k^2} - \frac{\omega_2 - \omega_0}{\alpha k^3}\right]q' - \frac{\beta\lambda}{\alpha k^2}p' - \frac{2\Omega}{\alpha k^4} = 0. \tag{8d}$$

Equations (5) and (8) thus constitute a six–parameter similarity reduction of the system (1) and, as we shall see in the next section, from this reduction an exact particular solution of the system (1) can be found.

At first sight, the similarity reduction defined by Eqs. (5) and (8) is the only reduction which can be obtained by the direct CK method. However, we must remember that this similarity reduction was obtained under the hypothesis that $z_x \neq 0$ and, as Lou has observed [10], new results can be obtained by considering that $z_x = 0$. Consequently, let us consider that $z_x = 0$ in Eqs. (4a) and (4b). In that case, these equations reduce to:

$$p'(bz_t) + p^2(-bb_x) + p(b_t - ab_x - ba_x + b_{xxx}) + q(\lambda d_x) +$$
$$+a_t - aa_x + a_{xxx} + \lambda c_x = 0, \tag{9a}$$

$$q'(dz_t) + q^2(-dd_x) + q(d_t - \delta d_x - cd_x - dc_x - \alpha d_{xxx}) + p(\beta\lambda b_x) +$$
$$+c_t - \delta c_x - cc_x - \alpha c_{xxx} + \beta\lambda a_x = 0, \tag{9b}$$

where we have written p and q instead of P and Q, respectively.

In order to determine the forms of the factors a, b, c and d, we can consider (without loss of generality) that $z(t) = t$, and we can take the coefficients of p' and q' as normalizing coefficients for Eqs. (9a) and (9b), respectively. In this way, demanding that the system (9) reduces to a system of ODEs for $p(t)$ and $q(t)$, we arrive at the conclusion that a, b, c and d must have the following forms:

$$a(x,t) = a_0(t)x, \tag{10a}$$

$$c(x,t) = c_0(t)x, \tag{10b}$$

$$b(x,t) = d(x,t) = 1, \tag{10c}$$

where each of the functions $a_0(t)$ and $c_0(t)$ can take two forms, namely:

$$a_0(t) = \begin{cases} 0 \\ 1/(t_u - t) \end{cases}, \tag{11a}$$

$$c_0(t) = \begin{cases} 0 \\ 1/(t_v - t) \end{cases}, \tag{11b}$$

where t_u and t_v are arbitrary constants. Therefore we obtain that u and v have the forms:

$$u(x,t) = a_0(t)x + p(t), \tag{12a}$$

$$v(x,t) = c_0(t)x + q(t), \tag{12b}$$

and consequently Eqs. (9a) and (9b) reduce to the linear equations:

$$p' - a_0 p = -\lambda c_0, \tag{13a}$$

$$q' - c_0 q = -\beta\lambda a_0 + \delta c_0. \tag{13b}$$

Equations (11) – (13) thus define an additional family of similarity reductions of the system (1). Taking into account that there are two possible forms for each of the functions $a_0(t)$ and $c_0(t)$, and leaving aside the trivial case $a_0(t) = c_0(t) = 0$, we can see that this family includes three types of similarity reductions. The first one corresponds to the choice:

$$a_0(t) = \frac{1}{t_u - t}, \tag{14a}$$

$$c_0(t) = \frac{1}{t_v - t}, \tag{14b}$$

the second type corresponds to:

$$a_0(t) = \frac{1}{t_u - t}, \tag{15a}$$

$$c_0(t) = 0, \tag{15b}$$

and finally, the third type corresponds to the choice:

$$a_0(t) = 0, \tag{16a}$$

$$c_0(t) = \frac{1}{t_v - t}. \tag{16b}$$

In the three cases, it is possible to obtain analytical solutions of the ordinary equations (13a) and (13b), and consequently from these reductions we can obtain additional similarity solutions of the system (1).

It is worth noticing that Lie's method of infinitesimal transformations just permits obtaining particular cases of the similarity reductions already found in this section (the determination of similarity reductions of the system (1) by Lie's procedure is presented in Appendix A).

3. *Similarity solutions*

A first similarity solution of the system (1) can be determined if the following values of the parameters k, Ω, ω_1 and ω_2 are taken in the similarity reduction defined by Eqs. (5) and (8):

$$k = (12)^{-\frac{1}{2}}, \tag{17a}$$

$$\Omega = 0, \tag{17b}$$

$$\omega_1 = \omega_0, \tag{17c}$$

$$\omega_2 = k\delta + \omega_0. \tag{17d}$$

With this choice of the four parameters, and defining two arbitrary constants Ω_0 and Ω_1 as follows:

$$\Omega_0 \equiv \frac{z_0}{k} \tag{18a}$$

$$\Omega_1 \equiv \frac{\omega_0}{k} \tag{18b}$$

the similarity reduction defined by (5) and (8) reduces to:

$$u(x,t) = p(z) - \Omega_1, \tag{19a}$$

$$v(x,t) = q(z) - \delta - \Omega_1, \tag{19b}$$

$$z(x,t) = 12^{-1/2}(x - \Omega_1 t + \Omega_0), \tag{19c}$$

$$p'' - 6p^2 + 12\lambda q = A_1, \tag{19d}$$

$$q'' + \frac{6}{\alpha}q^2 - \frac{12\beta\lambda}{\alpha}p = A_2, \tag{19e}$$

where Ω_0, Ω_1, A_1 and A_2 are arbitrary constants. If we now consider the particular case in which:

$$A_1 = -6\alpha^2\lambda^2 - 12\frac{\beta\lambda^2}{\alpha}, \tag{20a}$$

$$A_2 = 6\frac{\beta^2\lambda^2}{\alpha^3} + 12\beta\lambda^2, \tag{20b}$$

we can solve the coupled equations (19d) and (19e), and the following similarity solution of system (1) is obtained:

$$u(x,t) = \frac{12}{(x - \Omega_1 t + \Omega_0)^2} - \Omega_1 - \alpha\lambda, \tag{21a}$$

$$v(x,t) = -\frac{12\alpha}{(x - \Omega_1 t + \Omega_0)^2} - \Omega_1 - \delta - \frac{\beta\lambda}{\alpha}. \tag{21b}$$

If we now consider the similarity reduction defined by Eqs. (12), (13) and (14), the following particular solution can be obtained:

$$u(x,t) = \frac{\pm c_1 - x + \lambda t + \lambda(t_v - t_u) ln|t - t_v|}{t - t_u}, \tag{22a}$$

$$v(x,t) = \frac{\pm c_2 - x + (\beta\lambda - \delta)t + \beta\lambda(t_u - t_v) ln|t - t_u|}{t - t_v}, \tag{22b}$$

where t_u, t_v, c_1 and c_2 are arbitrary constants (c_1 and c_2 are positive constants). In Eq. (22a), the plus sign in front of c_1 corresponds to $t > t_u$, and the minus sign to $t < t_u$. Similarly, in Eq. (22b), the plus sign in front of c_2 corresponds to $t > t_v$, and the minus sign to $t < t_v$. This solution contains, as a particular case, the simpler solution:

$$u(x,t) = \frac{\pm c_1 - x + \lambda t}{t - t_0}, \tag{23a}$$

$$v(x,t) = \frac{\pm c_2 - x + (\beta\lambda - \delta)t}{t - t_0}, \tag{23b}$$

where c_1, c_2 and t_0 are arbitrary constants (c_1 and c_2 are positive), and the plus sign in front of c_1 and c_2 corresponds to $t > t_0$, and the minus sign to $t < t_0$.

In a similar way, from the similarity reduction defined by Eqs. (12), (13) and (15), the following particular solution is obtained:

$$u(x,t) = \frac{\pm c_1 - x}{t - t_u}, \tag{24a}$$

$$v(x,t) = \beta\lambda ln(\omega|t - t_u|), \tag{24b}$$

where $t_u, \omega > 0$, and $c_1 > 0$ are arbitrary constants, and the plus sign in front of c_1 corresponds to $t > t_u$, and the minus sign to $t < t_u$.

Finally, from the similarity reduction defined by Eqs. (12), (13) and (16), the following particular solution is obtained:

$$u(x,t) = \lambda ln(\omega|t - t_v|), \tag{25a}$$

$$v(x,t) = \frac{\pm c_2 - x - \delta t}{t - t_v}, \tag{25b}$$

where $t_v, \omega > 0$, and $c_2 > 0$ are arbitrary constants, and the plus sign in front of c_2 corresponds to $t > t_v$, and the minus sign to $t < t_v$.

The existence of logarithmic solutions such as (22), (24) and (25) evidences that the system (1) does not possess the Painlevé property for PDEs, as defined by Weiss, Tabor and Carnevale (WTC) [20]. The proof of this fact, using the WTC procedure, is given in Appendix B.

4. *Summary and final remarks*

In this paper we have obtained four similarity reductions for the system (1), and from these reductions five particular solutions were determined. The most important feature of the first of these reductions (defined by Eqs. (5) and (8)), is that it allows finding the travelling wave solutions for the system (1). On the other hand, a distinctive feature of the remaining three reductions (defined by Eqs. (12) – (16)), is that they lead to logarithmic solutions of system (1), thus evidencing that this system does not possess the Painlevé property for PDEs, as defined by WTC. It is worth noticing that only one of the particular solutions here determined (that given by Eq. (23)) could have been found by means of the classical Lie's method of infinitesimal transformations; the remaining four solutions are not obtainable by Lie's method (further details on Lie's procedure are given in Appendix A). As a final remark, we would like to observe that both, the similarity reductions and the particular solutions here determined, could be useful in the development and improvement of numerical schemes devised to obtain numerical solutions of the system (1).

Appendix A: Similarity reductions by Lie's method

To obtain similarity reductions of the system (1) by means of Lie's method of infinitesimal transformations we have to consider a one-parameter Lie group of transformations (OPLGT), acting on $(x, t, u, v)-$space, of the form:

$$x' = X(x,t,u,v;\epsilon) = x + \epsilon\xi(x,t,u,v) + O(\epsilon^2), \tag{A1a}$$

$$t' = T(x,t,u,v;\epsilon) = t + \epsilon\tau(x,t,u,v) + O(\epsilon^2), \tag{A1b}$$

$$u' = U(x,t,u,v;\epsilon) = u + \epsilon\eta(x,t,u,v) + O(\epsilon^2), \tag{A1c}$$

$$v' = V(x,t,u,v;\epsilon) = v + \epsilon\mu(x,t,u,v) + O(\epsilon^2). \tag{A1d}$$

Then we extend this OPLGT to another OPLGT, acting on $(x,t,u,v,\partial^1 u,\partial^1 v,\partial^2 u,$ $\partial^2 v,\partial^3 u,\partial^3 v)-$space, where $\partial^k u$ and $\partial^k v$ denote all kth order partial derivatives of u and v, respectively, defined by Eqs. (A1), and additional equations which define how the partial derivatives of u and v are transformed. These additional equations have the following forms:

$$(u_x)' = u_x + \epsilon\eta^x(x,t,u,v,\partial^1 u,\partial^1 v) + O(\epsilon^2), \tag{A2a}$$

$$(u_t)' = u_t + \epsilon\eta^t(x,t,u,v,\partial u^1,\partial v^1) + O(\epsilon^2), \tag{A2b}$$

$$(u_{xxx})' = u_{xxx} + \epsilon\eta^{xxx}(x,t,u,v,\partial u^1,\partial v^1,\partial u^2,\partial v^2,\partial u^3,\partial v^3) + O(\epsilon^2). \tag{A2c}$$

and similar expressions for $(v_x)', (v_t)'$ and $(v_{xxx})'$, which are obtained from Eqs. (A2) by replacing (u,v,η) by (v,u,μ). The forms of the extended infinitesimals η^x,η^t,η^{xxx}, μ^x,μ^t and μ^{xxx} can be determined by the procedure explained in Ref. 21.

In order to find similarity reductions for the system (1), we have to determine appropriate infinitesimals ξ,τ,η and μ, in such a way that the extended transformations (A1)–(A2) leave the surfaces in $(x,t,u,v,\partial^1 u,\partial^1 v$, $\partial^2 u,\partial^2 v,\partial^3 u,\partial^3 v)-$space defined by Eqs. (1), invariant. These two surfaces can be written in the form:

$$F^u(u,u_x,v_x,u_t,u_{xxx}) = u_t - uu_x + u_{xxx} + \lambda v_x = 0, \tag{A3a}$$

$$F^v(v,u_x,v_x,v_t,v_{xxx}) = v_t - \delta v_x - vv_x - \alpha v_{xxx} + \beta\lambda u_x = 0, \tag{A3b}$$

and therefore, for (A1) – (A2) to leave the surfaces (A3) invariant, it is necessary that:

$$X^{(3)}F^u = -\eta u_x - \eta^x u + \lambda\mu^x + \eta^t + \eta^{xxx} = 0, \tag{A4a}$$

$$X^{(3)}F^v = -\mu v_x + \beta\lambda\eta^x - \mu^x(\delta + v) + \mu^t - \alpha\mu^{xxx} = 0, \tag{A4b}$$

when Eqs. (A3) are satisfied, and where $X^{(3)}$ is the infinitesimal generator:

$$X^{(3)} = \eta\frac{\partial}{\partial u} + \mu\frac{\partial}{\partial v} + \eta^x\frac{\partial}{\partial u_x} + \mu^x\frac{\partial}{\partial v_x} + \eta^t\frac{\partial}{\partial u_t} + \mu^t\frac{\partial}{\partial v_t} +$$

$$+\eta^{xxx}\frac{\partial}{\partial u_{xxx}} + \mu^{xxx}\frac{\partial}{\partial v_{xxx}}. \tag{A5}$$

Observe that the conditions (A4) will hold when Eqs. (A3) are satisfied. Consequently, in order to write down the conditions (A4) in explicit form, it is not enough to substitute into them the expressions of the extended infinitesimals ($\eta^x, \eta^t, \ldots$, etc.), but previously we have to substitute Eqs. (A3) and their differential consequences into the expressions of these extended infinitesimals. Proceeding in this way, Eqs. (A4) transform into lengthy polynomial equations in the variables $u, v, \partial^1 u, \partial^1 v, \partial^2 u, \partial^2 v, \partial^3 u$ and $\partial^3 v$. Setting the coefficients of all the terms appearing in these equations equal to zero, we obtain the so-called determining equations, which is an overdetermined system of equations for the infinitesimals ξ, τ, η and μ. From these equations, it follows that the infinitesimals have the forms:

$$\xi = -\mu_0 t + \xi_0, \tag{A6a}$$

$$\tau = \tau_0, \tag{A6b}$$

$$\eta = \mu_0, \tag{A6c}$$

$$\mu = \mu_0, \tag{A6d}$$

where ξ, τ_0 and μ_0 are arbitrary constants.

Once knowing the infinitesimals, we can determine similarity reductions for the system (1) by considering the invariant surface conditions:

$$\xi(t) u_x + \tau_0 u_t = \mu_0, \tag{A7a}$$

$$\xi(t) v_x + \tau_0 v_t = \mu_0. \tag{A7b}$$

Leaving aside the trivial case $\mu_0 = \tau_0 = 0$, these equations lead to three types of similarity reductions of system (1), which are classified according to the values of μ_0 and τ_0:

- Type 1: $\mu_0 \neq 0,\ \tau_0 \neq 0$,
- Type 2: $\mu_0 = 0, \tau_0 \neq 0$,
- Type 3: $\mu_0 \neq 0, \tau_0 = 0$.

If $\mu_0 \neq 0$ and $\tau_0 \neq 0$ it can be proved that Eqs. (A7) imply that for u and v to be invariant surfaces of the Lie group of transformations (A1), they must be of the form:

$$u = \frac{2}{k} t - \frac{\omega}{k} + p(z), \tag{A8a}$$

$$v = \frac{2}{k} t - \frac{\omega}{k} + q(z), \tag{A8b}$$

where we have defined:

$$z(x,t) = kx + t^2 - \omega t + \frac{\omega^2}{4}, \tag{A9a}$$

$$k = 2\tau_0/\mu_0, \tag{A9b}$$

$$\omega = 2\xi_0/\mu_0, \tag{A9c}$$

and $p(z)$ and $q(z)$ are arbitrary functions. If we now substitute (A8) into (1), we conclude that u and v are invariant solutions of system (1) if $p(z)$ and $q(z)$ satisfy the following system of ordinary equations:

$$p''' - \frac{1}{k^2}pp' + \frac{\lambda}{k^2}q' + \frac{2}{k^4} = 0, \tag{A10a}$$

$$q''' + \frac{1}{\alpha k^2}qq' + \frac{\delta}{\alpha k^2}q' - \frac{\beta\lambda}{\alpha k^2}p' - \frac{2}{\alpha k^4} = 0, \tag{A10b}$$

where the primes indicate z derivatives. Equations (A8) – (A10) thus constitute the first type of similarity reduction of the system (1). We can see that this reduction is just a particular case of the similarity reduction defined by Eqs. (5) and (8), which was obtained in Section 2.

If we now consider that $\mu_0 = 0$ but $\tau_0 \neq 0$ (type 2), Eqs. (A7) lead to the conclusion that for u and v to be invariant surfaces of the Lie group (A1), they must be of the form:

$$u = p(z), \tag{A11a}$$

and

$$v = q(z), \tag{A11b}$$

where $z(x,t)$ is now defined as follows:

$$z(x,t) = \tau_0 x - \xi_0 t. \tag{A12}$$

Substituting Eqs. (A11) and (A12) into (1), we obtain that u and v are invariant solutions of the system (1) if p and q satisfy the following system of ordinary equations:

$$\tau_0^2 p'' - \frac{1}{2}p^2 - \frac{\xi_0}{\tau_0}p = -\lambda q \tag{A13a}$$

and

$$\alpha\tau_0^2 q'' + \frac{1}{2}q^2 + (\delta + \xi_0/\tau_0)q = \beta\lambda p. \tag{A13b}$$

Therefore, Eqs. (A11) – (A13) constitute a second type of similarity reduction for system (1). We can see that this reduction is also a particular case of the similarity reduction defined by Eqs. (5) and (8).

Finally, in the third case ($\mu_0 \neq 0, \tau_0 = 0$), the equations (A7) imply that for u and v to be invariant surfaces of the Lie group (A1) they must be of the form:

$$u = \frac{x}{\omega/2 - t} + p(t) \tag{A14a}$$

and

$$v = \frac{x}{\omega/2 - t} + q(t), \tag{A14b}$$

where ω is the constant defined by Eq. (A9c), and $p(t)$ and $q(t)$ are arbitrary functions. If we now substitute these expressions into (1), we find that u and v are invariant solutions of the system (1) if $p(t)$ and $q(t)$ satisfy the ordinary equations:

$$p' + \left(\frac{1}{t - \omega/2}\right)p = \frac{\lambda}{t - \omega/2} \tag{A15a}$$

and

$$q' + \left(\frac{1}{t - \omega/2}\right)q = \frac{\beta\lambda - \delta}{t - \omega/2}. \tag{A15b}$$

Equations (A14) – (A15) thus constitute a third type of similarity reduction for the system (1). We can see that this reduction is a particular case of the similarity reduction defined by Eqs. (12) – (14), which we found in Section 2. Solving Eq. (A15), we can obtain the particular solution (23) found in Section 3.

Appendix B: Painlevé analysis of the system (1)

In this appendix we prove that the system (1) does not possess the Painlevé property for PDEs, as defined by Weiss, Tabor and Carnevale [20]. To do so, we have to prove that the general solution of (1) cannot be expressed in the form:

$$u = \phi^{\alpha_1} \sum_{j=0}^{\infty} u_j \phi^j, \tag{B1a}$$

$$v = \phi^{\alpha_2} \sum_{j=0}^{\infty} v_j \phi^j, \tag{B1b}$$

where α_1 and α_2 are negative integers, and $\phi(x,t) = 0$ defines the singularity manifold of the system (1). In the following we will use the Kruskal ansatz for the form of the function ϕ, namely:

$$\phi(x,t) = x - \psi(t), \tag{B2}$$

which allows one to write $u_j = u_j(t)$ and $v_j = v_j(t)$ [20,22].

Substituting (B1) into (1), taking into account (B2), and requiring α_1 and α_2 to be negative numbers, lead to the conclusion that $\alpha_1 = \alpha_2 = -2$. Then, collecting all the terms containing ϕ^j in Eq. (1a), and setting the coefficient of the resulting ϕ^j−term equal to zero, we obtain the first recursion relation:

$$u_{j-3,t} - (j-4)\psi_t u_{j-2} - \sum_{k=0}^{j}(j-k-2)u_k u_{j-k}$$

$$+(j^3 - 9j^2 + 26j - 24)u_j + \lambda(j-4)v_{j-2} = 0. \qquad (B3a)$$

Proceeding in a similar way with Eq. (1b) we obtain a second recursion relation:

$$v_{j-3,t} - (j-4)\psi_t v_{j-2} - \delta(j-4)v_{j-2} - \sum_{k=0}^{j}(j-k-2)v_k v_{j-k}$$

$$-\alpha(j^3 - 9j^2 + 26j - 24)v_j + \beta\lambda(j-4)u_{j-2} = 0. \qquad (B3b)$$

These two recursion relations constitute a system of the form:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} u_j \\ v_j \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad (B4)$$

where:

$$A_{11} = j^3 - 9j^2 + 26j - 24 - (j-2)u_0 + 2u_0, \qquad (B5a)$$

$$A_{12} = 0, \qquad (B5b)$$

$$A_{21} = 0, \qquad (B5c)$$

$$A_{22} = -\alpha(j^3 - 9j^2 + 26j - 24) - (j-2)v_0 + 2v_0. \qquad (B5d)$$

The so-called "resonances" of the recursion relations are the roots of the equation:

$$A_{11}A_{22} - A_{12}A_{21} = 0 \qquad (B6)$$

and, if we take into account that $u_0 = 12$ and $v_0 = -12\alpha$, we find that Eq. (B6) has three double roots:

$$j = -1, 4, 6. \qquad (B7)$$

The fact that these are the only resonances already implies that the system (1) does not possess the Painlevé property, because these three resonances cannot account for the six undetermined functions which the expressions (B1) should contain, if the system (1) had the Painlevé property.

If we now substitute the positive resonances $j = 4, 6$ into the recursion relations we obtain a set of compatibility conditions which have to be satisfied for (B1) to be a valid local representation of a solution of the system (1). For $j = 4$ the recursion relations reduce to identities, but for $j = 6$ the recursion relations are transformed into the following equations:

$$v_4 = -\alpha\lambda u_4, \tag{B8}$$

$$\psi_t = (\lambda - 2)\frac{\beta}{\alpha} - \delta. \tag{B9}$$

Equation (B8) implies that only one of the functions, u_4 or v_4 , can be chosen arbitrarily, not both of them. On the other hand, Eq. (B9) implies that $\psi(t)$ is not arbitrary, but it must be of the form:

$$\psi(t) = \Omega_1 t - \Omega_0, \tag{B10}$$

where Ω_1 is the constant given in (B9), and Ω_0 is arbitrary. There are no conditions on the functions $u_6(t)$ and $v_6(t)$ and, therefore, these two functions are arbitrary.

We have thus found that the singular manifold expansions (B1) represent solutions of the system (1) if the functions $u_j(t)$ and $v_j(t)$ obey the recursion relations (B3), $\psi(t)$ has the form given in (B10), and u_4 (or v_4), u_6 and v_6 are arbitrary functions. The fact that these are the only undetermined functions in the expansions (B1) implies that these expansions do no constitute the general solution of the system (1) because, according to the Cauchy-Kowalevski theorem [23], the general solution of the system (1) should contain six arbitrary functions. In fact, from the results found in Section 3, we already know that there are particular solutions of the system (1) which involve logarithmic terms, and which, therefore, cannot be expressed in the form (B1).

If we set the arbitrary functions u_4, u_6 and v_6 equal to zero, the singular manifold expansions (B1) are truncated consistently, and we obtain the solution (21), which we found in Section 3.

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SLIČNOSNO SMANJENJE LINEARNO–VEZANOG KORTEWEG – de VRIESOVOG SUSTAVA

Primjenom Clarkson–Kruskalovog postupka dobiva se sličnosno smanjenje dvaju vezanih Korteweg – de Vriesovih jednadžbi, koje su nedavno proučavali Grimshaw i Malomed. Određena su racionalna i logaritamska partikularna rješenja tog sustava.