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On concatenations of Padovan and Perrin numbers

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Abstract. Here we find the Padovan and Perrin numbers that are concatenations of two terms of the other sequence. We also find the intersection between these ternary sequences.
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1. Introduction

Let $(P_n)_{n\geq 0}$ be the Padovan sequence given by the Fibonacci–like recurrence relation

$$P_{n+1} = P_{n-1} + P_{n-2}, \quad \text{for all} \quad n \geq 2,$$

with initial conditions $P_0 = P_1 = P_2 = 1$. The first few terms are

1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, 65, 86, 114, 151, 200, 265, 351, ...

The above definition is the one listed in Sloane's On–Line Encyclopedia of Integer Sequences (OEIS) [24] as A134816. These numbers were named by Stewart [25] after Richard Padovan, although Padovan attributted the sequence to Hans van der Laan (see Padovan [18]). The Perrin sequence $(R_n)_{n\geq 0}$ satisfies the same recurrence relation as the Padovan sequence but with different starting values. The first few terms for $n = 0, 1, \ldots$ are

3, 0, 2, 3, 2, 5, 5, 7, 10, 12, 17, 22, 29, 39, 51, 68, 90, 119, ... (OEIS A001608)

This sequence was discussed by Edouard Lucas [16] in 1878. It was later investigated by Raoul Perrin [19]. For more information on these sequences, see for example Anderson, Horadam, and Shannon [5].

We remind the reader that the concatenation of two or more numbers is the number formed by concatenating their numerals. Given positive integers $D_1, \ldots, D_t$, we write $\overline{D_1 \cdots D_t}$ for the integer whose decimal representation is the concatenation of the decimal representations of $D_1, \ldots, D_t$. In recent years, several authors have studied problems in this subject. For example, concatenations of two repdigit numbers (OEIS A010785) that are Fibonacci numbers (OEIS A000045), generalized Fibonacci numbers, Perrin numbers, Tribonacci numbers (OEIS A000073), generalized

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Lucas numbers, Padovan numbers, Narayana cows numbers (OEIS A000930), generalized Pell numbers, Lucas numbers (OEIS A000032), balancing numbers (OEIS A001109), and associated Pell numbers (OEIS A001333) have been treated separately in [1, 2, 7, 9, 10, 12, 13, 15, 20, 21, 22].

In the case of the concatenation of binary recurrent sequences, there is a general result due to Banks and Luca [6]. They showed that if $(U_n)_{n\geq 0}$ is a binary recurrent sequence of integers satisfying some mild hypothesis, then only finitely many terms U_n can be written as concatenations of two or more terms of the same sequence. In particular, they showed that the only Fibonacci numbers that are non–trivial concatenations of two other Fibonacci numbers are $13, 21, 55$. Recently, Alan [3] found the Fibonacci and Lucas numbers that are concatenations of two terms of the other sequence.

Inspired by these results, we study an analogue of the problem of Alan [3] with Padovan and Perrin numbers instead of Fibonacci and Lucas numbers, respectively; i.e., we determine all solutions of the Diophantine equations

$$\begin{aligned} P_n &= \overline{R_m R_k} \\ &= 10^d R_m + R_k \end{aligned} \tag{1}$$

and

$$\begin{aligned} R_n &= \overline{P_m P_k} \\ &= 10^d P_m + P_k \end{aligned} \tag{2}$$

in non–negative integers n, m, k, where d denotes the number of digits of R_k and P_k, respectively.

More precisely, we prove the following results.

Theorem 1. *The only Padovan numbers which are concatenations of two Perrin numbers are* $2, 3, 5, 7, 12, 37, 351$. *All of the representations of these numbers as concatenations of two Perrin numbers are given below.*

$$\begin{aligned} P_3 = P_4 = 2 &= \overline{R_1 R_2} = \overline{R_1 R_4}; \\ P_5 = 3 &= \overline{R_1 R_0} = \overline{R_1 R_3}; \\ P_7 = 5 &= \overline{R_1 R_5} = \overline{R_1 R_6}; \\ P_8 = 7 &= \overline{R_1 R_7}; \\ P_{10} = 12 &= \overline{R_1 R_9}; \\ P_{14} = 37 &= \overline{R_0 R_7} = \overline{R_3 R_7}; \\ P_{22} = 351 &= \overline{R_0 R_{14}} = \overline{R_3 R_{14}}. \end{aligned}$$

Corollary 1. *The only numbers that are both Padovan and Perrin with non–negative integer indices are* $2, 3, 5, 7, 12$. *More precisely, the only solutions of equation*

$$P_n = R_k \tag{3}$$

in non–negative integers n, k are those listed below.

$$\begin{aligned} P_3 = P_4 = 2 &= R_2 = R_4; \\ P_5 = 3 &= R_0 = R_3; \\ P_7 = 5 &= R_5 = R_6; \\ P_8 = 7 &= R_7; \\ P_{10} = 12 &= R_9. \end{aligned}$$

Theorem 2. *There are only seven Perrin numbers that are concatenations of two Padovan numbers, namely* $12, 17, 22, 29, 39, 51, 486$. *Their representations as concatenations of two Padovan numbers are listed below.*

$$\begin{aligned} R_9 &= 12 = \overline{P_0P_3} = \overline{P_0P_4} = \overline{P_1P_3} = \overline{P_1P_4} = \overline{P_2P_3} = \overline{P_2P_4}; \\ R_{10} &= 17 = \overline{P_0P_8} = \overline{P_1P_8} = \overline{P_2P_8}; \\ R_{11} &= 22 = \overline{P_3P_4} = \overline{P_4P_3} = \overline{P_3P_3} = \overline{P_4P_4}; \\ R_{12} &= 29 = \overline{P_3P_9} = \overline{P_4P_9}; \\ R_{13} &= 39 = \overline{P_5P_9}; \\ R_{14} &= 51 = \overline{P_7P_0} = \overline{P_7P_1} = \overline{P_7P_2}; \\ R_{22} &= 486 = \overline{P_6P_{17}}. \end{aligned}$$

Our argument is based on elementary properties of the Padovan and Perrin sequences together with five applications of linear forms in logarithms that help us to obtain absolute upper bounds for the variables of the Diophantine equations treated here. Since these bounds are large, we use reduction methods such as the Baker–Davenport method and an application of the LLL algorithm to reduce them, and we finish the work by computationally finding all the solutions of our equations in the ranges found.

2. The Padovan and Perrin sequences

We begin by recalling some properties of these ternary recurrence sequences. First, their characteristic polynomial is given by $\Psi(X) = X^3 - X - 1$. Denoting its zeros by ρ, β, γ, with ρ being the only real root, an analytic expression of the kth term of the Padovan and Perrin sequences can be given by

$$P_k = c_\rho \rho^k + c_\beta \beta^k + c_\gamma \gamma^k \quad \text{and} \quad R_k = \rho^k + \beta^k + \gamma^k, \tag{4}$$

respectively, see Vieira, Mangueira, Alves, and Catarino [26, sections 3.7 and 3.8]. Here

$$c_\rho = \frac{7\rho^2 + \rho + 3}{23}, \quad c_\beta = \frac{7\beta^2 + \beta + 3}{23} \quad \text{and} \quad c_\gamma = \overline{c_\beta}.$$

Moreover,

$$\rho = \sqrt[3]{\frac{9+\sqrt{69}}{18}} + \sqrt[3]{\frac{9-\sqrt{69}}{18}}$$

is called the plastic constant, and it is the smallest Pisot number (see Siegel [23]). Numerically,

$$1.32 < \rho < 1.33, \qquad 0.86 < |\beta| = |\gamma| = \rho^{-1/2} < 0.87,$$
$$0.72 < c_\rho < 0.73, \qquad 0.24 < |c_\beta| = |c_\gamma| = (23c_\rho)^{-1/2} < 0.25.$$

By induction, it can be shown that the Perrin sequence can be obtained from the Padovan sequence by

$$R_k = P_{k+1} + P_{k-10}. \tag{5}$$

In addition, it can also be shown by induction that

$$\rho^{k-2} \le P_k \le \rho^{k-1}, \quad \text{for all} \quad k \ge 4, \tag{6}$$

and

$$\rho^{k-2} \le R_k \le \rho^{k+1}, \quad \text{for all} \quad k \ge 2. \tag{7}$$

Let $\mathbb{L} = \mathbb{Q}(\rho, \beta)$ be the splitting field of the polynomial Ψ over $\mathbb{Q}$, which has degree $D = 6$. The Galois group of $\mathbb{L}$ over $\mathbb{Q}$ is given by

$$\mathrm{Gal}\,(\mathbb{L}/\mathbb{Q}) \simeq \{(1)\,, (\rho\beta)\,, (\rho\gamma)\,, (\beta\gamma)\,, (\rho\beta\gamma)\,, (\rho\gamma\beta)\} \simeq S_3.$$

Therefore, we can identify the automorphisms of $\mathrm{Gal}\,(\mathbb{L}/\mathbb{Q})$ with the permutations of the zeros of Ψ. For example, the permutation $(\rho\beta)$ corresponds to the automorphism $\sigma_{\rho\beta} : \rho \to \beta, \beta \to \rho$ ($\sigma_{\rho\beta} : c_\rho \to c_\beta, c_\beta \to c_\rho$).

3. Linear forms in logarithms

Here we give the general lower bound for linear forms in logarithms due to Matveev [17]. Let $\mathbb{K}$ be a number field of degree D over $\mathbb{Q}$, let $\alpha_1, \dots, \alpha_t$ be non–zero elements of $\mathbb{K}$, and let $b_1, \dots, b_t$ be integers. Set

$$\Lambda = \alpha_1^{b_1} \cdots \alpha_t^{b_t} - 1 \quad \text{and} \quad B \ge \max\{|b_1|, \dots, |b_t|\}.$$

Recall that we define the logarithmic height of an algebraic number α as follows. Let α be of degree d over $\mathbb{Q}$ with minimal primitive polynomial $\sum_{0 \le j \le d} a_j X^j$ in $\mathbb{Z}[X]$ with $a_d \ne 0$. Then, the logarithmic height $h(\alpha)$ of α is given by

$$h(\alpha) = \frac{1}{d}\Bigg(\log(|a_d|) + \sum_{j=1}^{d} \max\big\{\log\big|\alpha^{(j)}\big|, 0\big\}\Bigg),$$

where $\alpha = \alpha^{(1)}, \dots, \alpha^{(d)}$ are the conjugates of α. The following are some basic properties of this height that will be used later without reference:

$$h(\alpha) = h\big(\alpha^{(j)}\big), \quad h(\alpha_1 + \alpha_2) \le h(\alpha_1) + h(\alpha_2) + \log 2, \quad h(\alpha_1\alpha_2^{\pm 1}) \le h(\alpha_1) + h(\alpha_2),$$
$$h(\alpha^{\frac{p}{q}}) = \big|\tfrac{p}{q}\big| h(\alpha), \quad h\big(\tfrac{p}{q}\big) = \log\max\{|p|, q\} \quad \Big(\tfrac{p}{q} \in \mathbb{Q},\, q > 0,\, \gcd(p, q) = 1\Big).$$

Let $A_1, \dots, A_t$ be real numbers such that

$$A_j \ge \max\{Dh(\alpha_j), |\log\alpha^{(j)}|, 0.16\}, \quad 1 \le j \le t.$$

With this notation, the main result of Matveev [17] implies the following estimate.

Theorem 3. *Assume that $\Lambda \neq 0$ and $\mathbb{K} \subseteq \mathbb{R}$. We then have*

$$|\Lambda| > \exp\big(-1.4 \cdot 30^{t+3} t^{4.5} D^2 A_1 \cdots A_t (1+\log D)(1+\log B)\big).$$

4. Reduction tools

Next, we remind the Baker–Davenport reduction method from Bravo, Gómez, and Luca [8, Lemma 1], which is an immediate variation of a result due to Dujella and Pethö [14, Lemma 5(a)], which turns out to be useful in order to reduce the bounds arising from applying Theorem 3. For a real number x, we write $\|x\| = |x - \lfloor x \rceil|$, where $\lfloor x \rceil$ is the nearest integer to x.

Lemma 1. *Let κ be an irrational number, let M be a positive integer, and let p/q be a convergent of the continued fraction of κ such that $q > 6M$. Let A, B, μ be real numbers with $A > 0$ and $B > 1$. Set $\epsilon = \|\mu q\| - M\|\kappa q\|$. If $\epsilon > 0$, then there is no solution of the inequality*

$$0 < |s\kappa - r + \mu| < AB^{-w}$$

in positive integers r, s, w, with $s \leq M$ and $w \geq \log(Aq/\epsilon)/\log B$.

One of the most spectacular applications of the LLL algorithm is to linear forms in real numbers (for more details, see Cohen [11, Section 2.3.5]). Let $\lambda_1, \ldots, \lambda_n$ be real numbers, and let $x_1, \ldots, x_n$ be integers. Consider

$$\Gamma = x_1\lambda_1 + \cdots + x_n\lambda_n$$

and fix a (large) positive constant C. If (e_i) is the canonical basis of $\mathbb{R}^n$, for $i \leq n-1$ we set $\mathbf{b}_i = e_i + \lfloor C\lambda_i \rceil e_n$ and $\mathbf{b}_n = \lfloor C\lambda_n \rceil e_n$, and let Ω be the lattice generated by the $\mathbf{b}_i$.

Lemma 2. *Keep the above notation, and consider a reduced basis $\{\mathbf{v}_i\}$ of Ω with $\{\mathbf{v}_i^*\}$ its associated Gram–Schmidt basis. Let $X_1, \ldots, X_n$ be positive integers, set*

$$c_1 = \max_{1 \leq i \leq n} \frac{\|\mathbf{v}_1\|}{\|\mathbf{v}_i^*\|}, \quad d_\Omega = \frac{\|\mathbf{v}_1\|}{c_1}, \quad Q = \sum_{i=1}^{n-1} X_i^2 \quad \textit{and} \quad T = \frac{1}{2}\left(1 + \sum_{i=1}^{n} X_i\right).$$

If $|x_i| \leq X_i$ for all $i = 1, \ldots, n$ and $d_\Omega^2 \geq T^2 + Q$, then we have

$$|\Gamma| \geq \frac{\sqrt{d_\Omega^2 - Q} - T}{C}.$$

5. The proof of Theorem 1

First, by a computational search in the range $0 \leq k, m < 285$ and $1 \leq d \leq 36$ we find that the Padovan numbers that are concatenations of two Perrin numbers are those given in Theorem 1. Therefore, from now on we can assume that $\max\{k, m\} \geq 285$.

Assume that equation (1) holds. Note that we can write the number of digits of R_k as

$$d=\begin{cases}\lfloor \log_{10} R_k\rfloor+1, & \text{if } k\neq 1,\\[2ex] 1, & \text{if } k=1,\end{cases}$$

where $\lfloor x\rfloor$ denotes the greatest integer less than or equal to x. Then using (7) we obtain

$$d\le 1+\log_{10} R_k\le 1+\log_{10}\rho^{k+1}\le 1+(k+1)\log_{10}\rho<1+0.123(k+1),$$

so that

$$d<0.123k+1.123. \tag{8}$$

In addition,

$$R_k=10^{\log_{10} R_k}<10^d\le 10^{1+\log_{10} R_k}\le 10R_k. \tag{9}$$

Combining (9) with (1), (6) and (7) we get

$$\rho^{n-2}\le P_n=10^dR_m+R_k\le 10R_kR_m+R_k\le 10.5R_kR_m<\rho^{k+m+11},$$

and also

$$\rho^{n-1}\ge P_n=10^dR_m+R_k>R_kR_m+R_k\ge R_kR_m\ge\rho^{k+m-4}.$$

Therefore

$$m-3<n-k<m+13 \tag{10}$$

is valid for $k,m\ge 2$ and $n\ge 4$. We now rewrite (1) using (4) as

$$c_\rho\rho^n-10^d\rho^m=-(c_\beta\beta^n+c_\gamma\gamma^n)+(\beta^m+\gamma^m)10^d+R_k.$$

Dividing both sides of the above equation by $c_\rho\rho^n$ and taking absolute values on both of its sides, we obtain that

$$\begin{aligned}\left|10^dc_\rho^{-1}\rho^{m-n}-1\right|&\le\frac{2|c_\beta||\beta|^n}{c_\rho\rho^n}+\frac{2|\beta|^m10^d}{c_\rho\rho^n}+\frac{R_k}{c_\rho\rho^n}\\&<\frac{1}{\rho^{n-k}}\left(\frac{2}{\sqrt{23}c_\rho^{3/2}\rho^{\frac{n}{2}+k}}+\frac{20}{c_\rho\rho^{\frac{m}{2}-1}}+\frac{1}{c_\rho\rho^{-1}}\right)\\&<\frac{1.9}{\rho^{n-k}},\end{aligned}\tag{11}$$

where we used (7) and the facts that $\max\{k,m\}\ge 285$ and $10^d\le 10R_k$ by (9). Now we find a lower bound for the left–hand side of (11). To do this, we use Theorem 3. We put

$$t=3,\quad \alpha_1=10,\quad \alpha_2=c_\rho,\quad \alpha_3=\rho,\quad b_1=d,\quad b_2=-1,\quad b_3=m-n.$$

The number field containing $\alpha_1,\alpha_2,\alpha_3$ is $\mathbb{K}=\mathbb{Q}(\rho)$, which has degree $D=3$. Note that $d\le n-m$. Otherwise, by inequalities (8) and (10), it follows that

$$k-3<n-m<d<0.123k+1.123,$$

which in turn implies that $k \leq 4$. Then $d = 1$ and thus $n = m$. Then by (1) and (5) we get $P_n = 10(P_{n+1} + P_{n-10}) + R_k$, which is not possible. So we can take $B = n - m$. On the other hand, $\Lambda_1 = \alpha_1^{b_1}\alpha_2^{b_2}\alpha_3^{b_3} - 1 = 0$ implies that

$$c_\rho \rho^{n-m} = 10^d.$$

Conjugating this last relation by the automorphism $\sigma_{\rho\beta}$, and then taking absolute values on both sides of the resulting equality, we conclude that

$$0.24\rho^{-(n-m)/2} < 10^d,$$

which is false for any $n - m \geq 1$ and $d \geq 1$. Therefore $\Lambda_1 \neq 0$. Since $h(\alpha_1) = \log 10$, $h(\alpha_2) = (1/3)\log 23$ and $h(\alpha_3) = (1/3)\log\rho$, we take $A_1 = 3\log 10$, $A_2 = \log 23$ and $A_3 = \log\rho$. Theorem 3 applied to $|\Lambda_1|$ tells us that

$$|\Lambda_1| > \exp\big(-1.64715 \times 10^{13}(1 + \log(n-m))\big). \tag{12}$$

Combining inequalities (11) and (12) and then taking logarithms in the resulting inequality we obtain

$$n - k < 5.9 \times 10^{13}(1 + \log(n-m)). \tag{13}$$

We then rewrite (1) with the help of (4) as follows:

$$\rho^n(c_\rho - \rho^{k-n}) - 10^d R_m = -(c_\beta\beta^n + c_\gamma\gamma^n) + \beta^k + \gamma^k.$$

Dividing the above equation by $\rho^n(c_\rho - \rho^{k-n})$ and taking absolute values on both sides, we obtain that

$$\begin{aligned}
\left|10^d \frac{R_m}{c_\rho - \rho^{k-n}}\rho^{-n} - 1\right| &\leq \frac{1}{\rho^n(c_\rho - \rho^{k-n})}\big(|c_\beta||\beta|^n + |c_\gamma||\gamma|^n + |\beta|^k + |\gamma|^k\big)\\
&\leq \frac{1}{\rho^{3k/2}}\left(\frac{2(23c_\rho)^{-1/2}}{\rho^{\frac{n-k}{2}}(c_\rho\rho^{n-k} - 1)} + \frac{2}{c_\rho\rho^{n-k} - 1}\right)\\
&< \frac{3.1 \times 10^{-18}}{\rho^{3k/2}},
\end{aligned} \tag{14}$$

where we have used that $n - k \geq 147$ by (10) and the fact that $\max\{k, m\} \geq 285$. Again, we apply Theorem 3 to the left–hand side of the above inequality (14). For $\Lambda_2 = 10^d R_m(c_\rho - \rho^{k-n})^{-1}\rho^{-n} - 1$, we take

$$t = 3, \quad \alpha_1 = 10, \quad \alpha_2 = \frac{R_m}{c_\rho - \rho^{k-n}}, \quad \alpha_3 = \rho, \quad b_1 = d, \quad b_2 = 1, \quad b_3 = -n.$$

As in the first application of Theorem 3, we have $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{K} = \mathbb{Q}(\rho)$, $D = 3$, $A_1 = 3\log 10$ and $A_3 = \log\rho$. Assume $n < d$. Then $n < 0.123k + 1.123$ by (8) and thus $0.877n + 0.123m < 1.492$ from (10), which is impossible since $\max\{k, m\} \geq 285$. Therefore

$$d \leq n, \tag{15}$$

and we can choose $B = n$. Suppose $\Lambda_2 = \alpha_1^{b_1}\alpha_2^{b_2}\alpha_3^{b_3} - 1 = 0$. Then

$$10^d R_m = c_\rho \rho^n - \rho^k.$$

Conjugating this last relation by the automorphism $\sigma_{\rho\beta}$, and then taking absolute values on both sides of the resulting equality we get

$$10^d R_m < 0.25\rho^{-n/2} + \rho^{-k/2},$$

which is not possible for any $d \geq 1$, $n \geq 0$, and $\max\{k, m\} \geq 285$. Thus $\Lambda_2 \neq 0$. It remains to estimate A_2. To do this, note that

$$\begin{aligned} h(\alpha_2) &\leq h(R_m) + h(c_\rho - \rho^{k-n}) \\ &\leq \log R_m + h(c_\rho) + h(\rho^{k-n}) + \log 2 \\ &\leq (m+1)\log\rho + (1/3)\log 23 + (1/3)(n-k)\log\rho + \log 2 \\ &< ((4(n-k)+12)/3)\log\rho + (1/3)\log 23 + \log 2, \end{aligned} \tag{16}$$

where we have used (10). So we can choose $A_2 = (4(n-k)+12)\log\rho + 5.22$. Theorem 3 together with inequality (14) allows us to conclude that

$$k < 1.25 \times 10^{13}(1+\log n)((4(n-k)+12)\log\rho + 5.22). \tag{17}$$

Suppose that $k \leq m$. By (13) we have then that $n - k < 5.9 \times 10^{13}(1 + \log(n-k))$. This implies that $n - k < 2.14 \times 10^{15}$. That in turn together with (10) leads us to

$$n < 4.28 \times 10^{15}.$$

In the case where $m < k$, we have $(n-13)/2 < k$ by (10). Combining this inequality with inequalities (17) and (13) we get that

$$n < 2.5 \times 10^{13}(1+\log n)((4(5.9 \times 10^{13}(1+\log n)) + 12)\log\rho + 5.22),$$

from which it follows that

$$n < 8.6 \times 10^{30}.$$

Therefore, in any case, $n < 8.6 \times 10^{30}$ holds. We summarize what we have shown so far in the following lemma.

Lemma 3. *All solutions of equation* (1) *with* $\max\{k, m\} \geq 285$ *satisfy*

$$d < 0.123k + 1.123 \quad \textit{and} \quad k + m - 3 < n < 8.6 \times 10^{30}.$$

Let us first reduce the upper bounds of $n - k$ and m. Putting

$$\Gamma_1 := d\log 10 - (n-m)\log\rho - \log c_\rho, \tag{18}$$

inequality (11) can be rewritten as

$$|e^{\Gamma_1} - 1| < \frac{1.9}{\rho^{n-k}}.$$

Note that $|e^{\Gamma_1}-1|<1/2$ for all $n-k\ge 5$ (because $1.9/\rho^{n-k}<1/2$ for all $n-k\ge 5$). If $\Gamma_1>0$, then $0<\Gamma_1\le e^{\Gamma_1}-1<1.9/\rho^{n-k}$. If, on the other hand, $\Gamma_1<0$, then $e^{|\Gamma_1|}<2$, and hence $0<|\Gamma_1|\le e^{|\Gamma_1|}-1=e^{|\Gamma_1|}|e^{\Gamma_1}-1|<3.8/\rho^{n-k}$. Therefore,

$$0<|\Gamma_1|<\frac{3.8}{\rho^{n-k}}.$$

Substituting (18) in the previous inequality and dividing both sides of the resulting inequality by $\log\rho$, we obtain

$$0<|d\kappa-(n-m)+\mu|<AB^{-(n-k)}, \tag{19}$$

where

$$\kappa:=\frac{\log 10}{\log\rho},\quad \mu:=-\frac{\log c_\rho}{\log\rho},\quad A:=13.6,\quad B:=\rho.$$

By the Gelfond–Schneider theorem, κ is an irrational number. Note that $d\le n<8.6\times 10^{30}:=M$ by (15) and Lemma 3. In addition, we have that

$$\frac{p_{71}}{q_{71}}=\frac{2258853669333136386080922972687607}{2758589431534988744983614 80573513}$$

is the first convergent of the continued fraction of κ with denominator $q_{71}>5.16\times 10^{31}=6M$ and $\epsilon>0.367844$. Applying Lemma 1 to inequality (19) we get

$$n-k<\frac{\log(Aq_{71}/\epsilon)}{\log B}<278.477,$$

so $m\le 281$ by Lemma 3.

Now let us reduce the upper bound of k. Let

$$\Gamma_2=d\log 10-n\log\rho+\log(R_m/(c_\rho-\rho^{k-n})).$$

From inequality (14), we have that

$$|e^{\Gamma_2}-1|<\frac{3.1\times 10^{-18}}{\rho^{3k/2}}.$$

Note that the right–hand side above is less than $1/2$ because $\max\{k,m\}\ge 285$. Thus, arguing as before, the above inequality implies

$$0<|\Gamma_2|<\frac{6.2\times 10^{-18}}{\rho^{3k/2}},$$

which leads to

$$0<|d\kappa-n+\mu|<AB^{-3k/2}, \tag{20}$$

where we now apply Lemma 1 with the choices

$$\kappa:=\frac{\log 10}{\log\rho},\quad \mu:=\frac{\log(R_m/(c_\rho-\rho^{k-n}))}{\log\rho},\quad A:=2.20484\times 10^{-17},\quad B:=\rho.$$

As in the first application of Lemma 1, we take $M = 8.6 \times 10^{30}$ by (15) and Lemma 3. This time

$$\frac{p_{77}}{q_{77}} = \frac{10578403127486876833505529092325440 6}{12918708044783169584916763263610615}$$

is the first convergent of the continued fraction of κ such that $q_{77} > 6M$ and $\epsilon > 7.20451 \times 10^{-5}$ for each $m \in [0, 281] \setminus \{1\}$ and $n - k \in [147, m + 13]$. Accordingly,

$$\frac{3k}{2} < \frac{\log(Aq_{77}/\epsilon)}{\log B} < 158.425,$$

so that $k \leq 105$, which contradicts the fact that $\max\{k, m\} \geq 285$. Thus, equation (1) has no solutions for $\max\{k, m\} \geq 285$.

5.1. Special case: Padovan numbers that are Perrin numbers

If $m = 1$, equation (1) becomes equation (3). Using (6) and (7) in equation (3) we obtain

$$\rho^{k-2} \leq R_k = P_n \leq \rho^{n-1},$$

for all $k \geq 2$ and $n \geq 4$. Taking logarithms on both sides of the resulting inequality above, we obtain $k \leq n + 1$. A quick search by inspection reveals that the only solutions of the Diophantine equation (3) for $k \leq n + 1 \leq 43$ are those given in Corollary 1.

From now on, let us assume that $n > 42$. Equation (3) can be expressed as

$$c_\rho \rho^n - \rho^k = \beta^k + \gamma^k - c_\beta \beta^n - c_\gamma \gamma^n$$

using (4). Multiplying by $c_\rho^{-1} \rho^{-n}$ and taking absolute values on both sides of the equation above, we obtain

$$\left| c_\rho^{-1} \rho^{k-n} - 1 \right| < \frac{2.8}{\rho^n}. \tag{21}$$

Now, we apply Theorem 3 choosing

$$t = 2, \quad \alpha_1 = c_\rho, \quad \alpha_2 = \rho, \quad b_1 = -1, \quad b_2 = k - n.$$

Note that $\alpha_1, \alpha_2 \in \mathbb{K} = \mathbb{Q}(\rho)$, and therefore $D = 3$. From the first application of Theorem 3, we have $A_1 = \log 23$ and $A_2 = \log \rho$. We can take $B = n - k$ since $k \leq n + 1$. Suppose now that $\Lambda_3 = \alpha_1^{b_1} \alpha_2^{b_2} - 1 = 0$. Then $k - n = (\log c_\rho)/\log \rho$, which is not possible because the left–hand side is an integer, while the right–hand side is irrational. Therefore $\Lambda_3 \neq 0$. Theorem 3 tells us that

$$\left| c_\rho^{-1} \rho^{k-n} - 1 \right| > \exp\big(-1.28193 \times 10^{10} (1 + \log n)\big). \tag{22}$$

Putting inequalities (21) and (22) together, we obtain the following absolute upper bound for n.

Lemma 4. *If (n, k) is a non–negative integer solution of equation (3), then $k \leq n + 1 < 1.32 \times 10^{12}$.*

To apply Lemma 2 for

$$\Gamma_3 := (n-k)\log\rho + \log c_\rho,$$

let us first note that $0 < \Gamma_3 < e^{\Gamma_3} - 1 = |e^{\Gamma_3} - 1| = |\Lambda_3| < 2.8\rho^{-n} < 1/2$ holds if $n \geq 7$. Thus, (21) can be rewritten as

$$|\Gamma_3| < \frac{5.6}{\rho^n}. \tag{23}$$

Let $X_1 = X_2 := 1.32 \times 10^{12}$ and let $C = 1.3 \times 10^{25}$ be fixed. Therefore, $Q = 1.7424 \times 10^{24}$ and $T = 1.32 \times 10^{12}$. The LLL algorithm uses the lattice Ω generated by the columns of matrix

$$(\mathbf{b}_1, \mathbf{b}_2) = \begin{pmatrix} 1 & 0 \\ \lfloor C\log\rho \rceil & \lfloor C\log c_\rho \rceil \end{pmatrix},$$

and returns the following reduced basis of Ω:

$$(\mathbf{v}_1, \mathbf{v}_2) = \begin{pmatrix} -1305780173941 & 1040857895043 \\ 1611149606912 & 1956894474240 \end{pmatrix}.$$

Then, by the Gram–Schmidt orthogonalization, we get

$$(\mathbf{v}_1^*, \mathbf{v}_2^*) = \begin{pmatrix} -1305780173941 & \frac{681879081652270731455325478445069107 2}{430086491851008033364722 5} \\ & \\ 1611149606912 & \frac{5526390485568753601603697843367837696}{4300864918510080333647225} \end{pmatrix}.$$

Now, we estimate $c_1 = \max\{1, 1.01621\} = 1.01621$ and $d_\Omega = 2.04077 \times 10^{12}$. Since $|n-k| = n-k < n < 1.32 \times 10^{12}$ (by Lemma 4) and $d_\Omega^2 = 4.16473 \times 10^{24} \geq 3.4848 \times 10^{24} = T^2 + Q$, it follows from Lemma 2 that

$$|\Gamma_3| \geq 2.15016 \times 10^{-14}.$$

This previous lower bound for $|\Gamma_3|$ together with the upper bound given in (23) imply that $n \leq 118$. Using this new bound, Lemma 2 (which we do not detail here) can reduce the upper bound of n to 41. This contradicts the assumption that $n > 42$ and ends the proof of Theorem 1.

6. The proof of Theorem 2

This proof largely follows the line of argumentation given in the above proof of Theorem 1. We omit some details. When $0 \leq k, m < 285$ and $1 \leq d \leq 35$, by a computational search we see that the only solutions of (2) are those shown in Theorem 2. From now on, we will assume that $\max\{k, m\} \geq 285$. Assume that equation (2) holds. This time $d = \lfloor \log_{10} P_k \rfloor + 1$ for all k, and using (6) we obtain the following upper bound for d in terms of k:

$$d < 0.123k + 0.877. \tag{24}$$

Moreover, $P_k \leq 10P_k$, and combining this inequality with (2), (6) and (7) we obtain the following bounds for n in terms of k and m:

$$m - 5 < n - k < m + 9 \quad \text{for all} \quad k, m \geq 4 \quad \text{and} \quad n \geq 2. \tag{25}$$

We now use (4) to rewrite equation (2), after which we divide it by ρ^n and take absolute values, thus obtaining that

$$\begin{aligned} \left|10^d c_\rho \rho^{m-n} - 1\right| &\leq \frac{1}{\rho^{n-k}} \left(\frac{2}{\rho^{\frac{n}{2}+k}} + \frac{20}{\sqrt{23 c_\rho}\,\rho^{\frac{m}{2}-1}} + \frac{1}{\rho} \right) \\ &< \frac{0.8}{\rho^{n-k}}, \end{aligned} \tag{26}$$

where we used (6) and the facts that $\max\{k, m\} \geq 285$ and $10^d < 10P_k$. We apply Theorem 3 with the same choices of α_i's and b_i's as when we first applied it, except that now $b_2 = 1$. Again, inequalities (24) and (25) imply that $d \leq n - m$ and the proof that $\Lambda_4 := 10^d c_\rho \rho^{m-n} - 1$ is non–zero is similar to the proof that Λ_1 is non–zero, so the lower bound we obtain for $|\Lambda_4|$ is the same as obtained in (12) for $|\Lambda_1|$. Combining inequalities (12) and (26) we also obtain the same upper bound for $n-k$ given in (13).

We now rewrite (2) using (4) again, after which we divide it by $\rho^n(1 - c_\rho \rho^{k-n})$ and take absolute values, arriving at

$$\begin{aligned} \left|10^d \frac{P_m}{1 - c_\rho \rho^{k-n}} \rho^{-n} - 1\right| &\leq \frac{1}{\rho^{3k/2}} \left(\frac{2}{\rho^{\frac{n-k}{2}}(\rho^{n-k} - c_\rho)} + \frac{2}{\sqrt{23 c_\rho}(\rho^{n-k} - c_\rho)} \right) \\ &< \frac{9.7 \times 10^{-19}}{\rho^{3k/2}}, \end{aligned} \tag{27}$$

where we have used that $n - k \geq 145$ by (25) and the fact that $\max\{k, m\} \geq 285$. Next, we use Theorem 3 with the same choices as we did in its second application, only this time $\alpha_2 = P_m/(1 - c_\rho \rho^{k-n})$. This single change does not affect the choice of $\mathbb{K}$. Moreover, the proof that $\Lambda_5 := 10^d P_m (1 - c_\rho \rho^{k-n})^{-1} \rho^{-n} - 1$ is non–zero follows by a repetition of the argument used in the proof that Λ_2 is non–zero. Using (25) and the properties of the logarithmic height it can be seen that the upper bound of $h(\alpha_2)$ is the same as the one given in (16); therefore, A_2 does not change either. Combining the lower bound for $|\Lambda_5|$ given by Theorem 3 and inequality (27) we get the same upper bound for k in terms of n and k given in (17).

Studying both cases, namely $k \leq m$ and $m < k$, it can be noted that $n < 8.6 \times 10^{30}$ always holds. Summarizing we have proved the following result.

Lemma 5. *If* (d, k, m, n) *is a solution of equation* (2) *with* $\max\{k, m\} \geq 285$*, then*

$$d < 0.123k + 0.877 \quad \textit{and} \quad k + m - 5 < n < 8.6 \times 10^{30}.$$

We start again by reducing the upper bounds of $n - k$ and m. To do so, we rewrite (26) making

$$\Gamma_4 := d \log 10 - (n - m) \log \rho + \log c_\rho.$$

We then reason as we did before when we obtained (19) and arrive at this same inequality with only two changes, namely $\mu = \log c_\rho / \log \rho$ and $A = 5.7$. It can be seen that $d \leq n$ using inequalities (24) and (25) and the fact that $\max\{k, m\} \geq 285$, so we also take $M = 8.6 \times 10^{30}$ due to Lemma 5. As in the first application of Lemma 1 with the 71th convergent of the continued fraction of κ, we get that $q_{71} > 6M$ and $\epsilon > 0.367844$. This time, from Lemma 1 it follows that $n - k \leq 275$, and thus $m \leq 280$ by Lemma 5.

To reduce the upper bound of k, we rewrite (27) making

$$\Gamma_5 := d \log 10 - n \log \rho + \log(P_m/(1 - c_\rho \rho^{k-n})).$$

Now arguing as we did before to obtain (20) we arrive at this same inequality with only two modifications, namely $\mu = \log(P_m/(1-c_\rho\rho^{k-n}))/\log\rho$ and $A = 6.9 \times 10^{-18}$. Again, it suffices to take $M = 8.6 \times 10^{30}$ and p_{77}/q_{77} is the first convergent of the continued fraction of κ such that $q_{77} > 6M$ and $\epsilon > 8.36185 \times 10^{-5}$ for each $m \in [0, 280]$ and $n - k \in [145, m + 9]$. In this application of Lemma 1 we have that $k \leq 110$, which is not possible since $\max\{k, m\} \geq 285$. This completes the proof of Theorem 2.

Remark 1. *Our results together with those of Banks and Luca [6] invite us to ask whether only a finite number of terms of any recurrent ternary sequence can be written as concatenations of the terms of the same sequence with some mild restrictions. On the other hand, Altassan and Alan [4] recently determined Fibonacci numbers which are mixed concatenations of a Fibonacci number and a Lucas number. By a mixed concatenation of two non–negative integers* D_1 *and* D_2 *we mean* $\overline{D_1D_2}$ *and* $\overline{D_2D_1}$. *So it is natural to ask then also about the mixed concatenations of Padovan and Perrin numbers.*

We believe that our method will prove the following statements. We leave the details of these proofs to the interested reader.

Conjecture 1.

1) *The only Padovan numbers which are concatenations of two other Padovan numbers are* $12, 21, 37, 49, 265, 465$.

2) *The only Perrin number that is a non–trivial concatenation of two other Perrin numbers is* 22.

3) $12, 37, 151, 351$ *are the only Padovan numbers that are concatenations of a Padovan number and a Perrin number.*

4) *There are only three Padovan numbers that are non–trivial concatenations of a Perrin number and a Padovan number, and these are* $21, 37, 265$.

5) *All Perrin numbers that are concatenations of a Perrin number and a Padovan number are* $2, 3, 5, 7, 12, 22, 29, 39, 51$.

6) *There are only five Perrin numbers which are concatenations of a Padovan number and a Perrin number, namely* $10, 12, 17, 22, 90$.

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References

[1] A. Alahmadi, A. Altassan, F. Luca, H. Shoaib, *Fibonacci numbers which are concatenations of two repdigits*, Quaest. Math. **44**(2021), 281–290.

[2] A. Alahmadi, A. Altassan, F. Luca, H. Shoaib, *k–generalized Fibonacci numbers which are concatenations of two repdigits*, Glas. Mat. **56**(2021), 29–46.

[3] M. Alan, *On concatenations of Fibonacci and Lucas numbers*, Bull. Iranian Math. Soc. **48**(2022), 2725–2741.

[4] A. Altassan, M. Alan, *On mixed concatenations of Fibonacci and Lucas numbers which are Fibonacci numbers*, arXiv: 2206.13625v1 [math.NT] (2022).

[5] P. G. Anderson, A. F. Horadam, A. G. Shannon, *Properties of Cordonnier, Perrin and Van der Laan numbers*, Internat. J. Math. Ed. Sci. Tech. **37**(2006), 825–831.

[6] W. D. Banks, F. Luca, *Concatenations with binary recurrent sequences*, J. Integer Seq. **8**(2005), Art. 05.1.3.

[7] H. Batte, T. P. Chalebgwa, M. Ddamulira, *Perrin numbers that are concatenations of two repdigits*, Arab. J. Math. **11**(2022), 469–478.

[8] J. J. Bravo, C. A. Gómez, F. Luca, *Powers of two as sums of two k–Fibonacci numbers*, Miskolc Math. Notes **17**(2016), 85–100.

[9] E. F. Bravo, J. J. Bravo, *Tribonacci numbers with two blocks of repdigits*, Math. Slovaca **71**(2021), 267–274.

[10] E. F. Bravo, J. J. Bravo, C. A. Gómez, *Generalized Lucas numbers which are concatenations of two repdigits*, Results Math. **76**(2021), Art. 139.

[11] H. Cohen, *Number Theory Volume I: Tools and Diophantine equations*, Springer, New York, 2007.

[12] M. Ddamulira, *Padovan numbers that are concatenations of two distinct repdigits*, Math. Slovaca **71**(2021), 275–284.

[13] M. Ddamulira, P. Emong, G. I. Mirumbe, *Members of Narayana's cow sequence that are concatenations of two repdigits*, arXiv: 2210.00926v1 [math.NT] (2022).

[14] A. Dujella, A. Pethö, *A generalization of a theorem of Baker and Davenport*, Q. J. Math. **49**(1998), 291–306.

[15] R. Keskin, Z. Şiar, *k–Generalized Pell numbers which are concatenation of two repdigits*, Mediterr. J. Math. **19**(2022), Art. 180.

[16] É. Lucas, *Théorie des Fonctions Numériques Simplement Périodiques*, Amer. J. Math. **1**(1878), 197–240.

[17] E. M. Matveev, *An explicit lower bound for a homogeneous rational linear form in the logarithms of algebraic numbers II*, Izv. Ross. Akad. Nauk Ser. Mat. **64**(2000), 125–180, in Russian; English translation in Izv. Math. **64**(2000), 1217–1269.

[18] R. Padovan, *Dom Hans van der Laan: modern primitive*, Architectura & Natura Press, Amsterdam, 1994.

[19] R. Perrin, *Item* 1484, L'Intermédiare des Math. **6**(1899), 76–77.

[20] Y. Qu, J. Zeng, *Lucas numbers which are concatenations of two repdigits*, Mathematics **8**(2020), Art. 1360.

[21] S. G. Rayaguru, G. K. Panda, *Balancing numbers which are concatenation of two repdigits*, Bol. Soc. Mat. Mex. **26**(2020), 911–919.

[22] S. G. Rayaguru, G. K. Panda, Z. Şiar, *Associated Pell numbers which are repdigits or concatenation of two repdigits*, Bol. Soc. Mat. Mex. **27**(2021), Art. 54.

[23] C. L. Siegel, *Algebraic numbers whose conjugates lie in the unit circle*, Duke Math. J. **11**(1944), 597–602.

[24] N. J. A. Sloane, *The On–Line Encyclopedia of Integer Sequences (OEIS)*, 2023, available at `https://oeis.org`.

[25] I. Stewart, *Tales of a neglected number*, Sci. Am. **274**(1996), 102–103.

[26] R. Vieira, M. Mangueira, F. R. Alves, P. Catarino, *Application of the BenTaher–Rachidi method in numerical sequences*, Bol. Soc. Port. Mat. **80**(2022), 1–20.