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## Cyclic disjointness of Hamiltonian tours

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# Cyclic disjointness of Hamiltonian tours 

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#### Abstract

This papers studies a new notion of difference between Hamiltonian tours in the complete graph $K_{n}$ which we call cyclic disjointness: the distance between any two vertices in one tour must be different from their distance in the other tour. We state some theoretical results on cyclic disjointness and prove a main result of existence of pairs of cyclic disjoint tours dependent on $n$. We do this by showing equivalence to the toroidal $n$-queens problem. Next, we generalize our problem from pairs of cyclic disjoint tours to sets of pairwise cyclic disjoint tours. Finally, we provide a simple heuristic algorithm that gives a solution to the Peripatetic Salesman Problem, but with cyclic disjoint tours instead of edge-disjoint tours.


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## 1 Introduction

Optimization problems are an area in mathematics with many applications in real life. Combinatorial optimization is specifically useful for applications in logistics and planning. One typical example is that of the Traveling Salesman Problem (the TSP for short), where we want to find a tour visiting all points in some network exactly once while traveling the least distance. In the abstract version, this network is a graph consisting of a set of vertices and a set of weighted edges between the vertices, and the goal is to find the tour with the least total weight of all edges. This abstract version, or simple variants of it, has many applications and appears even in less obvious problems.

When solving or optimizing a real-world problem, sometimes not every aspect of that problem can be modeled mathematically. This limits the value of a single mathematically optimal solution. A user of an algorithm (for the traveling salesman problem) might have some additional practical insights into the problem and its solutions, and see some disadvantage of the optimal solution. In such a situation it is preferable to see some alternate solutions to the problem, and then decide which one would be best in real life.

The algorithm or solver could take the second best solution as an alternative, but, in the case of the TSP, it is likely that the resulting tour is almost exactly the same as the optimal tour, with only a small part of the tour being different. For a decision maker, this difference may in practice not be significant, and so we will want to find another solution that is 'more different'. It is clear we need some notion of diversity between tours.

An existing variant of TSP, the Peripatetic Salesman Problem (the PSP), first introduced by Krarup [1] has the objective to find two Hamiltonian tours with minimal total length or weight. Here the tours have to be edge disjoint. Two tours are edge disjoint when any edge of the graph is used by at most one of the two tours.

The purpose of this report is to look into a new notion of distinctness between Hamiltonian tours: cyclic disjointness. It is different to (and in fact stronger than) edge disjointness. Such a condition can then be at the basis of any diverse variant of a problem using Hamiltonian tours.

After some preliminaries in Section 2 , we introduce the condition of cyclic disjointness in Section 3. There we study some of its properties and the existence of pairs of cyclic disjoint Hamiltonian tours in graphs of certain size, both theoretically and computationally. To prove a main result on this existence, we first show equivalence to another problem, the toroidal $n$-queens problem.
In Section 4 we extend our problem: from pairs of cyclic disjoint tours to sets of pairwise cyclic disjoint tours. We study the cardinality of such a set, again depending on the size of the graph. For this we consider a very special class of Hamiltonian tours, which have a regular structure, to improve the bounds on this size. Finishing the section, we state the results of a search for our extended problem in the literature of the toroidal $n$-queens problem, and state our computational results.
Finally, in Section5 we state a simple, heuristic, algorithm for a slight variant of the PSP limited to two cyclic disjoint tours instead of edge disjoint tours.

## 2 Preliminaries and notation

Given a (simple undirected) graph $G$ with a set of vertices and a set of edges between them. A Hamiltonian tour in $G$ is a path that goes through each vertex exactly once and begins and ends in the same vertex, i.e. it is a cycle.

A graph does not necessarily contain a Hamiltonian tour, and the problem of finding out if it does is a hard problem itself [2]. So, for simplicity, in this report we only consider complete graphs, where every pair of vertices has an edge connecting them. The graph $K_{n}$ is the complete graph with $n$ vertices. In such a graph there always exist Hamiltonian tours. We label the vertices in the graph arbitrarily from 1 to $n$.

We define $\mathbb{Z}_{n}=\{1, \ldots, n\}$. We consider the bijection $\sigma: \mathbb{Z}_{n} \rightarrow \mathbb{Z}_{n}$ to be a Hamiltonian tour, with the permutation $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ being the sequence of the cities or vertices in the tour. Since the tour is a cycle, we consider a rotation of $k, \sigma(i+k), i \in \mathbb{Z}_{n}$, or a reflection $\sigma(-i)$, to be equivalent. Since $\sigma$ is a bijection, we denote its inverse $\sigma^{-1}$, such that for all $i, j \in \mathbb{Z}_{n}$ we have $\sigma^{-1}(j)=i \Longleftrightarrow \sigma(i)=j$.

Let $\sigma$ and $\tau$ be two Hamiltonian tours (or equivalently two arbitrary permutations of $\mathbb{Z}_{n}$ ). Then the function composition of $\sigma$ and $\tau$ is $\sigma \circ \tau$, where we first apply $\tau$ and then apply $\sigma$ to the result, i.e. $\forall i \in \mathbb{Z}_{n}:(\sigma \circ \tau)(i)=\sigma(\tau(i))$. Then we have that $\sigma \circ \tau$ also is a permutation, thus also a Hamiltonian tour.

Finally, we define $i d$ to be the trivial tour $(1,2, \ldots, n)$, i.e. the identity function $i d(x)=x$.

$$
\begin{aligned}
i d & =(1,2,3,4,5,6,7) \\
\sigma & =(1,6,4,3,5,7,2) \\
\sigma_{2} & =(5,7,2,1,6,4,3) \\
\sigma_{3} & =(1,2,7,5,3,4,6) \\
\sigma_{4} & =(5,3,4,6,1,2,7) \\
\tau & =(1,4,3,2,7,6,5) \\
\sigma \circ \tau & =(1,3,4,6,2,7,5)
\end{aligned}
$$

Figure 1: Examples of tours in $\mathbb{Z}_{7}$. First the identity tour $(i d)$. Secondly a tour $(\sigma)$, a version of it rotated by 4 , or 3 depending on the direction, $\left(\sigma_{2}\right)$, a reflected version $\left(\sigma_{3}\right)$, and a version that is both rotated by 4 , or 3 depending on direction, and reflected $\left(\sigma_{4}\right)$. Note that these are all equivalent. Finally a different tour $(\tau)$, and the composition of $\sigma$ and $\tau$ are shown.

## 3 Cyclic disjoint tours

As stated before, we will be looking at a strong sense of difference between two tours, namely that of cyclic disjointness: that given any two vertices in the graph, the distance between them in the first tour is different then their distance in the second tour.
Definition 1 (Distance in a tour). Given is a (Hamiltonian) tour $\sigma$ in a graph of $n$ vertices, and two vertices $v, w$. Let $i, j$ be the indices in the tour such that $v=\sigma(i), w=\sigma(j)$. Then we define the distance $\operatorname{dist}_{\sigma}(v, w)$ between $v$ and $w$ within tour $\sigma$ to be the smallest of $i-j \bmod n$ and $-i+j \bmod n$, i.e. $\operatorname{dist}_{\sigma}(v, w)=\min (i-j \bmod n, j-i \bmod n)$.
Since a tour is a cycle, when determining the distance between vertices with indices $i$ and $j$, we calculate $i-j$ modulo $n$. Secondly, since we can count the distance in one direction of the tour or the reverse direction, we also count $j-i$. As an example, consider the $\sigma=(1,6,4,3,5,7,2)$ from Figure 1 Then the distance between 1 and 4 clearly is 2 , so is the distance between 3 and 7 . If we consider 1 and 7 , their distance is not 5 , however: since a tour is a cycle, vertex 1 and 2 are also connected, so we go from point 7 to 2 and point 2 to 1 for a distance of 2 . In this case, using their indices, their position in the tour, 1 and 6 in this case, we see indeed $1-6 \bmod 7=2$.

Definition 2 (Cyclic disjointness). Given are two Hamiltonian tours $\sigma$ and $\tau$ in a graph $G$. Then they are called cyclic disjoint if for any two vertices $v, w \in G$ we have that $\operatorname{dist}_{\sigma}(v, w) \neq \operatorname{dist}_{\tau}(v, w)$.
Remark. Note that this means that cyclic disjointness is stronger than edge disjointness. Consider we have two cyclic disjoint tours, then we have that if an edge is in one of the tours, the two vertices it connects have distance 1. Since these two vertices must have a different distance in the other tour, the edge between them cannot be in the other tour. Thus the tours are also edge disjoint.

Given the definitions for distance and cyclic disjointness, we can list some basic properties:
Proposition 1. Let $\sigma, \tau$ be two cyclic disjoint tours, let $i d$ be the trivial tour $(1,2, \ldots, n)$, let $\alpha$ be an arbitrary bijection in $\mathbb{Z}_{n}$. Let $v, w$ be vertices in the graphs and $i, j \in \mathbb{Z}_{n}$ be two indices. Note that $\sigma \circ \tau$ is the function composition of $\sigma$ and $\tau$. Then we have the following:

1. $\operatorname{dist}_{\sigma}(v, w)=\operatorname{dist}_{i d}\left(\sigma^{-1}(v), \sigma^{-1}(w)\right)=\min \left(\sigma^{-1}(v)-\sigma^{-1}(w) \bmod n, \sigma^{-1}(w)-\sigma^{-1}(v) \bmod n\right)$
2. $\operatorname{dist}_{\alpha \circ \sigma}(v, w)=\operatorname{dist}_{\sigma}\left(\alpha^{-1}(v), \alpha^{-1}(w)\right)$
3. Let $\sigma^{\prime}(i)=\sigma(p+q i)$, with $p$ an integer and $q= \pm 1$, so $\sigma^{\prime}$ is $\sigma$ rotated and/or reflected and thus an equivalent tour. Then $\sigma^{\prime}$ is cyclic disjoint with $\tau$ if and only if $\sigma$ is cyclic disjoint with $\tau$.
4. If $\sigma$ and $\tau$ are cyclic disjoint, then also $\alpha \circ \sigma$ and $\alpha \circ \tau$ are cyclic disjoint. Note that in this case $\sigma \circ \alpha$ and $\tau \circ \alpha$ are not necessarily cyclic disjoint.

Proof. Properties 1 through 3 can easily be shown using the definitions of distance and cyclic disjointness and property 4 follows from 2 .

### 3.1 Existence: a first investigation

Since cyclic disjointness is a stronger property than edge disjointness, we might ask the question whether there even exist two cyclic disjoint tours in a graph? Or even in a complete graph? If there do exist such tours, then how many different ones are there? For a complete graph of $n$ vertices, there are $(n-1)$ !/2 possible distinct tours, so there are $\binom{(n-1)!/ 2}{2}$ possible pairs of tours, is it possible none of these pairs are cyclic disjoint?
Remark. We do not have to look for two arbitrary tours that are cyclic disjoint. If there are two such tours, then by Property 4 of Proposition 1 above we can transform them such that one of them is the identity tour $i d$, effectively relabeling the vertices of the graph. Therefore, for existence, it is sufficient to look for a tour that is cyclic disjoint with the trivial tour. Using this we can we can try to find a pair in $K_{n}$ for some small $n$.
Note that for $n \leq 4$ there is not even an edge disjoint pair of tours, so we consider only $n \geq 5$. So we first try for $K_{5}$. As the first tour of the pair we take the the trivial tour $i d$. If our second tour is to be cyclic disjoint, it must also be edge disjoint with $i d$. We can easily see the only edge disjoint tour is ( $1,3,5,2,4$ ), shown in Figure 2 Trying all the pairs of vertices, one indeed sees that these two tours are truly cyclic disjoint. Therefore, for $n=5$ we know cyclic disjoint tours exist in the complete graph.


Figure 2: $(1,2,3,4,5)$ and $(1,3,5,2,4)$, two cyclic disjoint tours in $K_{5}$.

We can try the same for $n=6$. Now there is more than one edge disjoint tour, so we can go step by step. If we start at vertex 1 , then the possible next vertices are 3,4 and 5 to ensure edge-disjointness. If we choose 3 , then the next one can be 5 or 6 . Say we choose 6 , next we choose $4,2,5$, and finally back to 1 , we get the tour $(1,3,6,4,2,5)$. This tour is indeed edge disjoint with the trivial tour, but not cyclic disjoint. For example, vertices 3 and 5 have distance 2 in this tour, but also in the trivial tour. There are more tours that can be found, a few of them can be seen in Figure 3 but none of these are cyclic disjoint with $i d$. Were we to exhaust this trial and error process, we would come to the conclusion that there does not exist any cyclic disjoint pair in $K_{6}$.

For larger $n$, trying tours by hand is less feasible. For $7,11,13$, one can fairly easily find a cyclic disjoint example, see Figure 4 For 8, 9, 10, 12, 14, 15, it would seem impossible. An exhaustive search by hand is out of the question at this point, so a definitive answer of existence is still lacking. Additionally, it is hard to infer a correct pattern from this sequence for which $n$ there exists a pair of cyclic disjoint tours. Existence only for odd $n$ is clearly not the case, since we cannot find a cyclic disjoint pair for 9 and 15 . A second possibility would be only prime $n$, but here $n=25$ is the first counterexample where there does exist a pair.

(a) Tours id and (1,3,6,4,2,5).

The distance between (for example) 3 and 5 is the same in both tours.

(b) Tours id and (1,4,2,6,3,5).

The distance between (for example) 1 and 3 is the same in both tours.

(c) Tours id and (1,4,2,5,3,6).

The distance between (for example) 4 and 6 is the same in both tours.

Figure 3: Different possible attempts to find a cyclic disjoint tour in $K_{6}$. All edge disjoint possibilities one can find are symmetries of these.

### 3.2 A computational search

Where trying by hand is not feasible anymore, we can turn to a computer and use it to search for cyclic disjoint pairs, or give us a guarantee there does not exist any pair for a certain $n$. Since there are $n$ ! possible tours, a naive search already becomes infeasible for small $n$, but even with good models and software the problem quickly grows.

As before, without loss of generality, we can take one of our tours to be the trivial tour, and limit our search for tours that are cyclic disjoint with that one. Aside from greatly reducing the solution space from only having to find one tour instead of two, this additionally has the advantage that the trivial tour is easy to reason about, due to the simplicity of its distance function $\operatorname{dist}_{i d}(i, j)=\min (i-j \bmod n, j-i \bmod n)$. Using this, the following lemma holds.

Lemma 1. Let $\sigma$ be a Hamiltonian tour in $K_{n}$. If for any distinct $i, j \in \mathbb{Z}_{n}$ we have that $\sigma(i)-\sigma(j) \equiv$ $\pm(i-j)(\bmod n)$, then $\sigma$ is not cyclic disjoint with the trivial tour id. The reverse is also true.

Proof. Let $i, j \in \mathbb{Z}_{n}$ be distinct such that $\sigma(i)-\sigma(j) \equiv i-j(\bmod n)$. Let $d=i-j$. Then $\sigma(i-j+j)=$ $\sigma(j+d)$, so we get $\sigma(j+d)-\sigma(j)=d$. Now let $\sigma(j)=v$ and $\sigma(j+d)=w$, these are two distinct vertices with $w-v=d$ and therefore

$$
\operatorname{dist}_{i d}(v, w)=\min (v-w \bmod n, w-v \bmod n)=\min (d \bmod n,-d \bmod n)=\min (d, n-d)
$$

It is also clear from the indices $j$ and $j+d$ that also $\operatorname{dist}_{\sigma}(v, w)=\min (d, n-d)$. Thus $\operatorname{dist}_{i d}(v, w)=$ $\operatorname{dist}_{\sigma}(v, w)$, and $\sigma$ and $i d$ are not cyclic disjoint.

Now let $i, j \in \mathbb{Z}_{n}$ be distinct such that $\sigma(i)-\sigma(j) \equiv j-i(\bmod n)$. Let $d=j-i$, and we get $\sigma(i+d)-\sigma(i)=d$, and the same argument as above holds. This proves the forward implication.
As for the other direction, let $\sigma$ not be cyclic disjoint with the trivial tour. Then there must be distinct $i, j \in \mathbb{Z}_{n}$ with $\sigma(i)=v$ and $\sigma(j)=w$ such that $\operatorname{dist}_{i d}(v, w)=\operatorname{dist}_{\sigma}(v, w)$. Let $d$ be this distance. Then we have $\operatorname{dist}_{\sigma}(v, w)=d$ and so $i-j=\min (d, n-d)$. From $\operatorname{dist}_{i d}(v, w)=d$ we get that $v-w=\min (d, n-d)$. Thus as a result

$$
\sigma(i)-\sigma(j)=v-w \equiv \pm d \equiv \pm(i-j) \quad(\bmod n)
$$

We can do a computational search using integer programming. The following model describes a tour that

(a) Two cyclic disjoint tours in $K_{7}$ : id and ( $1,3,5,7,2,4,6$ ).

(b) Two cyclic disjoint tours in $K_{11}$ : id and (1,4,7,10,2,5,8,11,3,6,9).


(c) Two cyclic disjoint tours in $K_{13}$ : id and
(1,5,9,13,4,8,12,3,7,11,2,6,10).
(d) These two tours in $K_{9}$, id and ( $1,3,5,7,9,2,4,6,8$ ), would seem cyclic disjoint, but for example vertices 2 and 5 have distance 3 in both tours.

Figure 4: Some examples of tours that can be found in $K_{n}$ for larger $n$. One might first expect that such a "regular" tour would also work for $n=9$, but this is not the case.
is cyclic disjoint with the trivial tour:

$$
\begin{array}{ll}
p_{i j} \in\{0,1\} & \forall i, j \in\{1, \ldots, n\} \\
\sum_{j=1}^{n} p_{i j}=1 & \forall i \in\{1, \ldots, n\} \\
\sum_{i=1}^{n} p_{i j}=1 & \forall j \in\{1, \ldots, n\} \\
p_{i j}+p_{k l} \leq 1 & \forall i, j, k, l \in\{1, \ldots, n\}: i<k,|i-k|=|j-l| \\
p_{i j}+p_{k l} \leq 1 & \forall i, j, k, l \in\{1, \ldots, n\}: i<k,|i-k+n|=|j-l| \\
p_{11}=1 & \tag{6}
\end{array}
$$

Here the variable $p_{i j}$ represents whether we place vertex $j$ on the $i$ th place of the tour or not, i.e. if $\sigma(i)=j$ with $\sigma$ as the resulting tour. Constraints 2 and 3 make sure each place in the tour gets assigned exactly one vertex, and each vertex is assigned exactly once. Constraints 4 and 5 are the constraints that enforce the cyclic disjointness with $(1, \ldots, n)$, with each constraint considering distances in one 'direction' of the tour. This implicitly uses Lemma 1, namely that if $j-l=\sigma(i)-\sigma(k) \equiv \pm(i-k)(\bmod n)$ holds we then must forbid $\sigma(i)=j$ and $\sigma(k)=l$ to simultaneously both be true. Finally, constraint 6 ensures that the first spot in the tour is vertex 1. This is to remove the symmetry in the starting point of the tour.

A computational search up to $n=41$ gave the following results. For all $n$ where $2 \nmid n$ and $3 \nmid n$ there exists at least one pair of cyclic disjoint tour in $K_{n}$. For $n \leq 11$ the tours that were found where very "regular", such as the tours in Figure 4 with a fixed increment between consecutive vertices of the tour.

Such tours will be significant later in the report, and are further described in Section 4.2 For $n>11$, however, the search resulted in more 'random' tours. One such tour, for $n=13$, is shown in Figure 5


Figure 5: A tour in $K_{13}$ cyclic disjoint with the trivial tour:

$$
(1,8,6,12,3,11,4,10,5,9,2,13,7)
$$

For $n \leq 26$ where $2 \mid n$ or $3 \mid n$ the solver indicated that the problem was infeasible, i.e. that there does not exist a pair of cyclic disjoint tours. For larger $n>26$ the search was not exhaustive, and stopped early due to time constraints. Note however, for what its worth, that the running time for different $n$ when a tour was found, was significantly shorter.
For software we used IBM CPLEX (version 22.1) both from within the AIMMS environment and directly using Python bindings using DOcplex.MP and IBM ILOG CPLEX Optimization Studio. Setting the "feasibility pump switch" to emphasize feasible solutions and increasing the "MIP heuristic effort" somewhat, from its default to 1 to 4 or 5 , increased the speed of finding a solution (if one could be find), but not neccesarily the speed of showing infeasibility.

These results strongly suggest that cyclic disjoint tours only exist when $n$ is not a multiple of 2 or 3 . Is this also the case for larger $n$ ? We will find out that this is indeed true, although the result has a perhaps surprising source.
Theorem 1. There exist no pair of cyclic disjoint tours in $K_{n}$ if $2 \mid n$ or $3 \mid n$.
This result will be proven in the next section, Section 3.3

### 3.3 An equivalent problem and existence result

The $n$-queens problem is a well-known problem where the goal is to place $n$ queens on an $n \times n$ chessboard such that none of the queens can capture another. A variant of this problem is that of the toroidal $n$-queens problem, sometimes called the modular $n$-queens or $n$-superqueens problem. A definition, introduction, and large overview of results and research of both the regular $n$-queens and the toroidal variant can be found in [3].

Where on a normal chessboard the diagonals stop at the edge of the board, in the toroidal variant we let them 'wrap around' the board, such that diagonals that usually stop at the top edge of the board continue at the bottom instead (and vice versa), and diagonals that stop at the right edge continue at the left (and vice versa). This results in exactly $2 n$ diagonals of length $n$ : $n$ diagonals from the top left to bottom right, and $n$ diagonals from bottom left to top right. By connecting opposing sides of the chessboard, the board becomes a torus, hence the name. Figure 6 shows an example of how the diagonals would run.

The goal of the (toroidal) $n$-queens problem is to place $n$ queens such that they do not capture each other. Because each queen can reach a lot more squares, placing $n$ of them is harder then the in the regular $n$-queens problem.

Remark. One property of this modular board is that, given a square in row $i$ and column $j$ (coordinate $(i, j)$ ), we can find which diagonals it belongs to. The top-left to bottom-right diagonal is uniquely identified by calculating $i-j \bmod n$, and the bottom-left to top-right diagonal by $i+j \bmod n$. Note how the modulo operator appears here, and we indeed get $2 n$ diagonals in total, $n$ for each direction. Figure 7 clearly shows what this looks like.


Figure 6: An example of the toroidal chessboard. The lines show where the queen can travel.

|  | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 0 | 7 | 6 | 5 | 4 | 3 | 2 | 1 |
| 2 | 1 | 0 | 7 | 6 | 5 | 4 | 3 | 2 |
| 3 | 2 | 1 | 0 | 7 | 6 | 5 | 4 | 3 |
| 4 | 3 | 2 | 1 | 0 | 7 | 6 | 5 | 4 |
| 5 | 4 | 3 | 2 | 1 | 0 | 7 | 6 | 5 |
| 6 | 5 | 4 | 3 | 2 | 1 | 0 | 7 | 6 |
| 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 | 7 |
| 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 |

(a) The top-left to bottom-right diagonals are all identified by subtracting the column index from the row index (modulo $n$ ).

|  | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 0 | 1 |
| 2 | 3 | 4 | 5 | 6 | 7 | 0 | 1 | 2 |
| 3 | 4 | 5 | 6 | 7 | 0 | 1 | 2 | 3 |
| 4 | 5 | 6 | 7 | 0 | 1 | 2 | 3 | 4 |
| 5 | 6 | 7 | 0 | 1 | 2 | 3 | 4 | 5 |
| 6 | 7 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 7 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 8 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 0 |

(b) The bottom-left to top-right diagonals are all identified by adding the column index and the row index (modulo $n$ ).

Figure 7: Using the column and row coordinates of a square, we can find to which diagonals it belongs. Again, note how the modular board 'wraps around'.

This means that if we have two queens, one on $\left(i_{1}, j_{1}\right)$ and the other on $\left(i_{2}, j_{2}\right)$, then they can attack each other if:

- they are on the same row, in that case $i_{1}=i_{2}$,
- they are on the same column, then $j_{1}=j_{2}$,
- they are on the same top-left to bottom-right diagonal, so $i_{1}-j_{1} \equiv i_{2}-j_{2}(\bmod n)$,
- or they are on the same bottom-left to top-right diagonal, so $i_{1}+j_{1} \equiv i_{2}+j_{2}(\bmod n)$.

Definition 3 (Solution to the toroidal $n$-queens problem). Let $n \in \mathbf{N}$. The set of $n$ squares $\left(i_{1}, j_{1}\right)$ through $\left(i_{n}, j_{n}\right)$ on the $n \times n$ toroidal chessboard, where $i$ is the row index and $j$ the column index, is a solution to the toroidal n-queens problem if we can place queens on all these $n$ squares and none can capture each other.

Let $\sigma: \mathbb{Z}_{n} \rightarrow \mathbb{Z}_{n}$ be a bijection. Then $\sigma$ is a solution to the toroidal $n$-queens problem if the set of $n$ squares $(i, \sigma(i))$ for $i \in \mathbb{Z}_{n}$ is a solution as defined above.
It is clear that a solution must be a bijection, otherwise there would be multiple queens in a column. Figure 8 shows an example solution for $n=5$.


Figure 8: A solution for the toroidal 5 -queens problem. Queens are on positions $(1,1),(2,4),(3,2),(4,5),(5,3)$.

It turns out that the problem of finding a tour that is cyclic disjoint with the trivial tour $(1, \ldots, n)$ is equivalent with finding a solution to the toroidal $n$-queens problem. Figure 9 has an example for the case of $n=7$.

Theorem 2 (Equivalence of cyclic disjointness and toroidal $n$-queens). Let $\sigma: \mathbb{Z}_{n} \rightarrow \mathbb{Z}_{n}$ be a bijection. Then the tour defined by $\sigma$ is cyclic disjoint with the trivial tour $(1, \ldots, n)$ if and only if $\sigma$ is a solution to the toroidal n-queens problem.

Proof. We prove that $\sigma$ is not cyclic disjoint with $i d$ if and only if $\sigma$ is not a solution to the toroidal $n$-queens problem.
Suppose $\sigma$ is not cyclic disjoint with the trivial tour $i d$. Then according to Lemma 1 there exist $i, j \in \mathbb{Z}_{n}$ such that $\sigma(i)-\sigma(j) \equiv \pm(i-j)(\bmod n)$. Then consider the squares on the $n$ by $n$ chessboard $(i, \sigma(i))$ and $(j, \sigma(j))$. If $\sigma$ is a solution to the $n$-queens problem, then these squares cannot be in the same diagonals. However, if $\sigma(i)-\sigma(j) \equiv \pm(i-j)(\bmod n)$ then $\sigma(i)-i \equiv \sigma(j)-j(\bmod n)$, so they are in the same top-left to bottom-right diagonal. In the other case, we have $\sigma(i)+i \equiv \sigma(j)+j(\bmod n)$, and they are on the same bottom-left to top-right diagonal. In any case, $\sigma$ is not a valid solution to the toroidal $n$-queens problem.

Now let $\sigma$ not be a valid solution to the toroidal $n$-queens problem, but still a bijection. Then there are two queens on $(i, \sigma(i))$ and $(j, \sigma(j))$ that are on the same diagonal. Then either $\sigma(i)-i \equiv \sigma(j)-j$ $(\bmod n)$ or $\sigma(i)+i \equiv \sigma(j)+j(\bmod n)$, and in either case we have $\sigma(i)-\sigma(j) \equiv \pm(i-j)(\bmod n)$ and by Lemma $1 \sigma$ is not cyclic disjoint with the trivial tour.

A known fact in the literature of the toroidal $n$-queens problem, is that solutions do not exist if $n$ is a multiple of 2 or 3 . If that is the case, we can only place at most $n-1$ or $n-2$ queens before all squares on the board can be attacked by the queens, so no new queens can be added at that point [4]. This result was first shown by Pólya in 1918 [5] with all subsequent direct proofs by different authors being essentially the same [3]. The proof uses the following property:

Lemma 2. If the bijection $\sigma: \mathbb{Z}_{n} \rightarrow \mathbb{Z}_{n}$ is a solution to the toroidal n-queens problem, then $\sigma+i d$ $(\bmod n)$ and $\sigma-i d(\bmod n)$ are also bijections.
From the toroidal $n$-queens perspective, this result follows directly from Definition 3, stating when $\sigma$ is a solution, and the conditions of when two queens can attack each other. Since $\sigma$ already is a bijection, all $\sigma(i)$ clearly are different for all $i$, and the queens are all on different rows and columns. Additionally, since none of the queens can be on the same diagonals, $i+\sigma(i)$ and $i-\sigma(i)$ must also all be different for all $i$. We can also look at the result from the perspective of cyclic disjointness, and use Lemma 1 to prove it:

Proof of Lemma 2, Let $i, j \in \mathbb{Z}_{n}$ be distinct. Since $\sigma$ is a solution to the toroidal $n$-queens problem, it is also cyclic disjoint with the trivial tour $(1, \ldots, n)$. Then following Lemma 1:

$$
\begin{array}{rlr}
\sigma(i)-\sigma(j) \not \equiv j-i & (\bmod n) \\
\sigma(i)+i \not \equiv \sigma(j)+j & (\bmod n) \\
(\sigma+i d)(i) \not \equiv(\sigma+i d)(j) & (\bmod n)
\end{array}
$$


(a) Shift of 2 , the red tour: $(1,3,5,7,2,4,6)$

(b) Shift of 3, the blue tour: $(1,4,7,3,6,2,5)$

(d) The shifted tours in $K_{7}$.

Figure 9: All possible different shifted tours of $K_{7}$, and their corresponding solution to the $n$-queens problem.

So we know $\sigma+i d$ is injective. Since $\mathbb{Z}_{n}$ is finite, $\sigma+i d$ must be a bijection. We also know $\sigma(i)-\sigma(j) \not \equiv i-j$ $(\bmod n)$, and the same way we find $\sigma-i d$ is injective and also bijective.

Using this property we continue to the real theorem.
Theorem 3 (Pólya). The toroidal n-queens problem has a solution if and only if $2 \nmid n$ and $3 \nmid n$.
Proof. Suppose there is a solution $\sigma$. Let $n$ be even. We know $\sum_{i=1}^{n} i=\frac{n(n+1)}{2}$. Since $\sigma$ and $\sigma+i d$ are bijections, summing $\sigma(i)$ or $(\sigma+i d)(i)$ instead of $i$ results in effectively summing the same numbers, but in a different order. Therefore

$$
\sum_{i=1}^{n} \sigma(i)=\sum_{i=1}^{n}(\sigma+i d)(i)=\sum_{i=1}^{n} i=\frac{n(n+1)}{2} .
$$

Furthermore, we know $\frac{n(n+1)}{2}=\frac{n^{2}}{2}+\frac{n}{2}$. The first term $\frac{n^{2}}{2}$ has a factor $n$ and is equivalent to 0 modulo $n$, so we are left with $\frac{n}{2}$. If $n$ were odd, we would have that $\operatorname{gcd}(n, 2)=1$, so 2 would have a multiplicative inverse and we would get $\frac{n}{2} \equiv 2^{-1} n \equiv 0(\bmod n)$. If $n$ is even, this is not the case, and therefore

$$
\sum_{i=1}^{n} \sigma(i)=\sum_{i=1}^{n}(\sigma+i d)(i)=\sum_{i=1}^{n} i=\frac{n(n+1)}{2} \equiv \frac{n}{2} \not \equiv 0 \quad(\bmod n) .
$$

However, then we have both

$$
\sum_{i=1}^{n}(\sigma+i d)(i) \equiv \frac{n}{2} \quad(\bmod n)
$$

and

$$
\sum_{i=1}^{n}(\sigma+i d)(i) \equiv \sum_{i=1}^{n} \sigma(i)+\sum_{i=1}^{n} i \equiv \frac{n}{2}+\frac{n}{2} \equiv 0 \quad(\bmod n) .
$$

And we have a contradiction. This finishes the case of $2 \mid n$.

Now let $n$ be odd and a multiple of 3 . We know $\sum_{i=1}^{n} i^{2}=\frac{n(n+1)(2 n+1)}{6}$. Then in the same manner, since $\sigma$ and $\sigma+i d$ and $\sigma-i d$ all are bijections, we have

$$
\sum_{i=1}^{n} \sigma(i)^{2}=\sum_{i=1}^{n}(\sigma(i)+i)^{2}=\sum_{i=1}^{n}(\sigma(i)-i)^{2}=\sum_{i=1}^{n} i^{2} \equiv \frac{n(n+1)(2 n+1)}{6} \equiv \frac{n}{6} \quad(\bmod n)
$$

Then we have

$$
\begin{aligned}
\sum_{i=1}^{n}(\sigma(i)+i)^{2}-\sum_{i=1}^{n}(\sigma(i)-i)^{2} & =0 \\
\sum_{i=1}^{n}\left(\sigma(i)^{2}+2 \sigma(i) i+i^{2}-\sigma(i)^{2}+2 \sigma(i) i-i^{2}\right) & =0 \\
2 \sum_{i=1}^{n} 2 \sigma(i) i & =0 \\
\sum_{i=1}^{n} 2 \sigma(i) i & =0
\end{aligned}
$$

We know $n \nmid 2$. If also $n \nmid 3$, then $\operatorname{gcd}(n, 6)=1$ and $\frac{n}{6} \equiv 6^{-1} n \equiv 0(\bmod n)$. But since $n \mid 3$, this is not the case. Additionally, this means $\frac{n}{3} \not \equiv 0(\bmod n)$. So we have both

$$
\sum_{i=1}^{n}(\sigma(i)+i)^{2} \equiv \frac{n}{6} \quad(\bmod n)
$$

and

$$
\sum_{i=1}^{n}(\sigma(i)+i)^{2} \equiv \sum_{i=1}^{n} \sigma(i)^{2}+\sum_{i=1}^{n} 2 \sigma(i) i+\sum_{i=1}^{n} i^{2} \equiv \frac{n}{6}+0+\frac{n}{6} \equiv \frac{n}{3} \quad(\bmod n)
$$

And again, we reach a contradiction. Thus if we have a solution, then $2 \nmid n$ or $3 \nmid n$.
The other implication is easy. If $2 \nmid n$ or $3 \nmid n$, then we take the tour $(1,3,5, \ldots, n-2, n, 2,4,6, \ldots, n-$ $3, n-1$ ). The distance between vertices $i$ and $j$ in this tour is the minimum of $\frac{i-j}{2} \equiv 2^{-1}(i-j) \bmod n$ and $2^{-1}(j-i) \bmod n$, this is different from the distance in the trivial tour, $\pm(i-j) \bmod n$. So this tour is cyclic disjoint with the trivial tour $(1, \ldots, n)$ and also a solution for the toroidal $n$-queens problem.

The proof of Theorem 1 that a pair of cyclic disjoint tours only exist in $K_{n}$ if $2 \nmid n$ and $3 \nmid n$, directly follows from Theorem 2 of equivalence of the problems and Theorem 3 of Pólya.

### 3.4 A constraint programming model

In Section 3.2 we described an integer programming (IP) model where we used binary decision variables $p_{i j}$ to represent whether we put vertex $j$ on the $i$ th spot of the tour. Using the insights of the previous section, especially Lemma 2 we can make a different model. This second model is stated both for the sake of completeness and since we will expand on this model and the earlier IP model in Section 4.4 later in the report.

$$
\begin{align*}
& x(i) \in \mathbb{Z}_{n} \quad \forall i \in \mathbb{Z}_{n}  \tag{7}\\
& \left|\left\{x(i) \mid i \in \mathbb{Z}_{n}\right\}\right|=n  \tag{8}\\
& \left|\left\{x(i)+i \bmod n \mid i \in \mathbb{Z}_{n}\right\}\right|=n  \tag{9}\\
& \left|\left\{x(i)-i \bmod \mid i \in \mathbb{Z}_{n}\right\}\right|=n \tag{10}
\end{align*}
$$

Note that $|\cdot|$ denotes the cardinality of the set in constraints (8) through 10). The variables $x(i)$ in this model denote the vertex we place on the $i$ th place in the tour (or the column $x(i)$ for row $i$ on which we place a queen). The three constraints (8) through (10) effectively enforce each of these sets by their cardinalities to be maximal, i.e. to have no duplicate elements, i.e. that all columns, diagonals in one directions, and diagonals in the other diagonals are different for all the queens. This is a common
way to model the $n$-queens problem, although in the regular, non-toroidal, version this is done without calculating modulo $n$ in constraint (9) and (10).
Constraint programming (CP) solvers can use AllDifferent constraints to efficiently reason about such a model. This formulation of the problem is therefore perfect for this, with only three of these constraints constituting the entire model.

We tried this model both using the IBM ILOG CP Optimizer (version 22.1), again both from within the AIMMS environment and directly using Python bindings using, this time using DOcplex.CP and IBM ILOG CPLEX Optimization Studio. In our experience, however, the CP solver is no faster than the IP solver with the model from earlier in the report.

## 4 Sets of cyclic disjoint tours

We now have a good view of cyclic disjoint tours, and specifically when they exist. We can extend our problem to finding triplets or larger sets of cyclic disjoint tours, instead of only pairs. The tours in such a set would have to be pairwise cyclic disjoint. In our analogy of a decision maker, this also is a natural next problem. If the decision maker is not satisfied with the two resulting tours for their optimization problem for whatever reason, he will ask for a third alternative. This third tour would have to be different from both the first and the second proposed tours, otherwise it would not be of value.

Finding such a set is harder than finding only a pair, so we can ask whether a set of three even exists for some $n$ ? Or a set of four? How large can such a set get for $n$ ? In other words, given $n$, what is the largest set of pairwise disjoint tours that we can we find?

Definition 4. Let $K_{n}$ be the complete graph of $n$ vertices, and $T$ the set of Hamiltonian tours in $K_{n}$. Let $T^{\prime} \subseteq T$ be such that any pair of tours in $T^{\prime}$ is cyclic disjoint, and let $T^{\prime}$ be of maximal cardinality. We define $f(n)=\left|T^{\prime}\right|$.

### 4.1 First bounds for the size

We already have results on pairs of tours, or sets of size two, so we can already make a first statement on the bounds of $f(n)$ using the results of Theorem 1 If $2 \mid n$ or $3 \mid n$ then $f(n)=1$; we can clearly take a set of a single tour for the lower bound of 1 , and a pair does not exist. Additionally, we know that if $2 \nmid n$ and $3 \nmid n$, pairs of cyclic disjoint tours exist, so $f(n) \geq 2$.

An upper bound can also easily be found using the amount of edges in a complete graph of size $n$.
Proposition 2. For any $n \geq 3$ we have $1 \leq f(n) \leq\left\lfloor\frac{n-1}{2}\right\rfloor$.
Proof. The lower bound is trivial, any single tour suffices. For the upper bound: there are $\frac{n(n-1)}{2}$ vertices in $K_{n}$. A single tour takes up $n$ edges, so there are at most $\frac{n-1}{2}$ pairwise edge-disjoint tours, which is an upper bound for $f(n)$.

We know, in the case of edge-disjoint tours, the upper bound can truly be reached for any $n$. A construction for decomposing $K_{n}$ into $(n-1) / 2$ tours for odd $n$ or into $n / 2-1$ tours and a 1-factor for even $n$ is attributed by Walecki, as described in [6. The question whether this bound can be reached for any $n$ with cyclic disjoint tours is addressed in the next section.

### 4.2 The shift construction

In this section we describe a construction of tours which can give us an increased lower bound. For smaller $n$, when searching for a tour cyclic disjoint with the the trivial tour, one probably finds that the resulting tour is very 'regular', with a fixed increment between consecutive vertices of the tour. Examples are shown in Figure 2 and Figure 4 Drawing the tours in this manner gives a graph that is symmetric under rotation.

Consider the trivial tour id

$$
(1, \ldots, n)
$$

We have that each consecutive number has an increase of 1 . We can also increase (or decrease) by a different number $s$ going to the next point, calculating modulo $n$ :

$$
(1,1+s, 1+2 s, \ldots)
$$

If we take $n=5$ and $s=2$, for example, we get $(1,3,5,2,4)$. We can also use $s=3$, giving us $(1,4,2,5,3)$. Note that this tour is actually the reverse of the previous one. We call such a tour with a regular difference between consecutive numbers a shifted tour, and this difference $s$ is its shift.

Definition 5. Given n, we define the shifted tour $\sigma$ with shift $s$ to be the tour in $K_{n}$ given by $\sigma(i)=$ $(i-1) s+1$.
Remark. By this definition, we consider the trivial tour to be a shifted tour with shift 1 . Note that all these shifted tours are transformations of the trivial tour. Any shifted tour $\sigma$ has $\sigma(i)=i d(p+k i \bmod n)=$ $p+k i \bmod n$ for some integers $p$ and $q$.


Figure 10: Different shifted tours for $K_{7}$. Note that these are all the possibilities for shifted tours in $K_{7}$.

Proposition 3. Given is n. A shifted tour with shift $s$ only exists when $n$ and $s$ are coprime, i.e. when $\operatorname{gcd}(n, s)=1$. The shifted tour of shift $s$ is equivalent to the tour with shift $n-s \equiv-s(\bmod n)$, namely the reverse.

Proof. Let $s \in \mathbb{Z}_{n}$ such that $\operatorname{gcd}(n, s)=d>1$ and consider the shifted tour $\sigma$ with shift $s$. Let $k=\frac{n}{d}$, then $\sigma(1+k) \equiv(1+k-1) s+1 \equiv k s+1 \equiv \frac{n}{d} s+1 \equiv n \frac{s}{d}+1 \equiv 1 \equiv \sigma(1)(\bmod n)$. Thus $\sigma$ is not a bijection and also not a Hamiltonian tour.
Now let $s \in \mathbb{Z}_{n}$ such that $\operatorname{gcd}(n, s)=1$, let $\sigma$ be the shifted tour with shift $s$, and $\hat{\sigma}$ the shifted tour with shift $n-s \equiv-s$. Then $\hat{\sigma}(i) \equiv(i-1)(n-s)+1 \equiv(1-i) s+1 \equiv((2-i)+1) s+1 \equiv \sigma(2-i)(\bmod n)$. So indeed $\sigma$ and $\hat{\sigma}$ are each other's reverse and are equivalent.

As an example of when the shift construction breaks down, take $n=9$ and let $\sigma$ be the shifted tour with shift $s=3$. In this case, $\operatorname{gcd}(n, s)=3$ and we get have $\sigma=(1,4,7,1,4,7,1,4,7)$, and the construction breaks down. Additionally, we have that the shifted tour with $s=4,(1,5,9,4,8,3,7,2,6)$, and shift 5 , $(1,6,2,7,3,8,4,9,5)$, are indeed equivalent and reverse.

In the next theorem, $\phi(n)$ is Euler's totient function, being the amount of integers $1 \leq i<n$ such that $\operatorname{gcd}(n, i)=1$. There are various identities and formulas for it, but only two relevant properties are used in this report. The following definition and identities are taken from [7.
Definition 6 (Euler's totient function). For $n \geq 1$, let $\phi(n)$ denote the number of positive integers not exceeding $n$ that are relatively prime to $n$.
Proposition 4. For $n>1, \phi(n)=n-1$ if and only if $n$ is prime.
Theorem 4. For $n>2, \phi(n)$ is an even integer.
Now we continue to the next theorem. This is the main result on shifted tours, stating the conditions when two shifted tours are cyclic disjoint.

Theorem 5. Given is $n \geq 3$. Then there exist $\phi(n) / 2$ distinct shifted tours in $K_{n}$. Two shifted tours with shifts $s$ and $t$ are cyclic disjoint if and only if $\operatorname{gcd}(s+t, n)=1$ and $\operatorname{gcd}(s-t, n)=1$.

Proof. Shifted tours only exist for shift $s \in \mathbb{Z}_{n}$ when $\operatorname{gcd}(s, n)=1$, by Proposition 3 so there are exactly $\phi(n)$ possibilities. Since for any $s$ the equivalent reverse tour with shift $n-s$ is also in this set, the amount of distinct shifted tours is $\phi(n) / 2$.

Let $\sigma, \tau$ be two different shifted tours, with shift $s$ and $t$ respectively. We prove that $\sigma$ and $\tau$ are not cyclic disjoint if and only if $\operatorname{gcd}(s+t, n) \neq 1$ or $\operatorname{gcd}(s-t, n) \neq 1$.
Suppose $\sigma$ and $\tau$ are not cyclic disjoint. Then there exists distinct $a, b \in \mathbb{Z}_{n}$ such that $\operatorname{dist}_{\sigma}(a, b)=$ $\operatorname{dist}_{\tau}(a, b)$. Let $i, j, k, l \in \mathbb{Z}_{n}$ such that

$$
\begin{aligned}
\sigma(i) & \equiv(i-1) s+1 \equiv a \equiv(k-1) t+1 \equiv \tau(k) \quad(\bmod n) \\
\sigma(j) & \equiv(j-1) s+1 \equiv b \equiv(l-1) t+1 \equiv \tau(l) \quad(\bmod n)
\end{aligned}
$$

And consequently, with $s^{-1}$ and $t^{-1}$ being the multiplicative inverses of $s$ and $t$ modulo $n$ :

$$
\begin{aligned}
i \equiv(a-1) s^{-1}+1 & (\bmod n) \\
j & \equiv(b-1) s^{-1}+1 \\
k & (\bmod n) \\
k & \equiv(a-1) t^{-1}+1 \\
l & (\bmod n) \\
l(b-1) t^{-1}+1 & (\bmod n)
\end{aligned}
$$

We have that either for both tours the distance is counted "in the same direction" or both in a different direction.

$$
\begin{aligned}
\operatorname{dist}_{\sigma}(a, b) & =\operatorname{dist}_{\tau}(a, b) & & \\
i-j & \equiv \pm(k-l) & & (\bmod n) \\
(a-1) s^{-1}+1-(b-1) s^{-1}-1 & \equiv \pm\left((a-1) t^{-1}+1-(b-1) t^{-1}-1\right) & & (\bmod n) \\
s^{-1}(a-b) & \equiv \pm t^{-1}(a-b) & & (\bmod n) \\
\left(s^{-1} \pm t^{-1}\right)(a-b) & \equiv 0 & & (\bmod n)
\end{aligned}
$$

Note that $s^{-1} \pm t^{-1} \equiv 0(\bmod n)$ is equivalent to $s \pm t \equiv 0(\bmod n)$. So we have $n \mid(s \pm t)(a-b)$. If either $a-b$ or $s \pm t$ is coprime to $n$, then the other has to be a multiple of $n$, implying that $s=t$ or $a=b$, respectively. This, however, would contradict our assumptions.
So we need $\operatorname{gcd}(a, b)=d<n$ and $k>1$ such that $n=k d$. Then $k \mid(s \pm t)$. But then $\operatorname{gcd}(s+t, n) \geq k$ or $\operatorname{gcd}(s-t, n) \geq k$ which also is a contradiction. This proves that the forward implication.

For the other direction, suppose $\operatorname{gcd}(s \pm t, n)=k>1$ and $d>1$ such that $n=k d$. We consider vertices 1 and $1+d$. The index of $1+d$ in $\sigma$ is $d s^{-1}+1$ and in $\tau$ it is $d t^{-1}+1$. Since $1=\sigma(1)=\tau(1)$, we have

$$
\operatorname{dist}_{\sigma}(1,1+d)=\min \left(d s^{-1}+1-1 \bmod n, 1-d s^{-1}-1 \bmod n\right)= \pm d s^{-1} \quad(\bmod n)
$$

Similarly, we have

$$
\operatorname{dist}_{\tau}(1,1+d)= \pm d t^{-1} \quad(\bmod n)
$$

Now since $k \mid s \pm t$, we have $d \cdot(s \pm t) \equiv n \equiv 0(\bmod n)$. And again, since $s^{-1} \pm t^{-1} \equiv 0(\bmod n)$ is equivalent to $s \pm t \equiv 0(\bmod n)$, this means $d s^{-1} \equiv \pm d t^{-1}$. But then dist ${ }_{\sigma}(1,1+d) \equiv \pm d s^{-1} \equiv \pm d t^{-1} \equiv$ $\operatorname{dist}_{\tau}(1,1+d)(\bmod n)$, so $\sigma$ and $\tau$ are not cyclic disjoint.

The following result is of course no new result at all, only the special case of Theorem 1 when limited to only shifted tours. Nevertheless, it is nice to see there is a short and direct proof for it.
Corollary 1. There exist no pair of cyclic disjoint shifted tours for $K_{n}$ if $2 \mid n$ or $3 \mid n$.
Proof. Consider $n$ even. A shifted tour only exists for odd shifts. The sum or difference between two odd numbers is even, so not coprime with 2 . So any pair of possible shifted tours are not cyclic disjoint.

Now consider $3 \mid n$. Shifted tours exist only if they have a shift of $3 k+1$ or $3 k+2$. If two tours have shifts $3 k+1$ and $3 l+1$ or have shifts $3 k+2$ and $3 l+2$, then the difference of the two shifts is $3(k-l)$ which is a multiple of 3 , so they are not cyclic disjoint. If two tours have shifts $3 k+1$ and $3 l+2$, then the sum of the shifts is $3(k+l+1)$, so they also are not cyclic disjoint.

Corollary 2. For $n>3$ prime, all $\frac{n-1}{2}$ distinct shifted tours are pairwise cyclic disjoint.
Proof. This result is fairly trivial. When $\operatorname{gcd}(s-t, n) \neq 1$, we have $s \equiv t$, so these would not be distinct. The other possibility $\operatorname{gcd}(s+t, n) \neq 1$ implies $s \equiv-t$, which would mean one tour is the reverse of the other, so also not distinct.

This corollary improves our bounds for $f(n)$ in two ways. Firstly, this means that in the case that $n$ is prime, the upper bound stated in Proposition 2 is reached. Secondly, in the case of composite numbers, we know there are still shifted tours that are cyclic disjoint with each other, and we can create a pairwise cyclic disjoint set with these.

Consider $n=25$, then we know the shifted tours with shifts 1 and 2 are cyclic disjoint, but a third shift is impossible, having either $s \pm 1 \equiv 5$ or $s \pm 2 \equiv 5$. The same is the case for any $n$ with 5 as smallest prime factor. In the case the smallest factor is 7 , we can for example take shifts of 1,2 and 3 , and these are pairwise cyclic disjoint. A fourth shifted tour can not be cyclic disjoint anymore, in the same way as with $n=5$. In general, if $m$ is the smallest prime factor of $n$, then taking the shifted tours with shifts of $1,2, \ldots, \frac{m-1}{2}$ gives us a set of pairwise cyclic disjoint tours. This gives us a new lower bound.
Proposition 5. Let $n \geq 3$ with $m$ as smallest prime factor. Then we have $1 \leq\left\lceil\frac{m-1}{2}\right\rceil \leq f(n) \leq\left\lfloor\frac{n-1}{2}\right\rfloor$. These bounds are tight when $2|n, 3| n$ or $n$ is prime.
In conclusion, the shift-construction has to potential to be very useful, but are also very limited. They can easily be reasoned about and always give us a set of pairwise cyclic disjoint tours, especially for prime $n$. However, for composite $n$ we do not know if there might be a larger such set. Additionally, they give us an easy construction to find cyclic disjoint tours for any given tour and use them in real applications, as we will do in Section 5 But there are only very few shifted tours, only $\phi(n) / 2$, limiting the usability.

### 4.3 Non-shifted tours

To make a more general statement about $f(n)$ and its bounds, we probably need to look at tours that do not follow the shift-construction and when these are cyclic disjoint. If we want to find a set of three pairwise cyclic disjoint tours we can of course take the trivial tour as one of the tours. We have the results from previous sections for finding two other tours $\sigma$ and $\tau$ that are cyclic disjoint with id. However, ensuring these $\sigma$ and $\tau$ are also cyclic disjoint with each other has not been covered in depth.

All our previous results in Section 3 focus on cyclic disjointness with the trivial tour, not between two arbitrary tours. Results we used such as Lemma 1 and Lemma 2 can only be used by using $\sigma^{-1} \circ \tau$ and $\tau^{-1} \circ \sigma$, since these are both cyclic disjoint to $i d$. We were not able to find any other results for cyclic disjointness between any two tours, nor did we find any properties for, or structures of, non-shifted tours we could use.

These non-shifted tours also exist as solutions for the toroidal $n$-queens problem, so we can search in its literature for these solutions. In the toroidal $n$-queens literature, the shifted tours are known as linear or regular solutions, and other tours are known as nonlinear or irregular solutions [3]. As described in Section 3.2, our computational results only returned such non-shifted tours for $n \geq 13$. Chandra 8] states that, indeed, non-linear solutions only exist for $n \geq 13$, and for $n=5,7,11$ only linear solutions, i.e. shifted tours, exist.

That means that for these small $n$, only the very few distinct shifted tours exist. For larger $n$, the number of tours that are cyclic disjoint with id grows rapidly, with consequently almost all of the tours being nonshifted tours. At $n=13$, this is only 174 distinct tours, but at $n=23$ this already is 2801088 , see Table 1 . These values come from relevant sequences on OEIS.org ${ }^{11}$ A007705 ("Number of ways of arranging $2 n+1$ nonattacking queens on a $2 n+1 \times 2 n+1$ toroidal board."), where for each tour the 'rotated' tours, with a different starting vertex, and their reverses are all counted separately, and equivalently A071607 ("Number of strong complete mappings of the cyclic group $\mathbb{Z}_{2 n+1}$ "), where the reverses of tours are

[^0]counted separately. A strong complete mapping is also equivalent to a solution of the toroidal $n$-queens problem [9].

| 5 | 7 | 11 | 13 | 17 | 19 | 23 | 25 | 29 |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| 1 | 2 | 4 | 174 | 4138 | 21592 | 2801088 | 39154500 | 10446845782 |
| 1 | 2 | 4 | 5 | 7 | 8 | 10 | 9 | 13 |

Table 1: Given $n$ in the top row, the middle row is the amount of distinct tours that are cyclic disjoint with the trivial tour in $K_{n}$, and the bottom row is how many of these tours are shifted tours.

As for any structure of these non-shifted tours, both Bruen and Dixon [10] and Kløve [11] state theorems to construct non-linear solutions, but neither of these theorems completely describe all non-linear solutions, i.e. there are non-linear solutions that cannot be created in one of these ways. There are articles making statements about non-linear solutions, such as Bell and Stevens [12, but these are limited to the classes of non-linear solutions constructed by the aforementioned theorems.

The equivalence between the toroidal $n$-queens problem and our problem breaks down when we consider cyclic disjointness between any two tours, in contrast to where one of the tours is the trivial tour. In the queens problem, disjointness between solutions as we require has no meaning. It is therefore unsurprising that we have found nothing in existing literature on that problem. The closest articles we have found that discusses anything about 'difference' between solutions are on orthogonal Latin squares (Latin squares are equivalent to $n$-queens solutions), specifically one from Shapiro [13], but the orthogonality here is quite different from our cyclic disjointness. The other is by Chandra [8] which describes sets of 'independent permutations' and tries to find the largest possible independent set of permutations of $\mathbb{Z}_{n}$. He also uses these to prove Pólya's theorem. He is able to state tight bounds for such sets, but again, his notion of independence is somewhat different than cyclic disjointness, so not directly applicable to our case.

### 4.4 Computational results

Considering the obstacles identified in the previous sections for improving the bounds of $f(n)$ for general $n$, we turn to computational experiments. The goal here is computationally finding the maximal size of pairwise cyclic disjoint sets for the $n$ where the bounds are not tight yet. The smallest such $n$ are 25,35 and 49 , being composite numbers having smallest prime factor 5,5 and 7 , respectively. We know that using shifted tours, we can find a set of two pairwise cyclic disjoint tours for 25 and 35 , and a set of three tours for 49 . We either want to find a larger set including non-shifted tours, or want to know if that is infeasible.

We make models for finding a set of three pairwise cyclic disjoint tours, $\{i d, \sigma, \tau\}$. As before, without loss of generality, we assume one of our tours to be the trivial tour. To create models for this we adapted our earlier models: the IP model from Section 3.2 and the CP model from Section 3.4

For the IP model, we essentially duplicate the entire model, with $q_{i j}$ to mirror the $p_{i j}$. Where $p_{i j}$ is the binary decision variable to decide whether $\sigma(i)=j, q_{i j}$ is the same but for the $\tau$. All the constraints to ensure $\sigma$ is cyclic disjoint with $i d$ are also added for $\tau$. Finally, we add the following constraint to ensure $\sigma$ and $\tau$ are cyclic disjoint:

$$
p_{i v}+p_{j w}+q_{k v}+q_{l w} \leq 3, \quad \forall i, j, k, l, v, w \in \mathbb{Z}_{n}: \quad v<w, \quad i-j \equiv \pm(k-l) \quad(\bmod n)
$$

For all pairs of vertices $v$ and $w$, if the distance between them in the one tour, $i-j$, would be the same as their distance in the other tour, $k-l$, we forbid assigning the cities in such a way in both of the tours at the same time.

For the CP model, we also duplicate the previous model. This time we have both an $x(i)$ denoting the vertex we place on spot $i$ of the second tour, and an $y(i)$ being the same for the third tour. Again, we add the same constraints for $y_{i}$ to ensure $y$ being cyclic disjoint with $i d$. Lastly, we add the following constraints to make $x$ and $y$ cyclic disjoint:

$$
\begin{aligned}
& \left|\left\{x^{-1}(y(i)) \mid i \in \mathbb{Z}_{n}\right\}\right|=n \\
& \left|\left\{x^{-1}(y(i))+i \quad \bmod n \mid i \in \mathbb{Z}_{n}\right\}\right|=n \\
& \left|\left\{x^{-1}(y(i))-i \quad \bmod \mid i \in \mathbb{Z}_{n}\right\}\right|=n
\end{aligned}
$$

Here we define $x^{-1}(i)$ to be the inverse of $x(i)$, i.e. for all $i, j \in \mathbb{Z}_{n}$ we have $x^{-1}(j)=i \Longleftrightarrow x(i)=j$. The CP solver we used had special constraints available for such an inverse relation, so the performance impact should not be too big. The new constraints above use the fact that $x^{-1} \circ y$ is cyclic disjoint with the trivial tour if $x$ and $y$ are cyclic disjoint, and consequently the result from Lemma 2 that $x^{-1} \circ y$, $\left(x^{-1} \circ y\right)+i d$, and $\left(x^{-1} \circ y\right)-i d$ are all bijections.

We use the same software as before: IBM CPLEX as MIP solver and IBM ILOG CP Optimizer as CP solver. Unlike with the models from earlier in the report, with these models the CP solver is faster than the IP solver. Unfortunately, it seems that already for the smallest value that is relevant, $n=25$, the problem is already too large for the solver to finish in reasonable time and resources on the available computer. For other values of $n$, where a set of size three is known to exist, the solver was able to find a solution fairly quickly. We let the solver run significantly longer for $n=25$, so we do suspect $f(25)=2$, meaning that the lower bound of $\frac{5-1}{2}=2$ provided by the shift-construction also is the true upper bound.
With no true guarantee that finding a third tour is infeasible, the final thing to test is whether there exists a set of three tours when we fix the first tour to be $i d$ and the second tour to be one of the shifted tours. Since this reduces the complexity of the problem, we tried this for all the shifted tours for both $n=25$ and $n=35$, and the problem always turned out to be infeasible. This means that if it is possible to find a set of three pairwise disjoint tours for $n=25$ or $n=35$ and we fix one of the tours to be the trivial tour, then the other two tours are both non-shifted tours.

Since $n=25$ already proves too hard for the computer, and due to time-limits, we have made no attempts for $n=49$.

In conclusion, given the computational results, we have the following conjectures. These are stated in decreasing order of certainty.
Conjecture 1. $f(25)=2$.
It would be very surprising if this turns out to be false.
Conjecture 2. $f(n)=2$ if 5 is the smallest prime factor of $n$.
Given the results for $n=35$, this would still be expected. However, since 25 only has factors of 5 , but 35 has a factor 7 , there is a difference. Additionally, 35 has a much larger amount of possible tours, so there might be some combination where three tours are pairwise cyclic disjoint.

Conjecture 3. $f(n)=\frac{m-1}{2}$, with $m$ being the smallest prime factor of $n$.
The only results this is based on is for when $m=5$, with larger values of $m$ resulting in a problem that is too big to really study by computational results. Were this to be true, it would imply that the construction of shifted tours is both easy and effective, however, it does also mean that the property of cyclic disjointness might be too strong for applications, except in the case that $n$ is prime.

## 5 An algorithm for the PSP with cyclic disjointness

As stated in the introduction, the Paripatetic Salesman Problem is a variant of the Traveling Salesman Problem, first stated by Krarup [1]. The objective is, given a weighted graph, to find two edge-disjoint Hamiltonian tours with minimal total length. In the literature there exist many approximation algorithms for it, such as [14] or [15], and also variants such as [16] or [17], but to the best of our knowledge, non considering cyclic disjointness.
We consider a slight variant to the PSP, limited to cyclic disjoint tours instead of edge-disjoint tours. Additionally, we require the graph to be the complete graph with $n$ vertices $K_{n}$, where $n$ is of course not a multiple of 2 or 3 . A solution to our variant is also a solution to the regular PSP, with two cyclic disjoint tours also being edge-disjoint. Such a solution is also likely not close to optimal in the regular PSP.

We state a simple greedy algorithm for finding a solution to this variant of the PSP. For the algorithm, we do not make any statements about the quality of the solution, but we do consider its running time.

Note that the distance we consider for the cyclic disjointness is not be the weights or costs of the edges, but the order of the vertices in the tour, so the same as we have been considering in the report up to now.

The cyclic disjointness of two tours therefore is completely independent from the weights of the edges or the total costs of the tours.

The algorithm uses shifted tours to find cyclic disjoint tours, since we know when these are cyclic disjoint and the structure of these tours are known

Remark. Shifted tours as described in Section 4.2 are defined and used as shifted 'versions' of the trivial tour, where the difference between subsequent vertices in the tour is a fixed shift. Another way to look at it is that not the vertices are incremented by a fixed amount, but the indices instead. For example, for a shift of 2 , the vertex at index $i+1$ in the shifted tours is the same as the vertex at $i+2$ in the trivial tour. For shifted tour $\sigma$ with shift $s$, we have $\sigma(i+1)=i d(i+s)$.

We can generalise this structure to any tour, regardless of the vertices in the original tour.
Definition 7. Let $\sigma$ be a tour in $K_{n}$. Then we define the shifted variant $\sigma_{s}$ of $\sigma$ with shift $s$ by $\sigma_{s}(i)=\sigma((i-1) s+1)$ for all $i \in \mathbb{Z}_{n}$.
Remark. If we take $\sigma=i d$ then we get the original definition of shifted tours as in Definition 5 Also note that, just as $i d$ was the shifted tour with shift 1 , we get that $\sigma_{1}=\sigma$, i.e. the shifted variant of a tour with shift 1 is simply the tour itself.

An example of a shifted variants of a tour is shown in Figure 11 .

$$
\begin{aligned}
i d=\sigma & =(1,2,3,4,5,6,7) \\
\tau & =(1,6,5,2,4,7,3) \\
\sigma_{2} & =(1,3,5,7,2,4,6) \\
\tau_{2} & =(1,5,4,3,6,2,7) \\
\sigma_{3} & =(1,4,7,3,6,2,5) \\
\tau_{3} & =(1,2,3,5,7,6,4)
\end{aligned}
$$

Figure 11: Examples of shifted variants in $\mathbb{Z}_{7}$. We take $\sigma=i d$, and we show shifted variants with shifts 2 and $3, \sigma_{2}$ and $\sigma_{3}$, equivalently being the shifted tours with shifts 2 and 3 . We take $\tau=(1,6,5,2,4,7,3)$ as an arbitrary other cycle, and similarly show $\tau_{2}$ and $\tau_{3}$. Notice how the structures of the shifts are the same.

Remark. Another way to view shifted variants is as follows. Given the trivial tour and a shifted tour $\sigma$ with shift $s$ that is cyclic disjoint with $i d$, we can 'relabel' the vertices, i.e. permute them with a permutation $\alpha$. Because $i d$ and $\sigma$ are cyclic disjoint, then we get by Property 4 of Proposition 1 that $\alpha \circ i d=\alpha$ and $\alpha \circ \sigma$ are also cyclic disjoint. Then we have $(\alpha \circ \sigma)(i)=\alpha(\sigma(i))=\alpha((i-1) s+1)=\alpha_{s}$, so by the definition above, $\alpha \circ \sigma$ is the shifted variant of shift $s$ of alpha. This again emphasises how the structure of shifted tours and shifted variants really is the same.
As a consequence we can state an equivalent theorem of Theorem 5 for shifted variants.
Theorem 6. Given is $n \geq 3$ and a tour $\sigma$ in $K_{n}$. Then there exist $\phi(n) / 2$ distinct shifted variants of $\sigma$. Two shifted variants with shifts $s$ and $t$ are cyclic disjoint if and only if $\operatorname{gcd}(s+t, n)=1$ and $\operatorname{gcd}(s-t, n)=1$.

Proof. Let $\sigma_{s}$ and $\sigma_{t}$ be two shifted variants of $\sigma$ with shifts $s$ and $t$, respectively, such that $\operatorname{gcd}(s+t, n)=1$ and $\operatorname{gcd}(s-t, n)=1$. Consider the tours $\sigma^{-1} \circ \sigma_{s}$ and $\sigma^{-1} \circ \sigma_{t}$. Then $\left(\sigma^{-1} \circ \sigma_{s}\right)(i)=\sigma^{-1}(\sigma((i-1) s+1))=$ $i d((i-1) s+1)=(i-1) s+1$, so $\sigma^{-1} \circ \sigma_{s}$ is the shifted tour with shift $s$. Similarly $\sigma^{-1} \circ \sigma_{t}$ is the shifted tour with shift $t$. By Theorem 5 we know $\sigma^{-1} \circ \sigma_{t}$ and $\sigma^{-1} \circ \sigma_{t}$ are cyclic disjoint. Then by Property 4 of Proposition 1 we know $\sigma_{s}$ and $\sigma_{t}$ are also cyclic disjoint.

### 5.1 The idea

The idea of the algorithm is to choose a fixed shift such that $i d$ and the shifted tour with that shift are cyclic disjoint, and then stick to finding a tour $x=\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ and its shifted variant with that shift. We start with both tours 'empty', i.e. no $x_{i}$ having an assigned vertex yet. Then, step by step,
we choose a spot on the tour, $x_{i}$, and fill it with one of the remaining vertices. When placing a new vertex somewhere on one of the tours, we then also know where to place that vertex on the other tour, because we work with a known shifted variant of a fixed shift. Additionally, we know that regardless of what the resulting tours are, they are cyclic disjoint, since they are shifted variants. Every step placing a new vertex, there are two choices to make (which we do greedily): which empty spot to pick, and which number to place there.

For example, suppose we have $n=7$ and choose the shift of 2 . We have to choose at least one starting vertex, so for simplicity, we put vertex 1 on the first spot. Then we get the following situation of our two tours:

$$
\begin{aligned}
& (1, a, b, c, d, e, f) \\
& (1, b, d, f, a, c, e)
\end{aligned}
$$

All the variables $a$ through $f$ are vertices that we can still choose. At this point we have to choose one of them and give it a value. This brings us at the first (greedy) choice. When one of the variables is given a value, we can not say anything yet about the length the final tours will get, except when an adjacent spot in one of the tours already has a value. In this case, only $a, f, b$ and $e$ are such vertices, and only if we choose one of these, we can consider the costs of edges (that is $(1, a),(1, f),(1, b)$ or $(1, e)$, respectively) for the total length of our two tours and consequently we can choose a vertex to minimize the objective value.

Our algorithm does the following: firstly, we consider all remaining open spots, and count for each of them how many adjacent spots already have a value in both tours. This value is between 0 and 4 . We then choose any spot where this value is maximal. If there are multiple possibilities we choose the first. In this manner we maximize at each step the number of edges that we know will surely be in the final tours and we therefore can minimize the cost for. For example, in the following situation, the next choice is either $b$ or $f$, both having two adjacent values. The spots of $c$ and $e$ both have 1 adjacent value, and $d$ would add no new edges.

$$
\begin{aligned}
& (1,4, b, c, d, e, f) \\
& (1, b, d, f, 4, c, e)
\end{aligned}
$$

Secondly, after we have chosen a spot and we know which edges will be added to the solution, we simply choose the remaining vertex to minimize the cost of these new edges. In the situation above, we choose the first option $b$. The remaining vertices that can be placed are $S=\{2,3,5,6,7\}$, and the edges in that will be added are $(4, b)$ and $(1, b)$. Then for $b$ we choose $\operatorname{argmin}_{i \in S}[c(4, i)+c(1, i)]$ where $c(\cdot)$ is the cost of an edge.

The choices we make here in these steps are not heuristic in nature, so there might be better options. Additionally, we separate the step for choosing a spot and choosing the vertex for that spot for simplicity, but these might also be combined in a way.

### 5.2 The algorithm

Our algorithm is described in pseudocode as Algorithm 1 As input it requires the size of the graph $n$, where $2 \nmid n$ and $3 \nmid n$, otherwise no pair of cyclic disjoint tours exist. It also needs a valid shift $s$ for the second tour, i.e. $\operatorname{gcd}(s+1, n)=\operatorname{gcd}(s-1, n)=\operatorname{gcd}(s, n)=1$. Since we take shifted variants 1 and $s$ of the resulting tour, and 2 or 3 cannot be a factor of $n$, a shift of $s=2$ is always a valid shift. Finally, we require a cost function for the edges $c(\cdot)$.
In the first lines we initialize a few variables. Firstly, $x$ is the first tour that we will be returning, we directly set $x(1)=1$. Next, $c$ keeps track of the cost, $S$ contains the vertices that are not used in the tour yet, so still have to be used. Finally, $I$ contains the indices where $x$ has already been assigned a vertex, in this case only index 1.
Next we come into the main loop, which we repeat until all vertices are in the tour, i.e. $S=\emptyset$. In line 6 we define $i$ to be the index of which the most neighbouring spots have been filled, $j-1$ and $j+1$ for the first tour, $j-s$ and $j+s$ for the shifted tour. We then add $i$ to $I$. In line 8 , we define $A$, which is the set of adjacent vertices that are assigned in the neighbouring spots of $i$. We use these vertices to calculate the cost of the edges that would be added if we put vertex $w$ on spot $i$. We choose $v$ to be the smallest

```
Algorithm 1 PSP with cyclic disjoint tours
Require: \(n \geq 5\), with \(2 \nmid n\) and \(3 \nmid n\)
Require: shift \(s \in \mathbb{Z}_{n}\) such that \(\operatorname{gcd}(s+1, n)=\operatorname{gcd}(s-1, n)=\operatorname{gcd}(s, n)=1\)
Require: edge cost function \(c(v, w)\), for vertices \(v, w \in \mathbb{Z}_{n}\)
    \(x(1) \leftarrow 1\)
    \(c \leftarrow 0\)
    \(S \leftarrow\{2, \ldots, n\}\)
    \(I \leftarrow\{1\}\)
    while \(S \neq \emptyset\) do
        \(i \leftarrow \operatorname{argmax}_{j \in \mathbb{Z}_{n} \backslash I}|I \cap\{j-1 j+1, j-s, j+s\}| \quad \triangleright\) calculation of indices is modulo \(n\)
        \(I \leftarrow I \cup\{i\}\)
        \(A \leftarrow\{x(j) \mid j \in\{i-1, i+1, i-s, i+s\}\} \quad \triangleright\) calculation of indices is modulo \(n\)
        \(v \leftarrow \operatorname{argmin}_{w \in S}\left(\sum_{u \in A} c(u, w)\right)\)
        \(S \leftarrow S \backslash\{v\}\)
        \(x(i) \leftarrow v\)
        \(c \leftarrow c+\sum_{u \in A} c(u, v)\)
    end while
    for \(i \in \mathbb{Z}_{n}\) do
        \(y(i) \leftarrow x((i-1) s+1 \bmod n)\)
    end for
    return \(x, y\), and \(c\)
```

such $w$ on line 9 , and remove it from the set of available vertices $S$. Finally, on line 11, we assign $x(i)$ to be $v$, and increase $c$ to keep track of the cost up to now.

Finally, after all vertices have been assigned and the tour is chosen, we create the shifted variant $y$ of $x$ on lines 14 and 15 , and return the tours and the total cost.
Claim 1. Algorithm 1 runs in $O\left(n^{2}\right)$ time.
Proof. Lines 1 through 4 are all $O(1)$. The main loop runs $n-1$ times. On line 6 , we need to check whether 4 indices are in $I$, and need to do that $|S|$ times to find the maximal argument $i$. Lines 7 and 8 are $O(1)$. In line 9 we have to sum up to four weights $|S|$ times to find the maximal argument $v$. Finally, lines 10 through 12 are $O(1)$ again. That means that for the total loop we have at most $\sum_{i=1}^{n}(4|S|+O(1)+4|S|+O(1))=O(n)+8 \frac{n(n+1)}{2}=O\left(n^{2}\right)$. After the loop it is $O(n)$ to create $y$. The entire algorithm therefore runs in $O\left(n^{2}\right)$ time.

Remark. In the current algorithm we calculate the amount of filled neighbouring spots per spot (line 6 in Algorithm 1) again every time the loop runs. It is likely faster to keep track of this count throughout the entire loop instead, and only updating it when we place new vertices. If we keep track of the count per spot in something like a heap, the extraction of the optimal $i$ on line 6 would be $O(1)$. We need to update the heap as soon as we choose the $i$, but that is 4 neighbours at a cost of $O(\log n)$ each (at least for simple heaps). This would not change the running time of $O\left(n^{2}\right)$ for the entire algorithm, but in practice it would probably be faster for large $n$.

The algorithm can easily be extended to the $m$-Peripatetic Salesman Problem, where we want to find $m$ disjoint tours with minimal total cost instead of only two tours. Then we simply use $m$ different shifted tours.

In conclusion, this above algorithm showcases how the shifted tours are easy to reason about, and how it could be used in a diverse optimization problem. The algorithm is very simple, however, so the results will probably be far from optimal.

## 6 Discussion

Cyclic disjointness is at first sight not applicable to any real-world problems, and we have not found a direct application of it. There are situations where one can imagine that two edge-disjoint tours might not be distinct enough to be used as diverse solutions for TSP in real life, for example in large and dense
graphs edge. But whether cyclic disjoint tours would be significantly better in such a case is hard to say. What effect the condition of different distances in such tours really has in large graphs is difficult to imagine.
Even when we want two Hamiltonian tours to have globally different orders, a relaxed variant of cyclic disjointness would probably be better in practice. Due to the theoretical results this is also necessary. The fact that cyclic disjoint tours only exist for (relatively) few $n$ greatly limits applications. The result for sets of pairwise cyclic disjoint tours only aggravates this.
As for further research, it would be interesting to see how a relaxation of cyclic disjointness would look like, and whether it would be usable in applications. Allowing a limited number of breaches of the condition, i.e. a maximum number of pairs of vertices for which the distance is equal in both tours, may change the existence conditions. How many of these breaches would we need to allow for pairs of disjoint tours to exist for all $n$ ?

Other research can of course be on the open problem of the bounds of $f(n)$, especially for composite $n$. It seems Pólya's argument with these sum identities in Theorem 3 is not (easily) generalizable to higher $n$ or prime factors of $n$ bigger than 3. Other directions for research that may lead to new insights are the structure of, or new results on, non-shifted cyclic tours or equivalently non-linear solutions of the toroidal $n$-queens problem. There already exists literature for the latter problem, and it is equivalent to surprisingly many other problems. A more thorough search might yield some results.
Finally, there can be improvements in the form of computational results. A better solver can perhaps give us full results for $f(n)$ for $n=25,35,49$ or even larger $n$ with larger smallest prime factors. Using such result one can strengthen or disprove our final conjectures in Section 4.4 A large survey on existing literature of $n$-queens [3] has a section on computation, and also in the context of equivalent problems there is work on it, such as [9].

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[^0]:    ${ }^{1}$ The On-Line Encyclopedia of Integer Sequences, http://oeis.org

