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Infinite product representation of solutions of indefinite problem with a finite number of arbitrary turning points

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Abstract. In this paper we consider the Sturm-Liouville equation

$$y'' + (\rho^2\phi^2(x) - q(x))y = 0 \tag{1}$$

on a finite interval I, say $I = [0, 1]$, under the assumption that I contains a finite number of arbitrary type turning points, which are zeros of ϕ in I. According to the four types of turning points, first we obtain the asymptotic forms of the solutions of (1), and then based on Hadamard's factorization theorem we use this asymptotic estimates to study the infinite product representation of solutions of such equations. Infinite product form of the solution has a basic application in studies of inverse spectral problems.

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1. Introduction

In the literature dealing with differential equations we encountered a large number of research papers studying the Sturm-Liouville equation, i.e.,

$$y'' + (\rho^2\phi^2(x) - q(x))y = 0, \quad 0 \le x \le 1, \tag{2}$$

where the functions ϕ^2 and q are referred to as coefficients of the problem, the function $\phi^2(x)$ as the weight, and $q(x)$ as the potential function, which are real valued functions in $[0, 1]$. We call the zeros of $\phi^2(x)$, assumed to be a discrete set, turning points or transition points (TP) of (2). Let us define $I_+(I_-))$ by the set of $x \in (0, 1)$, where $\phi^2(x) > 0(\phi^2(x) < 0)$ and $x \in I_0$ if $\phi^2(x) = 0$. The weight function $\phi^2(x)$ is said to be indefinite if I_+ and I_- each have a positive Lebesgue measure. In the vast majority of differential equations, especially in equations with variable coefficients, we can not obtain an exact solution, so we must resort to methods of approximation. One of the most important approximation methods are asymptotic methods, representing the solution by an asymptotic form. There

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are in-depth studies for the existence of the asymptotic solution of (2), depending on parameter ρ^2. In [20], Mingarelli has presented a historical review. There are fundamental roles of asymptotic techniques for solving differential equations of the form (2), and also in the analysis and development of the methods of modern applied mathematics and theoretical physics. Based on transformations to a first order system, an asymptotic theory has been developed for linear differential equations by some authors who started from the work of Birkohoff [2], where we see various diagonal transformations. The mentioned methods can be found in the books by Wasow [27, 28]. There are important innovations in studies on the asymptotic approximation of solutions of Sturm-Liouville equations, such as in the results of Olver [23], Doronidcyn [4], McKelvey [19], Kazarinoff [11], Langer [14], Dyachenko [5], Wazwaz [29], Tumanov [26], and Kheiri, Jodayree & Mingarelli [12].
On the other hand, an important issue is how to deal with the equations involving turning points. The importance of asymptotic methods in obtaining the solution of a Sturm-Liouville equation with multiple turning points has been realized by Olver ([23, 22, 21]), Heading [9], Leung [16], and Eberhard, Freiling & Schneider [6].
But, in asymptotic methods for solutions of differential or integral equations we can not obtain a closed form for the solution and this is a weakness of these methods. Indeed, in methods studying dual equations we need a closed form of the solution. On the other hand, for representing the solution as a closed form first we note that, based on the results of Halvorsen [7], any solution $y(x, \lambda)$ of (2) with a fixed set of initial conditions is an entire function of λ, of order α, where α does not exceed $\frac{1}{2}$, for each fixed $x \in [0, 1]$. Therefore, by the classical Hadamard factorization theorem [17]), one can express such solutions as an infinite product, which is a closed form of the solution, and so this provides us an effective tool for approximation purposes in the various applications. We can see some applications of such infinite product representations in [3, 13, 25].
In [18], we considered

$$y'' + (\rho^2\phi^2(x) - q(x))y = 0,$$

where $x \in I = [0, 1]$, subject to initial conditions $y(0, \lambda) = 1, y'(0, \lambda) = 0$, under the assumption that $q(x)$ is Lebesgue integrable on the interval $[0, 1]$, and also $\phi^2(x)$ has m zeros, which are called turning points of the problem, such that one of the turning points is of odd order, while the others are of even order. In other words, it is assumed that

$$\phi^2(x) = \phi_0(x) \prod_{v=1}^{m} (x - x_v)^{-\ell_v},$$

where $\ell_1 = 4k + 1$ and $\ell_v = 4k$, $v = 2, \ldots, m$, based on the notations of [6], x_1 was of Type IV, while $x_2, x_3, \ldots, x_v$ were of Type II. In [18], the solution $y(x, \lambda)$ of (2) with these assumptions is represented with an infinite product for any $x \in [0, 1]$. In this paper, we consider the same equation (2) on a finite interval I, say $I = [0, 1]$, subject to arbitrary initial conditions, say, for example, $y(0, \lambda) = 0, y'(0, \lambda) = 1$. The equation can also have an arbitrary number of turning points of any kind. Corresponding to the shape and form of the asymptotic distributions of eigenvalues we obtain the infinite product representation of the solution of (2) between two successive turning points.

2. Notations, fundamental solutions and preliminary results

To provide asymptotic fundamental solutions, let us consider

$$y'' + (\rho^2\phi^2(x) - q(x))y = 0, \qquad x \in I = [0,1], \tag{3}$$

where ϕ^2 and q are real functions, and $\rho^2 = \lambda$ is the spectral parameter. Suppose $q \in L_2[0,1]$, and without loss of generality, let q be a positive function, and also

$$\phi^2(x) = \phi_0(x)\prod_{v=1}^{m}(x - x_v)^{\ell_v}, \tag{4}$$

where $0 = x_0 < x_1 < x_2 < \cdots < x_m < 1 = x_{m+1}$, $\ell_v \in N$, $\phi_0(x) > 0$ for $x \in I = [0,1]$, and ϕ_0 is twice continuously differentiable on I. In other words, the equation has m turning points of order ℓ_v.

Turning points are classified into four different types according to the following definition: For $1 \leq v \leq m$, let $I_{v,\epsilon} = [x_{v-1} + \epsilon, x_{v+1} - \epsilon]$, then

$$T_v = \begin{cases} \text{I}, & \ell_v = 4k; \phi^2(x)(x - x_v)^{-l_v} < 0, \\ \text{II}, & \text{if } \ell_v = 4k; \phi^2(x)(x - x_v)^{-l_v} > 0, \\ \text{III}, & \text{if } \ell_v = 4k+1; \phi^2(x)(x - x_v)^{-l_v} < 0, \\ \text{IV}, & \text{if } \ell_v = 4k+1; \phi^2(x)(x - x_v)^{-l_v} > 0, \end{cases}$$

is called the type of x_v, where it is assumed that $x \in I_{v,\epsilon,}$.

Furthermore, we set

$$\mu_v = \frac{1}{2 + \ell_v},$$

and

$$\vartheta_v = \begin{cases} 2, & \text{if } T_v = \text{III}, \text{IV} \\ 1, & \text{if } T_v = \text{I}, \text{II} \end{cases}, \tag{5}$$

$$\delta_v = \begin{cases} 1, & \text{if } T_v = \text{II}, \text{IV} \\ 2, & \text{if } T_v = \text{I}, \text{III} \end{cases}. \tag{6}$$

Indeed, we define

$$[1] \equiv 1 + \mathrm{O}\left(\frac{1}{\lambda}\right), \text{ as } \lambda \to \infty$$

$$[\alpha] \equiv \alpha + \mathrm{O}(\rho^{-\sigma_0}), \qquad \alpha \in C, \qquad \text{and} \qquad \sigma_0 = \min\{\mu_1, \mu_2, \ldots, \mu_m\}.$$

For $\rho \in S_{-1}$

$$S_{-1} = \{\rho \mid \arg \rho \in [-\frac{\pi}{4}, 0]\},$$

it is shown in [6] that for each fixed $x \in I_{v,\epsilon}$, according to the type of x_v, there exists a fundamental system of solutions of (3) $\{Z^{T_v}_{v,1}(x,\rho), Z^{T_v}_{v,2}(x,\rho)\}$, which are as follows:

- Turning point of type I:

$$
\begin{aligned}
Z^I_{v,1}(x,\rho) &= \begin{cases} \mid \phi(x) \mid^{-\frac{1}{2}} \ e^{\rho \int_{x_v}^{x} |\phi(t)| dt}[1], & x_{v-1} < x < x_v \\ \mid \phi(x) \mid^{-\frac{1}{2}} \ \csc \pi\mu_v e^{\rho \int_{x_v}^{x} |\phi(t)| dt}[1], & x_v < x < x_{v+1} \end{cases}, \\
Z^I_{v,2}(x,\rho) &= \begin{cases} \mid \phi(x) \mid^{-\frac{1}{2}} \ e^{-\rho \int_{x_v}^{x} |\phi(t)| dt}[1], & x_{v-1} < x < x_v \\ \mid \phi(x) \mid^{-\frac{1}{2}} \ \sin \pi\mu_v e^{-\rho \int_{x_v}^{x} |\phi(t)| dt}[1], & x_v < x < x_{v+1} \end{cases}, \\
Z^I_{v,1}(x_v,\rho) &= \frac{\sqrt{2\pi}}{2}(\iota\rho)^{\frac{1}{2}-\mu_v} \csc \pi\mu_v e^{\imath\pi(-\frac{1}{4}+\frac{\mu_v}{2})} \frac{2^{\mu_v}\psi(x_v)}{\Gamma(1-\mu_v)}[1], \\
Z^I_{v,2}(x_v,\rho) &= \frac{\sqrt{2\pi}}{2}(\iota\rho)^{\frac{1}{2}-\mu_v} e^{\imath\pi(-\frac{1}{4}+\frac{\mu_v}{2})} \frac{2^{\mu_v}\psi(x_v)}{\Gamma(1-\mu_v)}[1],
\end{aligned}
$$

where

$$
\psi(x_1) = \lim_{x \to x_1} \phi^{-\frac{1}{2}}(x) \left\{ \int_{x_1}^{x} \phi(t) dt \right\}^{\frac{1}{2}-\mu_1},
$$

and we have

$$
\mathrm{W}(\rho) = \mathrm{W}(Z^I_{v,1}(x,\rho), Z^I_{v,2}(x,\rho)) = -2\rho[1],
$$

where $W(f(x), g(x)) := f(x)g'(x) - f'(x)g(x)$ denotes the Wronskian.

- Turning point of type II:

$$
\begin{aligned}
Z^{II}_{v,1}(x,\rho) &= \begin{cases} \mid \phi(x) \mid^{-\frac{1}{2}} \ e^{\imath\rho \int_{x_v}^{x} |\phi(t)| dt}[1], \quad x_{v-1} < x < x_v \\ \mid \phi(x) \mid^{-\frac{1}{2}} \ \csc \pi\mu_v \{ e^{\imath\rho \int_{x_v}^{x} |\phi(t)| dt}[1] + \imath \cos \pi\mu_v e^{-\imath\rho \int_{x_v}^{x} |\phi(t)| dt}[1] \}, \\ \qquad\qquad x_v < x < x_{v+1} \end{cases} \\
Z^{II}_{v,2}(x,\rho) &= \begin{cases} \mid \phi(x) \mid^{-\frac{1}{2}} \ \{ e^{-\imath\rho \int_{x_v}^{x} |\phi(t)| dt}[1] + \imath \cos \pi\mu_v e^{\imath\rho \int_{x_v}^{x} |\phi(t)| dt}[1] \}, \\ \qquad\qquad x_{v-1} < x < x_v \\ \mid \phi(x) \mid^{-\frac{1}{2}} \ \sin \pi\mu_v e^{-\imath\rho \int_{x_v}^{x} |\phi(t)| dt}[1], x_v < x < x_{v+1} \end{cases} \\
Z^{II}_{v,1}(x_v,\rho) &= \frac{\sqrt{2\pi}}{2}\rho^{\frac{1}{2}-\mu_v} \csc \pi\mu_v e^{\imath\pi(\frac{1}{4}-\frac{\mu_v}{2})} \frac{2^{\mu_v}\psi(x_v)}{\Gamma(1-\mu_v)}[1], \\
Z^{II}_{v,2}(x_v,\rho) &= \frac{\sqrt{2\pi}}{2}\rho^{\frac{1}{2}-\mu_v} e^{\imath\pi(\frac{1}{4}-\frac{\mu_v}{2})} \frac{2^{\mu_v}\psi(x_v)}{\Gamma(1-\mu_v)}[1], \\
W(\rho) &= W(Z^{II}_{v,1}(x,\rho), Z^{II}_{v,2}(x,\rho)) = -2\iota\rho[1].
\end{aligned}
$$

- Turning point of type III:

$$Z_{v,1}^{III}(x,\rho)=\begin{cases} \mid\phi(x)\mid^{-\frac{1}{2}}\ e^{\imath\rho\int_{x_v}^{x}|\phi(t)|dt}[1], & x_{v-1}<x<x_v\\ \frac{1}{2}\mid\phi(x)\mid^{-\frac{1}{2}}\ \csc\frac{\pi\mu_v}{2}e^{\rho\int_{x_v}^{x}|\phi(t)|dt+\frac{\iota\pi}{4}}[1], x_v<x<x_{v+1}\end{cases},$$

$$Z_{v,2}^{III}(x,\rho)=\begin{cases} |\phi(x)|^{-\frac{1}{2}}\ \{e^{-\imath\rho\int_{x_v}^{x}|\phi(t)|dt}[1]+\imath e^{\imath\rho\int_{x_v}^{x}|\phi(t)|dt}[1]\}, x_{v-1}<x<x_v\\ 2\mid\phi(x)\mid^{-\frac{1}{2}}\ \sin\frac{\pi\mu_v}{2}e^{-\rho\int_{x_v}^{x}|\phi(t)|dt+\frac{\iota\pi}{4}}[1], \quad x_v<x<x_{v+1}\end{cases}$$

$$Z_{v,1}^{III}(x_v,\rho)=\frac{\sqrt{2\pi}}{2}(\iota\rho)^{\frac{1}{2}-\mu_v}\csc\pi\mu_v\frac{2^{\mu_v}\psi(x_v)}{\Gamma(1-\mu_v)}[1],$$

$$Z_{v,2}^{III}(x_v,\rho)=\frac{\sqrt{2\pi}}{2}(\iota\rho)^{\frac{1}{2}-\mu_v}e^{\iota\frac{\pi\mu_v}{2}}\sec(\frac{\pi\mu_v}{2})\frac{2^{\mu_v}\psi(x_v)}{\Gamma(1-\mu_v)}[1],$$

$$W(\rho)=W(Z_{v,1}^{III}(x,\rho),Z_{v,2}^{III}(x,\rho))=-2\iota\rho[1].$$

- Turning point of type IV:

$$Z_{v,1}^{IV}(x,\rho)=\begin{cases} \mid\phi(x)\mid^{-\frac{1}{2}}\ e^{\rho\int_{x_v}^{x}|\phi(t)|dt}[1], \quad x_{v-1}<x<x_v\\ \frac{1}{2}csc\frac{\pi\mu_v}{2}\mid\phi(x)\mid^{-\frac{1}{2}}\ \{e^{\imath\rho\int_{x_v}^{x}|\phi(t)|dt-\imath\frac{\pi}{4}}[1]\\ \quad +e^{-\imath\rho\int_{x_v}^{x}|\phi(t)|dt+\imath\frac{\pi}{4}}[1]\}, \quad x_v<x<x_{v+1}\end{cases} \tag{7}$$

$$Z_{v,2}^{IV}(x,\rho)=\begin{cases} \mid\phi(x)\mid^{-\frac{1}{2}}\ e^{-\rho\int_{x_v}^{x}|\phi(t)|dt}[1], x_{v-1}<x<x_v\\ 2\sin\frac{\pi\mu_v}{2}\mid\phi(x)\mid^{-\frac{1}{2}}\ \{e^{-\imath\rho\int_{x_v}^{x}|\phi(t)|dt-\imath\frac{\pi}{4}}[1]\}, x_v<x<x_{v+1}\end{cases} \tag{8}$$

$$Z_{v,1}^{IV}(x_v,\rho)=\frac{\sqrt{2\pi}}{2}\rho^{\frac{1}{2}-\mu_v}\csc\pi\mu_v\frac{2^{\mu_v}\psi(x_v)}{\Gamma(1-\mu_v)}[1], \tag{9}$$

$$Z_{v,2}^{IV}(x_v,\rho)=\frac{\sqrt{2\pi}}{2}\rho^{\frac{1}{2}-\mu_v}e^{-\imath\frac{\pi\mu_v}{2}}\sec(\frac{\pi\mu_v}{2})\frac{2^{\mu_v}\psi(x_v)}{\Gamma(1-\mu_v)}[1], \tag{10}$$

$$W(\rho)=W(Z_{v,1}^{IV}(x,\rho),Z_{v,2}^{IV}(x,\rho))=-2\rho[1]. \tag{11}$$

Based on the result of Halvorsen [7], the solution $y(x,\lambda)$ of (3) under the initial conditions on y,y' at a point c, $0<c<x<1$, is an entire function of λ of order $\frac{1}{2}$ if $\int_c^x|\phi(t)|dt\neq 0$. On the other hand, an entire function of finite order l can be represented as

$$f(z)=z^m e^{g(z)}\prod_n(1-\frac{z}{a_n})e^{\frac{z}{a_n}+\frac{1}{2}(\frac{z}{a_n})^2+\cdots+\frac{1}{h}(\frac{z}{a_n})^h},$$

where a_n are its zeros arranged in an increasing order, $h\leq l$, $g(z)$ is a polynomial of degree q such that $q\leq l$, and m is the multiplicity of origin as a zero of $f(z)$. For example, we provide two well-known results for later uses:

$$\sinh z=z\prod_{m=1}^{\infty}(1+\frac{z^2}{m^2\pi^2}),$$

and

$$\sinh c\sqrt{z} = c\sqrt{z}\prod_{m=1}^{\infty}(1+\frac{zc^2}{m^2\pi^2}) = c\sqrt{z}\prod_{m=1}^{\infty}(1+\frac{z}{z_m^2}),$$

where $z_m = \frac{m\pi}{c}$, and the domain of the function $f(z) = z^{\frac{1}{2}}$ is the complement of a negative real axis $z \leq 0$, while the range of $z^{\frac{1}{2}}$ is the right half of the z plane with the imaginary axis excluded.
Moreover, for the entire function $J_v'(z)$ [1]

$$J_v'(z) = \frac{(z/2)^{v-1}}{2\Gamma(v)}\prod(1-\frac{z^2}{\tilde{j}_m^2}), \quad v > 0, \tag{12}$$

where

$$\tilde{j}_m^2 = m^2\pi^2 - \frac{m\pi^2}{2} + O(1), \tag{13}$$

are the positive zeros of $J_1'(z)$. By putting $z = c\sqrt{\lambda},\ \Gamma(1) = 1$ in (12) we obtain

$$J_1'(c\sqrt{\lambda}) = \frac{1}{2}\prod(1-\frac{\lambda c^2}{\tilde{j}_m^2}), \tag{14}$$

and similarly we have

$$J_1'(\iota c\sqrt{\lambda}) = \frac{1}{2}\prod(1+\frac{\lambda c^2}{\tilde{j}_m^2}). \tag{15}$$

3. Asymptotic form of the solution

We consider the following second order differential equation:

$$y'' + (\lambda\phi^2(x) - q(x))y = 0, \qquad t \in I = [0,1] \tag{16}$$

$$y(0,\lambda) = 0,\ y'(0,\lambda) = 1, \tag{17}$$

where $\lambda = \rho^2$ and we suppose that $\phi^2(x)$ is of the form (4), that is, the problem has m turning points $x_1, x_2, \ldots, x_m,$, in I, which are zeros of ϕ. The solution of the problem in $I_{1,\epsilon}$, can be obtained by applying the initial conditions to

$$y(x,\rho) = C_1(\rho)Z_{1,1}^{T_1}(x,\rho) + C_2(\rho)Z_{1,2}^{T_1}(x,\rho), \quad x \in I_{1,\epsilon}.$$

Consequently, in view of formulas $\{Z_{1,1}^{T_1}(x,\rho), Z_{1,2}^{T_1}(x,\rho)\}$ and $\{Z_{1,1}^{T_1}(x_1,\rho), Z_{1,2}^{T_1}(x_1,\rho)\}$, we have

$$y_{(0,x_1)}(x,\rho) = H(x,\rho)e^{(-1)^{\delta_1-1}(\iota)^{\delta_1}\rho\int_0^x|\phi(t)|dt}E_k(x,\rho),$$

$$y(x_1,\rho) = F(x_1,\mu_1,\rho)\csc\pi\mu_1 e^{(-1)^{\theta_1}(\iota)^{\vartheta_1}\rho\int_0^{x_1}|\phi(t)|dt}E_k(x_1,\rho),$$

where $H(x,\rho)$ and $F(x_1,\mu_1,\rho)$ are obtained using the fundamental solutions according to the type of turning point x_1 and applying the initial conditions, k is the same as in $\ell_1 = 4k$ or $\ell_1 = 4k+1$ and

$$E_k(x,\rho) = [1] + \sum_{n=1}^{\nu(x)} e^{\rho\alpha_k\beta_{kn}(x)}[b_{kn}(x)],$$

such that $\alpha_{-2}=\alpha_1=-1, \alpha_0=-\alpha_{-1}=\imath, \beta_{k\nu(x)}(x)\neq 0, 0<\delta\leq\beta_{k1}(x)<\beta_{k2}(x)<\cdots<\beta_{k\nu(x)}(x)\leq 2\,max\{\mathrm{R}_+(1),\mathrm{R}_-(1)\}$, where

$$\mathrm{R}_+(x)=\int_0^x\sqrt{max\{0,\phi^2(t)\}}dt, \tag{18}$$

$$\mathrm{R}_-(x)=\int_0^x\sqrt{max\{0,-\phi^2(t)\}}dt. \tag{19}$$

Furthermore, the integer-valued functions ν and b_{kn} for ϵ sufficiently small are constant in intervals $[0,x_1-\epsilon]$ and $[x_1+\epsilon,x_2-\epsilon]$. In the following lemma we obtain the asymptotic solution of the problem (16)-(17) in the interval $[x_v,x_{v+1})$ by a recurence relation, back from the solution in the interval $(0,x_1]$.

Lemma 1. *Let $y_{(x_v,x_{v+1})}(x,\rho)$ be the asymptotic solution of initial value problem (16) in the interval (x_v,x_{v+1}). Then*

$$y_{(0,x_1)}(x,\rho)=H(x,\rho)e^{(-1)^{\delta_1-1}(\iota)^{\delta_1}\rho\int_0^x|\phi(t)|dt}E_k(x,\rho)=A(x,\rho)E_k(x,\rho), \tag{20}$$

and for $x\in(x_v,x_{v+1})$, $v\geq 1$, we have

$$\begin{aligned}y_{(x_v,x_{v+1})}(x,\rho)=&\frac{1}{2^r}A(x_1,\rho)\prod_{x_i\leq x_v}\csc\frac{\pi\mu_i}{2^{\vartheta_i-1}}e^{(-1)^{\delta_v-1}(\iota)^{\delta_v}\rho\sum_{x_i<x_v}\int_{x_i}^{x_{i+1}}|\phi(t)|dt}\\&\times e^{\sum_{x_i\leq x_v:T_i=III,IV}(-1)^{\delta_v}\frac{\iota\pi}{4}}e^{(-1)^{\delta_v-1}(\iota)^{\delta_v}\rho\int_{x_v}^{x}|\phi(t)|dt}E_k(x,\rho),\end{aligned} \tag{21}$$

where r is the number of turning points $x_i\leq x_v$, which are of type III or IV. Furthermore,

$$y(x_1,\rho)=F(x_1,\mu_1,\rho)\csc\pi\mu_1 e^{(-1)^{\vartheta_1}(\iota)^{\vartheta_1}\rho\int_0^{x_1}|\phi(t)|dt}E_k(x_1,\rho), \tag{22}$$

and for $x_v>x_1$, turning point II or IV, we have

$$\begin{aligned}y(x_v,\rho)=&\frac{1}{2^s}F(x_v,\mu_v,\rho)e^{(2-\vartheta_v)\iota\pi(\frac14-\frac{\mu_v}{2})}e^{(-1)^{\vartheta_1}(\iota)^{\vartheta_1}\rho\int_0^{x_1}|\phi(t)|dt}\csc\pi\mu_v\prod_{x_i<x_v}\csc\frac{\pi\mu_i}{2^{\vartheta_i-1}}\\&\times\ e^{(-1)^{\delta_v-1}(\iota)^{\delta_v}\rho\sum_{x_i<x_v}\int_{x_i}^{x_{i+1}}|\phi(t)|dt}e^{\sum_{x_i<x_v:T_i=III,IV}(-1)^{\delta_v}\frac{\iota\pi}{4}}E_k(x_v,\rho).\end{aligned}$$

Similarly, for $x_v>x_1$, turning point of type I *or* III*, we have*

$$\begin{aligned}y(x_v,\rho)=&\frac{1}{2^s}F(x_v,\mu_v,\iota\rho)e^{(2-\vartheta_v)\iota\pi(-\frac14+\frac{\mu_v}{2})}e^{(-1)^{\vartheta_1}(\iota)^{\vartheta_1}\rho\int_0^{x_1}|\phi(t)|dt}\csc\pi\mu_v\prod_{x_i<x_v}\csc\frac{\pi\mu_i}{2^{\vartheta_i-1}}\\&\times\ e^{(-1)^{\delta_v-1}(\iota)^{\delta_v}\rho\sum_{x_i<x_v}\int_{x_i}^{x_{i+1}}|\phi(t)|dt}e^{\sum_{x_i<x_v:T_i=III,IV}(-1)^{\delta_v}\frac{\iota\pi}{4}}E_k(x_v,\rho).\end{aligned}$$

where s is the number of turning points $x_i<x_v$, which are of type III *or* IV.

Proof. Clearly, (20) and (22) are evident as mentioned above. Now we fix $x\in(x_1,x_2)$, then by using of $\{Z_{2,1}^{T_2}(x,\rho),Z_{2,2}^{T_2}(x,\rho)\}$ and Cramer's rule we can determine the connection coefficients $\mathrm{A}(\rho)$, $\mathrm{B}(\rho)$ such that

$$y_{(x_1,x_2)}(x,\rho)=\mathrm{A}(\rho)Z_{2,1}^{T_2}(x,\rho)+\mathrm{B}(\rho)Z_{2,2}^{T_2}(x,\rho).$$

The continuation of the solution $y(x,\lambda)$ to the interval (x_2,x_3) satisfies

$$y_{(x_2,x_3)}(x,\rho)=\mathrm{A}(\rho)Z_{2,1}^{T_2}(x,\rho)+\mathrm{B}(\rho)Z_{2,2}^{T_2}(x,\rho).$$

This procedure can be used to calculate the solution in the remaining intervals. Now we suppose that $y_{(x_i,x_{i+1})}(x,\rho)=\mathrm{D}_{(x_i,x_{i+1})}(x,\rho)E_k(x,\rho)$ is the asymptotic form of the solution of initial value problem (16)-(17) in the interval (x_i,x_{i+1}), $i=1,2,\ldots,m$, with $x_{m+1}=1$. By the above procedure and according to the fundamental solution form introduced in Section 2, we can easily show the following assertions:

i: If x_v is a turning point of type I, then

$$\mathrm{D}_{(x_v,x_{v+1})}(x,\rho)=\mathrm{D}_{(x_{v-1},x_v)}(x_v,\rho)\csc\pi\mu_v e^{\rho\int_{x_v}^{x}|\phi(t)|dt}.$$

ii: If x_v is a turning point of type II, then

$$\mathrm{D}_{(x_v,x_{v+1})}(x,\rho)=D_{(x_{v-1},x_v)}(x_v,\rho)\csc\pi\mu_v e^{\iota\rho\int_{x_v}^{x}|\phi(t)|dt}.$$

iii: If x_v is a turning point of type III, then

$$\mathrm{D}_{(x_v,x_{v+1})}(x,\rho)=\frac{1}{2}\mathrm{D}_{(x_{v-1},x_v)}(x_v,\rho)\csc\frac{\pi\mu_v}{2}e^{\rho\int_{x_v}^{x}|\phi(t)|dt+\frac{\iota\pi}{4}}.$$

iv: If x_v is a turning point of type IV, then

$$\mathrm{D}_{(x_v,x_{v+1})}(x,\rho)=\frac{1}{2}\mathrm{D}_{(x_{v-1},x_v)}(x_v,\rho)\csc\frac{\pi\mu_v}{2}e^{\iota\rho\int_{x_v}^{x}|\phi(t)|dt-\frac{\iota\pi}{4}}.$$

And this proves the lemma. □

Example 1. *Let $y(x,\rho)$ be the solution of (16)-(17) and suppose that the first turning point x_1 is of type IV (that is, of order $l_1=4k+1$) and other $m-1$ turning points are arbitrary. Using the fundamental solutions $\{Z_{1,1}^{IV}(x,\rho),Z_{1,2}^{IV}(x,\rho)\}$ one can get*

$$y(x,\rho)=\frac{1}{-2\rho}(Z_{1,1}^{IV}(0,\rho)Z_{1,2}^{IV}(x,\rho)-Z_{1,1}^{IV}(x,\rho)Z_{1,2}^{IV}(0,\rho)),\quad x\in(0,x_1);$$

then from (7) and (8) we conclude

$$y(x,\rho)=\frac{|\phi(x)\phi(0)|^{\frac{-1}{2}}}{2\rho}\{e^{\rho\int_0^x|\phi(t)dt}[1]-e^{-\rho\int_0^x|\phi(t)dt}[1]\},\quad x\in(0,x_1),$$

or

$$y(x,\rho)=\frac{|\phi(x)\phi(0)|^{\frac{-1}{2}}}{2\rho}e^{\rho\int_0^x|\phi(t)dt}E_k(x,\rho),\quad x\in(0,x_1).\tag{23}$$

Furthermore, by virtue of (9) and (10) we get

$$y(x_1,\rho)=\frac{|\phi(0)|^{-\frac{1}{2}}\sqrt{2\pi}\rho^{\frac{1}{2}-\mu_1}2^{\mu_1}\psi(x_1)\csc\pi\mu_1}{4\Gamma(1-\mu_1)}e^{\rho\int_0^{x_1}|\phi(t)|dt}E_k(x,\rho).$$

Using Lemma 1 we calculate

$$y(x,\rho)=\frac{|\phi(x)\phi(0)|^{\frac{-1}{2}}}{4\rho}\csc\frac{\pi\mu_1}{2}e^{\rho\int_0^{x_1}|\phi(t)dt+\iota\rho\int_{x_1}^{x}|\phi(t)dt-\frac{\iota\pi}{4}}E_k(x,\rho),\ x_1<x<x_2. \tag{24}$$

We note that (24) can also be obtained directly.
Now for a turning point of type IV, *using Lemma 1, the solution in the interval* (x_v,x_{v+1}) *is of the form*

$$\begin{aligned}y_{(x_v,x_{v+1})}(x,\rho)=&\frac{1}{2^k}\frac{|\phi(x)\phi(0)|^{\frac{-1}{2}}}{2\rho}\csc\frac{\pi\mu_1}{2}\csc\frac{\pi\mu_v}{2}e^{\rho\int_0^{x_1}|\phi(t)|dt-\frac{\iota\pi}{4}}\prod_{x_2\le x_i<x_v}\csc\frac{\pi\mu_i}{2^{\vartheta_i-1}}\\&\times e^{(-1)^{\delta_v-1}(\iota)^{\delta_v}\rho\sum_{x_i<x_v}\int_{x_i}^{x_{i+1}}|\phi(t)|dt}e^{\sum_{x_2\le x_i<x_v:T_i=III,IV}(-1)^{\delta_v}\frac{\iota\pi}{4}}\\&\times e^{\iota\rho\int_{x_v}^{x}|\phi(t)|dt-\frac{\iota\pi}{4}}E_k(x,\rho).\end{aligned} \tag{25}$$

Furthermore, for a turning point of type III, *we have*

$$\begin{aligned}y_{(x_v,x_{v+1})}(x,\rho)=&\frac{1}{2^k}\frac{|\phi(x)\phi(0)|^{\frac{-1}{2}}}{2\rho}\csc\frac{\pi\mu_1}{2}\csc\frac{\pi\mu_v}{2}e^{\rho\int_0^{x_1}|\phi(t)|dt-\frac{\iota\pi}{4}}\prod_{x_2\le x_i<x_v}\csc\frac{\pi\mu_i}{2^{\vartheta_i-1}}\\&\times e^{(-1)^{\delta_v-1}(\iota)^{\delta_v}\rho\sum_{x_i<x_v}\int_{x_i}^{x_{i+1}}|\phi(t)|dt}e^{\sum_{x_2\le x_i<x_v:T_i=III,IV}(-1)^{\delta_v}\frac{\iota\pi}{4}}\\&\times e^{\rho\int_{x_v}^{x}|\phi(t)|dt+\frac{\iota\pi}{4}}E_k(x,\rho).\end{aligned}$$

Similarly, for a turning point of type II, *we obtain*

$$\begin{aligned}y_{(x_v,x_{v+1})}(x,\rho)=&\frac{1}{2^r}\frac{|\phi(x)\phi(0)|^{\frac{-1}{2}}}{2\rho}\csc\frac{\pi\mu_1}{2}\csc\pi\mu_v e^{\rho\int_0^{x_1}|\phi(t)|dt-\frac{\iota\pi}{4}}\prod_{x_2\le x_i<x_v}\csc\frac{\pi\mu_i}{2^{\vartheta_i-1}}\\&\times e^{(-1)^{\delta_v-1}(\iota)^{\delta_v}\rho\sum_{x_i<x_v}\int_{x_i}^{x_{i+1}}|\phi(t)|dt}e^{\sum_{x_2\le x_i<x_v:T_i=III,IV}(-1)^{\delta_v}\frac{\iota\pi}{4}}\\&\times e^{\iota\rho\int_{x_v}^{x}|\phi(t)|dt}E_k(x,\rho).\end{aligned} \tag{26}$$

And for a turning point of type I, *we have*

$$\begin{aligned}y_{(x_v,x_{v+1})}(x,\rho)=&\frac{1}{2^r}\frac{|\phi(x)\phi(0)|^{\frac{-1}{2}}}{2\rho}\csc\frac{\pi\mu_1}{2}\csc\pi\mu_v e^{\rho\int_0^{x_1}|\phi(t)|dt-\frac{\iota\pi}{4}}\prod_{x_2\le x_i<x_v}\csc\frac{\pi\mu_i}{2^{\vartheta_i-1}}\\&\times e^{(-1)^{\delta_v-1}(\iota)^{\delta_v}\rho\sum_{x_i<x_v}\int_{x_i}^{x_{i+1}}|\phi(t)|dt}e^{\sum_{x_2\le x_i<x_v:T_i=III,IV}(-1)^{\delta_v}\frac{\iota\pi}{4}}\\&\times e^{\rho\int_{x_v}^{x}|\phi(t)|dt}E_k(x,\rho).\end{aligned}$$

Indeed, according to Lemma 1 for a turning point of type II *or* IV, *we obtain*

$$\begin{aligned}y(x_v,\rho)=&\frac{1}{2^s}\frac{|\phi(0)|^{-\frac{1}{2}}\sqrt{2\pi}\rho^{\frac{1}{2}-\mu_1}2^{\mu_1}\psi(x_1)\csc\pi\mu_v}{4\Gamma(1-\mu_1)}e^{\rho\int_0^{x_1}|\phi(t)|dt}e^{(2-\vartheta_v)\iota\pi(\frac{1}{4}-\frac{\mu_v}{2})}\\&\times\prod_{x_i<x_v}\csc\frac{\pi\mu_i}{2^{\vartheta_i-1}}\ e^{(-1)^{\delta_v-1}(\iota)^{\delta_v}\rho\sum_{x_i<x_v}\int_{x_i}^{x_{i+1}}|\phi(t)|dt}\\&\times e^{\sum_{x_i<x_v:T_i=III,IV}(-1)^{\delta_v}\frac{\iota\pi}{4}}E_k(x_v,\rho).\end{aligned} \tag{27}$$

And for a turning point of type I *or* III*, we have*

$$\begin{aligned} y(x_v,\rho) =& \frac{1}{2^s}\frac{|\phi(0)|^{-\frac{1}{2}}\sqrt{2\pi}(\iota\rho)^{\frac{1}{2}-\mu_1}2^{\mu_1}\psi(x_1)\csc\pi\mu_v}{4\Gamma(1-\mu_1)}e^{\rho\int_0^{x_1}|\phi(t)|dt}e^{(2-\vartheta_v)\iota\pi(-\frac{1}{4}+\frac{\mu_v}{2})} \\ & \times \prod_{x_i<x_v}\csc\frac{\pi\mu_i}{2^{\vartheta_i-1}}e^{(-1)^{\delta_v-1}(\iota)^{\delta_v}\rho\sum_{x_i<x_v}\int_{x_i}^{x_{i+1}}|\phi(t)|dt} \\ & \times e^{\sum_{x_i<x_v:T_i=III,IV}(-1)^{\delta_v}\frac{\iota\pi}{4}}E_k(x_v,\rho). \end{aligned} \tag{28}$$

4. Infinite product representation

Now let $y(x,\lambda)$ be the solution of initial value problem (16)-(17). For each fixed $x\in I$, the function $y(x,\lambda)$ has a set of zeros, $\{\lambda_n(x)\}$, such that $y(x,\lambda_n(x))=0$. Therefore, these zeros are correspond to the eigenvalues of the associated Dirichlet problem for equation (16) on $[0,x]$.
First, let $0<x<x_1$, where x_1 is the first turning point; then the Dirichlet problem associated with (16) on $[0,x]$ has a sequence of negative eigenvalues, $\{\lambda_n(x)\}$, if x_1 is of type I or IV, and a sequence of positive eigenvalues if x_1 is of type II or III, such that Hadamard's theorem yields:

$$y(x,\lambda)=B(x)\prod_{n=1}^{\infty}(1-\frac{\lambda}{\lambda_n(x)}), \tag{29}$$

where the constant $B(x)$ does not depend on λ but may depend on x. By substituting ρ by $i\rho$ in (20) and using ([15, Theorem 7]) or ([24, p. 26]) and [8] from $y(x,\lambda)=0$, we see that each function $\lambda_n(x)$ has the following asymptotic:

$$\sqrt{(-1)^{\delta_1+\vartheta_1}\lambda_n(x)}=\frac{n\pi}{\int_0^x|\phi(t)|dt}+O(\frac{1}{n}), \qquad 0<x<x_1. \tag{30}$$

In order to estimate $B(x)$ we rewrite (29) as follows:

$$\begin{aligned} y(x,\lambda) =& B(x)\prod_{n=1}^{\infty}(1-\frac{\lambda}{\lambda_n(x)}) \\ =& B(x)\prod\frac{\lambda_n(x)-\lambda}{\lambda_n(x)} \\ =& B_1(x)\prod\frac{\lambda-\lambda_n(x)}{z_n^2}, \end{aligned} \tag{31}$$

with

$$B_1(x)=B(x)\prod\frac{-z_n^2}{\lambda_n(x)},$$

where $z_n=\frac{n\pi}{R_-(x)}$.
We note that on any compact subinterval of $(0,x_1)$ the infinite product $\prod\frac{-z_m^2}{\lambda_m(x)}$ is

absolutely convergent since

$$\frac{-z_m^2}{\lambda_m(x)} = 1 + O(\frac{1}{m^2}),$$

where the function $\frac{-z_m^2}{\lambda_m(x)}$ continues and such that the O-term is uniformly bounded in x.
For $x = x_1$, from (22), the eigenvalues of Dirichlet's problem associated with (16) have the following distributions:

$$\sqrt{(-)^{\delta_1+\vartheta_1}\lambda_n(x_1)} = \frac{n\pi + (\frac{\pi\mu_1}{2} - \frac{\pi}{4})}{\int_0^{x_1} |\phi(t)|dt} + O(\frac{1}{n}). \tag{32}$$

Then

$$y(x_1, \lambda) = E(x) \prod_{n\geq 1} (1 - \frac{\lambda}{\lambda_n(x_1)}),$$

by Hadamard's theorem.
Now, due to the following distribution of positive zeros of the Bessel function of order μ_1, j_n, $n = 1, 2, \ldots$, [1]

$$\frac{-j_n^2}{R_-^2(x_1)\lambda_n(x_1)} = 1 + O(\frac{1}{n^2}),$$

the infinite product $\prod \frac{-j_n^2}{R_-^2(x_1)\lambda_n(x_1)}$ is absolutely convergent. Therefore we have

$$y(x_1, \lambda) = E_1(x) \prod \frac{(\lambda - \lambda_n(x_1))R_-^2(x_1)}{j_n^2},$$

where $E_1(x) = E(x) \prod \frac{-j_n^2}{R_-^2(x_1)\lambda_n(x_1)}$.

The Dirichlet problem associated with (16) on $[0, x]$, when $x \in (x_1, x_2)$, has a sequence of positive and negative eigenvalues, $\{(1 - \delta_{x_1,I})\sqrt{\lambda_{n1}(x)} = (1 - \delta_{x_1,I}) \cdot u_{n1}(x)\}_{n1=1}^{\infty}$, $\{(1 - \delta_{x_1,II})\sqrt{-\lambda_{n1}(x)} = (1 - \delta_{x_1,II})r_{n1}(x)\}_{n1=1}^{\infty}$, respectively, where δ is the Kronecker delta function. Without loss of generality, we assume that the function $r(x)$ changes its sign and as a result we will have both positive and negative eigenvalues. From the asymptotic solution (21) we have

$$u_{n1(x)} = \frac{n\pi - \frac{\pi}{4}}{\int_{x_1}^{x} |\phi(t)|dt} + O(\frac{1}{n}),$$

and

$$r_{n1}(x) = -\frac{n\pi - \frac{\pi}{4}}{\int_0^{x_1} |\phi(t)|dt} + O(\frac{1}{n}).$$

Similarly to the previous interval, by Hadamard's theorem, we have

$$y(x, \lambda) = F(x) \prod_{n=1}^{\infty} (1 - \frac{\lambda}{r_{n,1}(x)}) \prod_{n=1}^{\infty} (1 - \frac{\lambda}{u_{n,1}(x)}), \quad x_1 < x < x_2.$$

Due to the following distributions [1]:

$$\begin{aligned}\frac{\tilde{j}_n^2}{R_+^2(x)u_{n1}(x)} =&1+O(\frac{1}{n^2}),\\ \frac{-\tilde{j}_n^2}{R_-^2(x_1)r_{n1}(x)} =&1+O(\frac{1}{n^2}),\end{aligned}$$

where $\tilde{j}_n$, $n=1,2,\dots$, are the positive zeros of $J_1'(z)$, we conclude that the infinite products $\prod \frac{\tilde{j}_n^2}{R_+^2(x)u_{n1}(x)}$ and $\prod \frac{-\tilde{j}_n^2}{R_-^2(x_1)r_{n1}(x)}$ are absolutely convergent, $x_1<x<x_2$. Then, we may write

$$y(x,\lambda)=F_1(x)\prod\frac{(\lambda-r_{n1}(x))R_-^2(x_1)}{\tilde{j}_n^2}\prod\frac{R_+^2(x)(u_{n1}(x)-\lambda)}{\tilde{j}_n^2},$$

where

$$F_1(x)=F(x)\prod\frac{\tilde{j}_n^2}{R_+^2(x)u_{n1}(x)}\prod\frac{-\tilde{j}_n^2}{R_-^2(x_1)r_{n1}(x)}.$$

Now, let $x\in(x_v,x_{v+1})$, $x_v\geq x_2$, and x_v is a turning point of type II or IV. In this case, the Dirichlet problem associated to (16) on $[0,x]$ has a sequence of positive and negative eigenvalues, $\{\sqrt{\lambda_{nv}(x)}=u_{nv}(x)\}_{nv=1}^{\infty}$, $\{\sqrt{-\lambda_{nv}(x)}=r_{nv}(x)\}_{nv=1}^{\infty}$, respectively. From the asymptotic form of the solution, (21), we have

$$\begin{aligned}u_{nv}(x)=&\frac{n\pi-\sum_{x_1\leq x_i\leq x_v:T_i=\mathrm{III,IV}}(-1)^{\delta_v}\frac{\pi}{4}}{\sum_{x_i<x_v:T_i=\mathrm{IV,II}}\int_{x_i}^{x_{i+1}}|\phi(t)|dt+\int_{x_v}^{x}|\phi(t)|dt}+O(\frac{1}{n}),\\ r_{nv}(x)=&-\frac{n\pi-\sum_{x_1\leq x_i\leq x_v:T_i=\mathrm{III,IV}}(-1)^{\delta_v}\frac{\pi}{4}}{\sum_{x_i<x_v:T_i=\mathrm{III,I}}\int_{x_i}^{x_{i+1}}|\phi(t)|dt+\int_0^{x_1}|\phi(t)|dt}+O(\frac{1}{n}).\end{aligned}$$

For $x\in(x_v,x_{v+1})$, $x_v\geq x_2$ of type I or III, the Dirichlet problem associated with (16) on $[0,x]$ has a sequence of positive and negative eigenvalues with the following distributions:

$$u_{nv}(x)=\frac{n\pi-\sum_{x_1\leq x_i\leq x_v:T_i=\mathrm{III,IV}}(-1)^{\delta_v}\frac{\pi}{4}}{\sum_{x_i<x_v:T_i=\mathrm{IV,II}}\int_{x_i}^{x_{i+1}}|\phi(t)|dt}+O(\frac{1}{n}), \tag{33}$$

$$r_{nv}(x)=-\frac{n\pi-\sum_{x_1\leq x_i\leq x_v:T_i=\mathrm{III,IV}}(-1)^{\delta_v}\frac{\pi}{4}}{\sum_{x_i<x_v:T_i=\mathrm{III,I}}\int_{x_i}^{x_{i+1}}|\phi(t)|dt+\int_{x_v}^{x}|\phi(t)|dt+\int_0^{x_1}|\phi(t)|dt}+O(\frac{1}{n}). \tag{34}$$

By Hadamard's theorem for $x_v<x<x_{v+1}$, $x_v\geq x_2$ of type II or IV, we have

$$y(x,\lambda)=C(x)\prod_{n\geq 1}(1-\frac{\lambda}{r_{nv}(x)})(1-\frac{\lambda}{u_{nv}(x)}).$$

By the following distributions:

$$\begin{aligned}\frac{\tilde{j}_n^2}{R_+^2(x)u_{nv}(x)} =&1+O(\frac{1}{n^2}),\\ \frac{-\tilde{j}_n^2}{R_-^2(x_v)r_{nv}(x)} =&1+O(\frac{1}{n^2}),\end{aligned}$$

where $\tilde{j}_n$, $n = 1, 2, \ldots$ are the positive zeros of $J_1'(z)$, we conclude that the infinite products $\prod \frac{\tilde{j}_n^2}{R_+^2(x)u_{nv}(x)}$ and $\prod \frac{-\tilde{j}_n^2}{R_-^2(x_v)r_{nv}(x)}$ are absolutely convergent for each $x \in (x_v, x_{v+1})$. We write

$$y(x,\lambda) = C_v(x) \prod_{n\geq 1} \frac{(\lambda - r_{nv}(x))R_-^2(x_v)}{\tilde{j}_n^2} \prod_{n\geq 1} \frac{(u_{nv}(x) - \lambda)R_+^2(x)}{\tilde{j}_n^2},$$

with

$$C_v(x) = C(x) \prod_{n\geq 1} \frac{\tilde{j}_n^2}{R_+^2(x)u_{nv}(x)} \prod_{n\geq 1} \frac{-\tilde{j}_n^2}{R_-^2(x_v)r_{nv}(x)}.$$

Similarly, for a fixed $x \in (x_v, x_{v+1})$, where x_v is of type I or III,

$$y(x,\lambda) = A_v(x) \prod_{n\geq 1} \frac{(\lambda - r_{nv}(x))R_-^2(x)}{\tilde{j}_n^2} \prod_{n\geq 1} \frac{(u_{nv}(x) - \lambda)R_+^2(x_v)}{\tilde{j}_n^2},$$

where

$$A_v(x) = A(x) \prod_{n\geq 1} \frac{\tilde{j}_n^2}{R_+^2(x_v)u_{nv}(x)} \prod_{n\geq 1} \frac{-\tilde{j}_n^2}{R_-^2(x)r_{nv}(x)}.$$

Furthermore, for $x_2 \leq x_v \leq x_m$, we have

$$y(x_v,\rho) = M_v(x) \prod_{m\geq 1} \frac{(\lambda - r_{mv}(x_v))R_-^2(x_v)}{\tilde{j}_m^2} \prod_{m\geq 1} \frac{(u_{mv}(x_v) - \lambda)R_+^2(x_v)}{\tilde{r}_m^2}, \qquad (35)$$

where $\tilde{r}_m, m = 1, 2, \ldots$, is the sequence of positive zeros of $J_{\mu_v + \frac{3}{2}}$ and

$$u_{nv}(x_v) = \frac{n\pi + (\frac{\pi\mu_v}{2} - \sum_{x_1 \leq x_i < x_v : T_i = \mathrm{III},\mathrm{IV}} (-1)^{\delta_v} \frac{\pi}{4})}{\sum_{x_i < x_v : T_i = \mathrm{IV},\mathrm{II}} \int_{x_i}^{x_{i+1}} |\phi(t)| dt} + O(\frac{1}{n}), \qquad (36)$$

$$r_{nv}(x_v) = -\frac{n\pi - \sum_{x_1 \leq x_i < x_v : T_i = \mathrm{III},\mathrm{IV}} (-1)^{\delta_v} \frac{\pi}{4}}{\sum_{x_i < x_v : T_i = \mathrm{III},\mathrm{I}} \int_{x_i}^{x_{i+1}} |\phi(t)| dt + \int_0^{x_1} |\phi(t)| dt} + O(\frac{1}{n}). \qquad (37)$$

Now, we prove the following lemma.

Lemma 2. *Let $x \in (x_v, x_{v+1})$, and let x_v be a turning point of type* II *or* IV. *Suppose*

$$u_{mv}(x) = \frac{m^2\pi^2}{R_+^2(x)} - \frac{m\pi^2}{2R_+^2(x)} + O(1), \quad m \geq 1.$$

Let $\tilde{j}_n$, $n = 1, 2, \ldots$, be the positive zeros of $J_1'(z)$; then for fixed x in (x_v, x_{v+1}) the following infinite product is an entire function:

$$\prod_{m\geq 1} \frac{(u_{mv}(x) - \lambda)R_+^2(x)}{\tilde{j}_m^2}.$$

Moreover,

$$\prod_{m\geq 1}\frac{(u_{mv}(x)-\lambda)R_+^2(x)}{\tilde{j}_m^2}=2J_1'(\sqrt{\lambda}R_+(x))(1+O(\frac{\log n}{n})),$$

uniformly on the circles $|\lambda|=\frac{n^2\pi^2}{R_+^2(x)}$.

Proof. From (13) we can obtain

$$\sum_{m\geq 1}|\frac{(u_{mv}(x)-\lambda)R_+^2(x)}{\tilde{j}_m^2}-1|=\sum_{m\geq 1}|\frac{\lambda+O(1)}{\tilde{j}_m^2}|.$$

Therefore, on any bounded subset of a complex plane it is uniformly convergent and this yields that the infinite product converges to an entire function of λ, whose roots are precisely $u_{mv}(x)$, $m\geq 1$. Now by using of (14) we have

$$J_1'(\sqrt{\lambda}R_+(x))=\frac{1}{2}\prod(1-\frac{\lambda R_+^2(x)}{\tilde{j}_m^2}).$$

Consequently

$$\frac{\prod_{m\geq 1}\frac{(u_{mv}(x)-\lambda)R_+^2(x)}{\tilde{j}_m^2}}{\frac{1}{2}\prod(1-\frac{\lambda R_+^2(x)}{\tilde{j}_m^2})}=2\prod\frac{u_{mv}(x)-\lambda}{\frac{\tilde{j}_m^2}{R_+^2(x)}-\lambda}.$$

Furthermore,

$$\frac{u_{mv}(x)-\lambda}{\frac{\tilde{j}_m^2}{R_+^2(x)}-\lambda}-1=\frac{|u_{mv}(x)-\frac{\tilde{j}_m^2}{R_+^2(x)}|}{|\frac{\tilde{j}_m^2}{R_+^2(x)}-\lambda|}\leq\frac{|O(1)|}{|\frac{\tilde{j}_m^2}{R_+^2(1)}-|\lambda||}.$$

Therefore, on the circles $|\lambda|=\frac{n^2\pi^2}{R_+^2(1)}$ the uniform estimates

$$\frac{u_{mv}(x)-\lambda}{\frac{\tilde{j}_m^2}{R_+^2(x)}-\lambda}=\begin{cases}1+O(\frac{1}{n}), & \text{if m=n},\\ 1+O(\frac{1}{|m^2-n^2|}), & \text{if } m\neq n,\end{cases}$$

hold. But if a_{mn}, $m,n\geq 1$ are complex numbers satisfying

$$|a_{mn}|=O(\frac{1}{|m^2-n^2|}),\quad m\neq n,$$

then for each $n\geq 1$ we have (for more details, see [25, p. 165])

$$\prod_{m\geq 1, m\neq n}(1+a_{mn})=1+O(\frac{\log n}{n}).$$

Therefore,

$$\prod_{m\geq 1}\frac{u_{mv}(x)-\lambda}{\frac{\tilde{j}_m^2}{R_+^2(x)}-\lambda}) = (1+O(\frac{\log n}{n}))(1+O(\frac{1}{n})) = 1+O(\frac{\log n}{n}),$$

and hence

$$\prod_{m\geq 1}\frac{(u_{mv}(x)-\lambda)R_+^2(x)}{\tilde{j}_m^2} = 2J_1'(\sqrt{\lambda}R_+(x))(1+O(\frac{\log n}{n})).$$

□

Lemma 3. *Let $x\in(x_v,x_{v+1})$, and let x_v be a turning point of type* II *or* IV*. Suppose*

$$r_{mv}(x) = -\frac{m^2\pi^2}{R_-^2(x_v)} + \frac{m\pi^2}{2R_-^2(x_v)} + O(1), \quad m\geq 1.$$

Let $\tilde{j}_n$, $n=1,2,\ldots$, be the positive zeros of $J_1'(z)$; then for fixed x in (x_v,x_{v+1}) the following infinite product is an entire function:

$$\prod_{m\geq 1}\frac{(\lambda-r_{mv}(x))R_-^2(x_v)}{\tilde{j}_m^2}.$$

Furthermore,

$$\prod_{m\geq 1}\frac{(\lambda-u_{mv}(x))R_-^2(x_v)}{\tilde{j}_m^2} = 2J_1'(\iota\sqrt{\lambda}R_-(x_v))(1+O(\frac{\log n}{n})),$$

uniformly on the circles $|\lambda| = \frac{n^2\pi^2}{R_+^2(x)}$.

Proof. This follows from (15) and using the method of the preceding lemma. □

If $x\in(x_v,x_{v+1})$ and x_v is a turning point of type I or III, similar results can be obtained for the infinite products $\prod_{m\geq 1}\frac{(\lambda-r_{mv}(x))R_-^2(x)}{\tilde{j}_n^2}$ and $\prod_{m\geq 1}\frac{(u_{mv}(x)-\lambda)R_+^2(x_v)}{\tilde{j}_n^2}$.

Lemma 4. *Let $x\in(x_v,x_{v+1})$, and let x_v be a turning point of type* I *or* III*. Suppose*

$$u_{mv}(x_v) = \frac{m^2\pi^2}{R_+^2(x_v)} + \frac{m\pi^2(\mu_v+1)}{R_+(x_v)} + O(1), \quad m\geq 1.$$

Let $\tilde{r}_m$, $m=1,2,\ldots$ be the positive zeros of $J_{\mu_v+\frac{3}{2}}(z)$; then for fixed x in (x_v,x_{v+1}) the following infinite product is an entire function:

$$\prod_{m\geq 1}\frac{(u_{mv}(x_v)-\lambda)R_+^2(x_v)}{\tilde{r}_m^2}.$$

Moreover,

$$\prod_{m\geq 1}\frac{(u_{mv}(x_v)-\lambda)R_+^2(x_v)}{\tilde{r}_m^2} = 2^{\mu_v+\frac{3}{2}}\Gamma(\mu_v+\frac{5}{2})(R_+(x_v)\sqrt{\lambda})^{-(\mu_v+\frac{3}{2})} \times J_{\mu_v+\frac{3}{2}}(R_+(x_v)\sqrt{\lambda})(1+O(\frac{\log n}{n})).$$

Proof. It can be easily proved from the infinite product representation of $J_{\mu_v+\frac{3}{2}}(z)$ (see [1]) and using the method of Lemma 2. □

Now we can prove the following main results for the infinite product representation of solutions of (16)-(17).

Theorem 1. *Suppose $y(x,\lambda)$ is the solution of (16)-(17) for $0<x<x_1$. We have*

$$y(x,\lambda)=|\phi(x)\phi(0)|^{-\frac{1}{2}}R_-(x)\prod_{m\geq 1}\frac{\lambda-\lambda_m(x)}{z_m^2},$$

where $z_m=\frac{m\pi}{R_-(x)}$, $R_-(x)$ and $\lambda_m(x), m\geq 1$, were obtained in (19) and (30), respectively.

Proof. Using (31) and (23), we have

$$y(x,\lambda)=B_1(x)\prod\frac{\lambda-\lambda_m(x)}{z_m^2}=\frac{|\phi(x)\phi(0)|^{\frac{-1}{2}}}{2\rho}e^{\rho\int_0^x|\phi(t)dt}E_k(x,\rho).$$

Therefore, we obtain

$$B_1(x)=|\phi(x)\phi(0)|^{-\frac{1}{2}}R_-(x).$$

□

Similarly, we have the following theorem.

Theorem 2. *For $x=x_1$,*

$$y(x_1,\lambda)=\frac{|\phi(0)|^{\frac{-1}{2}}\psi(x_1)}{2\mu_1}R_-(x_1)^{\frac{1}{2}+\mu_1}\prod_{n\geq 1}\frac{(\lambda-\lambda_n(x_1))R_-^2(x_1)}{j_n^2},$$

where $R_-(x)$ is defined in (19), $j_n, n=1,2,\ldots$ represents the positive zeros of the Bessel functions of order μ_1, $\lambda_n(x_1)$ obtained in (32) and

$$\psi(x_1)=\lim_{x\to x_1}\phi^{-\frac{1}{2}}(x)\{\int_{x_1}^x\phi(t)dt\}^{\frac{1}{2}-\mu_1}.$$

Theorem 3. *Suppose $y(x,\lambda)$ is the solution of (16)-(17) for $x_v<x<x_{v+1}$, such that x_v is a turning point of type* II *or* IV*. We have*

$$y(x,\lambda)=\frac{1}{2^k}\frac{\pi}{4}|\phi(x)\phi(0)|^{-\frac{1}{2}}R_-^{\frac{1}{2}}(x_v)R_+^{\frac{1}{2}}(x)\csc\frac{\pi\mu_1}{2}\prod_{x_2\leq x_i\leq x_v}\csc\frac{\pi\mu_i}{2^{\vartheta_i-1}} \times\prod_{m\geq 1}\frac{(\lambda-r_{mv}(x))R_-^2(x_v)}{\tilde{j}_m^2}\prod_{m\geq 1}\frac{(u_{mv}(x)-\lambda)R_+^2(x)}{\tilde{j}_m^2},$$

where k is the number of turning points $x_i \leq x_v$, which are of type III *or* IV, $R_+(x), R_-(x), \vartheta_i, \{u_{mv}(x_v)\}$ *and* $\{r_{mv}(x_v)\}$ *were defined or obtained in (18), (19), (5), (36), (37), respectively.*

Proof. By making use of (14), (15) we find

$$J_1'(R_+(x)\sqrt{\lambda})J_1'(\iota R_-(x_v)\sqrt{\lambda}) = \frac{e^{R_-(x_v)\sqrt{\lambda}}}{\pi R_-^{\frac{1}{2}}(x_v)R_+^{\frac{1}{2}}(x)\sqrt{\lambda}}\{\cos(R_+(x)\sqrt{\lambda}-\frac{\pi}{4})+O(\frac{1}{\sqrt{\lambda}})\},$$

as $\lambda \to \infty$. By lemmas 2 and 3 on the circles $|\lambda| = min\{\frac{n^2\pi^2}{R_-^2(x_v)}, \frac{n^2\pi^2}{R_+^2(x)}\}$, we have

$$\begin{aligned}\prod_{m\geq 1}&\frac{(\lambda - r_{mv}(x))R_-^2(x_v)}{\tilde{j}_m^2}\prod_{m\geq 1}\frac{(u_{mv}(x)-\lambda)R_+^2(x)}{\tilde{j}_m^2}\\ &= \frac{4e^{R_-(x_v)\sqrt{\lambda}}}{\pi R_-^{\frac{1}{2}}(x_v)R_+^{\frac{1}{2}}(x)\sqrt{\lambda}}\{\cos(R_+(x)\sqrt{\lambda}-\frac{\pi}{4})+O(\frac{1}{\sqrt{\lambda}})\},\end{aligned}$$

as $\lambda \to \infty$. Now by applying the asymptotic expansion of $y(x,\lambda)$ in (25) or (26) we get

$$\begin{aligned}C_v(x,\lambda) =&\frac{y(x,\lambda)}{\prod_{m\geq 1}\frac{(\lambda - r_{mv}(x))R_-^2(x_v)}{\tilde{j}_m^2}\prod_{m\geq 1}\frac{(u_{mv}(x)-\lambda)R_+^2(x)}{\tilde{j}_m^2}}\\ =&\frac{1}{2^k}\frac{\pi}{4}|\phi(x)\phi(0)|^{-\frac{1}{2}}R_-^{\frac{1}{2}}(x_v)R_+^{\frac{1}{2}}(x)\csc\frac{\pi\mu_1}{2}\prod_{x_2\leq x_i\leq x_v}\csc\frac{\pi\mu_i}{2^{\vartheta_i - 1}}.\end{aligned}$$

□

Similarly, we can prove the following theorem.

Theorem 4. *Suppose $y(x,\lambda)$ is the solution of (16)-(17) for $x_v < x < x_{v+1}$, such that x_v is a turning point of type* I *or* III. *We have*

$$\begin{aligned}y(x,\lambda) =&\frac{1}{2^k}\frac{\pi}{4}|\phi(x)\phi(0)|^{-\frac{1}{2}}R_-^{\frac{1}{2}}(x)R_+^{\frac{1}{2}}(x_v)\csc\frac{\pi\mu_1}{2}\prod_{x_2\leq x_i\leq x_v}\csc\frac{\pi\mu_i}{2^{\vartheta_i - 1}}\\ &\times\prod_{m\geq 1}\frac{(\lambda - r_{mv}(x))R_-^2(x)}{\tilde{j}_m^2}\prod_{m\geq 1}\frac{(u_{mv}(x)-\lambda)R_+^2(x_v)}{\tilde{j}_m^2},\end{aligned}$$

where k is the number of turning points $x_i \leq x_v$, which are of type III *or* IV, *and* $R_+(x), R_-(x), \vartheta_i, \{u_{mv}(x)\}$ *and* $\{r_{mv}(x)\}$ *were defined or obtained in (18), (19), (5), (33), (34), respectively.*

Theorem 5. *For $x = x_v$ $x_v \geq x_2$, we have*

$$\begin{aligned}y(x_v,\lambda) =&\frac{1}{2^s}\frac{\sqrt{\pi}\Gamma(\mu_v)|\varphi(0)|^{-\frac{1}{2}}}{4\Gamma(\mu_v+\frac{1}{2})}\psi(x_v)R_+^{\mu_v}(x_v)R_-^{\frac{1}{2}}(x_v)\csc\frac{\pi\mu_1}{2}\prod_{x_2\leq x_i\leq x_{v-1}}\csc\frac{\pi\mu_i}{2^{\vartheta_i - 1}}\\ &\times\prod_{m\geq 1}\frac{(\lambda - r_{mv}(x_v))R_-^2(x_v)}{\tilde{j}_m^2}\prod_{m\geq 1}\frac{(u_{mv}(x_v)-\lambda)R_+^2(x_v)}{\tilde{r}_m^2},\end{aligned}$$

where s is the number of turning points $x_i < x_v$, which are of type III *or* IV, *and* $R_+(x), R_-(x), \vartheta_i, \{u_{mv}(x_v)\}$ *and* $\{r_{mv}(x_v)\}$ *were defined or obtained in (18), (19), (5), (36), (37), respectively.*

Proof. From ([1, §9.2.1]) we can obtain

$$J_{\mu_v+\frac{3}{2}}(\sqrt{\lambda}R_+(x_v))J_1'(R_-(x_v) = \frac{1}{2\pi\sqrt{R_+(x_v)R_-(x_v)}}e^{\rho R_-(x_v)+\iota\rho R_+(x_v)-\frac{\pi\mu_v}{2}\iota}[1].$$

So, from (35), (27) or (28), lemmas 2 and 3 we see that

$$\begin{aligned} M_v(x) =& \frac{y(x_v,\lambda)}{\prod_{m\geq 1}\frac{(\lambda-r_{mv}(x_v))R_-^2(x_v)}{\tilde{j}_m^2}\prod_{m\geq 1}\frac{(u_{mv}(x_v)-\lambda)R_+^2(x_v)}{\tilde{r}_m^2}} \\ =& \frac{\sqrt{\pi}\Gamma(\mu_v)|\varphi(0)|^{-\frac{1}{2}}}{4\Gamma(\mu_v+\frac{1}{2})}\psi(x_v)R_+^{\mu_v}(x_v)R_-^{\frac{1}{2}}(x_v)\prod_{x_1\leq x_i\leq x_{v-1}}\csc\frac{\pi\mu_i}{2^{\vartheta_i-1}}. \end{aligned}$$

$\square$

This completes a canonical display of the solution of (16)-(17) for different modes of turning points.

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