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$\mathcal{I}^h$-convergence and convergence of positive series*

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Abstract. In 1827, L. Olivier proved a result about the speed of convergence to zero of the terms of convergent positive series with nonincreasing terms, the so-called Olivier's theorem (see [17]). T. Šalát and V. Toma in [20] made the remark that the monotonicity condition in Olivier's theorem can be dropped if the convergence of the sequence (na_n) is weakened by means of the notion of $\mathcal{I}$-convergence for an appropriate ideal $\mathcal{I}$. Results of this type are called a modified Olivier's theorem.
In connection with this, we will study the properties of summable ideals $\mathcal{I}^h$, where $h\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a function such that $\sum_{n\in\mathbb{N}} h(n) = +\infty$ and $\mathcal{I}^h = \left\{A \subsetneq \mathbb{N} : \sum_{n\in A} h(n) < +\infty\right\}$. We show that $\mathcal{I}^h$-convergence and $\mathcal{I}^{h*}$-convergence are equivalent. This is not valid in general.
Further, we also show that a modified Olivier's theorem is not valid for summable ideals $\mathcal{I}^h$ in general. We find sufficient conditions for a real function $h\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that a modified Olivier's theorem remains valid for the ideal $\mathcal{I}^h$.
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1. Introduction

We recall the basic definitions and connections that will be used throughout this paper. Let $\mathbb{N}$ be the set of all positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $\mathbb{R}^+$ the set of all positive real numbers. A system $\mathcal{I}$, $\emptyset \neq \mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal, provided that $\mathcal{I}$ is additive ($A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$) and hereditary ($A \in \mathcal{I}$, $B \subset A$ implies $B \in \mathcal{I}$). The ideal is called nontrivial if $I \neq 2^{\mathbb{N}}$. If $\mathcal{I}$ is a nontrivial ideal, then $\mathcal{I}$ is called admissible if it contains the singletons ($\{n\} \in \mathcal{I}$ for every $n \in \mathbb{N}$). The fundamental notation shall be used is $\mathcal{I}$-convergence introduced in the paper [14] (see also [5], where $\mathcal{I}$-convergence is defined by means of the dual notion to the ideal so-called filter). The notion of $\mathcal{I}$-convergence corresponds to the natural generalization of the notion of statistical convergence (see [8, 19]).

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Definition 1. *Let (x_n) be a sequence of real (complex) numbers. We say that the sequence $\mathcal{I}$-converges to a number L, and write $\mathcal{I}-\lim x_n = L$, if for each $\varepsilon > 0$ the set $A_\varepsilon = \{n : |x_n - L| \geq \varepsilon\}$ belongs to the ideal $\mathcal{I}$.*

In what follows, we assume that $\mathcal{I}$ is an admissible ideal. Then for every sequence (x_n) we immediately have that $\lim_{n\to\infty} x_n = L$ (classic limit) implies that (x_n) also $\mathcal{I}$-converges to a number L, but the opposite is not true. In other words, for an admissible ideal $\mathcal{I}$ we have $\mathcal{I}_{fin} \subseteq \mathcal{I}$, where $\mathcal{I}_{fin}$ is the ideal of all finite subsets of $\mathbb{N}$ and $\mathcal{I}_{fin}$-convergence coincides with the usual convergence.

Let $\mathcal{I}_d = \{A \subseteq \mathbb{N} : d(A) = 0\}$, where $d(A)$ is the asymptotic density of $A \subseteq \mathbb{N}$ ($d(A) = \lim_{n\to\infty} \frac{\#\{a \leq n \,:\, a\in A\}}{n}$, where $\#M$ denotes the cardinality of the set M). The usual $\mathcal{I}_d$-convergence is called statistical convergence. For $0 < q \leq 1$, the ideal $\mathcal{I}_c^{(q)} = \left\{A \subset \mathbb{N} : \sum_{a\in A} a^{-q} < \infty\right\}$ is an admissible ideal. The ideal $\mathcal{I}_c^{(1)} = \left\{A \subset \mathbb{N} : \sum_{a\in A} \frac{1}{a} < \infty\right\}$ is usually denoted by $\mathcal{I}_c$.

$\mathcal{I}$-convergence satisfies usual axioms of convergence i.e., the uniqueness of the limit, the arithmetical properties, etc. The class of all $\mathcal{I}$-convergent sequences is a linear space (see [14]).

The claim in the following proposition is a trivial fact about preservation of the limit.

Proposition 1 (see [14])**.** *Let $\mathcal{I}_1, \mathcal{I}_2$ be admissible ideals such that $\mathcal{I}_1 \subseteq \mathcal{I}_2$. If $\mathcal{I}_1 - \lim x_n = L$, then $\mathcal{I}_2 - \lim x_n = L$.*

Whenever $0 < q < q' < 1$, we get

$$\mathcal{I}_{fin} \subsetneq \mathcal{I}_c^{(q)} \subsetneq \mathcal{I}_c^{(q')} \subseteq \mathcal{I}_c \subseteq \mathcal{I}_d. \tag{1}$$

For a function $h\colon \mathbb{R}^+ \to \mathbb{R}^+$, such that $\sum_{n\in\mathbb{N}} h(n) = \infty$ and $\sum_{n\in\emptyset} h(n) = 0$, an ideal $\mathcal{I}^h = \left\{A \subset \mathbb{N} : \sum_{n\in A} h(n) < \infty\right\}$ is called a summable ideal. For any function h, the ideal $\mathcal{I}^h$ is admissible, so $\mathcal{I}_{fin} \subseteq \mathcal{I}^h$.

Another type of convergence related to an ideal $\mathcal{I}$, the so-called $\mathcal{I}^*$-convergence, was defined in papers [13] and [14].

Definition 2. *Let $\mathcal{I}$ be an admissible ideal on $\mathbb{N}$. A sequence (x_n) of real (complex) numbers is said to be $\mathcal{I}^*$-convergent to L if there exists a set $H \in \mathcal{I}$ such that for $M = \mathbb{N} \setminus H = \{m_1 < m_2 < \cdots\}$ we have $\lim_{k\to\infty} x_{m_k} = L$, where the limit is in the usual sense.*

It is easy to see that for an admissible ideal $\mathcal{I}$ we have that $\mathcal{I}^*$-convergence implies $\mathcal{I}$-convergence. The converse is not true (see [14], where the authors give a characterization of ideals $\mathcal{I}$, for which $\mathcal{I}$- and $\mathcal{I}^*$-convergence are equivalent by means of the property (AP)).

Definition 3. *An ideal (not necessarily admissible) $\mathcal{I} \subset 2^{\mathbb{N}}$ is said to satisfy the condition (AP) if for every countable family of mutually disjoint sets $\{A_1, A_2, \dots\}$ belonging to $\mathcal{I}$ there exists a countable family of sets $\{B_1, B_2, \dots\}$ such that the symmetric difference $A_j \triangle B_j$ is finite for $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}$.*

The property (AP) is similar to the property (APO) (see [6, 9] and [18]). All ideals in (1) have the property (AP). There exist many examples of an ideal that does not have the property (AP) (see e.g. [3, 14]).

Proposition 2 (see [14])**.** *The statement* $\mathcal{I}^*-\lim x_n = L$ *follows from* $\mathcal{I}-\lim x_n = L$ *if and only if* $\mathcal{I}$ *satisfies the property (AP).*

An ideal $\mathcal{I}$ (not necessarily admissible) is called a $\mathcal{P}$-ideal if for each sequence (A_n) of sets belonging to $\mathcal{I}$ there exists a set $A_\infty \in \mathcal{I}$ such that $A_n \setminus A_\infty$ is finite for all $n \in \mathbb{N}$.

The notions of $\mathcal{P}$-ideal and ideal with the (AP) property coincide (see [4]).

In [17], Olivier proved the so-called Olivier's Theorem about the speed of convergence to zero of the terms of convergent positive series with nonincreasing terms. Specifically, if (a_n) is a nonincreasing positive sequence and $\sum_{n=1}^{\infty} a_n < \infty$, then $\lim_{n\to\infty} na_n = 0$ (see also [1, 12]). In [20], authors made a remark that the monotonicity condition in Olivier's theorem can be dropped if the convergence of the sequence (na_n) is weakened by means of the notion of $\mathcal{I}$-convergence. They proved that for every positive real sequence (a_n) such that $\sum_{n=1}^{\infty} a_n < \infty$, we have $\mathcal{I}_c - \lim na_n = 0$.

In [11], there is a similar result for the ideals $\mathcal{I}_c^{(q)}$ $(0 < q \leq 1)$. For every positive real sequence (a_n) such that $\sum_{n=1}^{\infty} a_n^q < \infty$ for $0 < q \leq 1$, we have $\mathcal{I}_c^{(q)} - \lim na_n = 0$. The stronger condition of convergence of positive series also results in the stronger convergence property of the summands.

Results of this type are called a modified Olivier's theorem. In [2, 7, 15] and [16], there is an extension of the results in [20]. Moreover, in [16], there is a nice historical context of the object of our research.

In connection with the above results, we will study the properties of summable ideals $\mathcal{I}^h$ for a function $h\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $\sum_{n\in\mathbb{N}} h(n) = \infty$. We will show that the notions $\mathcal{I}^h$- and $\mathcal{I}^{h*}$-convergence are equivalent. It is clear that a modified Olivier's theorem is in general not valid for summable ideals.

If we limit ourselves to a large class of ideals $\mathcal{I}_c^g$ for a function $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $\sum_{n\in\mathbb{N}} \frac{1}{g(n)} = \infty$ and $\mathcal{I}_c^g = \left\{A \subset \mathbb{N} \,:\, \sum_{a\in A} \frac{1}{g(a)} < \infty\right\}$ we will find sufficient conditions for the real function g for the modified Olivier's Theorem to remain valid.

2. Olivier's theorem for ideals $\mathcal{I}_c^g$

First of all, we prove some properties of summable ideals. Let $h\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a function with the following properties:

$$\sum_{n\in\mathbb{N}} h(n) = \infty \quad \text{and} \quad \sum_{n\in\emptyset} h(n) = 0.$$

Then the system

$$\mathcal{I}^h = \left\{A \subset \mathbb{N} \,:\, \sum_{a\in A} h(a) < \infty\right\}$$

is an admissible ideal, so $\mathcal{I}_{fin} \subseteq \mathcal{I}^h$. The ideal $\mathcal{I}^h$ is called a summable ideal. It is easy to see that for a constant function $h(x) = c$, $x \in \mathbb{R}^+$ we have $\mathcal{I}_{fin} = \mathcal{I}^h$, and we also obtain the same for function $h(x) = x$, $x \in \mathbb{R}^+$.

More interesting for our purposes are the admissible ideals $\mathcal{I}^h$ such that $\mathcal{I}^h \neq \mathcal{I}_{fin}$, i.e., they contain an infinite subset of $\mathbb{N}$.

The following theorem gives a characterization of such ideals.

Theorem 1. *$\mathcal{I}^h \neq \mathcal{I}_{fin}$ if and only if $\liminf_{n\to\infty} h(n) = 0$.*

Proof. Suppose $\mathcal{I}^h \neq \mathcal{I}_{fin}$. Then there exists an infinite set $M = \{m_1 < m_2 < \cdots\} \subsetneq \mathbb{N}$ such that

$$\sum_{k=1}^{\infty} h(m_k) < \infty.$$

From this we see that $\lim_{k\to\infty} h(m_k) = 0$; since h is positive, we have $\liminf_{n\to\infty} h(n) = 0$.

Suppose that $\liminf_{n\to\infty} h(n) = 0$. Then there exists a set $M = \{m_1 < m_2 < \cdots\} \subset \mathbb{N}$ such that $\lim_{k\to\infty} h(m_k) = 0$. It means that we can construct an infinite set $M' = \{m_{k_1} < m_{k_2} < \cdots\} \subseteq M$ with property $h(m_{k_i}) < \frac{1}{2^i}$ for every $i \in \mathbb{N}$. Consequently, we have

$$\sum_{m_{k_i}\in M'} h(m_{k_i}) < \sum_{i=1}^{\infty} \frac{1}{2^i},$$

therefore, the infinite set M' belongs to $\mathcal{I}^h$ and so $\mathcal{I}^h \neq \mathcal{I}_{fin}$. □

There exist positive functions $g, h\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $g \neq h$ and $\mathcal{I}^g = \mathcal{I}^h$. The following theorem gives sufficient conditions for functions $g, h\colon \mathbb{R}^+ \to \mathbb{R}^+$ to valid $\mathcal{I}^g = \mathcal{I}^h$.

Theorem 2. *Let $g, h\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $\sum_{n=1}^{\infty} g(n) = \sum_{n=1}^{\infty} h(n) = \infty$. If*

$$0 < \liminf_{n\to\infty} \frac{g(n)}{h(n)} \leq \limsup_{n\to\infty} \frac{g(n)}{h(n)} < \infty,$$

then $\mathcal{I}^g = \mathcal{I}^h$.

Proof. The condition

$$0 < \limsup_{n\to\infty} \frac{g(n)}{h(n)} < \infty$$

implies that there exists such real number $K > 0$ that for every $n \in \mathbb{N}$ we have

$$0 < \frac{g(n)}{h(n)} \leq K,$$

therefore, $g(n) \leq Kh(n)$. Let $M \in \mathcal{I}^h$. Then $\sum_{n\in M} h(n) < \infty$. Immediately, we have

$$\sum_{n\in M} g(n) \leq K \sum_{n\in M} h(n) < \infty,$$

therefore, $M \in \mathcal{I}^g$ and so $\mathcal{I}^h \subseteq \mathcal{I}^g$.

Analogously using the condition

$$0 < \liminf_{n\to\infty} \frac{g(n)}{h(n)} < \infty,$$

we obtain $\mathcal{I}^g \subseteq \mathcal{I}^h$. □

Problem 1. *It would also be interesting to have the necessary conditions for the functions $g, h\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $\mathcal{I}^g = \mathcal{I}^h$.*

The next theorem shows that $\mathcal{I}^h$- and $\mathcal{I}^{h*}$-convergence are equivalent. See also [10], where it is proved that each summable ideal is $\mathcal{P}$-ideal, thus it has the property (AP) that is a sufficient and necessary condition for $\mathcal{I}^h$- and $\mathcal{I}^{h*}$-convergence to be equivalent.

Theorem 3. *Let $h\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a real function. Then $\mathcal{I}^h$- and $\mathcal{I}^{h*}$-convergence coincide.*

Proof. It sufficies to show that for any sequence (x_n) of real numbers such that $\mathcal{I}^h - \lim x_n = L$, there exists a set $M = \{m_1 < m_2 < \cdots\} \subseteq \mathbb{N}$ such that $\mathbb{N}\setminus M \in \mathcal{I}^h$ and $\lim_{k\to\infty} x_{m_k} = L$. Without loss of generality, we can assume that (x_n) is not convergent in the usual sense, but it is $\mathcal{I}^h$-convergent. For any positive integer k, let $\varepsilon_k = \frac{1}{2^k}$ and

$$A_k = \left\{ n \in \mathbb{N} \;:\; |x_n - L| \geq \frac{1}{2^k} \right\}.$$

It is clear that $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots$, and there exists $n_0 \in \mathbb{N}$ such that A_{n_0} is an infinite set. As $\mathcal{I}^h - \lim x_n = L$, we have $A_k \in \mathcal{I}^h$, i.e., $\sum_{n\in A_k} h(n) < \infty$.

Therefore, there exists an infinite sequence $n_1 < n_2 < \cdots < n_k < \cdots$ of positive integers such that for every $k = 1, 2, \ldots$ we have

$$\sum_{\substack{n>n_k\\ n\in A_k}} h(n) < \frac{1}{2^k}.$$

Put

$$H = \bigcup_{n=1}^{\infty} \left[(n_k, n_{k+1}\rangle \cap A_k \right].$$

Then

$$\begin{aligned}\sum_{n\in H} h(n) &\leq \sum_{\substack{n>n_1\\ n\in A_1}} h(n) + \sum_{\substack{n>n_2\\ n\in A_2}} h(n) + \cdots + \sum_{\substack{n>n_k\\ n\in A_k}} h(n) + \cdots \\ &< \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^k} + \cdots \\ &< \infty.\end{aligned}$$

Thus, $H \in \mathcal{I}^h$. Put $M = \mathbb{N} \setminus H = \{m_1 < m_2 < \cdots < m_k < \cdots\}$ and we show that $\lim_{k\to\infty} x_{m_k} = L$.

Let $\varepsilon > 0$. Choose $k_0 \in \mathbb{N}$ such that $\frac{1}{2^{k_0}} < \varepsilon$. Let $m_k > m_{k_0}$. Then m_k belongs to some interval $(n_j, n_{j+1}\rangle$ where $j \geq k_0$, and does not belong to $\mathbb{N} \setminus A_j$ $(j \geq k_0)$. Hence m_k belongs to $\mathbb{N} \setminus A_j$ and then $|x_{m_k} - L| < \varepsilon$ for every $m_k > m_{k_0}$, thus $\lim_{k\to\infty} x_{m_k} = L$. □

Corollary 1. *Ideals $\mathcal{I}^h$ for a real function $h\colon \mathbb{R}^+ \to \mathbb{R}^+$ have the property (AP).*

The next proposition shows that all bounded real sequences are not $\mathcal{I}^h$-convergent.

Proposition 3. *Let $h\colon \mathbb{R}^+ \to \mathbb{R}^+$. Then there exists a bounded real sequence (x_n) that is not $\mathcal{I}^h$-convergent.*

Proof. Since $\sum_{n\in\mathbb{N}} h(n) = \infty$, there exists a decomposition of $\mathbb{N}$ into two sets N_1 and N_2 such that

$$\sum_{n\in N_1} h(n) = \sum_{n\in N_2} h(n) = \infty.$$

For instance, let (n_i) be a sequence of nonnegative integers such that

$$h(n_{i-1}+1) + h(n_{i-1}+2) + \cdots + h(n_i) > 1.$$

Define

$$N_1 = \bigcup_{\substack{i\in\mathbb{N}_0 \\ i \text{ is odd}}} \{n \ : \ n_{i-1} < n \leq n_i\},$$
$$N_2 = \bigcup_{\substack{i\in\mathbb{N}_0 \\ i \text{ is even}}} \{n \ : \ n_{i-1} < n \leq n_i\}.$$

It is clear that $N_1, N_2 \notin \mathcal{I}^h$.

Define a sequence (x_n) as follows:

$$x_n = \begin{cases} 0 & \text{if } n \in N_1, \\ 1 & \text{if } n \in N_2. \end{cases}$$

The sequence (x_n) is real bounded sequence which is not $\mathcal{I}^h$-convergent. □

Corollary 2. *An ideal $\mathcal{I}^h$ for any real function $h\colon \mathbb{R}^+ \to \mathbb{R}^+$ is not a maximal ideal.*

Proof. It follows from Theorem 2.2 in [13] that an admissible ideal $\mathcal{I}$ is the maximal ideal if and only if each bounded real sequence (x_n) is $\mathcal{I}$-convergent. On the basis of the previous proposition, we have a contradiction. □

It is a natural question whether summable ideals $\mathcal{I}^h$ for a function $h\colon \mathbb{R}^+ \to \mathbb{R}^+$ can be used in a modified Olivier's theorem in the following way:

If $\sum_{n\in\mathbb{N}} h(a_n)$ is a convergent positive series for a function $h\colon \mathbb{R}^+ \to \mathbb{R}^+$ and for a positive sequence (a_n), then $\mathcal{I}^h - \lim na_n = 0$.

It is easy to see that such modified Olivier's theorem is not fulfilled in general. Consider a function $h\colon \mathbb{R}^+ \to \mathbb{R}^+$, $h(x) = x^2$ and the sequence (a_n), $a_n = \frac{1}{n}$ for $n \in \mathbb{N}$.

Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a function such that

$$\sum_{n\in\mathbb{N}} \frac{1}{g(n)} = \infty \quad \text{and} \quad \sum_{n\in\emptyset} \frac{1}{g(n)} = 0. \tag{2}$$

Then the system of subsets of $\mathbb{N}$, which denotes $\mathcal{I}_c^g = \left\{A \subset \mathbb{N} \ : \ \sum_{n\in A} \frac{1}{g(n)} < \infty\right\}$ is again an admissible ideal. Ideals $\mathcal{I}_c^g$ seem to be more convenient for a modified Olivier's theorem.

If we put a function $g\colon \mathbb{R}^+ \to \mathbb{R}^+$, $g(x) = x$, we have the same result as in [20] for a function $g(x) = x^q$ for $0 < q \le 1$ we obtain the same result as in [11].

The following example shows that a modified Olivier's theorem is not valid in general for an arbitrary function $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ and an associated ideal $\mathcal{I}_c^g$ with the function g having properties (2).

Example 1. *Put $g\colon \mathbb{R}^+ \to \mathbb{R}^+$, $g(x) = \log_2(x+1)$. It is easy to see that the function g is an increasing function such that*

$$\sum_{n=1}^{\infty} \frac{1}{\log_2(n+1)} = \infty \quad \textit{and} \quad \liminf_{n\to\infty} \frac{1}{\log_2(n+1)} = 0.$$

We show only that $\sum_{n=1}^{\infty} \frac{1}{\log_2(n+1)} = \infty$. It is easy to see that for all $x > 1$ we have $\log_2(x+1) < x$, and so $\frac{1}{x} < \frac{1}{\log_2(x+1)}$. Using integrals for the last inequality, we obtain

$$\infty = \int_1^{\infty} \frac{1}{x}\mathrm{d}x < \int_1^{\infty} \frac{1}{\log_2(x+1)}\mathrm{d}x.$$

Hence $\sum_{n=1}^{\infty} \frac{1}{\log_2(n+1)} = \infty$. The ideal

$$\mathcal{I}_c^{\log_2(x+1)} = \left\{A \subset \mathbb{N} \ : \ \sum_{a\in A} \frac{1}{\log_2(a+1)} < \infty\right\}$$

is the admissible ideal, for which a modified Olivier's theorem is not valid. It sufficies to find a positive sequence (a_n) such that $\sum_{n=1}^{\infty} \log_2(a_n+1) < \infty$, but $\mathcal{I}_c^{\log_2(x+1)} - \lim n a_n \neq 0$. Take the set $B = \left\{2^k - 1 \ : \ k \in \mathbb{N}\right\}$ and consider the following positive sequence (a_n):

$$a_n = \begin{cases} \frac{1}{n} & \textit{if } n \in B, \\ \frac{1}{2^n} & \textit{if } n \in \mathbb{N} \setminus B. \end{cases}$$

Let us count

$$\begin{aligned} \sum_{n=1}^{\infty} \log_2(n+1) &= \sum_{n\in B} \log_2(n+1) + \sum_{n\in\mathbb{N}\setminus B} \log_2(n+1) \\ &= \sum_{k=1}^{\infty} \log_2\left(\frac{1}{2^k-1} + 1\right) + \sum_{n\in\mathbb{N}\setminus B} \log_2\left(\frac{1}{2^n} + 1\right). \end{aligned}$$

First, we show that the series $\sum_{k=1}^{\infty} \log_2\left(\frac{1}{2^k-1}+1\right)$ *is convergent. From the inequality*

$$0 < \log_2(x+1) < 2x,$$

for all $x \in \mathbb{R}^+$ *we have*

$$\log_2\left(\frac{1}{2^k-1}+1\right) < \frac{2}{2^k-1}.$$

Since the series $\sum_{k=1}^{\infty} \frac{1}{2^k-1}$ *is convergent, we also see that the series* $\sum_{k=1}^{\infty} \log_2\left(\frac{1}{2^k-1}+1\right)$ *is convergent.*

In the same way, we also show convergence of the series $\sum_{n\in\mathbb{N}\setminus B} \log_2\left(\frac{1}{2^n}+1\right)$. *We will show that* $\mathcal{I}_c^{\log_2(x+1)} - \lim na_n \neq 0$. *By using Definition 1 for any ideal* $\mathcal{I}$, *we have that a real sequence* (x_n) *is* $\mathcal{I}$*-convergent to zero if for each* $\varepsilon > 0$ *the set* $A_\varepsilon = \{n \in \mathbb{N} : |x_n| \geq \varepsilon\}$ *belongs to the ideal* $\mathcal{I}$. *In our case, it means that for* $\varepsilon = 1$ *and the sequence* (na_n) *the set* $A_{\varepsilon=1} = \{n \in \mathbb{N} : na_n \geq 1\}$ *belongs to* $\mathcal{I}_c^{\log_2(x+1)}$. *It sufficies to realize that* $A_{\varepsilon=1} \supseteq B$ *and* $B \notin \mathcal{I}_c^{\log_2(x+1)}$. *Count*

$$\sum_{n\in B} \frac{1}{\log_2(x+1)} = \sum_{k=1}^{\infty} \frac{1}{\log_2(2^k-1+1)} = \sum_{k=1}^{\infty} \frac{1}{\log_2 2^k} = \sum_{k=1}^{\infty} \frac{1}{k} = \infty.$$

The next theorem gives a sufficient condition for a real function $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that a modified Olivier's theorem is true for an associated ideal $\mathcal{I}_c^g$ with the function g.

Theorem 4. *Let a function* $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ *have the following properties:*

(*i*) g *is nondecreasing,*

(*ii*) $g(nt) \leq g(n)g(t)$ *for all* $n \in \mathbb{N}$ *and* $t \in \mathbb{R}^+$.

If $\sum_{n=1}^{\infty} g(a_n)$ *is a convergent series for a positive sequence* (a_n), *then* $\mathcal{I}_c^g - \lim na_n = 0$.

Proof. We proceed by contradiction. Then there exists a positive sequence (a_n) with $\sum_{n=1}^{\infty} g(a_n) < \infty$ such that the equality $\mathcal{I}_c^g - \lim na_n = 0$ does not hold. Then there exists $\varepsilon_0 > 0$ for which $A_{\varepsilon_0} = \{n \in \mathbb{N} : na_n \geq \varepsilon_0\} \notin \mathcal{I}_c^g$. Hence from the definition of ideal $\mathcal{I}_c^g$ we get $\sum_{n\in A_{\varepsilon_0}} \frac{1}{g(n)} = \infty$. For $n \in A_{\varepsilon_0}$ we have $na_n \geq \varepsilon_0$. Using properties (i) and (ii) we have

$$0 < g(\varepsilon_0) \leq g(na_n) \leq g(n)g(a_n),$$
$$g(\varepsilon_0)\frac{1}{g(n)} \leq g(a_n) \text{ for every } n \in A_{\varepsilon_0}.$$

Therefore,

$$\infty = g(\varepsilon_0) \sum_{n\in A_{\varepsilon_0}} \frac{1}{g(n)} \leq \sum_{n\in A_{\varepsilon_0}} g(a_n).$$

Therefore, it must also be $\sum_{n=1}^{\infty} g(a_n) = \infty$, and this is a contradiction. □

Problem 2. *To find the necessary condition for a function* $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ *such that a modified Olivier's theorem is true for an associated ideal* $\mathcal{I}_c^g$ *with the function* g.

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