BACHELOR

Counting minimal prime ideals

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Counting minimal prime ideals

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Abstract

An overview of the properties of a Sym(n)-orbit of an ideal is provided using the properties of a group action. The egg-colouring problem is explained and used for some combinatorial problems. Properties of minimal prime ideals are explored and the connection between a prime ideal and a domain is explained. The connection between a real polynomial ring and a complex polynomial ring through an irreducible polynomial is explored. Using this connection, the notion of minimal prime ideals that contain an ideal that is generated by functions with multiple distinct irreducible polynomials in a real polynomial ring is introduced. Some properties of a quasi-polynomial and some properties of the degree of a polynomial are provided. The number of Sym(n)-orbits of the minimal prime ideals that contain an ideal that is generated by functions $f(x_i)$ that factor into purely linear polynomials in $K[x_1, ..., x_n]$ and consequently also those that contain an ideal that is generated by functions $f(x_i)$ in $\mathbb{C}[x_1, ..., x_n]$, is found. Then the number of Sym(n)-orbits of the minimal prime ideals that contain an ideal that is generated by functions $f(x_i)$ that factor into irreducible quadratic polynomials in $\mathbb{R}[x_1, ..., x_n]$, followed by those that contain an ideal that is generated by an arbitrary function $f(x_i)$ in $\mathbb{R}[x_1, ..., x_n]$, is found. The latter expression is then proven to be quasi-polynomial and the maximum degree of this expression is found.

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Chapter 1

Introduction

In a branch of abstract algebra, namely ring theory, there exists the concept of rings. Significant research has yet been done within this branch with a focus on rings. However, the effort made regarding ideals has items which can be added. Within this branch of abstract algebra, an ideal of a ring is a special kind of subset of the ring [5].

Let $K[x_1, ..., x_n]$ be a polynomial ring over the real or complex numbers and let I be an ideal in $K[x_1, ..., x_n]$. A minimal prime ideal containing I, which is not prime itself, is a prime ideal $p \supseteq I$ such that there is no prime ideal q such that $I \subseteq q \subsetneq p$. In this project, we will try to count the number of Sym(n)-orbits of the minimal prime ideals containing I.

To achieve this, firstly in Chapter 2, we will introduce the Sym(n)-orbits of ideals, which will illustrate the existence of ideals which are the same under permutations of the indices in the variables $x_1, ..., x_n$. Secondly, the concept of a minimal prime ideal will be explained in Chapter 3. Then in Chapter 4 we will be given an ideal $I \subset \mathbb{R}[x_1, ..., x_n]$ that is generated by $f(x_i) = p_1 \cdot ... \cdot p_d$ that factors into d quadratic irreducible polynomials. We will then show how to construct minimal prime ideals $p \subset \mathbb{R}[x_1, ..., x_n]$ that contain I that correspond to the minimal prime ideals $p' \subset \mathbb{C}[x_2, ..., x_n]$, using the ring homomorphism $\mathbb{R}[x_1, ..., x_n]/(f(x_1), ..., f(x_n)) \cong \mathbb{C}[x_2, ..., x_n]/(f(x_2), ..., f(x_n))$. We will then explain how any irreducible polynomial in $\mathbb{R}[x_1, ..., x_n]$ can be factored as a product of two linear terms in $\mathbb{R}[x_1, ..., x_n]/(f(x_i))$. The last thing that will be explained in Chapter 4 is the change of a degree in a sum using the findings of Leonhard Euler (1707-1783) and Colin Maclaurin (1698 - 1746), which will be used in Chapter 6. Finally in Chapter 5 and 6, we will use the findings of the previous Chapters and some new findings to find the number of Sym(n)-orbits of the minimal prime ideals that contain an ideal which consists of $f(x_i)$ which consist of irreducible factors in $\mathbb{C}[x_1, ..., x_n]$ and $\mathbb{R}[x_1, ..., x_n]$ respectively.

An easy and elementary example of what will be explored in this project is finding the zeroes of a function. If we take a couple of the same functions, but with different variables (ie. $f(x_1), f(x_2), ..., f(x_n)$), we know that all these functions have the same set of roots. If we have n-tuples of solutions, we will see the collection of these n-tuples forms a grid. This example will be explained in Chapter 3.

Now especially in Chapter 2, Chapter 3 and Chapter 4, but also in the other two Chapters, we will use terms which have been introduced in the course Algebra and Discrete Mathematics. For a full overview of these subjects we refer you to the lecture notes of this course by Chloe Martindale [5]. We also introduce the concept of a quasi-polynomial for a small component of Chapter 6, which for the interested readers will be explained somewhat throughout by Petr Lisoněk [4] or a more compact explanation can be found on the Wikipedia page [7].

Chapter 2

Sym(n)-orbits

2.1 Group actions

Here we will give some insight of what group actions are about, which we will later use to define the Sym(n)-orbit of an ideal.

Definition 2.1.1. *Let G be a group with identity element e and X be a set. A group action α from G on X is a function $\alpha : G \times X \to X$ such that for a $\pi, \sigma \in G$ and a $f \in X$ we have:*

1. $\alpha(e, f) = f$.
2. $\alpha(\pi \cdot \sigma, f) = \alpha(\pi, \alpha(\sigma, f))$.

Note: we define $gx := \alpha(g, x)$ from now on.

Lemma 2.1.2. *We let $G = Sym(n)$ with identity element (1) and $X = k[x_1, ..., x_n]$. Let $\pi \in G$ and $f \in X$. We define: $\pi f = \pi f(x_1, x_2, ..., x_n) = f(x_{\pi(1)}, x_{\pi(2)}, ..., x_{\pi(n)})$. We have that the induced map $\alpha : G \times X \to X$ is a group action.*

Proof. We have that

$$(1)f = (1)f(x_1, x_2, ..., x_n) = f(x_1, x_2, ..., x_n) = f.$$

So we have that α satisfies the first requirement of a group action and we also have that:

$$\begin{aligned}\pi(\sigma f) &= \pi(\sigma f(x_1, x_2, ..., x_n)) \\ &= \pi f(x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(n)}) \\ &= f(x_{\pi(\sigma(1))}, x_{\pi(\sigma(2))}, ..., x_{\pi(\sigma(n))}).\end{aligned}$$

Now by the definition of a permutation [1] we have that

$$x_{\pi(\sigma(i))} = x_{(\pi\circ\sigma)(i)}.$$

So we get that:

$$f(x_{\pi(\sigma(1))}, x_{\pi(\sigma(2))}, ..., x_{\pi(\sigma(n))}) = f(x_{(\pi\circ\sigma)(1)}, x_{(\pi\circ\sigma)(2)}, ..., x_{(\pi\circ\sigma)(n)}) = (\pi \circ \sigma)f.$$

Which proves that α satisfies second property of a group action. So we have that $\alpha : G \times X \to X$ is a group action. □

2.2 The Sym(n)-orbit

In this section we will give the definition of the Sym(n)-orbit of an ideal, which will be an important element in the proofs of the theorems in Chapter 5 and Chapter 6.

It will show how two ideals can be the same under permutation, thus showing the similarities to each other.

Definition 2.2.1. *Let $X = k[x_1, ..., x_n]$ be a ring and let I be an ideal in X. Let $G = Sym(n)$ act on X as above. Then the Sym(n)-orbit of an ideal I is $\{\pi(I)|\pi \in Sym(n)\}$.*

Definition 2.2.2. *The Sym(n)-orbit of an element $f \in X = k[x_1, x_2, ..., x_n]$ is: Orb(f)=$\{\pi(f)|\pi \in Sym(n)\}$.*

Remark 2.2.3. *When we have that $g \in Orb(f) \iff \exists\pi \in Sym(n) : \pi(f) = g$, we also have that $f \in Orb(g) \iff \exists\pi \in Sym(n) : \pi(g) = f$. Now since we take our first statement to be true, in particular we have that $g = \pi(f) \iff f = \pi^{-1}(g)$.*

Lemma 2.2.4. *Let $X = k[x_1, ..., x_n]$ be a ring. If $I \subseteq X$ is an ideal and $\pi \in$ Sym(n), then $\pi(I)$ is an ideal.*

Proof. We know that I is an ideal, so if we have that $f \in I$ and $g \in X$, then $fg \in I$. We also have that for $f, g \in I$, $f + g \in I$. So we now have to prove that $\pi(I)$ is an ideal, thus we have to prove that:

1. $\forall f, g \in \pi(I) : f + g \in \pi(I)$.
2. $\forall f \in \pi(I), g \in X : fg \in \pi(I)$.

To prove the first statement we let $f = \pi(h)$ and $g = \pi(k)$ for some $h, k \in I$. Since I is an ideal, we have that $h + k \in I$. So we also have that $\pi(h + k) \in \pi(I)$. Since we have that:

$$\begin{aligned}\pi(h+k) &= (h+k)(x_{\pi(1)}, ..., x_{\pi(n)}) \\ &= h(x_{\pi(1)}, ..., x_{\pi(n)}) + k(x_{\pi(1)}, ..., x_{\pi(n)}) \\ &= \pi(h) + \pi(k) = f + g.\end{aligned}$$

We have that $f + g \in \pi(I)$.

To prove the second statement we first note that

$$\pi(f) \cdot \pi(g) = f(x_{\pi(1)}, ..., x_{\pi(n)}) \cdot g(x_{\pi(1)}, ..., x_{\pi(n)}) = fg(x_{\pi(1)}, ..., x_{\pi(n)}) = \pi(fg).$$

We also use the property that

$$\pi(\sigma(f)) = (\pi \circ \sigma)(f).$$

So now if we have an $f \in \pi(I)$, there is an $h \in I$, such that $f = \pi(h)$. We also have that there is a $g \in X$ and a $k \in X$, such that $g = \pi(k)$. Since we have that $hk \in I$ by the properties of an ideal, we have that $\pi(hk) \in \pi(I)$. We also have that $\pi(hk) = \pi(h)\pi(k) = fg$. So $fg \in \pi(I)$. Since k was taken arbitrarily, g is an arbitrary element of X. We also had taken f arbitrarily. So we have that $\forall f \in \pi(I), g \in X : fg \in \pi(I)$. Thus we have that if $I \subseteq X$ is an ideal and $\pi \in \mathrm{Sym}(n)$, then $\pi(I)$ is an ideal. □

We can now talk about the Sym(n)-orbit of an ideal I.

Notation 2.2.5. *If we have that two ideals I and J are in the same Sym(n)-orbit we sometimes denote this as $I \sim J$.*

2.3 The egg colouring problem

Suppose one would like to colour 5 eggs and they have 3 colours of paint to their disposal. In how many different ways can these eggs be coloured, given that the difference between, for example, two yellow painted eggs cannot be seen.

Lemma 2.3.1. *Let there be k different colours and n eggs, with $k, n \in \mathbb{N}$. Then $f(k,n) = \binom{n+k-1}{k-1}$ is the formula for the number of different ways these eggs can be coloured when we have the following requirements:*

- *Eggs of the same colour are indistinguishable.*
- *All eggs are to be coloured with the available colours.*
- *Not every available colour has to be used.*

Proof. Using the explanation from the book "Discrete Wiskunde" [3], we prove this lemma.
We take the colours to be baskets in which we can put the eggs and between these baskets we put walls. So we get that there are $k-1$ walls between our k colours. One of the ways to colour the eggs would be given by placing the n eggs in the k baskets. As an example we take 5 eggs and 3 colours, one way to colour the eggs would be:

$$\text{egg}, \text{egg}, \text{wall}, \text{egg}, \text{egg}, \text{wall}, \text{egg}.$$

Here we see that there are 2 eggs in the basket of colour 1, 2 eggs in the basket of colour 2 and 1 egg in the basket of colour 3. We also see that we have created a series of 5+2=7 objects, 5 eggs and 2 walls. The number of such series is the number of ways to choose 5 positions for the eggs from the 7 positions total, or the number of ways to choose 2 positions for the walls from the 7 positions total.
So now considering our example, if we were to have k colours and n eggs, we would create a series of $n+k-1$ objects, n eggs and $k-1$ walls. Now the number of such series that we can create is the number of ways we can choose the position for the $k-1$ walls in the series of $n+k-1$ total positions. So we get the formula

$$f(k,n) = \binom{n+k-1}{k-1}$$

for the number of ways to colour n eggs with k colours. □

We will see the egg colouring problem in various forms in later chapters.

Chapter 3

Minimal prime ideals

We will now give the definition of a minimal prime ideal and prove some properties of I together with R/I for some ring R and some ideal $I \subset R$. On top of that, we will give two examples of the minimal prime ideals that contain a given ideal I.

Lemma 3.0.1. *Let R be a ring and let I be an ideal in R, then we have that I is prime if and only if R/I is a domain.*

Proof. Suppose that I is prime. Let $x, y \in R/I$. Then there are elements $a, b \in R$ such that $x = a(modI)$ and $y = b(modI)$. Now suppose that $xy = 0$, but $x \neq 0$, so we have $a \notin I$. We get:

$$0 = xy = (a(modI))(b(modI)) = ab(modI) \Rightarrow ab(modI) = 0.$$

So now we have that $ab \in I$ and since I is prime, we have $b \in I$. But then we have $y = b(modI) = 0$ in R/I. So we have that R/I is a domain.

Now suppose R/I is a domain. Let $a, b \in R$ so that $ab \in I$ and suppose $a \notin I$. Let $x = a(modI)$ and $y = b(modI)$. Then we get $xy = ab(modI) = 0$. Since we had $x \neq 0$ in R/I and we have R/I is a domain, we get that $y = 0$ in R/I. But then we have that $b \in I$ and thus we have that I is prime. □

Definition 3.0.2. *Let R be a ring and let I be an ideal in R. A prime ideal $p \subset R$ containing I is minimal when we have the property that for any prime ideal $J \subset R$ containing I, if we have $I \subseteq J \subseteq p$, then $J = p$.*

Example 3.0.3. *Suppose we have the function $f(x) \in \mathbb{R}[x]$, where $f(x) = x^2 + 5x + 6$. We know this function can be factored and that we get $f(x) = (x+3)(x+2)$. If we want to know for which values of x we get $f(x) = 0$, we get that $x = -2$ and $x = -3$. So now if we have an ideal $I = (f(x)))$, we have that the zero set, $\{-2, -3\}$ of this ideal corresponds to the union of the points $x + 2$ and $x + 3$. So we get that the minimal prime ideals that contain I are $(x + 2)$ and $(x + 3)$.*

Example 3.0.4. *Suppose we have the function $f(x) \in \mathbb{R}[x]$, where $f(x) = x^2 - 1$. We know this function can be factored and that we get $f(x) = (x-1)(x+1)$. So $f(x) = 0$ for $x = 1$ and $x = -1$. So if we have an ideal $I = (x_1^2 - 1, x_2^2 - 1) \subset \mathbb{R}[x_1, x_2]$, we have that the zero set of this ideal, $\{(1,1), (-1,1), (1,-1), (-1,-1)\}$, corresponds to unions of the points $x_1 - 1, x_1 + 1, x_2 - 1$ and $x_2 + 1$. So we get that the minimal prime ideals that contain I are $p_1 = (x_1 - 1, x_2 - 1)$, $p_2 = (x_1 + 1, x_2 - 1)$, $p_3 = (x_1 - 1, x_2 + 1)$ and $p_4 = (x_1 + 1, x_2 + 1)$. We also see that $\pi(p_2) = p_3$ for $\pi = (1,2)$, so we have that p_2 and p_3 are in the same Sym(n)-orbit, which gives us that there are 3 Sym(n)-orbits of the minimal prime ideals containing I.*

We will show how one can calculate the number of Sym(n)-orbits in any ideal $I \subset \mathbb{K}[x_1, ..., x_n]$ for $K \in \{\mathbb{C}, \mathbb{R}\}$ in the later Chapters 5 and 6.

We will now explain an important property of R/I, where R is a ring and I is a maximal ideal. This is an important property that will be used to find the minimal prime ideals of ideals generated by an irreducible polynomial in $\mathbb{R}[x]$.

Lemma 3.0.5. *Let R be a ring and let I be an ideal in R, then we have that I is maximal if and only if R/I is a field.*

Proof. I is maximal means that if we have that if $I \subset J \Rightarrow J = (1) = R$.

$'' \Rightarrow ''$ Let I be a maximal ideal.
Let $a(modI) \in R/I$ be a non-zero element, so we have $a \notin I$, since $0 \in I, a \neq 0$.
Now let $J = \{ra + g | r \in R, g \in I\}$. We have:

1. $0 = 0a + 0 \in J$.
2. Let $r_1a + g_1, r_2a + g_2 \in J$.
   Then we have that $(r_1a + g_1) - (r_2a + g_2) = (r_1 - r_2)a + (g_1 - g_2) \in J$.
3. Let $x \in R$ and $ra + g \in J$.
   Then $x(ra + g) = (xr)a + xg \in J$ and $(ra + g)x = (rx)a + gx \in J$.

So we have that J follows the properties of an ideal of R. We also have that if $g \in I$, then $g = 0a + g \in J$. So we have that $I \subseteq J$ and since $a = 1a + 0 \in J$, $a \notin I$, we have $I \subset J$ and $I \neq J$. Since I is maximal, we have $J = R$. Now since we have that $1 \in R = J$, we have that there are an $b \in R$, $f \in I$ such that:

$$1 = ba + f \Rightarrow 1(modI) = ba(modI) = (b(modI)) \cdot (a(modI)).$$

So we have that every non-zero element $a(modI) \in R/I$ has an inverse, so R/I is a field.

$'' \Leftarrow ''$ Let R/I be a field.
So we have $0(modI), 1(modI) \in R/I$. Therefore $I \neq R$. Now for I to be a maximal ideal we have to have that for an ideal J of R for which $I \subset J \Rightarrow J = R$.

Let J be an ideal of R with $I \subset J$. Let $a \in J, a \notin I$. Since $a \notin I, a(modI) \neq 0(modI)$. Now since R/I is a field and $a(modI)$ is non-zero, $a(modI)$ has an inverse. So there is a $b \in R$ such that $(a(modI))(b(modI)) = ab(modI) = 1(modI)$. So there is a $f \in I$ such that $ab + f = 1$ in R/I.
We now have that $ba \in J$ since $a \in J$. We also have $ba + f \in J$, since $I \subset J$. So we have $1 \in J$. So then we have $J = R$. So we have that I is a maximal ideal. $\square$

Chapter 4

Ring homomorphisms and the degree of polynomials

4.1 A ring homomorphism

In this section we will show the ring homomorphism from $\mathbb{R}[x_1, ..., x_n]/(p_1, ..., p_n)$ to $\mathbb{C}[x_2, ..., x_n]/(p_2, ..., p_n)$, where p_i is an irreducible polynomial in $\mathbb{R}[x_i]$. This will then be used to show how to construct minimal prime ideals $p \subset \mathbb{R}[x_1, ..., x_n]$ that contains I that correspond with the minimal prime ideals $p' \subset \mathbb{C}[x_2, ..., x_n]$. We will then explain how any irreducible polynomial in $\mathbb{R}[x_1, ..., x_n]$ can be factored as a product of two linear terms in $\mathbb{R}[x_1, ..., x_n]/(p_i)$.

Lemma 4.1.1. *Let $p = ax^2 + bx + c$ be an irreducible polynomial in $\mathbb{R}[x]$ with $a \neq 0$ then we have that*

$$\mathbb{R}[x]/(p) \cong \mathbb{C}.$$

Proof. We have that any element in $\mathbb{R}[x]/(p)$ has a representative of the form $dx + e$ for $d, e \in \mathbb{R}$. We have that $p = ax^2 + bx + c$ is irreducible, so we have that the ideal is maximal, since now the quotient is a field, which is in particular a domain.

We have to show that $\mathbb{R}[x]/(p) \cong \mathbb{C}$. We let ϕ be the ring homomorphism from $\mathbb{R}[x]$ to $\mathbb{C}$ given by $f(x) \to f(\frac{-b+\sqrt{b^2-4ac}}{2a})$, where $\frac{-b+\sqrt{b^2-4ac}}{2a} \in \mathbb{C}$. This means that we will evaluate a polynomial in $\mathbb{R}[x]$ at $\frac{-b+\sqrt{b^2-4ac}}{2a}$.
We will take $z = \alpha + \beta i = \frac{-b+\sqrt{b^2-4ac}}{2a}$ with $\beta \neq 0$ as a complex zero of the polynomial $p = ax^2 + bx + c$.

We now show that this is a ring homomorphism. Let $p, q \in \mathbb{R}[x]$, then we have:

$$\begin{aligned}
\phi(p+q) &= (p+q)(z) \\
&= p(z) + q(z) = \phi(p) + \phi(q) \\
\phi(pq) &= (pq)(z) \\
&= p(z)q(z) = \phi(p)\phi(q).
\end{aligned}$$

So ϕ is a ring homomorphism.

We now show that it is surjective. First we note that $b^2 - 4ac < 0$, since we have that p is irreducible in $\mathbb{R}[x]$. So we have that there has to be a complex solution for the equation $p = 0$. We know that $\mathbb{C}$ can be identified with the vector space $\mathbb{R}^2$ by treating each complex number $z = a + bi$ as a vector $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$. We have that the image of ϕ contains $u\phi(x) + v$ for all real numbers u and v. Since $\phi(x)$ is not real, the image contains a 2 dimensional subspace which is then necessarily equal to $\mathbb{C}$. Thus the map is surjective.

We now show that the kernel is $\langle ax^2+bx+c\rangle$:
Since we have that z is a complex zero of ax^2+bx+c, it is clear that ax^2+bx+c is in $\ker(\phi)$ and so we have that $\langle ax^2+bx+c\rangle \subseteq \ker(\phi)$. To show that also $\langle ax^2+bx+c\rangle \supseteq \ker(\phi)$ we do the following:
Suppose there is a $p \in \ker(\phi)$. By dividing p by ax^2+bx+c using polynomial long division we get that $p=(ax^2+bx+c)q+r$ for polynomials r of a degree less than 2. Now we have that $0=\phi(p)=p(z)=r(z)$. But since the only polynomial in $\mathbb{R}[x]$ with degree less than two with $r(z)=0$ is the zero polynomial, we get that $r=0$. This gives us that $p=(ax^2+bx+c)q$, so $p\in\langle ax^2+bx+c\rangle$. So we have that $\langle ax^2+bx+c\rangle=\ker(\phi)$.

So we have that $\mathbb{R}[x]/(ax^2+bx+c)\cong\mathbb{C}$. □

Corollary 4.1.2. *$\mathbb{R}[x_1,...,x_n]/(p_1,...,p_n)$ for $n\in\mathbb{N}$ and for an arbitrary irreducible polynomial $p_i=a(x_i)^2+bx_i+c\in\mathbb{R}[x_1,...,x_n]$ with a degree of 2 is isomorphic to $\mathbb{C}[x_2,...,x_n]/(p_2,...,p_n)$.*

Proof.

$$\mathbb{R}[x_1,...,x_n]/(p_1,...,p_n)\cong(\mathbb{R}[x_1,...,x_n]/(p_1))/(p_2,...,p_n)\cong\mathbb{C}[x_2,...,x_n]/(p_2,...,p_n).$$

Where the first congruency follows from simple modular rules and the second is a direct application of Lemma 4.1.1. □

What follows from this is that prime ideals in $\mathbb{R}[x_1,...,x_n]$ that contain $I\subset\mathbb{R}[x_1,...,x_n]$ which contain an irreducible polynomial of degree 2 in the variable x_1, correspond exactly with the prime ideals in $\mathbb{C}[x_2,...,x_n]$ that contain this same I.

Analogously it also follows that prime ideals in $\mathbb{R}[x_1,...,x_n]$ that contain $I\subset\mathbb{R}[x_1,...,x_n]$ which contain an irreducible polynomial of degree 2 in the variable x_i, correspond exactly with the prime ideals in $\mathbb{C}[x_1,...,x_{i-1},x_{i+1},...,x_n]$ that contain $I\subset\mathbb{R}[x_1,...,x_n]$, which we will use in Chapter 6.

Lemma 4.1.3. *Suppose $f_1(x)=x^2+a_1x+b_1, f_2(x)=x^2+a_2x+b_2,...,f_d(x)=x^2+a_dx+b_d$ are pairwise distinct irreducible polynomials in $\mathbb{R}[x_1,...,x_n]$. Now suppose we have the ideal $I=((f_1(x_1))^{e_1}(f_2(x_1))^{e_2}...(f_d(x_1))^{e_d},...,(f_1(x_n))^{e_1}(f_2(x_n))^{e_2}...(f_d(x_n))^{e_d})$. Fix $i\in[n]$. A minimal prime ideal p that contains I and $x_i^2+a_1x_i+b_1$, should contain at least one of x_j-x_i, x_j+x_i+a, $x_j-e_{1,2}x_i-e_{2,2}$, $x_j-e_{3,2}x_i-e_{4,2}$, $x_j-e_{1,3}x_i-e_{2,3}$, $x_j-e_{3,3}x_i-e_{4,3}$,..., $x_j-e_{1,d}x_i-e_{2,d}$ and $x_j-e_{3,d}x_i-e_{4,d}$ for all $j\neq i$ in $\{1,...,n\}$, where for $k\in\{2,...,d\}$ we have*

$$e_{1,k}=\sqrt{\frac{a_k^2-4b_k}{a_1^2-4b_1}};$$

$$e_{2,k}=\frac{-a_k+\sqrt{\frac{a_k^2-4b_k}{a_1^2-4b_1}}a_1}{2};$$

$$e_{3,k}=-\sqrt{\frac{a_k^2-4b_k}{a_1^2-4b_1}}=-e_{1,k};$$

$$e_{4,k}=\frac{-a_k-\sqrt{\frac{a_k^2-4b_k}{a_1^2-4b_1}}a_1}{2}=e_{2,k}-e_{1,k}a_1.$$

Proof. Without loss of generality we can prove this for the case where $d=2$. The general case can be done completely analogously.

Suppose that $d = 2$, we then have that $f_1(x) = x^2 + a_1x + b_1$ and $f_2(x) = x^2 + a_2x + b_2$ are distinct irreducible polynomials. We then also have the ideal $I = ((f_1(x_1))^{e_1}(f_2(x_1))^{e_2}, ..., (f_1(x_n))^{e_1}(f_2(x_n))^{e_2})$. So in this case a minimal prime ideal p that contains I and $x_i^2 + a_1x_i + b_1$, should contain at least one of $x_j - x_i$, $x_j + x_i + a_1$, $x_j - e_{1,2}x_i - e_{2,2}$ and $x_j - e_{3,2}x_i - e_{4,2}$ for all $j \neq i$ in $\{1, ..., n\}$, where

$$\begin{aligned}
e_{1,2} &= \sqrt{\frac{a_2^2 - 4b_2}{a_1^2 - 4b_1}};\\
e_{2,2} &= \frac{-a_2 + \sqrt{\frac{a_2^2-4b_2}{a_1^2-4b_1}}a_1}{2};\\
e_{3,2} &= -\sqrt{\frac{a_2^2 - 4b_2}{a_1^2 - 4b_1}} = -e_{1,2};\\
e_{4,2} &= \frac{-a_2 - \sqrt{\frac{a_2^2-4b_2}{a_1^2-4b_1}}a_1}{2} = e_{2,2} - e_{1,2}a_1.
\end{aligned}$$

To prove this case we proceed as follows.

We have that by assumption $f_1(x_i) \in p$, we then have, for p to contain I, that either $f_1(x_j)$ or $f_2(x_j)$ is contained by p for all $j \in [n] \setminus \{i\}$. In the case that $f_1(x_j)$ is contained in p, we have that $f_1(x_j) - f_1(x_i) = x_j^2 - x_i^2 + a_1(x_j - x_i) = (x_j - x_i)(x_j + x_i + a_1)$ and we can conclude that a minimal prime ideal p contains at least one of $x_j - x_i$ and $x_j + x_i + a_1$.

Now the case that we have that $f_2(x_j)$ is contained in p. Since we know that $f_2(x_j)$ factorises in $\mathbb{C}$, we know that it factorises in $\mathbb{R}[x_i]/(f_1(x_i))$ by Lemma 4.1.1. We now have that all classes in $\mathbb{R}[x_i]/(f_1(x_i))$ have elements of the form $sx_i + t$. So if $f_2(x_j)$ factorises over $\mathbb{C}$ as $(x_j - z_1)(x_j - z_2)$, then z_1 corresponds with a form of $e_{1,2}x_i + e_{2,2}$ and z_2 with $e_{3,2}x_i + e_{4,2}$. So we find that $(x_j - e_{1,2}x_i - e_{2,2})(x_j - e_{3,2}x_i - e_{4,2}) = f_2(x_j)(mod f_1(x_i))$. So we get $(x_j - e_{1,2}x_i - e_{2,2})(x_j - e_{3,2}x_i - e_{4,2}) - f_2(x_j) = hf_1(x_i)$. We have that the degree on the left hand side of this equation is 2, thus the degree of h is 0.
So we have that $f_2(x_j)(mod f_1(x_i)) = (x_j - e_{1,2}x_i - e_{2,2})(x_j - e_{3,2}x_i - e_{4,2})$ for some $e_{1,2}, e_{2,2}, e_{3,2}, e_{4,2} \in \mathbb{R}$. $e_{1,2}$ and $e_{3,2}$ are non-zero, since $f_2(x_j) = 0$ has no real solutions. So we get that for an $h \in \mathbb{R}$ we have:

$$\begin{aligned}
x_j^2 + a_2x_j + b_2 - h \cdot (x_i^2 + a_1x_i + b_1) &= (x_j - e_{1,2}x_i - e_{2,2})(x_j - e_{3,2}x_i - e_{4,2})\\
x_j^2 + a_2x_j + b_2 - hx_i^2 - ha_1x_i - hb_1 &= x_j^2 - e_{3,2}x_ix_j - e_{4,2}x_j - e_{1,2}x_ix_j + e_{1,2}e_{3,2}x_i^2\\
&\quad + e_{1,2}e_{4,2}x_i - e_{2,2}x_j + e_{2,2}e_{3,2}x_i + e_{2,2}e_{4,2}\\
x_j^2 + a_2x_j - hx_i^2 - ha_1x_i - hb_1 + b_2 &= x_j^2 + (-e_{4,2} - e_{2,2})x_j + e_{1,2}e_{3,2}x_i^2 + (e_{1,2}e_{4,2}\\
&\quad + e_{2,2}e_{3,2})x_i + e_{2,2}e_{4,2} + (-e_{3,2} - e_{1,2})x_ix_j\\
a_2x_j - hx_i^2 - ha_1x_i - hb_1 + b_2 &= (-e_{4,2} - e_{2,2})x_j + e_{1,2}e_{3,2}x_i^2 + (e_{1,2}e_{4,2}\\
&\quad + e_{2,2}e_{3,2})x_i + e_{2,2}e_{4,2} + (-e_{3,2} - e_{1,2})x_ix_j.
\end{aligned}$$

From this we get the following equations

$$
\begin{aligned}
&-e_{3,2}-e_{1,2}=0 \Rightarrow e_{3,2}=-e_{1,2};\\
&a_2=-e_{4,2}-e_{2,2};\\
&-h=e_{1,2}e_{3,2} \Rightarrow -h=-e_{1,2}^2 \Rightarrow h=e_{1,2}^2 \Rightarrow e_{1,2}=\sqrt{h};\\
&-ha_1=e_{1,2}e_{4,2}+e_{2,2}e_{3,2} \Rightarrow -ha_1=\sqrt{h}(e_{4,2}-e_{2,2}) \Rightarrow \sqrt{h}a_1=e_{2,2}-e_{4,2};\\
&-hb_1+b_2=e_{2,2}e_{4,2};\\
&a_2+\sqrt{h}a_1=-e_{4,2}-e_{2,2}+e_{2,2}-e_{4,2} \Rightarrow a_2+\sqrt{h}a_1=-2e_{4,2} \Rightarrow e_{4,2}=-\frac{a_2+\sqrt{h}a_1}{2};\\
&a_2=-e_{4,2}-e_{2,2} \Rightarrow a_2=-e_{2,2}+\frac{a_2+\sqrt{h}a_1}{2} \Rightarrow e_{2,2}=\frac{-a_2+\sqrt{h}a_1}{2};\\
&-hb_1+b_2=-\frac{a_2+\sqrt{h}a_1}{2}\cdot\frac{-a_2+\sqrt{h}a_1}{2}=-\frac{ha_1^2-a_2^2}{4};\\
&-hb_1+\frac{ha_1^2}{4}=\frac{a_2^2}{4}-b_2 \Rightarrow h(-b_1+\frac{a_1^2}{4})=\frac{a_2^2}{4}-b_2 \Rightarrow h=\frac{\frac{a_2^2-4b_2}{4}}{\frac{a_1^2-4b_1}{4}}=\frac{a_2^2-4b_2}{a_1^2-4b_1}.
\end{aligned}
$$

So we get that:

$$
\begin{aligned}
e_{1,2}&=\sqrt{\frac{a_2^2-4b_2}{a_1^2-4b_1}};\\
e_{2,2}&=\frac{-a_2+\sqrt{\frac{a_2^2-4b_2}{a_1^2-4b_1}}a_1}{2};\\
e_{3,2}&=-\sqrt{\frac{a_2^2-4b_2}{a_1^2-4b_1}}=-e_{1,2};\\
e_{4,2}&=\frac{-a_2-\sqrt{\frac{a_2^2-4b_2}{a_1^2-4b_1}}a_1}{2}=e_{2,2}-e_{1,2}a_1.
\end{aligned}
$$

Now let p' be the ideal $(x_i^2+a_1x_i+b_1, h_j | j\in[n]\setminus\{i\})$ for $h_j\in\{x_j-x_i, x_j+x_i+a_1, x_j-e_{1,2}x_i-e_{2,2}, x_j-e_{3,2}x_i-e_{4,2}\}$, since we have that all f_j are either equal to x_j-x_i, $x_j+x_i+a_1$, $x_j-e_{1,2}x_i-e_{2,2}$ or $x_j-e_{3,2}x_i-e_{4,2}$, we have that $\mathbb{R}[x_1,...,x_n]/p'\cong\mathbb{R}[x_i]/(x_i^2+ax_i+b)\cong\mathbb{C}$, where $x_i^2+ax_i+b$ is irreducible. Since the latter is a domain, we know that p' is indeed prime.

So we have that a minimal prime ideal p that contains I and $x_i^2+a_1x_i+b_1$, should contain at least one of x_j-x_i, $x_j+x_i+a_1$, $x_j-e_{1,2}x_i-e_{2,2}$ and $x_j-e_{3,2}x_i-e_{4,2}$ for all $j \in [n] \setminus \{i\}$, where

$$e_{1,2} = \sqrt{\frac{a_2^2 - 4b_2}{a_1^2 - 4b_1}};$$

$$e_{2,2} = \frac{-a_2 + \sqrt{\frac{a_2^2-4b_2}{a_1^2-4b_1}}a_1}{2};$$

$$e_{3,2} = -\sqrt{\frac{a_2^2 - 4b_2}{a_1^2 - 4b_1}} = -e_{1,2};$$

$$e_{4,2} = \frac{-a_2 - \sqrt{\frac{a_2^2-4b_2}{a_1^2-4b_1}}a_1}{2} = e_{2,2} - e_{1,2}a_1.$$

□

4.2 The degree of a sum of polynomials

If we have a sum from 1 to n of polynomials of degree d, we can show that the sum is a polynomial in n, with a degree of $d+1$.

Lemma 4.2.1. *Let $f(x) = x^d$ be a polynomial of degree d and consider*

$$g(n) = \sum_{k=0}^{n} f(k)$$

for $n \geq 0$. Then we have that $g(n)$ is a polynomial in n of degree $d+1$.

Proof. By the Euler-Maclaurin formula [2], we get the following

$$\sum_{k=0}^{n} f(k) = \int_0^{(n+1)} f(x)dx - \frac{(n+1)^d}{2} + \sum_{j=1}^{\infty} \frac{B_{2j}}{(2j)!}(f^{(2j-1)}(n+1) - f^{(2j-1)}(0)).$$

Here we have that B_j are Bernoulli numbers and $f^{(j)}(x)$ is the j^{th} derivative of $f(x)$. Now since $f(x)$ is a polynomial, the terms

$$\sum_{j=1}^{\infty} \frac{B_{2j}}{(2j)!}(f^{(2j-1)}(n+1) - f^{(2j-1)}(0))$$

are all zero when $2j - 1 > d$. Now since $\int_0^{n+1} f(x)dx$ is of degree $d+1$ and all the other terms in the above formula have lower degrees, we get that $\sum_{k=0}^{n} x^d$ is a polynomial in n and has a degree of $d+1$. □

Corollary 4.2.2. *Suppose $f(x)$ is an arbitrary polynomial of degree d, now consider*

$$g(n) = \sum_{k=0}^{n} f(k)$$

for $n \geq 0$. Then we have that $g(n)$ is a polynomial in n of degree $d+1$.

Proof. If we have a polynomial $f(x)$ of degree d, it can be written as

$$f(x) = c_d x^d + c_{d-1} x^{d-1} + ... + c_1 d + c_0.$$

We have here that $c_d, ..., c_0$ are constants and c_d is nonzero. We now have that

$$\sum_{k=0}^{n} f(k) = (\sum_{k=0}^{n} c_d x^d) + (\sum_{k=0}^{n} c_{d-1} x^{d-1}) + ... + (\sum_{k=0}^{n} c_0).$$

Now we can see easily, using 4.2.1, that the degree of the first sum is $d+1$, the degree of the second sum is d, etc. Thus we have that $g(n) = \sum_{k=0}^{n} f(k)$ is a polynomial in n of degree $d+1$. $\square$

Chapter 5

Counting in $\mathbb{C}[x_1, ..., x_n]$

Here we will find the number of Sym(n)-orbits of the minimal prime ideals that contain an ideal $I \subset K[x_1, ..., x_n]$, where K is a field, which is generated by functions that can be decomposed into linear polynomial factors. Then we will see that when K is equal to $\mathbb{C}$ this covers all ideals that are generated by the Sym(n)-orbit of a single univariate polynomial in $\mathbb{C}[x_1, ..., x_n]$.
Let $f(x) \in K[x]$. Suppose $f(x)$ is decomposed as $(x - a_1)^{e_1}(x - a_2)^{e_2}...(x - a_d)^{e_d}$, where $a_1 \neq a_2 \neq ... \neq a_d \in K$ and $e_1, ..., e_d \in \mathbb{N}$.
We now have an ideal $I = (f(x_1), f(x_2), ..., f(x_n)) \subset K[x_1, ..., x_n]$.

Proposition 5.0.1. *The number of minimal prime ideals p in $K[x_1, ..., x_n]$ that contain I is d^n.*

Proof. We have $I = (f(x_1), f(x_2), ..., f(x_n))$ and we also have that $f(x) = (x - a_1)^{e_1}(x - a_2)^{e_2}...(x - a_d)^{e_d}$. An ideal is prime when it has the property that for any $ab \in I$, either a or b is in I. Since linear polynomials are prime, we have that a p that is generated by linear polynomials is prime. So for a minimal prime ideal p to contain I, we have to have that one of the factors $x_i - a_j$ for all $i \in [n], j \in [d]$ should be contained in p. Since all the linear polynomials $x_i - a_j$ are divisors of $f(x_i)$, the minimal prime ideal that consists of the polynomials $x_i - a_j$, where every $i \in [n]$ is only used once, contains I.
So now for p to contain I we should choose one of the d factors of every $f(x_i)$, so we have n choices of d options, so in total there are d^n minimal prime ideals that contain I. □

Theorem 5.0.2. *The number of Sym(n)-orbits of the minimal prime ideals containing*

$$I = (f(x_1), f(x_2), ..., f(x_n))$$

is equal to

$$\binom{n+d-1}{d-1}.$$

Proof. To find the number of Sym(n)-orbits we have to select a j for every i in $x_i - a_j$. This gives us that we have to divide all n over the d different j, as stated above. We have that the Sym(n)-orbits make sure that there is no difference between x_i, so order does not matter. This means the ideal $(x_1 - 2, x_2 - 3)$ is in the same Sym(n)-orbit as the ideal $(x_1 - 3, x_2 - 2)$ since one x_i has been given a 3 and one has been given a 2 in both ideals. So now we have an egg-colouring problem as given in 2.3.1, where we have n eggs and d colours. This gives us that there are $f(d, n) = \binom{n+d-1}{d-1}$ Sym(n)-orbits of the minimal prime ideals that contain I. □

Corollary 5.0.3. *If we take our polynomial ring* $K[x_1, ..., x_n]$ *to be* $\mathbb{C}[x_1, ..., x_n]$*, we see that any polynomial* $f(x) \in \mathbb{C}[x]$ *can be decomposed as* $(x - a_1)^{e_1}(x - a_2)^{e_2}...(x - a_d)^{e_d}$ *and thus any ideal*

$$I = (f(x_1), f(x_2), ..., f(x_n)) \subset \mathbb{C}[x_1, x_2, ..., x_n]$$

has d^n *minimal prime ideals containing* I *and has* $\binom{n+d-1}{d-1}$ *Sym(n)-orbits of the minimal prime ideals* p *containing* I*.*

Chapter 6

Counting in $\mathbb{R}[x_1, ..., x_n]$

Here we first make a base case, namely the ideals are generated by polynomials $f(x_i)$ which can be factored into a single irreducible polynomial in $\mathbb{R}[x_i]$ to the power of e, which we will see in section 6.1. We then use this base case to find a general expression by induction for the ideals which are generated by functions $f(x_i)$ which can be factored into an arbitrary number of irreducible polynomials in $\mathbb{R}[x_i]$ in section 6.2. This expression can then be upgraded, in section 6.3, to the general expression for the number of Sym(n)-orbits of the minimal prime ideals that contain an ideal in $\mathbb{R}[x_1, ..., x_n]$ that is generated by $f(x_i)$ which can be factored into an arbitrary number of irreducible linear and quadratic polynomials in $\mathbb{R}[x_1, ..., x_n]$, we then prove that this expression is quasi-polynomial and we give the maximum degree of this expression.

6.1 A single quadratic irreducible polynomial

Let $f(x) \in \mathbb{R}[x]$. Suppose $f(x)$ is decomposed as $(x^2 + ax + b)^e$, where we have that $x^2 + ax + b$ is irreducible and where $a, b \in \mathbb{R}$ and $e \in \mathbb{N}$. We now consider the ideal $I_{n,1} = (f(x_1), f(x_2), ..., f(x_n)) \subset \mathbb{R}[x_1, ..., x_n]$ (here we have that the 1 in $I_{n,1}$ stands for the fact that $f(x)$ decomposes into a single irreducible polynomial, which will become clearer in section 6.2).

Lemma 6.1.1. *Suppose p is a minimal prime ideal containing $I_{n,1}$. Fix $i \in [n] := \{1, 2, ..., n\}$. Then:*

(a) For every $j \in [n] \setminus \{i\}$, there is an $f_j \in \{x_j - x_i, x_j + x_i + a\}$ such that $p = (x_i^2 + ax_i + b, f_j | j \in [n] \setminus \{i\})$.

(b) If there is a j such that $f_j = x_j - x_i$, then $p = (x_j^2 + ax_j + b, g_k | k \in [n] \setminus \{j\})$, where $g_k \in \{x_k - x_j, x_k + x_j + a\}$ such that for all $k \neq i, j$:

$$f_k = x_k - x_i \iff g_k = x_k - x_j.$$

(c) If there is a j such that $f_j = x_j + x_i + a$, then $p = (x_j^2 + ax_j + b, g_k | k \in [n] \setminus \{j\})$, where $g_k \in \{x_k - x_j, x_k + x_j + a\}$ such that for all $k \neq i, j$: $f_k = x_k - x_i \iff g_k = x_k + x_j + a$.

Proof. (a) If the ideal p contains $I_{n,1} = ((x_1^2 + ax_1 + b)^e, ..., (x_n^2 + ax_n + b)^e)$ and $x_i^2 + ax_i + b$, then p contains at least one of $x_j - x_i$ and $x_j + x_i + a$ for all $j \neq i$, because $(x_j - x_i)(x_j + x_i + a) = x_j^2 + x_i x_j + ax_j - x_i x_j - x_i^2 - ax_i = (x_j^2 + ax_j + b) - (x_i^2 + ax_i + b)$, which is a linear combination of $x_i^2 + ax_i + b$ and $x_j^2 + ax_j + b$, which shows $x_j^2 + ax_j + b(mod x_i^2 + ax_i + b)$.

Let p' be the ideal $(x_i^2 + ax_i + b, f_j | j \in [n] \setminus \{i\})$, for $f_j \in \{x_j - x_i, x_j + x_i + a\}$. Since we have that all f_j are either equal to $x_j - x_i$ or to $x_j + x_i + a$, we have that $\mathbb{R}[x_1, ..., x_n]/p' \cong \mathbb{R}[x_i]/(x_i^2 + ax_i + b) \cong \mathbb{C}$, where $x_i^2 + ax_i + b$ is irreducible. Since the latter is a domain, we know that p' is indeed prime.

So in general we get that a minimal prime ideal p that contains $I_{n,1}$ is $p = (x_i^2 + ax_i + b, f_j | j \in [n] \setminus \{i\})$, where $f_j \in \{x_j - x_i, x_j + x_i + a\}$.

(b) If we have a $p = (x_i^2 + ax_i + b, f_j | j \in [n] \setminus \{i\})$ in which we have a j such that $f_j = x_j - x_i$, then we can do a couple operations inside of this ideal.
We then have that for every $k \in [n] \setminus \{i, j\}$ such that $f_k = x_k - x_i$, we can rewrite f_k so that we get

$$f_k = x_k - x_i - (x_j - x_i) = x_k - x_j = g_k.$$

We also have that for every $k \in [n] \setminus \{i, j\}$ such that $f_k = x_k + x_i + 1$, we can rewrite f_k so that we get

$$f_k = x_k + x_i + 1 + (x_j - x_i) = x_k + x_j + 1 = g_k.$$

We also have that we can rewrite $f_j = x_j - x_i - 2(x_j - x_i) = x_i - x_j = g_i$. We also have that since $f_j = x_j - x_i$, we can rewrite $x_i^2 + ax_i + b$ so that we get

$$x_i^2 + ax_i + b + (x_i + x_j + a)(x_j - x_i) = x_j^2 + ax_j + b.$$

So in total we have that if we have a $p = (x_i^2 + ax_i + b, f_j | j \in [n] \setminus \{i\})$ in which we have a j such that $f_j = x_j - x_i$, we can rewrite this p, such that we get $p = (x_j^2 + ax_j + b, g_k | k \in [n] \setminus \{j\})$, where $g_k \in \{x_k - x_j, x_k + x_j + a\}$ and we also have seen that $\forall k \neq i, j$: $f_k = x_k - x_i \iff g_k = x_k - x_j$ and we also have $f_k = x_k + x_i + a \iff g_k = x_k + x_j + a$.

(c) If we have a $p = (x_i^2 + ax_i + b, f_j | j \in [n] \setminus \{i\})$ in which we have a j such that $f_j = x_j + x_i + a$, then we can do a couple operations inside of this ideal.
We then have that for every $k \in [n] \setminus \{i, j\}$ such that $f_k = x_k - x_i$, we can rewrite f_k so that we get

$$f_k = x_k - x_i + (x_j + x_i + a) = x_k + x_j + a = g_k.$$

We also have that for every $k \in [n] \setminus \{i, j\}$ such that $f_k = x_k + x_i + a$, we can rewrite f_k so that we get

$$f_k = x_k + x_i + a - (x_j + x_i + a) = x_k - x_j = g_k.$$

We also have that we can rewrite $f_j = x_j + x_i + a = x_i + x_j + a = g_i$. We also have that since $f_j = x_j + x_i + a$, we can rewrite $x_i^2 + ax_i + b$ so that we get

$$x_i^2 + ax_i + b + (x_j - x_i)(x_j + x_i + a) = x_j^2 + ax_j + b.$$

So in total we have that if we have a $p = (x_i^2 + ax_i + b, f_j | j \in [n] \setminus \{i\})$ in which we have a j such that $f_j = x_j + x_i + a$, we can rewrite this p, such that we get $p = (x_j^2 + ax_j + b, g_k | k \in [n] \setminus \{j\})$, where $g_k \in \{x_k - x_j, x_k + x_j + a\}$

and we also have seen that $\forall k \neq i, j$: $f_k = x_k - x_i \iff g_k = x_k + x_j + a$ and $f_k = x_k + x_i + a \iff g_k = x_k - x_j$.

$\square$

Example 6.1.2. *Suppose we have that $I_{3,1} = (f(x_1), f(x_2), f(x_3)) \subset \mathbb{R}[x_1, x_2, x_3]$, where we assume that each $x_i^2 + ax_i + b$ is irreducible in $\mathbb{R}[x_1, x_2, x_3]$. We now have for any p that contains $I_{3,1}$:*

- *We have for each $j \in [n] \setminus \{i\}$, there is an $f_j \in \{x_j - x_i, x_j + x_i + a\}$ such that $p = (x_i^2 + ax_i + b, f_j | j \in [n] \setminus \{i\})$.*
- *If there is a j: $f_j = x_j - x_i$, then $p = (x_j^2 + ax_j + b, g_k | k \in [n] \setminus \{j\})$, where $g_k \in \{x_k - x_j, x_k + x_j + a\}$ such that for all $k \neq i, j$: $f_k = x_k - x_i \iff g_k = x_k - x_j$.*
- *If there is a j: $f_j = x_j + x_i + a$, then $p = (x_j^2 + ax_j + b, g_k | k \in [n] \setminus \{j\})$, where $g_k \in \{x_k - x_j, x_k + x_j + a\}$ such that for all $k \neq i, j$: $f_k = x_k - x_i \iff g_k = x_k + x_j + a$.*

So now if we write down all p which contain an $x_i^2 + ax_i + b$ we get the following:

$$\begin{aligned}
p_1 &= (x_1^2 + ax_1 + b, x_2 - x_1, x_3 - x_1);\\
p_2 &= (x_1^2 + ax_1 + b, x_2 - x_1, x_3 + x_1 + a);\\
p_3 &= (x_1^2 + ax_1 + b, x_2 + x_1 + a, x_3 - x_1);\\
p_4 &= (x_1^2 + ax_1 + b, x_2 + x_1 + a, x_3 + x_1 + a);\\
p_5 &= (x_2^2 + ax_2 + b, x_1 - x_2, x_3 - x_2);\\
p_6 &= (x_2^2 + ax_2 + b, x_1 - x_2, x_3 + x_2 + a);\\
p_7 &= (x_2^2 + ax_2 + b, x_1 + x_2 + a, x_3 - x_2);\\
p_8 &= (x_2^2 + ax_2 + b, x_1 + x_2 + a, x_3 + x_2 + a);\\
p_9 &= (x_3^2 + ax_3 + b, x_1 - x_3, x_2 - x_3);\\
p_{10} &= (x_3^2 + ax_3 + b, x_1 + x_3 + a, x_2 - x_3);\\
p_{11} &= (x_3^2 + ax_3 + b, x_1 - x_3, x_2 + x_3 + a);\\
p_{12} &= (x_3^2 + ax_3 + b, x_1 + x_3 + a, x_2 + x_3 + a).
\end{aligned}$$

And here we see that we can say that a few of these are the same ideal under permutation and are thus in the same Sym(n)-orbit. This means there is a permutation of the indices in the variables x_1, x_2 and x_3 such that $p_i \sim p_j$ for two of the ideals p_i, p_j given above. For the permutations $(1,2)$ and $(1,3)$ we respectively see that p_1 is in the same Sym(n)-orbit as p_5 and p_9, p_2 is in the same Sym(n)-orbit as p_6 and p_{10}, p_3 is in the same Sym(n)-orbit as p_7 and p_{11} and p_4 is in the same Sym(n)-orbit as p_8 and p_{12}. And due to the second and third property above combined with permutations we can also say that p_2 is in the same Sym(n)-orbit as p_3 and p_4.

Now to show why p_2 is in the same Sym(n)-orbit as p_4, we have to use the third property given above. We will show it below using the properties of an ideal.

$$\begin{aligned}
&(x_1^2 + ax_1 + b, x_2 - x_1, x_3 + x_1 + a) \\
&= (x_1^2 + ax_1 + b + (x_3 - x_1)(x_3 + x_1 + a), x_2 - x_1 + x_3 + x_1 + a, x_1 + x_3 + a) \\
&= (x_1^2 + ax_1 + b + x_3^2 - x_1^2 + ax_3 - ax_1, x_2 + x_3 + a, x_1 + x_3 + a) \\
&= (x_3^2 + ax_3 + b, x_2 + x_3 + a, x_1 + x_3 + a) \\
&= (x_3^2 + ax_3 + b, x_1 + x_3 + a, x_2 + x_3 + a) \\
&= (x_1^2 + ax_1 + b, x_2 + x_1 + a, x_3 + x_1 + a).
\end{aligned}$$

Here we used in the first equality that $x_3 - x_1 \in \mathbb{R}[x_1, x_2, x_3]$ and $x_3 + x_1 + a \in p$, thus $(x_3 - x_1)(x_3 + x_1 + a) \in p$ and in the last equality we used that p_4 is the same as p_{12} under permutation. It can be shown similarly that p_3 is in the same Sym(n)-orbit as p_4. Thus we have that $p_1 \sim p_5 \sim p_9$ and $p_2 \sim p_3 \sim p_4 \sim p_6 \sim p_7 \sim p_8 \sim p_{10} \sim p_{11} \sim p_{12}$. So we have that there are 2 Sym(n)-orbits of the minimal prime ideals p that contain $I_{3,1}$.

As seen in the example above, we get that the total number of minimal prime ideals p that contain the ideal $I_{3,1} \subset \mathbb{R}[x_1, ..., x_3]$ is bigger than the number of Sym(n)-orbits of the minimal prime ideals that contain $I_{3,1}$. We will show below that this is the case for any $I_{n,1}$. We will do this after we have made a representation that makes counting the number of Sym(n)-orbits of the minimal prime ideals p that contain $I_{n,1}$ easier.

Definition 6.1.3. *If p is the ideal $(x_i^2 + ax_i + b, f_j | j \in [n] \setminus \{i\})$, where $f_j \in \{x_j - x_i, x_j + x_i + a\}$, we say it is of type $(i, k, n-k-1)$. We have here that $k = \#j \neq i : f_j = x_j - x_i$ and where $n - k - 1 = \#j \neq i : f_j = x_j + x_i + a$.*

Lemma 6.1.4. *Suppose p, q are minimal prime ideals containing $I_{n,1}$. Now suppose p is of type $(i, k, n-k-1)$. Then p, q are in the same Sym(n)-orbit if there is a j such that q is of type $(j, k, n-k-1)$; moreover, in this the case q is of type $(i, k, n-k-1)$ or $(i, n-k-2, k+1)$.*

Proof. We want to know that if p is of type $(i, k, n-k-1)$ and q is of type $(j, l, n-l-1)$ whether they are in the same Sym(n)-orbit. This will be the case if and only if either $l = k$ or $l = n-k-2$.

Now since p is of type $(i, k, n-k-1)$, there are $i_1, ..., i_k, i'_1, ..., i'_{n-k-1} \neq i$ which are pairwise distinct, such that $p = (x_i^2 + ax_i + b, x_{i_1} - x_i, ..., x_{i_k} - x_i, x_{i'_1} + x_i + a, ..., x_{i'_{n-k-1}} + x_i + a)$.

Now if q is is of type $(j, k, n-k-1)$, then likewise there are $j_1, ..., j_k, j'_1, ..., j'_{n-k-1} \neq j$ which are pairwise distinct, such that $q = (x_j^2 + ax_j + b, x_{j_1} - x_j, ..., x_{j_k} - x_j, x_{j'_1} + x_j + a, ..., x_{j'_{n-k-1}} + x_j + a)$.

Now take the permutation σ of $1, ..., n$ that sends i to j, i_f to j_f for $f \in \{1, ..., k\}$, and i'_m to j'_m for $m \in \{1, ..., n-k-1\}$. Then $\sigma(p) = q$, so we have that p of type $(i, k, n-k-1)$ and q of type $(j, k, n-k-1)$ are in the same Sym(n)-orbit.

From here it can clearly be seen that if p is of type $(i, k, n-k-1)$ and q is of type $(i, k, n-k-1)$, that they are in the same Sym(n)-orbit.

Now if q is of type $(j, n-k-2, k+1)$, then there are $j_1, ..., j_{n-k-2}, j'_1, ..., j'_{k+1}$ such that $q = (x_j^2 + ax_j + b, x_{j_1} - x_j, ..., x_{j_{n-k-2}} - x_j, x_{j'_1} + x_j + a, ..., x_{j'_{k+1}} + x_j + a)$.

Now from the third statement of Lemma 6.1.1 we find that $q = (x_{j'_1}^2 + ax_{j'_1} + b, x_{j_1} + x_{j'_1} + a, ..., x_{j_{n-k-2}} + x_{j'_1} + a, x_{j'_2} - x_{j'_1}, ..., x_{j'_{k+1}} - x_{j'_1}, x_j + x_{j'_1} + a)$. This can be reordered to $q = (x_{j'_1}^2 + ax_{j'_1} + b, x_{j'_2} - x_{j'_1}, ..., x_{j'_{k+1}} - x_{j'_1}, x_{j_1} + x_{j'_1} + a, ..., x_{j_{n-k-2}} + x_{j'_1} + a, x_j + x_{j'_1} + a)$ and we find that we now have that q is of type $(j'_1, k, n-k-1)$. As seen above, we now know that p and q are in the same Sym(n)-orbit.

From here it can clearly be seen that if p is of type $(i, k, n-k-1)$ and q is of type $(i, n-k-2, k+1)$, that they are in the same Sym(n)-orbit.

Now if p is of type $(i, k, n-k-1)$ and q is of type $(j, l, n-l-1)$, then without loss of generality, namely by using a permutation of the indices in the variables $x_1, ..., x_n$, we have that p is of type $(1, k, n-k-1)$, where $p = (x_1^2 + ax_1 + b, x_2 - x_1, ..., x_{k+1} - x_1, x_{k+2} + x_1 + a, ..., x_n + x_1 + a)$ and q is of type $(j, l, n-l-1)$.
If p and q were to be in the same Sym(n)-orbit, we would have a permutation σ, where $p = \sigma(q)$. If this were the case we would have that p is of type $(1, k, n-k-1)$ and $\sigma(q)$ is of type $(1, l, n-l-1)$ or $(1, n-l-2, l+1)$, depending on whether $x_{\sigma^{-1}(1)} - x_j$ is in q or $x_{\sigma^{-1}(1)} + x_j + a$ is in q.
Without loss of generality in the case that q is of type $(1, l, n-l-1)$, we have that $p = (x_1^2 + ax_1 + b, x_2 - x_1, ..., x_{k+1} - x_1, x_{k+2} + x_1 + a, ..., x_n + x_1 + a)$ and that $\sigma(q) = (x_1^2 + ax_1 + b, x_{j_1} - x_1, ..., x_{j_l} - x_1, x_{j'_1} + x_1 + a, ..., x_{j'_{n-l-2}} + x_1 + a)$, where all j_i and j'_i are pairwise distinct and not equal to 1. The case that q is of type $(1, n-l-2, l+1)$ is treated similarly.

Suppose there is a position i, such that there is $x_i - x_1$ in p and $x_i + x_1 + a$ in $\sigma(q)$. Then since we have that $\sigma(q) = p$, we get that $x_i + x_1 + a \in p$. So we also get that $x_i + x_1 + a - (x_i - x_1) = 2x_1 + a \in p \Rightarrow \frac{1}{2}(2x_1 + a) = x_1 + \frac{a}{2} \in p$. Then now since $x_1^2 + ax_1 + b \in p$, we have that $(-\frac{a}{2})^2 + a \cdot -\frac{a}{2} + b = -\frac{a^2}{4} + b \in p$. We have that $-\frac{a^2}{4} + b \neq 0$, since $x_1^2 + ax_1 + b$ has no real zeroes. So we have that p contains a non-zero constant, so we have that $p = \mathbb{R}$. This is a contradiction with the fact that p is a minimal prime ideal. So we have that l should be equal to k in this case, in the case that q is of type $(1, n-l-2, l+1)$, we find that $n-l-2$ is equal to k and hence l is equal to $n-k-2$.

So we have that if p, q are minimal prime ideals containing $I_{n,1}$ and p is of type $(i, k, n-k-1)$. Then p, q are in the same Sym(n)-orbit when we have that there is a j such that q is of type $(j, k, n-k-1)$, moreover, in this the case q is of type $(i, k, n-k-1)$ or $(i, n-k-2, k+1)$. □

Lemma 6.1.5. *The number of Sym(n)-orbits of the minimal prime ideals p containing $I_{n,1}$ is equal to:*

$$\frac{n}{2} + 1 \text{ for even } n;$$
$$\frac{n-1}{2} + 1 \text{ for odd } n.$$

Proof. First we look at the fact that p and q are in the same Sym(n)-orbit if we have that p is of type $(i, k, n-k-1)$ and q is of type $(j, k, n-k-1)$ for some i, j, k. This that leaves us with n Sym(n)-orbits, as the values that k can take ranges from 0 to $n-1$.

Now we look at the fact that p and q are in the same Sym(n)-orbit if p is of type $(i, k, n-k-1)$ and q is of type $(i, n-k-2, k+1)$ for some i, j, k. So now we look at the values that k can take. We have that k can go from 0 to $n-1$.

Now for an even n, if we have a p with type $(i, k, n-k-1)$ with a given value of k between $\frac{n}{2} - 1$ and $n-2$, it lies in the same Sym(n)-orbit as an ideal of type

$(i, l, n - l - 1)$ where $l = n - k - 2$ lies between 0 and $\frac{n}{2} - 1$. If we have a p of type $(i, n - 1, 0)$, then any ideal in its Sym(n)-orbit will also be of type $(i, n - 1, 0)$. These Sym(n)-orbits are distinct by Lemma 6.1.4 So in total we get that there are $\frac{n}{2} + 1$ distinct Sym(n)-orbits for even n.

Now for an odd n, if we have a p with type $(i, k, n - k - 1)$ with a given value of k between $\frac{n-1}{2}$ and $n - 2$, it lies in the same Sym(n)-orbit as an ideal of type $(i, l, n - l - 1)$ where $l = n - k - 2$ lies between 0 and $\frac{n-1}{2} - 1$. If we have a p of type $(i, n - 1, 0)$, then any ideal in its Sym(n)-orbit will also be of type $(i, n - 1, 0)$. These Sym(n)-orbits are distinct by Lemma 6.1.4. So in total we get that there are $\frac{n-1}{2} + 1$ Sym(n)-orbits for odd n.

So for any $n \in \mathbb{N}$, we have that there are

$$\frac{n}{2} + 1 \text{ for even } n;$$
$$\frac{n-1}{2} + 1 \text{ for odd } n.$$

Sym(n)-orbits of the minimal prime ideals p containing $I_{n,1} = ((x_1^2 + ax_1 + b)^e, ..., (x_n^2 + ax_n + b)^e)$. □

6.2 Multiple quadratic irreducible polynomials

Let $f(x) \in \mathbb{R}[x]$. Suppose $f(x)$ is decomposed as $(x^2 + a_1x + b_1)^{y_1}(x^2 + a_2x + b_2)^{y_2}...(x^2 + a_dx + b_d)^{y_d}$, where we have that all $x^2 + a_1x + b_1$,...,$x^2 + a_dx + b_d$ are distinct and irreducible and where all $a_i, b_i \in \mathbb{R}$ and the exponents $y_i \in \mathbb{N}$ for all $i \in \{1, ..., d\}$. We now consider the ideal $I_{n,d} = (f(x_1), f(x_2), ..., f(x_n)) \subset \mathbb{R}[x_1, ..., x_n]$ (here we have that the d in $I_{n,d}$ stands for the fact that $f(x)$ decomposes into powers of d distinct irreducible polynomials).

Lemma 6.2.1. *Suppose p is a minimal prime ideal containing $I_{n,d}$. Fix $i \in [n] := \{1, 2, ..., n\}$. If p contains $x_i^2 + a_1x_i + b_1$; then*

(a) *For every $j \in [n] \setminus \{i\}$, there is an $f_j \in \{x_j - x_i, x_j + x_i + a_1, x_j - e_{1,2}x_i - e_{2,2}, x_j - e_{3,2}x_i - e_{4,2}, x_j - e_{1,3}x_i - e_{2,3}, x_j - e_{3,3}x_i - e_{4,3}, ..., x_j - e_{1_d}x_i - e_{2,d}, x_j - e_{3,d}x_i - e_{4,d}\}$, where $e_{1,m} = \sqrt{\frac{a_m^2 - 4b_m}{a_1^2 - 4b_1}}$, $e_{2,m} = \frac{-a_m + \sqrt{\frac{a_m^2 - 4b_m}{a_1^2 - 4b_1}}a_1}{2}$, $e_{3,m} = -e_{1,m}$, $e_{4,m} = e_{2,m} - e_{1,m}a_1$ such that $p = (x_i^2 + a_1x_i + b_1, f_j | j \in [n] \setminus \{i\})$.*

(b) *If there is a j such that $f_j = x_j - x_i$, then $p = (x_j^2 + a_1x_j + b_1, g_k | k \in [n] \setminus \{j\})$, where $g_k \in \{x_k - x_j, x_k + x_j + a_1, x_k - e_{1,m}x_j - e_{2,m}, x_k - e_{3,m}x_j - e_{4,m}\}$ for all $m \in \{2, ..., d\}$ such that $f_j = x_j - x_i \iff g_i = x_i - x_j$ and for all $k \neq i, j$, for all $m \in \{2, ..., d\}$: $f_k = x_k - x_i \iff g_k = x_k - x_j$, $f_k = x_k + x_i + a_1 \iff g_k = x_k + x_j + a_1$, $f_k = x_k - e_{1,m}x_i - e_{2,m} \iff g_k = x_k - e_{1,m}x_j - e_{2,m}$ and $f_k = x_k - e_{3,m}x_i - e_{4,m} \iff g_k = x_k - e_{3,m}x_j - e_{4,m}$.*

(c) *If there is a j such that $f_j = x_j + x_i + a_1$, then $p = (x_j^2 + a_1x_j + b_1, g_k | k \in [n] \setminus \{j\})$, where $g_k \in \{x_k - x_j, x_k + x_j + a_1, x_k - e_{1,m}x_j - e_{2,m}, x_k - e_{3,m}x_j - e_{4,m}\}$ for all $m \in \{2, ..., d\}$ such that $f_j = x_j + x_i + a_1 \iff g_i = x_i + x_j + a_1$ and for all $k \neq i, j$, for all $m \in \{2, ..., d\}$: $f_k = x_k - x_i \iff g_k = x_k + x_j + a_1$, $f_k = x_k + x_i + a_1 \iff g_k = x_k - x_j$, $f_k = x_k - e_{1,m}x_i - e_{2,m} \iff g_k = x_k - e_{3,m}x_j - e_{4,m}$, $f_k = x_k - e_{3,m}x_i - e_{4,m} \iff g_k = x_k - e_{1,m}x_j - e_{2,m}$.*

Proof. (a) If the ideal p contains $I_n = (f(x_1), f(x_2), ..., f(x_n))$, we have that for any p that contains $x_i^2 + a_1 x_i + b_1$ and I_n, it contains at least one of $x_j - x_i$, $x_j + x_i + a_1$, $x_j - e_{1,m}x_i - e_{2,m}$ and $x_j - e_{3,m}x_i - e_{4,m}$ for all $m \in \{2, ..., d\}$ and for all $j \neq i$, because by 4.1.3, we have that if p contains $x_j^2 + a_1 x_j + b_1$, we have that it contains one of $x_j - x_i$ and $x_j + x_i + a_1$ and if p contains $x_j^2 + a_m x_j + b_m$, for an $m \in \{2, ..., d\}$, we have that it contains one of $x_j - e_{1,m}x_i - e_{2,m}$ and $x_j - e_{3,m}x_i - e_{4,m}$.

As in 4.1.3 we also see that p is prime.

So in general we get that a minimal prime ideal p that contains $I_{n,d}$ is $p = (x_i^2 + a_1 x_i + b_1, f_j | j \in [n] \setminus \{i\})$, where $f_j \in \{x_j - x_i, x_j + x_i + a_1, x_j - e_{1,m}x_i - e_{2,m}, x_j - e_{3,m}x_i - e_{4,m}\}$ for $m \in \{2, ..., d\}$.

(b) If we have a $p = (x_i^2 + a_1 x_i + b_1, f_j | j \in [n] \setminus \{i\})$ in which we have a j such that $f_j = x_j - x_i$, then we can do a couple operations inside of this ideal.
We then have that for every $k \in [n] \setminus \{i, j\}$ such that $f_k = x_k - x_i$, we can rewrite f_k so that we get

$$f_k = x_k - x_i - (x_j - x_i) = x_k - x_j = g_k.$$

We also have that for every $k \in [n] \setminus \{i, j\}$ such that $f_k = x_k + x_i + a_1$, we can rewrite f_k so that we get

$$f_k = x_k + x_i + a_1 + (x_j - x_i) = x_k + x_j + a_1 = g_k.$$

We also have that for every $k \in [n] \setminus \{i, j\}$ such that $f_k = x_k - e_{1,m}x_i - e_{2,m}$, we can rewrite f_k so that we get

$$f_k = x_k - e_{1,m}x_i - e_{2,m} - e_{1,m}(x_j - x_i) = x_k - e_{1,m}x_j - e_{2,m} = g_k.$$

We also have that for every $k \in [n] \setminus \{i, j\}$ such that $f_k = x_k - e_{3,m}x_i - e_{4,m}$, we can rewrite f_k so that we get

$$f_k = x_k - e_{3,m}x_i - e_{4,m} - e_{3,m}(x_j - x_i) = x_k - e_{3,m}x_j - e_{4,m} = g_k.$$

We also have that we can rewrite $f_j = x_j - x_i - 2(x_j - x_i) = x_i - x_j = g_i$. We also have that since $f_j = x_j - x_i$, we can rewrite $x_i^2 + a_1 x_i + b_1$ so that we get

$$x_i^2 + a_1 x_i + b_1 + (x_i + x_j + a_1)(x_j - x_i) = x_j^2 + a_1 x_j + b_1.$$

So in total we have that if we have a $p = (x_i^2 + a_1 x_i + b_1, f_j | j \in [n] \setminus \{i\})$ in which we have a j such that $f_j = x_j - x_i$, we can rewrite this p, such that we get $p = (x_j^2 + a_1 x_j + b_1, g_k | k \in [n] \setminus \{j\})$, where $g_k \in \{x_k - x_j, x_k + x_j + a_1, x_k - e_{1,m}x_j - e_{2,m}, x_k - e_{3,m}x_j - e_{4,m}\}$ for all $m \in \{2, ..., d\}$. We also have seen that $f_j = x_j - x_i \iff g_i = x_i - x_j$ and $\forall k \neq i, j$, for all $m \in \{2, ..., d\}$: $f_k = x_k - x_i \iff g_k = x_k - x_j$, $f_k = x_k + x_i + a_1 \iff g_k = x_k + x_j + a_1$, $f_k = x_k - e_{1,m}x_i - e_{2,m} \iff g_k = x_k - e_{1,m}x_j - e_{2,m}$ and $f_k = x_k - e_{3,m}x_i - e_{4,m} \iff g_k = x_k - e_{3,m}x_j - e_{4,m}$.

(c) If we have a $p = (x_i^2 + a_1 x_i + b_1, f_j | j \in [n] \setminus \{i\})$ in which we have a j such that $f_j = x_j + x_i + a_1$, then we can do a couple operations inside of this ideal.

We then have that for every $k \in [n] \setminus \{i, j\}$ such that $f_k = x_k - x_i$, we can rewrite f_k so that we get

$$f_k = x_k - x_i + (x_j + x_i + a_1) = x_k + x_j + a_1 = g_k.$$

We also have that for every $k \in [n] \setminus \{i, j\}$ such that $f_k = x_k + x_i + a_1$, we can rewrite f_k so that we get

$$f_k = x_k + x_i + a_1 - (x_j + x_i + a_1) = x_k - x_j = g_k.$$

We also have that for every $k \in [n] \setminus \{i, j\}$ such that $f_k = x_k - e_{1,m}x_i - e_{2,m}$, we can rewrite f_k so that we get

$$\begin{aligned} f_k &= x_k - e_{1,m}x_i - e_{2,m} - e_{3,m}(x_j + x_i + a_1) \\ &= x_k - e_{1,m}x_i + e_{1,m}x_i - e_{3,m}x_j - e_{2,m} - e_{3,m}a_1 \\ &= x_k - e_{3,m}x_j - e_{2,m} + e_{1,m}a_1 \\ &= x_k - e_{3,m}x_j - (e_{2,m} - e_{1,m}a_1) \\ &= x_k - e_{3,m}x_j - e_{4,m} = g_k. \end{aligned}$$

We also have that for every $k \in [n] \setminus \{i, j\}$ such that $f_k = x_k - e_{3,m}x_i - e_{4,m}$, we can rewrite f_k so that we get

$$\begin{aligned} f_k &= x_k - e_{3,m}x_i - e_{4,m} - e_{1,m}(x_j + x_i + a_1) \\ &= x_k - e_{3,m}x_i + e_{3,m}x_i - e_{1,m}x_j - e_{4,m} - e_{1,m}a_1 \\ &= x_k - e_{1,m}x_j - (e_{4,m} + e_{1,m}a_1) \\ &= x_k - e_{1,m}x_j - e_{2,m} = g_k. \end{aligned}$$

We also have that we can rewrite $f_j = x_j + x_i + a_1 = x_i + x_j + a_1 = g_i$. We also have that since $f_j = x_j + x_i + a_1$, we can rewrite $x_i^2 + a_1x_i + b_1$ so that we get

$$x_i^2 + a_1x_i + b_1 + (x_j - x_i)(x_j + x_i + a_1) = x_j^2 + a_1x_j + b_1.$$

So in total we have that if we have a $p = (x_i^2 + a_1x_i + b_1, f_j | j \in [n] \setminus \{i\})$ in which we have a j such that $f_j = x_j + x_i + a_1$, we can rewrite this p, such that we get $p = (x_j^2 + a_1x_j + b_1, g_k | k \in [n] \setminus \{j\}$, where $g_k \in \{x_k - x_j, x_k + x_j + a_1, x_k - e_{1,m}x_j - e_{2,m}, x_k - e_{3,m}x_j - e_{4,m}\}$ for all $m \in \{2, ..., d\}$. We also have seen that $f_j = x_j + x_i + a_1 \iff g_i = x_i + x_j + a_1$ and $\forall k \neq i, j$, for all $m \in \{2, ..., d\}$: $f_k = x_k - x_i \iff g_k = x_k + x_j + a_1$, $f_k = x_k + x_i + a_1 \iff g_k = x_k - x_j$, $f_k = x_k - e_{1,m}x_i - e_{2,m} \iff g_k = x_k - e_{3,m}x_j - e_{4,m}$ and $f_k = x_k - e_{3,m}x_i - e_{4,m} \iff g_k = x_k - e_{1,m}x_j - e_{2,m}$.

□

Example 6.2.2. *Suppose we have that $I_{3,2} = (f(x_1), f(x_2), f(x_3)) \subset \mathbb{R}[x_1, x_2, x_3]$, where we have that each $f(x_i)$ can be factored in $\mathbb{R}[x_1, x_2, x_3]$ in the following way $f(x_i) = (x_i^2 + ax_i + b)^{y_1}(x_i^2 + cx_i + d)^{y_2}$. Fix $i \in \{1, 2, 3\}$. We now have for any p that contains $x_i^2 + ax_i + b$ (for simplicity, in this example $e_{i,2} := e_i$ and $a_1 := a, b_1 := b, a_2 := c, b_2 := d$):*

- *We have for each* $j \in [n] \setminus \{i\}$, *there is an* $f_j \in \{x_j - x_i, x_j + x_i + a, x_j - e_1 x_i - e_2, x_j - e_3 x_i - e_4\}$, *where* $e_1 = \sqrt{\frac{c^2-4d}{a^2-4b}}$, $e_2 = \frac{-c+\sqrt{\frac{c^2-4d}{a^2-4b}}a}{2}$, $e_3 = -e_1$, $e_4 = e_2 - e_1 a$ *such that* $p = (x_i^2 + ax_i + b, f_j | j \in [n] \setminus \{i\})$.
- *If there is a j:* $f_j = x_j - x_i$, *then* $p = (x_j^2 + ax_j + b, g_k | k \in [n] \setminus \{j\})$, *where* $g_k \in \{x_k - x_j, x_k + x_j + a, x_k - e_1 x_j - e_2, x_k - e_3 x_j - e_4\}$ *such that* $f_j = x_j - x_i \iff g_i = x_i - x_j$ *and for all* $k \neq i, j$: $f_k = x_k - x_i \iff g_k = x_k - x_j$, $f_k = x_k + x_i + a \iff g_k = x_k + x_j + a$, $f_k = x_k - e_1 x_i - e_2 \iff g_k = x_k - e_1 x_j - e_2$ *and* $f_k = x_k - e_3 x_i - e_4 \iff g_k = x_k - e_3 x_j - e_4$.
- *If there is a j:* $f_j = x_j + x_i + a$, *then* $p = (x_j^2 + ax_j + b, g_k | k \in [n] \setminus \{j\})$, *where* $g_k \in \{x_k - x_j, x_k + x_j + a, x_k - e_1 x_j - e_2, x_k - e_3 x_j - e_4\}$ *such that* $f_j = x_j + x_i + a \iff g_i = x_i + x_j + a$ *for all* $k \neq i, j$: $f_k = x_k - x_i \iff g_k = x_k + x_j + a$, $f_k = x_k + x_i + a \iff g_k = x_k - x_j$, $f_k = x_k - e_1 x_i - e_2 \iff g_k = x_k - e_3 x_j - e_4$, $f_k = x_k - e_3 x_i - e_4 \iff g_k = x_k - e_1 x_j - e_2$.

So now if we write down all p which contain an $x_i^2 + ax_i + b$ *we get the following:*

$$
\begin{aligned}
p_1 &= (x_1^2 + ax_1 + b, x_2 - x_1, x_3 - x_1);\\
p_2 &= (x_1^2 + ax_1 + b, x_2 - x_1, x_3 + x_1 + a);\\
p_3 &= (x_1^2 + ax_1 + b, x_2 - x_1, x_3 - e_1 x_1 - e_2);\\
p_4 &= (x_1^2 + ax_1 + b, x_2 - x_1, x_3 - e_3 x_1 - e_4);\\
p_5 &= (x_1^2 + ax_1 + b, x_2 + x_1 + a, x_3 - x_1);\\
p_6 &= (x_1^2 + ax_1 + b, x_2 + x_1 + a, x_3 + x_1 + a);\\
p_7 &= (x_1^2 + ax_1 + b, x_2 + x_1 + a, x_3 - e_1 x_1 - e_2);\\
p_8 &= (x_1^2 + ax_1 + b, x_2 + x_1 + a, x_3 - e_3 x_1 - e_4);\\
p_9 &= (x_1^2 + ax_1 + b, x_2 - e_1 x_1 - e_2, x_3 - x_1);\\
p_{10} &= (x_1^2 + ax_1 + b, x_2 - e_1 x_1 - e_2, x_3 + x_1 + a);\\
p_{11} &= (x_1^2 + ax_1 + b, x_2 - e_1 x_1 - e_2, x_3 - e_1 x_1 - e_2);\\
p_{12} &= (x_1^2 + ax_1 + b, x_2 - e_1 x_1 - e_2, x_3 - e_3 x_1 - e_4);\\
p_{13} &= (x_1^2 + ax_1 + b, x_2 - e_3 x_1 - e_4, x_3 - x_1);\\
p_{14} &= (x_1^2 + ax_1 + b, x_2 - e_3 x_1 - e_4, x_3 + x_1 + a);\\
p_{15} &= (x_1^2 + ax_1 + b, x_2 - e_3 x_1 - e_4, x_3 - e_1 x_1 - e_2);\\
p_{16} &= (x_1^2 + ax_1 + b, x_2 - e_3 x_1 - e_4, x_3 - e_3 x_1 - e_4);
\end{aligned}
$$

$$p_{17} = (x_2^2 + ax_2 + b, x_1 - x_2, x_3 - x_2);$$
$$p_{18} = (x_2^2 + ax_2 + b, x_1 - x_2, x_3 + x_2 + a);$$
$$p_{19} = (x_2^2 + ax_2 + b, x_1 - x_2, x_3 - e_1x_2 - e_2);$$
$$p_{20} = (x_2^2 + ax_2 + b, x_1 - x_2, x_3 - e_3x_2 - e_4);$$
$$p_{21} = (x_2^2 + ax_2 + b, x_1 + x_2 + a, x_3 - x_2);$$
$$p_{22} = (x_2^2 + ax_2 + b, x_1 + x_2 + a, x_3 + x_2 + a);$$
$$p_{23} = (x_2^2 + ax_2 + b, x_1 + x_2 + a, x_3 - e_1x_2 - e_2);$$
$$p_{24} = (x_2^2 + ax_2 + b, x_1 + x_2 + a, x_3 - e_3x_2 - e_4);$$
$$p_{25} = (x_2^2 + ax_2 + b, x_1 - e_1x_2 - e_2, x_3 - x_2);$$
$$p_{26} = (x_2^2 + ax_2 + b, x_1 - e_1x_2 - e_2, x_3 + x_2 + a);$$
$$p_{27} = (x_2^2 + ax_2 + b, x_1 - e_1x_2 - e_2, x_3 - e_1x_2 - e_2);$$
$$p_{28} = (x_2^2 + ax_2 + b, x_1 - e_1x_2 - e_2, x_3 - e_3x_2 - e_4);$$
$$p_{29} = (x_2^2 + ax_2 + b, x_1 - e_3x_2 - e_4, x_3 - x_2);$$
$$p_{30} = (x_2^2 + ax_2 + b, x_1 - e_3x_2 - e_4, x_3 + x_2 + a);$$
$$p_{31} = (x_2^2 + ax_2 + b, x_1 - e_3x_2 - e_4, x_3 - e_1x_2 - e_2);$$
$$p_{32} = (x_2^2 + ax_2 + b, x_1 - e_3x_2 - e_4, x_3 - e_3x_2 - e_4);$$

$$p_{33} = (x_3^2 + ax_3 + b, x_1 - x_3, x_2 - x_3);$$
$$p_{34} = (x_3^2 + ax_3 + b, x_1 + x_3 + a, x_2 - x_3);$$
$$p_{35} = (x_3^2 + ax_3 + b, x_1 - e_1x_3 - e_2, x_2 - x_3);$$
$$p_{36} = (x_3^2 + ax_3 + b, x_1 - e_3x_3 - e_4, x_2 - x_3);$$
$$p_{37} = (x_3^2 + ax_3 + b, x_1 - x_3, x_2 + x_3 + a);$$
$$p_{38} = (x_3^2 + ax_3 + b, x_1 + x_3 + a, x_2 + x_3 + a);$$
$$p_{39} = (x_3^2 + ax_3 + b, x_1 - e_1x_3 - e_2, x_2 + x_3 + a);$$
$$p_{40} = (x_3^2 + ax_3 + b, x_1 - e_3x_3 - e_4, x_2 + x_3 + a);$$
$$p_{41} = (x_3^2 + ax_3 + b, x_1 - x_3, x_2 - e_1x_3 - e_2);$$
$$p_{42} = (x_3^2 + ax_3 + b, x_1 + x_3 + a, x_2 - e_1x_3 - e_2);$$
$$p_{43} = (x_3^2 + ax_3 + b, x_1 - e_1x_3 - e_2, x_2 - e_1x_3 - e_2);$$
$$p_{44} = (x_3^2 + ax_3 + b, x_1 - e_3x_3 - e_4, x_2 - e_1x_3 - e_2);$$
$$p_{45} = (x_3^2 + ax_3 + b, x_1 - x_3, x_2 - e_3x_3 - e_4);$$
$$p_{46} = (x_3^2 + ax_3 + b, x_1 + x_3 + a, x_2 - e_3x_3 - e_4);$$
$$p_{47} = (x_3^2 + ax_3 + b, x_1 - e_1x_3 - e_2, x_2 - e_3x_3 - e_4);$$
$$p_{48} = (x_3^2 + ax_3 + b, x_1 - e_3x_3 - e_4, x_2 - e_3x_3 - e_4).$$

And here we see that we can say that a few of these are the same ideal under permutation and are thus in the same Sym(n)-orbit. This means there is a permutation over the indices of the variables x_1, x_2 and x_3 such that $p_i \sim p_j$ for two of the ideals p_i, p_j given above. We see that $p_m \sim p_{m+16}$ for the permutation $(1, 2)$ and that $p_m \sim p_{m+32}$ for the permutation $(1, 3)$ (we also have that some ideals are actually the same due to the second and third property above, for example 6.1, 6.17 *and* 6.33*). And due to the second and third property above* *combined with permutations we can also say looking solely at 6.1 to 6.16, that*

$$\begin{aligned} 6.1 &\textit{ is single};\\ 6.2 \sim 6.5 &\sim 6.6;\\ 6.3 &\sim 6.9;\\ 6.4 &\sim 6.13;\\ 6.7 \sim 6.8 \sim 6.10 &\sim 6.14;\\ 6.11 &\textit{ is single};\\ 6.12 &\sim 6.15;\\ 6.16 &\textit{ is single}. \end{aligned}$$

So we get that we have 8 Sym(n)-orbits of the minimal prime ideals that contain $I_{3,2}$ and $x_i^2 + ax_i + b$.

We again see the sort of case that happened in 6.1.2. We get that the total number of minimal prime ideals p that contain the ideal $I_{3,2} \subset \mathbb{R}[x_1, ..., x_3]$ is bigger than the number of Sym(n)-orbits of the minimal prime ideals that contain $I_{3,2}$. We will show below that this is the case for any $I_{n,d}$. We will do this after we have made a representation that makes counting the number of Sym(n)-orbits of the minimal prime ideals p that contain $I_{n,d}$ easier.

Definition 6.2.3. *If p is the ideal $(x_i^2 + a_1x_i + b_1, f_j | j \in [n] \setminus \{i\})$, where $f_j \in \{x_k - x_j, x_k + x_j + a_1, x_k - e_{1,m}x_j - e_{2,m}, x_k - e_{3,m}x_j - e_{4,m}\}$ for all $m \in \{2, ..., d\}$, we say it is of type $(i, k_1, l_1, k_2, l_2, ..., k_d, l_d)$. We have here that $k_1 = \#j \neq i : f_j = x_j - x_i$, $l_1 = \#j \neq i : f_j = x_j + x_i + a_1$ and where, for $m \in \{2, ..., d\}$: $k_m = \#j \neq i : f_j = x_j - e_{1,m}x_i - e_{2,m}$ and $l_m = \#j \neq i : f_j = x_j - e_{3,m}x_i - e_{4,m}$.*

Lemma 6.2.4. *Suppose p, q are minimal prime ideals containing $I_{n,d}$. Now suppose p is of type $(i, k_1, l_1, k_2, l_2, ..., k_d, l_d)$. Then p, q are in the same Sym(n)-orbit if there is a j such that q is of type $(j, k_1, l_1, k_2, l_2, ..., k_d, l_d)$; moreover, in this case q is of type $(i, k_1, l_1, k_2, l_2, ..., k_d, l_d)$ or $(i, l_1 - 1, k_1 + 1, l_2, k_2, ..., l_d, k_d)$.*

Proof. We want to know that if p is of type $(i, k_1, l_1, k_2, l_2, ..., k_d, l_d)$ and q is of type $(j, s_1, t_1, s_2, t_2, ..., s_d, t_d)$ whether they are in the same Sym(n)-orbit. This will be the case if and only if either $s_m = k_m$ and $t_m = l_m$ for all $m \in \{1, ..., d\}$ or $s_1 = l_1 - 1$, $t_1 = k_1 + 1$, $s_m = l_m$ and $t_m = k_m$ for all $m \in \{2, ..., d\}$.

So assume that p is of type $(i, k_1, l_1, k_2, l_2, ..., k_d, l_d)$. First (a) we will prove that if q is of type $(i, k_1, l_1, k_2, l_2, ..., k_d, l_d)$, then p and q are in the same Sym(n)-orbit. Then (b) we will prove that if q is of type $(i, l_1 - 1, k_1 + 1, l_2, k_2, ..., l_d, k_d)$, then p and q are in the same Sym(n)-orbit. At last (c) we will prove that if q is of type $(j, s_1, t_1, s_2, t_2, ..., s_d, t_d)$, then q is not in the same Sym(n)-orbit as p if we do not have that either $s_m = k_m$ and $t_m = l_m$ for all $m \in \{1, ..., d\}$ or $s_1 = l_1 - 1$, $t_1 = k_1 + 1$, $s_m = l_m$ and $t_m = k_m$ for all $m \in \{2, ..., d\}$.

(a) Since p is of type $(i, k_1, l_1, k_2, l_2, ..., k_d, l_d)$, there are $i_{k_1,1}, ..., i_{k_1,k_1}, i_{l_1,1}, ..., i_{l_1,l_1}$, $i_{k_2,1}, ..., i_{k_2,k_2}, i_{l_2,1}, ..., i_{l_2,l_2}, ..., i_{k_d,1}, ..., i_{k_d,k_d}, i_{l_d,1}, ..., i_{l_d,l_d} \neq i$ which are pairwise distinct, such that $p = (x_i^2 + a_1x_i + b_1, x_{i_{k_1,1}} - x_i, ..., x_{i_{k_1,k_1}} - x_i, x_{i_{l_1,1}} + x_i + a_1, ..., x_{i_{l_1,l_1}} + x_i + a_1, x_{i_{k_2,1}} - e_{1,2}x_i - e_{2,2}, ..., x_{i_{k_2,k_2}} - e_{1,2}x_i - e_{2,2}, x_{i_{l_2,1}} - e_{3,2}x_i - e_{4,2}, ..., x_{i_{l_2,l_2}} - e_{3,2}x_i - e_{4,2}, ..., x_{i_{k_d,1}} - e_{1,d}x_i - e_{2,d}, ..., x_{i_{k_d,k_d}} - e_{1,d}x_i - e_{2,d}, x_{i_{l_d,1}} - e_{3,d}x_i - e_{4,d}, ..., x_{i_{l_d,l_d}} - e_{3,d}x_i - e_{4,d})$.

Now if q is is of type $(j, k_1, l_1, k_2, l_2, ..., k_d, l_d)$, then likewise there are $j_{k_1,1}, ..., j_{k_1,k_1}$, $j_{l_1,1}, ..., j_{l_1,l_1}, j_{k_2,1}, ..., j_{k_2,k_2}, j_{l_2,1}, ..., j_{l_2,l_2}, ..., j_{k_d,1}, ..., j_{k_d,k_d}, j_{l_d,1}, ..., j_{l_d,l_d} \neq j$ which are pairwise distinct, such that $q = (x_j^2 + a_1x_j + b_1, x_{j_{k_1,1}} - x_j, ..., x_{j_{k_1,k_1}} - x_j, x_{j_{l_1,1}} + x_j + a_1, ...,$

$x_{j_{l_1,l_1}} + x_j + a_1, x_{j_{k_2,1}} - e_{1,2}x_j - e_{2,2}, ..., x_{j_{k_2,k_2}} - e_{1,2}x_j - e_{2,2}, x_{j_{l_2,1}} - e_{3,2}x_j - e_{4,2}, ..., x_{j_{l_2,l_2}} - e_{3,2}x_j - e_{4,2}, ..., x_{j_{k_d,1}} - e_{1,d}x_j - e_{2,d}, ..., x_{j_{k_d,k_d}} - e_{1,d}x_j - e_{2,d}, x_{j_{l_d,1}} - e_{3,d}x_j - e_{4,d}, ..., x_{j_{l_d,l_d}} - e_{3,d}x_j - e_{4,d})$.

Now take the permutation σ of $1, ..., n$ that sends i to j, $i_{k_m,f}$ to $j_{k_m,f}$ for $f \in \{1, ..., k_m\}$ and $i_{l_m,g}$ to $j_{l_m,g}$ for $g \in \{1, ..., l_m\}$ for $m \in \{1, ..., d\}$. Then $\sigma(p) = q$, so we have that p of type $(i, k_1, l_1, k_2, l_2, ..., k_d, l_d)$ and q of type $(j, k_1, l_1, k_2, l_2, ..., k_d, l_d)$ are in the same Sym(n)-orbit. From here it can clearly be seen that if p is of type $(i, k_1, l_1, k_2, l_2, ..., k_d, l_d)$ and q is of type $(i, k_1, l_1, k_2, l_2, ..., k_d, l_d)$, that they are in the same Sym(n)-orbit.

(b) If q is of type $(j, l_1 - 1, k_1 + 1, l_2, k_2, ..., l_d, k_d)$, then there are $j_{k_1,1}, ..., j_{k_1,l_1-1}, j_{l_1,1}, ..., j_{l_1,k_1+1}, j_{k_2,1}, ..., j_{k_2,l_2}, j_{l_2,1}, ..., j_{l_2,k_2}, ..., j_{k_d,1}, ..., j_{k_d,l_d}, j_{l_d,1}, ..., j_{l_d,k_d} \neq j$ (note that the first spot in the subtext denotes the placement in the type in this specific case, the second spot in the subtext denotes the count), such that $q = (x_j^2 + a_1x_j + b_1, x_{j_{k_1,1}} - x_j, ..., x_{j_{k_1,l_1-1}} - x_j, x_{j_{l_1,1}} + x_j + a_1, ..., x_{j_{l_1,k_1+1}} + x_j + a_1, x_{j_{k_2,1}} - e_{1,2}x_j - e_{2,2}, ..., x_{j_{k_2,l_2}} - e_{1,2}x_j - e_{2,2}, x_{j_{l_2,1}} - e_{3,2}x_j - e_{4,2}, ..., x_{j_{l_2,k_2}} - e_{3,2}x_j - e_{4,2}, ..., x_{j_{k_d,1}} - e_{1,d}x_j - e_{2,d}, ..., x_{j_{k_d,l_d}} - e_{1,d}x_j - e_{2,d}, x_{j_{l_d,1}} - e_{3,d}x_j - e_{4,d}, ..., x_{j_{l_d,k_d}} - e_{3,d}x_j - e_{4,d})$

Now from the third statement of Lemma 6.2.1 we find that $q = (x_{j_{l_1,1}}^2 + a_1x_{j_{l_1,1}} + b_1, x_{j_{k_1,1}} + x_{j_{l_1,1}} + a_1, ..., x_{j_{k_1,l_1-1}} + x_{j_{l_1,1}} + a_1, x_{j_{l_1,2}} - x_{j_{l_1,1}}, ..., x_{j_{l_1,k_1+1}} - x_{j_{l_1,1}}, x_{j_{k_2,1}} - e_{3,2}x_{j_{l_1,1}} - e_{4,2}, ..., x_{j_{k_2,l_2}} - e_{3,2}x_{j_{l_1,1}} - e_{4,2}, x_{j_{l_2,1}} - e_{1,2}x_{j_{l_1,1}} - e_{2,2}, ..., x_{j_{l_2,k_2}} - e_{1,2}x_{j_{l_1,1}} - e_{2,2}, ..., x_{j_{k_d,1}} - e_{3,d}x_{j_{l_1,1}} - e_{4,d}, ..., x_{j_{k_d,l_d}} - e_{3,d}x_{j_{l_1,1}} - e_{4,d}, x_{j_{l_d,1}} - e_{1,d}x_{j_{l_1,1}} - e_{2,d}, ..., x_{j_{l_d,k_d}} - e_{1,d}x_{j_{l_1,1}} - e_{2,d}, x_j + x_{j_{l_1,1}} + a_1)$

This can be reordered so that we find $q = (x_{j_{l_1,1}}^2 + a_1x_{j_{l_1,1}} + b_1, x_{j_{l_1,2}} - x_{j_{l_1,1}}, ..., x_{j_{l_1,k_1+1}} - x_{j_{l_1,1}}, x_{j_{k_1,1}} + x_{j_{l_1,1}} + a_1, ..., x_{j_{k_1,l_1-1}} + x_{j_{l_1,1}} + a_1, x_j + x_{j_{l_1,1}} + a_1, x_{j_{l_2,1}} - e_{1,2}x_{j_{l_1,1}} - e_{2,2}, ..., x_{j_{l_2,k_2}} - e_{1,2}x_{j_{l_1,1}} - e_{2,2}, x_{j_{k_2,1}} - e_{3,2}x_{j_{l_1,1}} - e_{4,2}, ..., x_{j_{k_2,l_2}} - e_{3,2}x_{j_{l_1,1}} - e_{4,2}, ..., x_{j_{l_d,1}} - e_{1,d}x_{j_{l_1,1}} - e_{2,d}, ..., x_{j_{l_d,k_d}} - e_{1,d}x_{j_{l_1,1}} - e_{2,d}, x_{j_{k_d,1}} - e_{3,d}x_{j_{l_1,1}} - e_{4,d}, ..., x_{j_{k_d,l_d}} - e_{3,d}x_{j_{l_1,1}} - e_{4,d})$ and we find that we now have that q is of type $(x_{j_{l_1,1}}, k_1, l_1, k_2, l_2, ..., k_d, l_d)$. As seen above, we now know that p and q are in the same Sym(n)-orbit. So we have that p of type $(i, k_1, l_1, k_2, l_2, ..., k_d, l_d)$ and q of type $(j, l_1 - 1, k_1 + 1, l_2, k_2, ..., l_d, k_d)$ are in the same Sym(n)-orbit. From here it can clearly be seen that if p is of type $(i, k_1, l_1, k_2, l_2, ..., k_d, l_d)$ and q is of type $(i, l_1 - 1, k_1 + 1, l_2, k_2, ..., l_d, k_d)$, that they are in the same Sym(n)-orbit.

(c) Now if p is of type $(i, k_1, l_1, k_2, l_2, ..., k_d, l_d)$ and q is of type $(j, s_1, t_1, s_2, t_2, ..., s_d, t_d)$, then without loss of generality, namely by using a permutation of the indices of the variables $x_1, ..., x_n$, we have that p is of type $(1, k_1, l_1, k_2, l_2, ..., k_d, l_d)$, where $p = (x_1^2 + a_1x_1 + b_1, x_2 - x_1, ..., x_{k_1+1} - x_1, x_{k_1+2} + x_1 + a_1, ..., x_{k_1+l_1+1} + x_1 + a_1, x_{k_1+l_1+2} - e_{1,2}x_1 - e_{2,2}, ...,$
$x_{k_1+l_1+k_2+1} - e_{1,2}x_1 - e_{2,2}, x_{k_1+l_1+k_2+2} - e_{3,2}x_1 - e_{4,2}, ..., x_{k_1+l_1+k_2+l_2+1} - e_{3,2}x_1 - e_{4,2}, ..., x_{(\sum_{u=1}^{d-1} k_u+l_u)+2} - e_{1,d}x_1 - e_{2,d}, ..., x_{(\sum_{u=1}^{d-1} k_u+l_u)+k_d+1} - e_{1,d}x_1 - e_{2,d}, x_{(\sum_{u=1}^{d-1} k_u+l_u)+k_d+2} - e_{3,d}x_1 - e_{4,d}, ..., x_n - e_{3,d}x_1 - e_{4,d})$.
Now if p and q were to be in the same Sym(n)-orbit, we would have a permutation σ, where $p = \sigma(q)$.
Now if this were the case we would have that p is of type $(1, k_1, l_1, k_2, l_2, ..., k_d, l_d)$ and $\sigma(q)$ is of type $(1, s_1, t_1, s_2, t_2, ..., s_d, t_d)$ or $(1, t_1 - 1, s_1 + 1, t_2, s_2, ..., t_d, s_d)$, depending on whether $x_{\sigma^{-1}(1)} - x_j$ is in q or $x_{\sigma^{-1}(1)} + x_j + a_1$ is in q.

Without loss of generality in the case that q is of type $(1, s_1, t_1, s_2, t_2, ..., s_d, t_d)$, we have that $p = (x_1^2 + a_1x_1 + b_1, x_2 - x_1, ..., x_{k_1+1} - x_1, x_{k_1+2} + x_1 + a_1, ..., x_{k_1+l_1+1} + x_1 + a_1, x_{k_1+l_1+2} - e_{1,2}x_1 - e_{2,2}, ..., x_{k_1+l_1+k_2+1} - e_{1,2}x_1 - e_{2,2}, x_{k_1+l_1+k_2+2} - e_{3,2}x_1 - e_{4,2}, ...,$

$x_{k_1+l_1+k_2+l_2+1} - e_{3,2}x_1 - e_{4,2}, ..., x_{(\sum_{u=1}^{d-1} k_u+l_u)+2} - e_{1,d}x_1 - e_{2,d}, ..., x_{(\sum_{u=1}^{d-1} k_u+l_u)+k_d+1} - e_{1,d}x_1 - e_{2,d}, x_{(\sum_{u=1}^{d-1} k_u+l_u)+k_d+2} - e_{3,d}x_1 - e_{4,d}, ..., x_n - e_{3,d}x_1 - e_{4,d})$ and that $\sigma(q) = (x_1^2 + a_1x_1 + b_1, x_{1_{k_1,1}} - x_1, ..., x_{1_{k_1,k_1}} - x_1, x_{1_{l_1,1}} + x_1 + a_1, ..., x_{1_{l_1,l_1}} + x_1 + a_1, x_{1_{k_2,1}} - e_{1,2}x_1 - e_{2,2}, ..., x_{1_{k_2,k_2}} - e_{1,2}x_1 - e_{2,2}, x_{1_{l_2,1}} - e_{3,2}x_1 - e_{4,2}, ..., x_{1_{l_2,l_2}} - e_{3,2}x_1 - e_{4,2}, ..., x_{1_{k_d,1}} - e_{1,d}x_1 - e_{2,d}, ..., x_{1_{k_d,k_d}} - e_{1,d}x_1 - e_{2,d}, x_{1_{l_d,1}} - e_{3,d}x_1 - e_{4,d}, ..., x_{1_{l_d,l_d}} - e_{3,d}x_1 - e_{4,d})$, where all $1_{k_m,i}$ and $1_{l_m,i}$ for all $m \in \{1, ..., d\}$ are pairwise distinct and not equal to 1. The case that q is of type $(1, t_1 - 1, s_1 + 1, t_2, s_2, ..., t_d, s_d)$ is treated similarly.

Suppose now that there is a position i, such that there is $x_i - x_1$ in p and $x_i + x_1 + a_1$ in $\sigma(q)$. Then since we have that $\sigma(q) = p$, we get that $x_i + x_1 + a_1 \in p$. So we also get that $x_i + x_1 + a_1 - (x_i - x_1) = 2x_1 + a_1 = x_1 + \frac{a_1}{2} \in p$. Then now since $x_1^2 + a_1x_1 + b_1 \in p$, we have that $(-\frac{a_1}{2})^2 + a_1 \cdot -\frac{a_1}{2} + b_1 = -\frac{a_1^2}{4} + b_1 \in p$. We have that $-\frac{a_1^2}{4} + b_1 \neq 0$, since $x_1^2 + a_1x_1 + b_1$ has no real zeroes. So we have that p contains a non-zero constant, so we have that $p = \mathbb{R}$. This is a contradiction with the fact that p is a minimal prime ideal. Analogously, we can show that on this position i, there is no $x_i + e_{1,m}x_1 + e_{2,m}$ or $x_i + e_{3,m}x_1 + e_{4,m}$ for $m \in \{2, ..., d\}$. So we have that s_1 should be greater or equal to k_1. Analogously it can be shown that t_1 should be greater or equal to l_1. Also analogously it can be shown that s_m should be greater or equal to k_m and t_m should be greater or equal to l_m for $m \in \{2, ..., d\}$ in this case. From the fact that $k_1 + l_1 + k_2 + l_2 + ... + k_d + l_d + 1 = n$ and $s_1 + t_1 + s_2 + t_2 + ... + s_d + t_d + 1 = n$, we get that $k_1 + l_1 + k_2 + l_2 + ... + k_d + l_d = s_1 + t_1 + s_2 + t_2 + ... + s_d + t_d$ and from this we get that $s_1 = k_1, t_1 = l_1, s_m = k_m$ and $t_m = l_m$, for $m \in \{2, ..., d\}$. In the case that q is of type $(1, t_1 - 1, s_1 + 1, t_2, s_2, ..., t_d, s_d)$, we find that $t_1 - 1$ is equal to k_1 and $s_1 + 1$ is equal to l_1, we also find that t_m is equal to k_m and s_m is equal to l_m for $m \in \{2, ..., d\}$.

So we have that if p, q are minimal prime ideals containing $I_{n,d}$ and p is of type $(i, k_1, l_1, k_2, l_2, ..., k_d, l_d)$. Then p, q are in the same Sym(n)-orbit when we have that there is a j such that q is of type $(j, k_1, l_1, k_2, l_2, ..., k_d, l_d)$, moreover, in this case q is of type $(i, k_1, l_1, k_2, l_2, ..., k_d, l_d)$ or $(i, l_1 - 1, k_1 + 1, l_2, k_2, ..., l_d, k_d)$. □

Theorem 6.2.5. *The number of Sym(n)-orbits of the minimal prime ideals p containing $I_{n,d}$ is equal to*

$$\sum_{u=1}^{d} (\frac{1}{2}(\binom{n+2u-2}{2u-1} + \binom{n+2u-3}{2u-2} + \binom{\frac{n-4}{2}+u}{u-1})) \qquad \textit{for even } n;$$

$$\sum_{u=1}^{d} (\frac{1}{2}(\binom{n+2u-2}{2u-1} + \binom{n+2u-3}{2u-2})) \qquad \textit{for odd } n.$$

Proof. First we note the fact that p and q are in the same Sym(n)-orbit if we have that p is of type $(i, k_1, l_1, k_2, l_2, ..., k_d, l_d)$ and q is of type $(j, k_1, l_1, k_2, l_2, ..., k_d, l_d)$ for some $i, j, k_1, l_1, k_2, l_2, ..., k_d, l_d$. 6.2.1 So now to count the number of Sym(n)-orbits, we look at the number of Sym(n)-orbits of p where p is of the form $(1, k_1, l_1, k_2, l_2, ..., k_d, l_d)$ and where we have that $\sum_{i=1}^{d} k_i + l_i = n - 1$. We look at the fact that we have d pairwise distinct irreducible quadratic polynomials. So p contains either $f_1(x_i)$, $f_2(x_i)$,..., $f_d(x_i)$ for each x_i. So p contains n_1 $f_1(x_i)$, n_2 $f_2(x_i)$, ..., n_d $f_d(x_i)$. We have that n_1 has values in the range from 0 to n. Now when we have that $n_1 = 0$, we get that $\sum_{u=2}^{d} n_d = n$. This gives us the number of Sym(n)-orbits for the minimal prime ideals that contain $I_{n,d-1}$, since these ideals are generated by $f(x_i)$ that consist of powers of $d - 1$ distinct irreducible quadratic polynomials.

To find an equation for the number of Sym(n)-orbits of the minimal prime ideals that contain $I_{n,d}$, we will use induction. Suppose that for $d \in \mathbb{N}$ and $d \geq 1$, we have that the number of Sym(n)-orbits of the minimal prime ideals p that contain $I_{n,d}$ is equal to

$$\sum_{u=1}^{d}(\frac{1}{2}(\binom{n+2u-2}{2u-1}+\binom{n+2u-3}{2u-2}+\binom{\frac{n-4}{2}+u}{u-1})) \qquad \text{for even } n;$$
$$\sum_{u=1}^{d}(\frac{1}{2}(\binom{n+2u-2}{2u-1}+\binom{n+2u-3}{2u-2})) \qquad \text{for odd } n.$$

We now take as base case $d = 1$, then we get that the number of Sym(n)-orbits of the minimal prime ideals p that contain $I_{n,d}$ is equal to

$$\sum_{u=1}^{1}(\frac{1}{2}(\binom{n+2u-2}{2u-1}+\binom{n+2u-3}{2u-2}+\binom{\frac{n-4}{2}+u}{u-1})) \qquad \text{for even } n;$$
$$\sum_{u=1}^{1}(\frac{1}{2}(\binom{n+2u-2}{2u-1}+\binom{n+2u-3}{2u-2})) \qquad \text{for odd } n.$$

Which is equal to

$$\begin{aligned}
\frac{1}{2}(\binom{n+2-2}{2-1}+\binom{n+2-3}{2-2}+\binom{\frac{n-4}{2}+1}{1-1})) &= \frac{1}{2}(\binom{n}{1}+\binom{n+-1}{0}+\binom{\frac{n-2}{2}}{0}) \\
&= \frac{n}{2}+1 \text{ for even } n; \\
\frac{1}{2}(\binom{n+2-2}{2-1}+\binom{n+2-3}{2-2}) &= \frac{1}{2}(\binom{n}{1}+\binom{n-1}{0}) \\
&= \frac{n-1}{2}+1 \text{ for odd } n.
\end{aligned}$$

Which is correct by 6.1.5.

Now for our induction step, we assume that the induction hypothesis is correct for $d = w - 1$. For our induction hypothesis to stand, we should have that $d = w$ also holds.

We look at the fact that we have w quadratic polynomials. So p contains one of $f_1(x_i)$, $f_2(x_i)$,..., $f_w(x_i)$ for each x_i. So p contains n_1 $f_1(x_i)$, n_2 $f_2(x_i)$, ..., n_w $f_w(x_i)$. We have that n_1 has values in the range from 0 to n. Now when we have that $n_1 = 0$, we get that $\sum_{u=2}^{w} n_u = n$. This gives us the number of Sym(n)-orbits for the minimal prime ideals that contain $I_{n,w-1}$ which is generated by $f(x)$ which consists of powers of $w - 1$ distinct irreducible quadratic polynomials.
So if we have that $n_1 = 0$, we get that there are

$$\sum_{u=1}^{w-1}(\frac{1}{2}(\binom{n+2u-2}{2u-1}+\binom{n+2u-3}{2u-2}+\binom{\frac{n-4}{2}+u}{u-1})) \qquad \text{for even } n;$$
$$\sum_{u=1}^{w-1}(\frac{1}{2}(\binom{n+2u-2}{2u-1}+\binom{n+2u-3}{2u-2})) \qquad \text{for odd } n,$$

Sym(n)-orbits.

Now for $n_1 > 0$, without loss of generality we take that the minimal prime ideals p are of type $(1, k_1, l_1, k_2, l_2, ..., k_w, l_w)$. We have three possibilities that we will count. We can count the Sym(n)-orbits of the minimal prime ideals p using the fact that $0 \leq l_1 \leq n-1$. We will count:

1. The Sym(n)-orbits where $l_1 = 0$.
2. The Sym(n)-orbits where $l_1 > 0, k_1 = l_1 - 1$ and $k_m = l_m$ for $m \in \{2, ..., w\}$.
3. The Sym(n)-orbits which consists of two types of minimal prime ideals following 6.2.1.

In the first case, we have an egg colouring problem 2.3.1, where we have $n-1$ eggs and $2w-1$ colours, since we have to divide the $n-1$ x_i, where $i \neq 1$ over the $2w-1$ different options that x_i can take when leaving $x_i + x_1 + a_1$, aka the option that is counted by l_1, out. So to count the number of Sym(n)-orbits, we have to calculate $f(2w-1, n-1) = \binom{n-1+2w-1-1}{2w-1-1} = \binom{n+2w-3}{2w-2}$.

In the second case, we count the prime ideals where, we have to have that n is even, so we have that $n-1$ is odd. Then we have to have that $n_1 - 1$ is odd and n_m for $m \in \{2, ..., w\}$ is even. So in this egg colouring problem, we first fix one egg to have colour l_1. After this we want to colour the eggs in pairs of two, either l_1 and k_1, k_2 and l_2, ..., k_w and l_w so that we keep the conditions of this case. So now we get an egg colouring problem with w colours (n_1, n_2,..., n_w) and $\frac{n-2}{2}$ eggs, since the eggs will be coloured in pairs of 2. So we get that there are $f(w, \frac{n-2}{2}) = \binom{\frac{n-2}{2}+w-1}{w-1} = \binom{\frac{n-4}{2}+w}{w-1}$.

Now in the third case, we can count the Sym(n)-orbits where we have that if $p = (1, k_1, l_1, k_2, l_2, ..., k_w, l_w)$ and $q = (1, l_1 - 1, k_1 + 1, l_2, k_2, ..., l_w, k_w)$. To do this, first we will count the total possibilities of how the $n-1$ eggs can be divided among the $2w$ colours $k_1, l_1, k_2, l_2, ..., k_w, l_w$. Then we will subtract the cases where the switch from the third item of 6.2.1 cannot be made, so for odd n we will subtract the first case and for even n we subtract the first and second case. Then after this subtraction is made, the total will be divided by two, since there are two p with a fixed indicator 1 that switch to each other in one Sym(n)-orbit in this case. So in total we get that there are

$$\begin{aligned}
&\frac{1}{2}(f(2w, n-1) - f(2w-1, n-1) - f(w, \frac{n-2}{2})) \\
&= \frac{1}{2}(\binom{n+2w-2}{2w-1} - \binom{n+2w-3}{2w-2} - \binom{\frac{n-4}{2}+w}{w-1}) && \text{for even } n; \\
&\frac{1}{2}(f(2w, n-1) - f(2w-1, n-1) - f(w, \frac{n-2}{2})) \\
&= \frac{1}{2}(\binom{n+2w-2}{2w-1} - \binom{n+2w-3}{2w-2}) && \text{for odd } n,
\end{aligned}$$

Sym(n)-orbits in the third case.

So in total we get that the number of Sym(n)-orbits of the minimal prime ideals containing $I = (f(x_1), f(x_2), ..., f(x_n))$, where we have that $f(x_i)$ consists of d

distinct irreducible quadratic polynomials is

$$\begin{aligned}
&\sum_{u=1}^{w-1}(\frac{1}{2}(\binom{n+2u-2}{2u-1}+\binom{n+2u-3}{2u-2}+\binom{\frac{n-4}{2}+u}{u-1}))+\frac{1}{2}(\binom{n+2w-2}{2w-1}\\
&-\binom{n+2w-3}{2w-2}-\binom{\frac{n-4}{2}+w}{w-1})+\binom{n+2w-3}{2w-2}+\binom{\frac{n-4}{2}+w}{w-1} \quad \text{for even } n;\\
&\sum_{u=1}^{w-1}(\frac{1}{2}(\binom{n+2u-2}{2u-1}+\binom{n+2u-3}{2u-2}))+\frac{1}{2}(\binom{n+2w-2}{2w-1}\\
&-\binom{n+2w-3}{2w-2})+\binom{n+2w-3}{2w-2} \quad \text{for odd } n.
\end{aligned}$$

Which equals

$$\begin{aligned}
&\sum_{u=1}^{w-1}(\frac{1}{2}(\binom{n+2u-2}{2u-1}+\binom{n+2u-3}{2u-2}+\binom{\frac{n-4}{2}+u}{u-1}))\\
&+\frac{1}{2}(\binom{n+2w-2}{2w-1}+\binom{n+2w-3}{2w-2}+\binom{\frac{n-4}{2}+w}{w-1}) && \text{for even } n;\\
&\sum_{u=1}^{w-1}(\frac{1}{2}(\binom{n+2u-2}{2u-1}+\binom{n+2u-3}{2u-2}))\\
&+\frac{1}{2}(\binom{n+2w-2}{2w-1}+\binom{n+2w-3}{2w-2}) && \text{for odd } n.
\end{aligned}$$

Which equals

$$\begin{aligned}
&\sum_{u=1}^{w}(\frac{1}{2}(\binom{n+2u-2}{2u-1}+\binom{n+2u-3}{2u-2}+\binom{\frac{n-4}{2}+u}{u-1})) && \text{for even } n;\\
&\sum_{u=1}^{w}(\frac{1}{2}(\binom{n+2u-2}{2u-1}+\binom{n+2u-3}{2u-2})) && \text{for odd } n.
\end{aligned}$$

Thus the hypothesis for $d = w - 1 + 1 = w$ also holds.

So we have that the number of Sym(n)-orbits of the minimal prime ideals p containing $I_{n,d}$ is equal to

$$\begin{aligned}
&\sum_{u=1}^{d}(\frac{1}{2}(\binom{n+2u-2}{2u-1}+\binom{n+2u-3}{2u-2}+\binom{\frac{n-4}{2}+u}{u-1})) && \text{for even } n;\\
&\sum_{u=1}^{d}(\frac{1}{2}(\binom{n+2u-2}{2u-1}+\binom{n+2u-3}{2u-2})) && \text{for odd } n.
\end{aligned}$$

□

Remark 6.2.6. *We have that the equation in that we found in 6.2.5 can be reduced, which we will show here. The original equation looks as follows.*

$$\begin{aligned}
&\sum_{u=1}^{d}(\frac{1}{2}(\binom{n+2u-2}{2u-1}+\binom{n+2u-3}{2u-2}+\binom{\frac{n-4}{2}+u}{u-1})) && \textit{for even } n;\\
&\sum_{u=1}^{d}(\frac{1}{2}(\binom{n+2u-2}{2u-1}+\binom{n+2u-3}{2u-2})) && \textit{for odd } n.
\end{aligned}$$

We will first look at the equation for odd n:

$$\begin{aligned}
&\sum_{u=1}^{d}(\frac{1}{2}(\binom{n+2u-2}{2u-1}+\binom{n+2u-3}{2u-2}))\\
&=\frac{1}{2}\sum_{u=0}^{d-1}(\binom{n+2(u+1)-2}{2(u+1)-1}+\binom{n+2(u+1)-3}{2(u+1)-2}) \qquad \textit{by index shift}\\
&=\frac{1}{2}\sum_{u=0}^{d-1}(\binom{n+2u}{2u+1}+\binom{n+2u-1}{2u}).
\end{aligned}$$

Now when we write out this equation, we get

$$\begin{aligned}
&\frac{1}{2}\sum_{u=0}^{d-1}(\binom{n+2u}{2u+1}+\binom{n+2u-1}{2u})\\
&=\frac{1}{2}(\binom{n}{1}+\binom{n-1}{0}+\binom{n+2}{3}+\binom{n+1}{2}+\binom{n+4}{5}+\binom{n+3}{4}+\ldots\\
&\quad+\binom{n+2(d-1)}{2(d-1)+1}+\binom{n+2(d-1)-1}{2(d-1)}).
\end{aligned}$$

When we first solve this for brackets and then reorder we will see that we get

$$\begin{aligned}
&\frac{1}{2}(\binom{n}{1}+\binom{n-1}{0}+\binom{n+2}{3}+\binom{n+1}{2}+\binom{n+4}{5}+\binom{n+3}{4}+\ldots\\
&\quad+\binom{n+2d-2}{2d-1}+\binom{n+2d-3}{2d-2})\\
&=\frac{1}{2}(\binom{n-1}{0}+\binom{n}{1}+\binom{n+1}{2}+\binom{n+2}{3}+\binom{n+3}{4}+\binom{n+4}{5}+\ldots\\
&\quad+\binom{n+2d-3}{2d-2}+\binom{n+2d-2}{2d-1})\\
&=\frac{1}{2}\sum_{u=0}^{2d-1}\binom{n-1+u}{u}.
\end{aligned}$$

Finally, by the Christmas Stocking Theorem (CST) [6] which states that $\sum_{i=0}^{k-1}\binom{n+i}{i}=\binom{k+n}{k-1}$, *we see that*

$$\frac{1}{2}\sum_{u=0}^{2d-1}\binom{n-1+u}{u}=\frac{1}{2}\binom{n-1+2d}{2d-1}=\frac{1}{2}\binom{n+2d-1}{n}.$$

Now we will look at the equation for even n:

$$\begin{aligned}
&\sum_{u=1}^{d}(\frac{1}{2}(\binom{n+2u-2}{2u-1}+\binom{n+2u-3}{2u-2})+\binom{\frac{n-4}{2}+u}{u-1}))\\
&=\frac{1}{2}(\sum_{u=1}^{d}(\binom{n+2u-2}{2u-1}+\binom{n+2u-3}{2u-2}))+\sum_{u=1}^{d}\binom{\frac{n-4}{2}+u}{u-1})\\
&=\frac{1}{2}(\binom{n+2d-1}{n}+\sum_{u=1}^{d}\binom{\frac{n-4}{2}+u}{u-1})\\
&=\frac{1}{2}(\binom{n+2d-1}{n}+\sum_{u=0}^{d-1}\binom{\frac{n-4}{2}+u+1}{u}) && \text{by index shift}\\
&=\frac{1}{2}(\binom{n+2d-1}{n}+\sum_{u=0}^{d-1}\binom{\frac{n-2}{2}+u}{u})\\
&=\frac{1}{2}(\binom{n+2d-1}{n}+\binom{\frac{n-2}{2}+d}{d-1}) && \text{by CST}\\
&=\frac{1}{2}(\binom{n+2d-1}{n}+\binom{\frac{n}{2}+d-1}{\frac{n}{2}}).
\end{aligned}$$

So we get that the equation that we found in 6.2.5 is equal to

$$\begin{aligned}
&\frac{1}{2}(\binom{n+2d-1}{n}+\binom{\frac{n}{2}+d-1}{\frac{n}{2}}) && \text{for even } n;\\
&\frac{1}{2}(\binom{n+2d-1}{n}) && \text{for odd } n.
\end{aligned}$$

Remark 6.2.7. *When we take a look at example 6.2.2, we see that we have 8 Sym(n)-orbits for the minimal prime ideals that contain $I_{3,2}$ and $x_i^2+ax_i+b$. Then looking at example 6.1.2 we see that we will have 2 Sym(n)-orbits for the minimal prime ideals that contain $I_{3,2}$ and not $x_i^2+ax_i+b$. Thus in total we have that there are 10 Sym(n)-orbits for the minimal prime ideals p that contain $I_{3,2}=((x_1^2+ax_1+b)^{y_1}(x_1^2+cx_1+d)^{y_2},(x_2^2+ax_2+b)^{y_1}(x_2^2+cx_2+d)^{y_2},(x_3^2+ax_3+b)^{y_1}(x_3^2+cx_3+d)^{y_2})\subset\mathbb{R}[x_1,x_2,x_3]$.*

Looking at the equation we got at 6.2.6 and we take $d=2$ and $n=3$, we get that there are 10 Sym(n)-orbits for the minimal prime ideals that contain $I_{3,2}$, which is the same as we found earlier.

6.3 The minimal prime ideals containing $I \in \mathbb{R}[x_1, ..., x_n]$

Theorem 6.3.1. *The number of Sym(n)-orbits of the minimal prime ideals p containing $I_{n,d,v}=(f(x_1),f(x_2),...,f(x_n))\subset\mathbb{R}[x_1,...,x_n]$, where each $f(x_i)$ can be factored as $(x_i^2+a_1x_i+b_1)^{y_1}(x_i^2+a_2x_i+b_2)^{y_2}...(x_i^2+a_dx_i+b_d)^{y_d}(x_i-c_1)^{e_1}(x_i-c_2)^{e_2}...(x_i-c_v)^{e_v}$, where all $x_i^2+a_1x_i+b_1$, $x_i^2+a_2x_i+b_2,...,x_i^2+a_dx_i+b_d$ are distinct and irreducible in $\mathbb{R}[x_1,...,x_n]$, where $c_1\neq c_2\neq...\neq c_v$ are all in $\mathbb{R}$ and where the exponents $y_1,...,y_d,e_1,...,e_d\in\mathbb{N}$, is equal to*

$$\begin{aligned}
&\sum_{z=0,z\text{ even}}^{n}(\binom{n-z+v-1}{v-1}\cdot\frac{1}{2}(\binom{z+2d-1}{z}+\binom{\frac{z}{2}+d-1}{\frac{z}{2}}))\\
&+\sum_{z=0,z\text{ odd}}^{n}(\binom{n-z+v-1}{v-1}\cdot\frac{1}{2}(\binom{z+2d-1}{z})).
\end{aligned}$$

Proof. For a minimal prime ideal p to contain $I_{n,d,v}$, we have to have that p contains one of $x_i^2 + a_m x_i + b_m$ or $x_i - c_h$ for $m \in \{1, ..., d\}$ and $h \in \{1, ..., v\}$ for each $i \in \{1, ..., n\}$.

So now we should count over the linear and quadratic polynomials that p contains. After dividing the linear and quadratic polynomials, we can use the equations found in 5.0.2 and 6.2.6. Now if we have that p contains z irreducible quadratic polynomials in a single variable, we have that p contains $n - z$ linear polynomials in a single variable. So to count the number of Sym(n)-orbits, we have to plug the equations found in 5.0.2 and 6.2.6 into the following sum: $\sum_{z=0}^{n}$ linear orbits $\cdot$ quadratic orbits, which will split into two sums, one counting over the even z and one over the odd z, because of the division between even and odd n in the equation found in 6.2.6. So we get that there are

$$\sum_{z=0, z \text{ even}}^{n} \left(\binom{n-z+v-1}{v-1} \cdot \frac{1}{2} \left(\binom{z+2d-1}{z} + \binom{\frac{z}{2}+d-1}{\frac{z}{2}} \right) \right) + \sum_{z=0, z \text{ odd}}^{n} \left(\binom{n-z+v-1}{v-1} \cdot \frac{1}{2} \left(\binom{z+2d-1}{z} \right) \right)$$

Sym(n)-orbits of the minimal prime ideals p that contain $I_{n,d,v}$. □

Lemma 6.3.2. *The degree of the number of Sym(n)-orbits of the minimal prime ideals containing $I_{n,d,v}$ as found in 6.3.1 as a function of n is equal to $v + 2d - 1$.*

Proof. The degree of a polynomial is the highest power that is taken in the polynomial, so if we look at the aspects of the polynomial found in 6.3.1, we see that the highest power that the polynomial in $(n - z)$:

$$\binom{n-z+v-1}{v-1},$$

can take is $v - 1$. We also see that the highest power that the polynomials in z:

$$\frac{1}{2} \left(\binom{z+2d-1}{z} + \binom{\frac{z}{2}+d-1}{\frac{z}{2}} \right) \text{ and } \frac{1}{2} \left(\binom{z+2d-1}{z} \right),$$

can take is $2d - 1$.
So we find that we have a polynomial of degree $v - 1$ in $n - z$, which we multiply with one of the polynomials of degree $2d - 1$ in z. This gives us the two polynomials

$$\binom{n-z+v-1}{v-1} \cdot \frac{1}{2} \left(\binom{z+2d-1}{z} + \binom{\frac{z}{2}+d-1}{\frac{z}{2}} \right)$$

and

$$\binom{n-z+v-1}{v-1} \cdot \frac{1}{2} \left(\binom{z+2d-1}{z} \right).$$

These two functions in z, if we take n to be a constant, have a degree equal to the sum of the two degrees earlier found, which is equal to $(v - 1) + (2d - 1)$. Now the degree of number of Sym(n)-orbits of the minimal prime ideals that contain $I_{n,d,v}$ as found in 6.3.1 is at most equal to $(v - 1) + (2d - 1) + 1 = v + 2d - 1$ by 4.2.2. □

Lemma 6.3.3. *The number of Sym(n)-orbits of the minimal prime ideals containing $I_{n,d,v}$ as found in 6.3.1 forms a quasi-polynomial.*

Proof. A quasi-polynomial can be written as $q(k) = c_d(k)k^d + c_{d-1}(k)k^{d-1} + ... + c_0(k)$. We can rewrite the equation that we found in the following way:

$$\begin{aligned}
&\sum_{z=0, z \text{ even}}^{n} (\binom{n-z+v-1}{v-1} \cdot \frac{1}{2}(\binom{z+2d-1}{z} + \binom{\frac{z}{2}+d-1}{\frac{z}{2}})) \\
&+ \sum_{z=0, z \text{ odd}}^{n} (\binom{n-z+v-1}{v-1} \cdot \frac{1}{2}(\binom{z+2d-1}{z})) \\
&= \sum_{z=0}^{n} (\binom{n-z+v-1}{v-1} \cdot \frac{1}{2}(\binom{z+2d-1}{z} + q(z)\binom{\frac{z}{2}+d-1}{\frac{z}{2}})).
\end{aligned}$$

We have here that $q(z)$ is 1 for z even and 0 for z odd. We now have that

$$h(z) = \frac{1}{2}(\binom{z+2d-1}{z} + q(z)\binom{\frac{z}{2}+d-1}{\frac{z}{2}})$$

is a quasi-polynomial. Then if we look at the multiplication of $\binom{n-z+v-1}{v-1}$ and $h(z)$, we clearly see there will still be a quasi-polynomial, as we multiply a polynomial with a quasi-polynomial and quasi-polynomials are closed under multiplication [4].

At last we have a summation of quasi-polynomials as we have

$$\sum_{z=0}^{n} (\binom{n-z+v-1}{v-1} \cdot \frac{1}{2}(\binom{z+2d-1}{z} + q(z)\binom{\frac{z}{2}+d-1}{\frac{z}{2}})).$$

Thus by the fact that quasi-polynomials are closed under addition [4], we still have a quasi-polynomial, but now the quasi-polynomial is in n, with at most the degree we found in 6.3.2. $\square$

Chapter 7

Discussion

In this project we proved a couple of statements relating to minimal prime ideals and we derived an expression for the number of Sym(n)-orbits of the minimal prime ideals that contain an ideal I in $\mathbb{R}[x_1, ..., x_n]$. We also looked at whether this expression is a quasi-polynomial or not and which degree it can have. There is still further research that can be done, building on top of what we found, and we will now give some examples.

The smallest extension that can be done is to prove whether the degree of the polynomial found in 6.3.1 is actually equal to $v + 2d - 1$.

Another possible extension would be to find the number of Sym(n)-orbits in $\mathbb{Q}[x_1, ..., x_n]$, by first finding a good representation for the minimal prime ideals that contain an ideal that consists of $f(x_i)$ which can not be factored into linear or quadratic polynomials in $\mathbb{Q}[x_i]$ and then using combinatorics and induction like in 6.1.5 and 6.2.6 to find an expression for the number of Sym(n)-orbits of the minimal prime ideals that contain this ideal. Then by using this expression and the expressions found in 5.0.2 and 6.2.5, we can find an expression for the number of Sym(n)-orbits of the minimal prime ideals that contain an ideal that consists of $f(x_i)$ that can be factored into irreducible terms in $\mathbb{Q}[x_i]$. As an addition to this, an expression can be found for the number of Sym(n)-orbits of the minimal prime ideals that contain an ideal that consists of $f(x_i)$ that can be factored into irreducible terms in any polynomial ring $\mathbb{K}[x_i]$.

What also can be investigated further, is whether there is an approach to find the number of Sym(n)-orbits of the minimal prime ideals that contain $I_{n,d}$ as in Section 6.2, that leads directly to the equations we found in 6.2.6. We now used an approach that led to a sum which could be transformed into the equation found in 6.2.6, but we have not found a direct approach yet. If a direct approach is found, this approach could maybe also be used for the extensions above.

Regarding the polynomial rings that we have already explored, we note that an addition to these could be the addition of multiple variables. With this we mean that we can find an expression for the number of Sym(n)-orbits of the minimal prime ideals that contain an ideal that consists of $f(x_i, x_j)$ that can be factored into irreducible terms in $\mathbb{C}[x_i, x_j]$ and other polynomial rings. This then could be expanded to ideals that consist of functions with n variables.

Investigating these additions and the expressions we already found could be interesting for research purposes in the fields of ring theory.

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