Math. Commun. **28**(2023), 141–152

Sturm's theorems for generalized derivative and generalized Sturm-Liouville problem

Prashant K. Pandey[1], Rajesh K. Pandey[1,*] and Om P. Agrawal[2]

[1] *Department of Mathematical Sciences, Indian Institute of Technology (BHU) Varanasi, Varanasi-221 005, UP India*
[2] *Mechanical Engineering and Energy Processes, Southern Illinois University, Carbondale, IL-62 901, Illinois, USA*

Received May 21, 2022; accepted February 2, 2023

Abstract. The present paper defines the Sturm separation and Sturm comparison theorems for the generalized derivative. The generalized derivative is defined with respect to the weight function and another function. Further, we define the generalized Sturm-Liouville problem (GSLP) and analyze the properties of the GSLP such that the eigenvalues of the GSLP are real, and for distinct eigenvalues, the associated eigenfunctions are orthogonal. Moreover, using a variational approach, we show that the GSLP has infinite eigenvalues.

AMS subject classifications: 34B05, 34B24, 34B30

Keywords: generalized derivative, generalized Sturm-comparison theorem, generalized Sturm-Liouville problem

1. Introduction

This paper deals with a linear differential equation for generalized derivatives associated with some suitable boundary conditions. The generalized derivatives are defined with the help of the weight function and another function [1, 7]. For a classical differential equation, Sturm proved the Sturm separation theorem and the Sturm comparison theorem and also discussed their applications (e.g. application to Bessel's equation) [4, 3]. These two theorems represent the behavior of zeros of the solutions of the differential equation. Further, researchers have developed Sturm's theorem for linear and non-linear differential equations and studied the oscillation criteria [16, 11, 12]. Leighton and Nehari [10] studied the oscillation of solutions to the fourth-order differential equation. Recently, Sturm's theorems have been extended for the conformable fractional derivative [14]. In this paper, we state and prove Sturm's theorem for generalized derivatives defined using the weight function and another function. Firstly, we define and study the zeros of the solutions of the generalized differential equation. Furthermore, we define the Sturm-Liouville problem (SLP) for generalized derivatives. SLPs have been extensively studied by several authors, for a detailed study we refer to [4, 5, 15]. Kong and Zettl [9] discussed the continuity and differentiability of eigenvalues and eigenfunctions. In [2], the authors

*Corresponding author. *Email address:* `prashantkp92@gmail.com` (P. K. Pandey), `rkpandey.mat@iitbhu.ac.in` (R. K. Pandey), `oagrawal@siu.edu` (O. P. Agrawal)

studied the singular dissipative boundary value transmission problem and showed the completeness of eigenfunctions. There are many papers, various methods, and approaches referring to the regular and singular SLP, not only for integer-order but also for fractional-order [8, 13]. Here, we try to develop Sturm's theorems for the GSLP and study the properties of the regular GSLP.

This paper contains Sturm's theorems for generalized derivatives. We show the Sturm separation theorem and the Sturm comparison theorem for generalized derivatives. We also define the SLP for generalized derivatives and study the properties of the GSLP such as the eigenvalues of the GSLP, which are real, and for distinct eigenvalues their respective eigenfunctions are orthogonal. Moreover, similarly to the classical SLP, we apply direct variational methods to the GSLP that contains infinite eigenvalues and corresponding eigenfunctions are unique.

We summarize this paper as follows: generalized derivatives and their properties are provided in Section 2; Section 3 and Section 4 cover the main results of the paper: Sturm's theorem for a generalized derivative (Theorem 1 and Theorem 2), the GSLP and their eigenvalue and eigenfunction properties (theorems 3-6), and the existence of infinite eigenvalues (Theorem 7).

2. Preliminaries

This section covers the definitions and properties of generalized derivatives. Let $\psi(x) > 0$ and monotonic increasing function on $(a, b]$, and $\psi'(x)(\neq 0)$ continuous on (a, b). Moreover, $w(x) \neq 0$ is the weight function.

Definition 1 (see [1])**.** *The left-weighted generalized derivative of a function $\phi(x)$ with respect to another function $\psi(x)$ $(\psi'(x) \neq 0)$ and with weight $w(x)$ is defined as*

$$^{L}D_{[\psi,w]}\phi(x) = [w(x)]^{-1}\left[\left(\frac{1}{\psi'(x)}D_x\right)(w(x)\phi(x))\right]. \tag{1}$$

The right-weighted generalized derivative of a function $\phi(x)$ with respect to another function $\psi(x)$ $(\psi'(x) \neq 0)$ and with weight $w(x)$ is defined as

$$^{R}D_{[\psi,w]}\phi(x) = [w(x)]\left[\left(\frac{-1}{\psi'(x)}D_x\right)([w(x)]^{-1}\phi(x))\right].$$

Here, $(D_x = \frac{d}{dx})$, $^{L}D_{[\psi,w]}$ and $^{R}D_{[\psi,w]}$ are derivative operators D and $-D$, respectively. But, these derivatives contain another function $\psi(x)$ and weight $w(x)$, therefore we call them generalized (integer-order) derivatives.

Property 1. *The following formulas are integration by parts for generalized derivatives:*

$$\int_a^b \psi'(x)\phi(x)\,^{L}D_{[\psi,w]}\xi(x)dx = \int_a^b \psi'(x)\xi(x)\,^{R}D_{[\psi,w]}\phi(x)dx + \phi(x)\xi(x)\Big|_a^b,$$
$$\int_a^b \psi'(x)\phi(x)\,^{R}D_{[\psi,w]}\xi(x)dx = \int_a^b \psi'(x)\xi(x)\,^{L}D_{[\psi,w]}\phi(x)dx - \phi(x)\xi(x)\Big|_a^b. \tag{2}$$

3. Conversion of the second-order generalized differential into Sturm-Liouville form

Any second-order generalized linear differential equation can be converted into Sturm-Liouville form. Here, we consider the left generalized differential equation (similarly, we can consider the right generalized derivative) of the form:

$$a_0(x)\,{}^L D_{[\psi,w]}\left({}^L D_{[\psi,w]}\phi(x)\right) + a_1(x)\,{}^L D_{[\psi,w]}\phi(x) + a_2(x)\phi(x) = 0. \tag{3}$$

Dividing both sides of Eq. (3) by $a_0(x)$, we get

$${}^L D_{[\psi,w]}\left({}^L D_{[\psi,w]}\phi(x)\right) + \frac{a_1(x)}{a_0(x)}\,{}^L D_{[\psi,w]}\phi(x) + \frac{a_2(x)}{a_0(x)}\phi(x) = 0. \tag{4}$$

Now, we multiply by $\mu(x)$ in Eq. (4):

$$\mu(x)\,{}^L D_{[\psi,w]}\left({}^L D_{[\psi,w]}\phi(x)\right) + \mu(x)\frac{a_1(x)}{a_0(x)}\,{}^L D_{[\psi,w]}\phi(x) + \mu(x)\frac{a_2(x)}{a_0(x)}\phi(x) = 0.$$

Using definition (1) and rearranging the terms, we obtain

$$\mu(x)D_x\left(\frac{1}{\psi'(x)}D_x(\Phi(x))\right) + \mu(x)\frac{a_1(x)}{a_0(x)}D_x(\Phi(x)) + \mu(x)\frac{a_2(x)}{a_0(x)}\psi'(x)\Phi(x) = 0, \tag{5}$$

where $\Phi(x) = w(x)\phi(x)$. Again, simplifying Eq. (5)

$$\begin{aligned}\mu(x)D_x\big(D_x(\Phi(x))\big) + \psi'(x)\left(\mu(x)\frac{a_1(x)}{a_0(x)} - \mu(x)\frac{\psi''(x)}{(\psi'(x))^2}\right)D_x(\Phi(x))& \\ + \mu(x)\frac{a_2(x)}{a_0(x)}(\psi'(x))^2\,\Phi(x) = 0.&\end{aligned} \tag{6}$$

The first two terms can be combined into an exact derivative $D_x\big(\mu(x)D_x(\Phi(x))\big)$ if

$$D_x\big(\mu(x)\big) = \psi'(x)\left(\mu(x)\frac{a_1(x)}{a_0(x)} - \mu(x)\frac{\psi''(x)}{(\psi'(x))^2}\right),$$

and it gives

$$\mu(x) = \frac{1}{\psi'(x)}e^{\int \psi'(x)\frac{a_1(x)}{a_0(x)}\,dx}.$$

Therefore, for this $\mu(x) = \frac{1}{\psi'(x)}e^{\int \psi'(x)\frac{a_1(x)}{a_0(x)}\,dx}$ (which is also called the integrating factor), we rewrite Eq. (6) as

$$D_x\big(\mu(x)D_x(\Phi(x))\big) + \mu(x)\frac{a_2(x)}{a_0(x)}(\psi'(x))^2\,\Phi(x) = 0. \tag{7}$$

Now, Eq. (7) can be put into the form:

$${}^L D_{[\psi,w]}\left(p(x)\,{}^L D_{[\psi,w]}\phi(x)\right) + q(x)\phi(x) = 0, \tag{8}$$

where, $p(x) = \psi'(x)\mu(x) \equiv e^{\int \psi'(x)\frac{a_1(x)}{a_0(x)}\,dx}$, $\quad q(x) = p(x)\frac{a_2(x)}{a_0(x)}$.

This Eq. (8) is in the form of the generalized Sturm-Liouville operator for the left derivative.

4. Sturm's theorems

The next two theorems, which are Sturm's theorems for the generalized derivative, tell us about the zeros of the solutions of the following equation in form of the left generalized derivative1.

$$ {}^{L}D_{[\psi,w]}\left(p(x)\,{}^{L}D_{[\psi,w]}\phi(x)\right)+q(x)\phi(x)=0, \tag{9}$$

where $p(x)>0$ and $p(x),p'(x)$ and $q(x)$ are continuous on (a,b).

Remark 1. *Similarly, we can consider the right generalized derivative in place of the left generalized derivative in Eq.* (9) *and study Sturm's theorems.*

Theorem 1 (Sturm separation theorem)**.** *Let* $\phi_1(x)$ *and* $\phi_2(x)$ *be linearly independent solutions of the equation*

$$ {}^{L}D_{[\psi,w]}\left(p(x)\,{}^{L}D_{[\psi,w]}\phi(x)\right)+q(x)\phi(x)=0 \tag{10}$$

on $[a,b]$*. Assume that* $p(x)>0$ *on* $[a,b]$*; if* b_1 *and* b_2 *are consecutive zeros of* $\phi_1(x)$*, then* $\phi_2(x)$ *has exactly one zero in* (b_1,b_2)*.*

Proof. Using definition (1), Eq. (10) can be converted into

$$[w(x)]^{-1}\left[\frac{1}{\psi'(x)}D_x\left(\frac{p(x)}{\psi'(x)}D_x(w(x)\phi(x))\right)\right]+q(x)\phi(x)=0.$$

Rearranging the above equation, we have

$$D_x\left(\frac{p(x)}{\psi'(x)}D_x(w(x)\phi(x))\right)+q(x)\psi'(x)w(x)\phi(x)=0. \tag{11}$$

Taking $\Phi(x)=w(x)\phi(x), P(x)=\frac{p(x)}{\psi'(x)}$ and $Q(x)=q(x)\psi'(x)$, Eq. (11) is transformed into

$$D_x\big(P(x)D_x(\Phi(x))\big)+Q(x)\Phi(x)=0. \tag{12}$$

This Eq. (12) is a well-known second-order (classical) differential equation. Furthermore, $\Phi(x)$ satisfies all the conditions of $\phi(x)$, i.e. if b_1 and b_2 are consecutive zeros of $\phi_1(x)$, which means $\phi_1(b_1)=0$ and $\phi_1(b_2)=0$. Therefore, $\Phi_1(b_1)=\Phi_1(b_2)=0$. Now, following [3, 15], we get our proof. □

Theorem 2 (Sturm comparison theorem)**.** *Consider the differential equations:*

$$ {}^{L}D_{[\psi,w]}\left(p(x)\,{}^{L}D_{[\psi,w]}\phi(x)\right)+q_1(x)\phi(x)=0, \tag{13}$$

$$ {}^{L}D_{[\psi,w]}\left(p(x)\,{}^{L}D_{[\psi,w]}\xi(x)\right)+q_2(x)\xi(x)=0, \tag{14}$$

where q_1 *and* q_2 *are continuous and* $q_1(x)\le q_2(x)$*. Assume that* $p(x)>0$ *on* $[a,b]$*, and* $\phi(x)$ *and* $\xi(x)$ *are solutions of* (13) *and* (14)*, respectively, on an interval* $[a,b]$*. Then between any two consecutive zeros* b_1 *and* b_2 *of* $\phi(x)$*, there exists at least one zero of* $\xi(x)$*, unless* $q_1(x)\equiv q_2(x)$ *on* (b_1,b_2)*.*

Proof. Using definition (1) in Eq. (13) and Eq. (14) and rearranging them, we get the transformed equations

$$D_x\big(P(x)D_x(\Phi(x))\big) + Q_1(x)\Phi(x) = 0, \tag{15}$$

$$D_x\big(P(x)D_x(\Psi(x))\big) + Q_2(x)\Psi(x) = 0, \tag{16}$$

where $\Phi(x) = w(x)\phi(x), \Psi(x) = w(x)\xi(x), P(x) = \frac{p(x)}{\psi'(x)}, Q_1(x) = q_1(x)\psi'(x)$ and $Q_2(x) = q_2(x)\psi'(x)$. Since eqs. (15) and (16) are in the form of classical differential equations, then for the remaining proof, we follow [3, 15]. □

4.1. Few examples

Consider a particular case of Eq. (9) for $p(x) = 1$ and $q(x) = 1$,

$${}^{L}D_{[\psi,w]}\left({}^{L}D_{[\psi,w]}\phi(x)\right) + \phi(x) = 0. \tag{17}$$

This equation gives us lots of choices to select the weight $w(x)$ and another function $\psi(x)$. For different $w(x)$ and $\psi(x)$ we get different equations. Here, we choose a few $w(x)$ and $\psi(x)$ and analyze the behavior of zeros.

Example 1. *Let $\psi(x) = x$ and $w(x) = e^x$, and substituting in Eq. (17) we get*

$$D_x\big(D_x(e^x\phi(x))\big) + e^x\phi(x) = 0. \tag{18}$$

We observe that, $\phi_1 = e^{-x}sin(x)$ and $\phi_2 = e^{-x}cos(x)$ are two linearly independent solutions of Eq. (18). Then from Sturm's separation theorem, if b_1 and b_2 are two zeros of ϕ_1, ϕ_2 has exactly one zero between zeros of ϕ_1, see Figure 1.

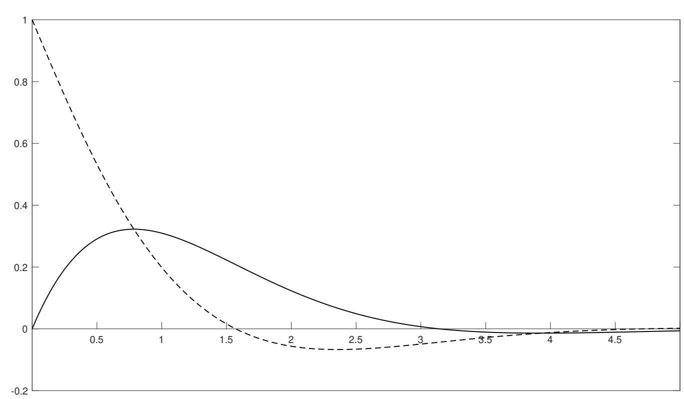

Figure 1: *Graph of $\phi_1 = e^{-x}sin(x)$ (solid) and $\phi_2 = e^{-x}cos(x)$ (dashed)*

Example 2. *Let $\psi(x) = e^x$ and $w(x) = e^x$, substituting in Eq. (18) and after simplification we have*

$$e^{-x}D_x\big(D_x(e^x\phi(x))\big) - e^{-x}D_x(e^x\phi(x)) + e^x\phi(x) = 0. \tag{19}$$

Note that $\phi_1 = e^{-x} sin(e^x)$ *and* $\phi_2 = e^{-x} cos(e^x)$ *are two linearly independent solutions of Eq. (19). Using Sturm's separation theorem, if* b_1 *and* b_2 *are two zeros of* ϕ_1, ϕ_2 *has a zero between zeros of* ϕ_1, *see Figure 2.*

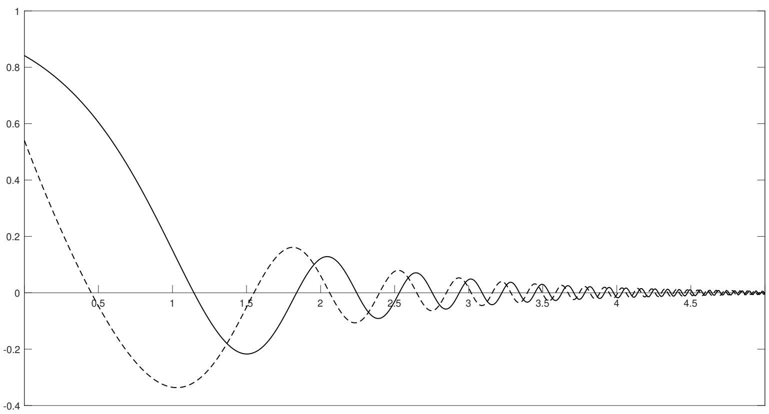

Figure 2: *Graph of* $\phi_1 = e^{-x} sin(e^x)$ *(solid) and* $\phi_2 = e^{-x} cos(e^x)$ *(dashed)*

Example 3. *Let* $\psi(x) = \frac{x^3}{3} + x$ *and* $w(x) = \frac{1}{x^2+1}$; *then from Eq. (17) we have*

$$D_x \left(\frac{1}{x^2+1} D_x \left(\frac{\phi(x)}{x^2+1} \right) \right) + \phi(x) = 0. \tag{20}$$

Since $\phi_1 = (x^2+1) sin(\frac{x^3}{3}+x)$ *and* $\phi_2 = (x^2+1) cos(\frac{x^3}{3}+x)$ *are linearly independent and solutions of Eq. (20), from Sturm's separation theorem, for any two successive zeros of* ϕ_1, *there is exactly one zero of* ϕ_2, *see Figure 3.*

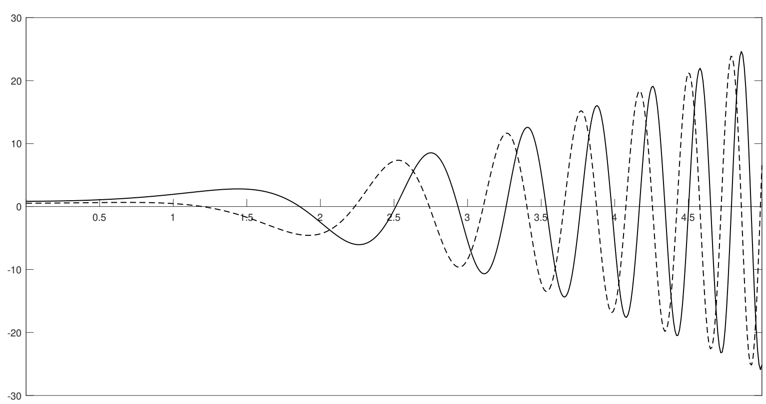

Figure 3: *Graph of* $\phi_1 = (x^2+1) sin(\frac{x^3}{3}+x)$ *(solid) and* $\phi_2 = (x^2+1) cos(\frac{x^3}{3}+x)$ *(dashed)*

Example 4 (Application to Bessel's differential equation)**.** *Consider Bessel's differential equation of order β in terms of the generalized derivative*

$$x^2\, {}^L D_{[\psi,w]} \left({}^L D_{[\psi,w]}\phi(x)\right) + x\, {}^L D_{[\psi,w]}\phi(x) + (x^2-\beta^2)\phi(x) = 0, \quad 0 < x < \infty. \tag{21}$$

This Eq. (21) is a particular case of Eq. (3). Now, using definition (1) and rearranging the terms, we get

$$x^2 D_x \left(\frac{1}{\psi'(x)} D_x\Phi(x)\right) + xD_x\Phi(x) + (x^2-\beta^2)\psi'(x)\Phi(x) = 0,$$

where $\Phi(x) = w(x)\phi(x)$. If we take $\psi(x) = x$, then the above equation becomes

$$x^2 D_x \left(D_x\Phi(x)\right) + xD_x\Phi(x) + (x^2-\beta^2)\Phi(x) = 0. \tag{22}$$

Since $w(x) \neq 0$, then the zeros in Eq. (22) and the zeros in classical Bessel's equation [3] are the same. Therefore, following [3] and using the Sturm comparison theorem, we conclude that:

Case 1: *If $0 \leq \beta < 1/2$, then each solution of Bessel's equation in every interval $\subset (0,\infty)$ of length π has at least one zero.*

Case 2: *If $\beta > 1/2$, then every solution of Bessel's equation can have at most one zero in every interval $\subset (0,\infty)$ of length π.*

Case 3: *If $\beta = 1/2$, the distance between the consecutive zeros of each non-trivial solution of Bessel's equation is π.*

5. Regular generalized Sturm-Liouville problem

5.1. Regular GSLP of kind I

A regular GSLP-I is defined as

$${}^R D_{[\psi,w]} \left(p(x)\, {}^L D_{[\psi,w]}\phi(x)\right) + q(x)\phi(x) = \lambda\phi(x), \tag{23}$$

where $p(x) \neq 0, p'(x), q(x)$, and $\psi'(x)$ are continuous in $[a,b]$. Since $\psi'(x) \neq 0$, we multiply by $\psi'(x)$ in Eq. (23) and define Eq. (23) with the following notation:

$$\mathcal{L}_1 := \psi'(x)\, {}^R D_{[\psi,w]} \left(p(x)\, {}^L D_{[\psi,w]}(\cdot)\right) + \psi'(x)q(x),$$

as

$$\mathcal{L}_1\phi(x) = \lambda\psi'(x)\phi(x), \tag{24}$$

and boundary conditions

$$c_1\phi(a) + c_2\, p(x)\, {}^L D_{[\psi,w]}\phi(x)\big|_{x=a} = 0, \tag{25}$$

$$d_1\phi(b) + d_2\, p(x)\, {}^L D_{[\psi,w]}\phi(x)\big|_{x=b} = 0, \tag{26}$$

where $c_1^2 + c_2^2 \neq 0, d_1^2 + d_2^2 \neq 0$.

Theorem 3. *The eigenvalues of GSLP-I* (24) *subject to the boundary conditions* (25)-(26) *are real.*

Proof. By using integration by part formula (2), we observe that the following relation holds:

$$\begin{aligned}\int_a^b \phi(x)\mathcal{L}_1\xi(x)dx = &\int_a^b q(x)\psi'(x)\phi(x)\xi(x)dx \\ &+ \int_a^b \psi'(x)p(x)\,{}^L D_{[\psi,w]}\xi(x)\,{}^L D_{[\psi,w]}\phi(x)dx \\ &- \phi(x)\left(p(x)\,{}^L D_{[\psi,w]}\xi(x)\right)\Big|_a^b.\end{aligned} \tag{27}$$

Assume that λ is an eigenvalue of problem (24)-(26) corresponding to eigenfunction ϕ. The following equality holds for ϕ and its complex conjugate $\bar{\phi}$:

$$\mathcal{L}_1\phi(x) = \lambda\psi'(x)\phi(x), \tag{28}$$

$$c_1\phi(a) + c_2\, p(x)\,{}^L D_{[\psi,w]}\phi(x)\big|_{x=a} = 0, \tag{29}$$

$$d_1\phi(b) + d_2\, p(x)\,{}^L D_{[\psi,w]}\phi(x)\big|_{x=b} = 0, \tag{30}$$

$$\mathcal{L}_1\bar{\phi}(x) = \bar{\lambda}\psi'(x)\bar{\phi}(x), \tag{31}$$

$$c_1\bar{\phi}(a) + c_2\, p(x)\,{}^L D_{[\psi,w]}\bar{\phi}(x)\big|_{x=a} = 0, \tag{32}$$

$$d_1\bar{\phi}(b) + d_2\, p(x)\,{}^L D_{[\psi,w]}\bar{\phi}(x)\big|_{x=b} = 0, \tag{33}$$

with $c_1^2 + c_2^2 \neq 0$ and $d_1^2 + d_2^2 \neq 0$. Now, we multiply (28) by $\bar{\phi}$ and (31) by ϕ and subtract them as:

$$(\lambda - \bar{\lambda})\psi'(x)\phi(x)\bar{\phi}(x) = \bar{\phi}(x)\mathcal{L}_1\phi(x) - \phi(x)\mathcal{L}_1\bar{\phi}(x). \tag{34}$$

Integrating Eq. (34) over $[a, b]$ and using relation (27), together with boundary conditions (29)-(30) and (32)-(33), we get

$$(\lambda - \bar{\lambda})\int_a^b \psi'(x)|\phi(x)|^2 dx = 0.$$

Since $\phi(x)$ is non-trivial and $\psi'(x) \neq 0$, $\lambda = \bar{\lambda}$. □

Theorem 4. *The eigenfunctions corresponding to different eigenvalues of the GSLP-I are orthogonal with respect to the function* $\psi'(x)$ *on* $[a, b]$ *that is,*

$$\int_a^b \psi'(x)\phi_1(x)\phi_2(x)dx = 0, \qquad \lambda_1 \neq \lambda_2,$$

where the eigenfunctions $\phi_j(x)$ *correspond to eigenvalues* λ_j, *for* $j = 1, 2$.

Proof. Given that, for two distinct eigenvalues, respective eigenfunctions $j = 1, 2$ satisfy GSLP-I

$$\mathcal{L}_1\phi_j(x) = \lambda_j\, \psi'(x)\phi_j(x), \tag{35}$$

$$c_1\phi_j(a) + c_2\, p(x)\, {}^L D_{[\psi,w]}\phi_j(x)\big|_{x=a} = 0, \tag{36}$$

$$d_1\phi_j(b) + d_2\, p(x)\, {}^L D_{[\psi,w]}\phi_j(x)\big|_{x=b} = 0. \tag{37}$$

In Eq. (35), we multiply by $\phi_2(x)$ for $j = 1$ and by $\phi_1(x)$ for $j = 2$, subtract the results and obtain the following relation:

$$(\lambda_1 - \lambda_2)\psi'(x)\phi_1(x)\phi_2(x) = \phi_2(x)\mathcal{L}_1\phi_1(x) - \phi_1(x)\mathcal{L}_1\phi_2(x). \tag{38}$$

Now, we integrate Eq. (38) over $[a, b]$ and using relation (27), we obtain:

$$\begin{aligned}(\lambda_1 - \lambda_2)\int_a^b \psi'(x)\phi_1(x)\phi_2(x)dx = {} & \phi_1(x)\, p(x)\, {}^L D_{[\psi,w]}\phi_2(x)\big|_{x=b} \\ & - \phi_2(x)p(x)\, {}^L D_{[\psi,w]}\phi_1(x)\big|_{x=b} \\ & - \phi_1(x)p(x)\, {}^L D_{[\psi,w]}\phi_2(x)\big|_{x=a} \\ & + \phi_2(x)p(x)\, {}^L D_{[\psi,w]}\phi_1(x)\big|_{x=a}.\end{aligned}$$

From boundary conditions (36)-(37), it follows that

$$(\lambda_1 - \lambda_2)\int_a^b \psi'(x)\phi_1(x)\phi_2(x)dx = 0.$$

Since $\lambda_1 \neq \lambda_2$, we have,

$$\int_a^b \psi'(x)\phi_1(x)\phi_2(x)dx = 0.$$

□

5.2. Regular GSLP of kind II

A regular GSLP-II is defined as

$${}^L D_{[\psi,w]}\left(p(x)\, {}^R D_{[\psi,w]}\phi(x)\right) + q(x)\phi(x) = \lambda\phi(x), \tag{39}$$

where $p(x) \neq 0, p'(x), q(x)$, and $\psi'(x)$ are continuous in $[a, b]$. Since $\psi'(x) \neq 0$, we multiply by $\psi'(x)$ in Eq. (39) and define Eq. (39) with the following notation:

$$\mathcal{L}_2 := \psi'(x)\, {}^L D_{[\psi,w]}\left(p(x)\, {}^R D_{[\psi,w]}(\cdot)\right) + \psi'(x)q(x),$$

as

$$\mathcal{L}_2\phi(x) = \lambda\psi'(x)\phi(x), \tag{40}$$

with boundary conditions

$$c_1\phi(a) + c_2\, p(x)\, {}^R D_{[\psi,w]}\phi(x)\big|_{x=a} = 0, \tag{41}$$

$$d_1\phi(b) + d_2\, p(x)\, {}^R D_{[\psi,w]}\phi(x)\big|_{x=b} = 0, \tag{42}$$

where $c_1^2 + c_2^2 \neq 0, d_1^2 + d_2^2 \neq 0$.

Theorem 5. *The eigenvalues of GSLP-II* (40) *subject to the boundary conditions* (41)-(42) *are real.*

Theorem 6. *The eigenfunctions corresponding to different eigenvalues of the GSLP-II are orthogonal with respect to the function $\psi'(x)$ on $[a,b]$ that is,*

$$\int_a^b \psi'(x)\phi_1(x)\phi_2(x)dx = 0, \qquad \lambda_1 \neq \lambda_2,$$

where eigenfunctions $\phi_j(x)$ correspond to eigenvalues λ_j, for $j = 1, 2$.

5.3. Existence of eigenvalue for the GSLP

The Sturm-Liouville equation in the form of the generalized derivative is defined as

$$-\,{}^L D_{[\psi,w]}\left(p(x)\,{}^L D_{[\psi,w]}\phi(x)\right) + q(x)\phi(x) = \lambda\phi(x), \tag{43}$$

subject to the boundary conditions

$$\phi(a) = \phi(b) = 0, \tag{44}$$

where $p(x) > 0$ and $p(x), p'(x), q(x)$, and $\psi'(x)$ are continuous.

Using definition (1) in Eq. (43), and after simplification, we have

$$-D_x\left(\frac{p(x)}{\psi'(x)}D_x(w(x)\phi(x))\right) + q(x)\psi'(x)w(x)\phi(x) = \lambda\psi'(x)w(x)\phi(x).$$

Substituting $w(x)\phi(x) = \Phi(x)$, $\frac{p(x)}{\psi'(x)} = P(x)$ and $q(x)\psi'(x) = Q(x)$, Eq. (43) is transformed into a standard SLP:

$$-D_x\big(P(x)D_x(\Phi(x))\big) + Q(x)\Phi(x) = \lambda\psi'(x)\Phi(x), \tag{45}$$

and this time the boundary conditions becomes

$$\Phi(a) = \Phi(b) = 0. \tag{46}$$

Theorem 7. *GSLP* (43)-(44) *has infinite eigenvalues, is arranged as $\lambda^{(1)} < \lambda^{(2)} < \dots$, and to each eigenvalue $\lambda^{(n)}$ there corresponds a unique (up to a constant factor) eigenfunction $\phi^{(n)}$.*

Proof. First, we convert GSLP (43)-(44) into a standard SLP of the form (45)-(46). Now, the problem of minimizing the functional for Eq. (45) becomes

$$F[\Phi(x)] = \int_a^b \big(P(x)(D_x(\Phi(x)))^2 + Q(x)(\Phi(x))^2\big)dx, \tag{47}$$

subject to boundary conditions (41) and the subsidiary condition

$$I[\Phi(x)] = \int_a^b \psi'(x)(\Phi(x))^2 dx = 1. \tag{48}$$

Therefore, if $\Phi(x)$ is a solution of the variational problem (47)-(48), then it is also a solution of (45)-(46). We take $a = 0$ and $b = \pi$ for simplification, and using the Ritz method, we choose the following approximate solution of (46)-(48):

$$\Phi_n(x) = \frac{1}{\sqrt{\psi'(x)}} \sum_{k=1}^{n} \alpha_k \, sin(kx).$$

Now, for the complete proof of the theorem, we follow all steps similarly to [6]. □

Now, the generalized form of the Sturm-Liouville equation from Eq. (45) can be written as follows:

$$D_x\big(P(x)D_x(\Phi(x))\big) + \big(\lambda\psi'(x) - Q(x)\big)\Phi(x) = 0. \tag{49}$$

This Eq. (49) is similar to Eq. (12). Thus, we can also analyze the behavior of the zeros of solutions of the generalized Sturm-Liouville equation.

6. Conclusion

We defined and proved the Sturm comparison theorem for generalized derivatives. We showed an application of the Sturm comparison theorem to the generalized Bessel's equation. We also introduced two classes of regular GSLP. Further, we showed that the eigenvalues of the GSLP are real, and corresponding to distinct eigenvalues, the eigenfunctions are orthogonal. Finally, we showed that the GSLP has infinite eigenvalues and corresponding eigenfunctions exist such that the sequence of eigenvalues is increasing.

Acknowledgement

The authors thank the reviewers for their valuable comments incorporated in the manuscript.

References

[1] O. P. Agrawal, *Some generalized fractional calculus operators and their applications in integral equations*, Fract. Calc. Appl. Anal. **15**(2012), 700–711.

[2] B. P. Allahverdiev, E. Bairamov, E. Ugurlu, *Eigenparameter dependent Sturm-Liouville problems in boundary conditions with transmission conditions*, J. Math. Anal. Appl. **401**(2013), 388–396.

[3] M. A. Al-Gwaiz *Sturm-Liouville theory and its applications*, Springer, London, 2008.

[4] W. O. Amrein, A. M. Hinz, D. B. Pearson, *Sturm-Liouville Theory: Past and Present*, Birkhäuser, Basel, 2005.

[5] E. A. Coddington, N. Levinson *Theory of ordinary differential equations*, Tata McGraw-Hill Education, New Delhi, 1955.

[6] I. M. Gelfand, S. V. Fomin, *Calculus of variations*, Dover Publications Inc., New York, 2000.

[7] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.

[8] M. Klimek, O. P. Agrawal, *Fractional Sturm-Liouville problem*, Comput. Math. Appl. **66**(2013), 795–812.

[9] Q. Kong, A. Zettl *Eigenvalues of regular Sturm-Liouville problems*, J. Differential Equations **131**(1996), 1–19.

[10] W. Leighton, Z. Nehari, *On the oscillation of solutions of self-adjoint linear differential equations of the fourth order*, Trans. Amer. Math. Soc. **89**(1958) 325–377.

[11] Z. Nehari, *On the zeros of solutions of second-order linear differential equations*, Amer. J. Math. **76**(1954), 689–697.

[12] Z. Nehari, *Oscillation criteria for second-order linear differential equations*, Trans. Amer. Math. Soc. **85**(1957), 428–445.

[13] P. K. Pandey, R. K. Pandey, O. P. Agrawal, *Variational approximation for fractional Sturm-Liouville problem*, Fract. Calc. Appl. Anal. **23**(2020), 861–874.

[14] M. Pospíšil, L. Pospíšilova Škripková *Sturms theorems for conformable fractional differential equations*, Math. Commun. **21**(2016), 273–281.

[15] A. Zettl, *Sturm-Liouville Theory*, Math. Surveys Monogr. **121**, AMS, Providence, RI, 2010.

[16] R. K. Zhuang, H. W. Wu *Sturm comparison theorem of solution for second order nonlinear differential equations*, Appl. Math. Comput. **162**(2005), 1227–1235.