BACHELOR

Optimising the Occupancy of Theatres under Minimal Distance Constraints

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Abstract

This report was written to provide the findings of an investigation into the maximisation of the rate of seats that can be occupied in a theatre, while taking minimal distance rules into account. Both finite and infinite theatres were analysed, and it was found that, for any minimal distance, the rate of seats that can be occupied in an infinite theatre provides a lower bound to the rate of seats that can be filled in a finite theatre with equivalent distance between consecutive rows and seats. In this report, methods are provided that allow to calculate the maximal number of seats that can be occupied in any finite theatre exactly. Moreover, an upper and a lower bound are given to the rate of seats that can be occupied in an infinite theatre. The results for a number of examples are given, in which it can be seen that the minimal distance rules significantly lowers the rate of seats that can be filled in a theatre.

Contents

1 Introduction

The coronavirus has influenced peoples' lives all around the world. Many countries have taken a variety of measures to prevent the spread of the virus. In the Netherlands, most of these measures have not been in place for the entire duration of this COVID-affected time, but have been put in place in response to the increase or decrease of the number of infections. However, one measure that has been in place for the majority of the past two years is the rule of social distancing (*Coronavirus tijdlijn — Rijksoverheid.nl*, n.d.). This measure prevents people from being too close to each other, in order to limit possible infections. Social distancing has a big impact on everyday life in many ways, one of which is the reduced number of people that can fit in a certain room, such as a theatre. Blom et al. (2020) discusses methods to determine the maximum number of people that can fit in the Music Building in Eindhoven (MBE), taking the social distancing rules in the Netherlands into account. This paper is used as the basis of the current project. In contrast with the paper by Blom et al. (2020), in which the objective is to mathematically optimise the occupancy of the MBE specifically, this report addresses the more general case by considering arbitrary theatres and minimal distances.

The report will be structured as follows. In Section 2, the problem description is given. To do so, an analysis is made of the situations that are investigated. In Section 3, the linear optimisation methods used to find solutions to the problem statement are explained. Furthermore, an upper bound is described for a specific case where the theatre considered is infinitely large. In Section 4, the corresponding lower bound is analysed. Section 5 consists of various results to the problem statement. The findings and the methods used are being discussed in Section 6. Finally, in Section 7 the final conclusions are drawn.

2 Problem Description

2.1 Situational Analysis

The objective considered in this project regards the maximisation of the number of people that are seated in a theatre under minimum distance constraints. The theatre that is considered has the following properties. First, it is assumed that the spacing between consecutive rows is constant. This distance is denoted by b. Similarly, the spacing between consecutive seats within a row is constant, and this distance is denoted by a. Furthermore, it is assumed that consecutive rows are shifted relative to each other by half a seat, which is customary in many theatres. A seat is referred to with coordinates, where the first entry is the row number, and the second entry is the seat number. For instance, seat $(4, 6)$ is seat number 6 in row number 4. Thus, all seats can be referred to using a pair of integers. The situation as described is visualised is Figure 1. Here, the dots represent the centers of seats. In this example, the blue arrows show the distance between consecutive rows and the distance between consecutive seats. These distances a and b are both chosen to be 1 in this case, but they may be different from each other in other examples.

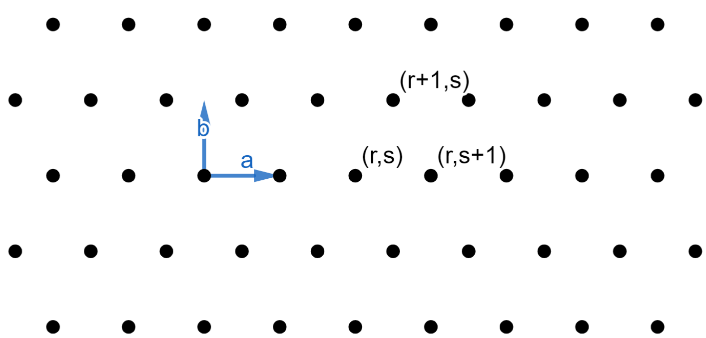

Figure 1: Example of a theatre setup.

2.2 Distance Function

For any theatre with a seating setup satisfying the properties as explained in the situational analysis, a function can be found that gives the distance between any two seats in the theatre. This function will be crucial in solving the problem. Throughout this report, this function will be referred to as the distance function (Blom et al., 2020). Let $s \in \mathbb{Z}^2$ and $t \in \mathbb{Z}^2$ be two seats within a theatre. Then we can write $s = (s_r, s_s)$ and $t = (t_r, t_s)$, where the first coordinate is the row number and the second is the seat number. As a convention, in the remainder of the report for any seat the subscript r will be used for the entry corresponding to the row number, and the subscript s will be for the entry corresponding to the seat number. Note that due to the shift between rows, the two integer entries do not directly indicate the seat's coordinates in the Euclidean space, but are merely two numbers that are associated to a particular seat within the theatre. Thus, to find the Euclidean distance between any two seats, the shift between consecutive rows and the seat and row spacing of the theatre, denoted by a and b, need to be taken into account. In order to find the Euclidean distance, we use Pythagoras' Theorem. A right-angled triangle can be created within the theatre with s and t as two of the three corners. We can easily complete the right-angled triangle by adding a "vertical" line between the rows of s and t and a "horizontal" line within the row. Then this "vertical" distance between the rows of seats s and t is simply $(s_r - t_r) \cdot b$. The rows shift by half a seat for consecutive rows, so the "horizontal" distance is given by $(s_s - t_s + \frac{1}{2}(s_r - t_r)) \cdot a$.

Using Pythagoras, we find the distance function

$$d(s,t) = \sqrt{(s_s - t_s + \frac{1}{2}(s_r - t_r))^2 a^2 + (s_r - t_r)^2 b^2}.$$

2.3 Theatre Setup and Seating Arrangements

Before we can use the distance function to analyse the distance between seats within a theatre, we first need to be able to mathematically specify the shape and size of the theatres that will be examined exactly. To do this, recall that any seat within the theatre may be indicated by its row and seat number. Thus, to the seats of any particular theatre satisfying the assumptions as described in the situational analysis, a set containing two dimensional, integer-valued elements can be associated. Such a set will be called $\mathcal{S}$ throughout the rest of the report. So, any set of seats is represented by a set $\mathcal{S} \subset \mathbb{Z}^2$. Therefore, the exact setup of any theatre can exactly be described by a set $\mathcal{S} \subset \mathbb{Z}^2$, and the parameters a and b. As an example, the theatre represented by the set $\mathcal{S} = \{s \subset \mathbb{Z}^2 : d(\mathbf{0}, s) \leq 2\}$, and the parameters $a = b = 1$ are depicted in Figure 2, where $\mathbf{0}$ represents the seat with row and seat number 0.

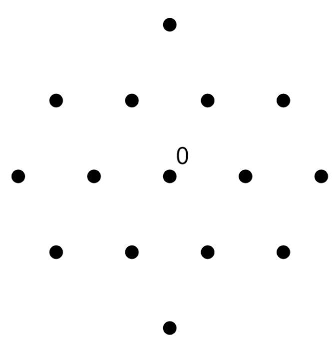

Figure 2: Theatre setup determined by $\mathcal{S}$, a and b.

Within a theatre, we are interested in the seats that are occupied. To this end, the concept of a seating arrangement is introduced. Let some set $\mathcal{S} \subset \mathbb{Z}^2$ be fixed. Then a seating arrangement is a set containing all seats within $\mathcal{S}$ that are occupied. So, a subset $\mathcal{A} \subseteq \mathcal{S} \subset \mathbb{Z}^2$ is a seating arrangement if each $s \in \mathcal{A}$ indicates an occupied seat. Therefore, if a set $\mathcal{S}$, parameters a and b, and a seating arrangement $\mathcal{A}$ are fixed, the theatre and the occupied seats within the theatre are exactly determined. In Figure 3, an example is given using $\mathcal{S} = \{s \subset \mathbb{Z}^2 : d(0, s) \leq 2\}$, $a = b = 1$, and $\mathcal{A} = \{s \in \mathcal{S} : d(0, s) > 3/2\}$. Here, the green dots represent occupied seats, and the red dots represent empty seats.

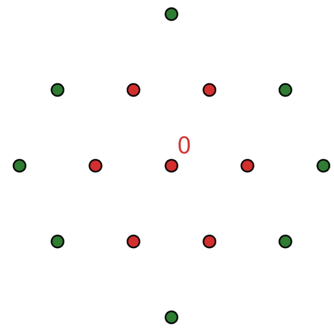

Figure 3: Theatre setup determined by $\mathcal{S}$, a and b.

2.4 Constraints

Now that all components necessary to exactly determine a theatre setup and a seating arrangement have been explained, we can start to consider the restrictions that need to be put on the seating arrangements such that we can distinguish between arrangements that are of interest and arrangements that are not. In this report, the seating arrangements that are of interest are those containing only elements associated to seats that are separated by at least some minimal distance. We call this minimal distance the forbidden distance, and will refer to it with the letter c. In this report, no exceptions to the minimum distance rule are considered. So, a situation where certain people, such as family members, may sit next to each other is not taken into account. The forbidden distance determines the safety of a seating arrangement. This concept of safety is made clear in the following definition.

> Definition 1: Let $\mathcal{S} \subset \mathbb{Z}^2$ be fixed. Then a seating arrangement $\mathcal{A} \subseteq \mathcal{S}$ is called (a, b, c)-safe if and only if $d(s,t) > c$ for all $s, t \in \mathcal{A}$ such that $s \neq t$.

As a consequence, a seating arrangement $\mathcal{A} \subseteq \mathcal{S}$ in which some $s \in \mathcal{A}$ is occupied can only be safe if all seats $t \in \mathcal{S}$ such that $d(s,t) \leq c$ are empty. An example of this is given in Figure 4. However, if seat s is not occupied, there are multiple ways of distributing the seats in the same region around s safely, an example of which is given in Figure 5.

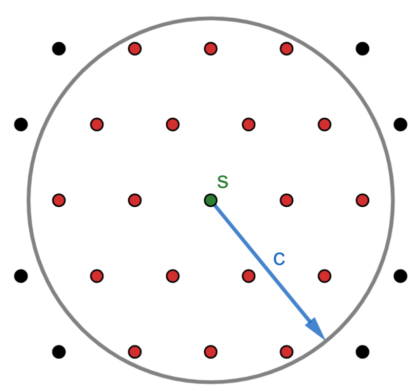

Figure 4: Example of an (a, b, c)-safe arrangement $\mathcal{A}$ around occupied seat $s \in \mathcal{A}$.

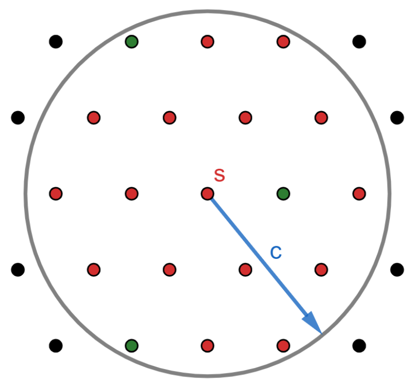

Figure 5: Example of an (a, b, c)-safe seating arrangement $\mathcal{A}$ around empty seat $s \in \mathcal{A}$.

2.5 Occupancy and Occupancy Rate

Suppose that we have fixed a set $\mathcal{S} \subset \mathbb{Z}^2$, and parameters a and b. So, we have exactly determined the characteristics of some theatre. Then for every seating arrangement $\mathcal{A} \subseteq \mathcal{S}$ a size can be determined. This size is denoted by $|\mathcal{A}|$, and equals the amount of elements of $\mathcal{A}$. This notion of size is used to define the occupancy of a theatre.

Definition 2: The occupancy of a theatre is defined by

$$q(\mathcal{S}, a, b, c) := \max \{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{S}, \mathcal{A} \text{ is } (a, b, c)\text{- safe}\} .$$

In other words, the occupancy of a theatre is the maximum amount of seats that can safely be occupied. Thus, the occupancy depends on the choices for $\mathcal{S}$, a, b and c. Using this notion of occupancy, we define the occupancy rate to be the maximum amount of seats in a theatre that can safely be occupied, divided by the total amount of seats within that theatre.

Definition 3: Suppose that $\mathcal{S} \subset \mathbb{Z}^2$ is finite. Then the occupancy rate of a theatre is defined by

$$r(\mathcal{S}, a, b, c) := \frac{q(\mathcal{S}, a, b, c)}{|\mathcal{S}|} = \frac{\max \{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{S}, \mathcal{A} \text{ is } (a, b, c)\text{- safe}\}}{|\mathcal{S}|}.$$

Note that this definition for the occupancy rate is well-defined if the size of $\mathcal{S}$ is finite. Therefore, we require the set $\mathcal{S}$ to be finite. However, for reasons that will soon become apparent, we would also like to investigate infinitely large theatres, besides finitely large theatres. In order to do so, we define a specific example of a set of seats $\mathcal{S} \subset \mathbb{Z}^2$. Let $\mathcal{S}_k \subset \mathbb{Z}^2$ be defined by

$$\mathcal{S}_k := \left\{ s \in \mathbb{Z}^2 : d((s_r, 0), (\mathbf{0})) \leq \frac{1}{2}k, d((s_r, s_s), (s_r, -\frac{1}{2}s_r)) \leq \frac{1}{2}k, k \in \mathbb{R} \right\}.$$

Then the set $\mathcal{S}_k$ can be used to refer to a theatre that contains the seats within a square area around seat $\mathbf{0}$. An example of a theatre represented by the set $\mathcal{S}_k$ is given in Figure 6. By changing the value of k, we can adjust the size of the theatre.

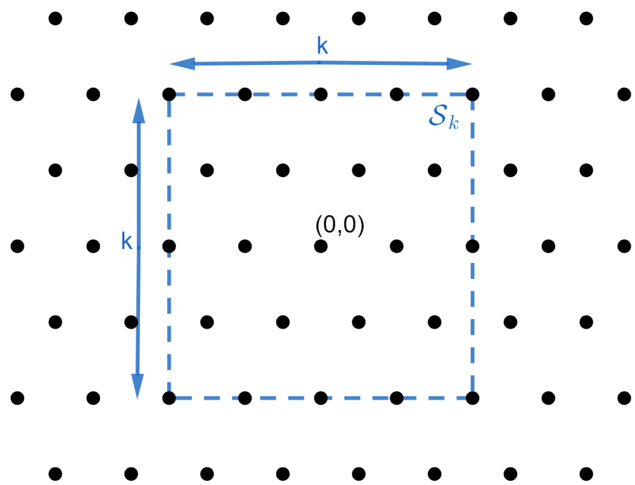

Figure 6: Example of a set $\mathcal{S}_k$.

We can use the set $\mathcal{S}_k \subset \mathbb{Z}^2$ to define the occupancy rate for all theatres, even for those that are infinitely large.

Definition 4: Let $\mathcal{S} \subseteq \mathbb{Z}^2$ be fixed. Then the occupancy rate of a theatre is defined by

$$r(\mathcal{S}, a, b, c) := \lim_{k\to\infty} \frac{q((\mathcal{S}\cap\mathcal{S}_k), a, b, c)}{|(\mathcal{S}\cap\mathcal{S}_k)|}$$
$$= \lim_{k\to\infty} \frac{\max\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{S}\cap\mathcal{S}_k, \mathcal{A} \text{ is } (a,b,c)\text{- safe}\}}{|(\mathcal{S}\cap\mathcal{S}_k)|}.$$

Note that Definition 4 is equal to Definition 3 for finite $\mathcal{S} \subset \mathbb{Z}^2$. This can be seen in the following way. Suppose $\mathcal{S} \subset \mathbb{Z}^2$ is finite. Then there exists a $k_0 \in \mathbb{R}$ such that $\mathcal{S} \subseteq \mathcal{S}_k$ for all $k \geq k_0$. So for finite $\mathcal{S}$

$$\lim_{k\to\infty} \frac{\max\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{S}\cap\mathcal{S}_k, \mathcal{A} \text{ is } (a,b,c)\text{- safe}\}}{|(\mathcal{S}\cap\mathcal{S}_k)|}$$
$$= \frac{\max\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{S}\cap\mathcal{S}_{k_0}, \mathcal{A} \text{ is } (a,b,c)\text{- safe}\}}{|(\mathcal{S}\cap\mathcal{S}_{k_0})|}$$
$$= \frac{\max\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{S}, \mathcal{A} \text{ is } (a,b,c)\text{- safe}\}}{|\mathcal{S}|}.$$

So, for finite $\mathcal{S} \subset \mathbb{Z}^2$, Definition 4 is equivalent to Definition 3. Therefore, we can conclude that Definition 4 is well-defined for all $\mathcal{S} \subseteq \mathbb{Z}^2$. A specific example of these infinite theatres are the theatres that include all seats corresponding to all elements of $\mathbb{Z}^2$. We define

$$\mathcal{S}_\infty := \mathbb{Z}^2.$$

Then, a theatre represented by parameters a, b, and the set $\mathcal{S}_\infty$ includes seat s for all $s \in \mathbb{Z}^2$. We use the set $\mathcal{S}_\infty$ to state the following theorem.

Theorem 1: For all finite $\mathcal{S} \subset \mathbb{Z}^2$ and all $a, b, c \geq 0$ we have

$$r(\mathcal{S}, a, b, c) \geq r(\mathcal{S}_\infty, a, b, c).$$

This theorem will be proven in Section 3.6. By Theorem 1, for certain fixed a, b and c, the occupancy rate of the theatre with the set of seats $\mathcal{S}_\infty$ provides a lower bound to the occupancy rate of any finite theatre. Therefore, it is useful to investigate the occupancy rates of infinite theatres $\mathcal{S}_\infty$, for various a, b and c.

2.6 Problem Statement

The objective of this report is to investigate the occupancy rate $r(\mathcal{S}, a, b, c)$ for various sets of seats $\mathcal{S}$, parameters a and b, and forbidden distances c. Thus, we want to find solutions to the following maximisation problem.

$$r(\mathcal{S}, a, b, c) := \lim_{k\to\infty} \frac{q((\mathcal{S} \cap \mathcal{S}_k), a, b, c)}{|\mathcal{S} \cap \mathcal{S}_k|} = \lim_{k\to\infty} \frac{\max\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{S} \cap \mathcal{S}_k, \mathcal{A} \text{ is } (a, b, c)\text{- safe}\}}{|\mathcal{S} \cap \mathcal{S}_k|}. \quad (1)$$

In particular, we want to investigate the occupancy rates of theatres with the set of seats $\mathcal{S}_\infty$, in order to find lower bounds to problem (1). Thus, we want to find solutions to the following maximisation problem.

$$r(\mathcal{S}_\infty, a, b, c) := \lim_{k\to\infty} \frac{q((\mathcal{S}_\infty \cap \mathcal{S}_k), a, b, c)}{|\mathcal{S}_\infty \cap \mathcal{S}_k|} = \lim_{k\to\infty} \frac{\max\{|\mathcal{A}| : \mathcal{A} \subseteq \bigcup_{i=1}^{k} \mathcal{S}_i, \mathcal{A} \text{ is } (a, b, c)\text{- safe}\}}{|\mathcal{S}_k|}. \quad (2)$$

3 Methods and Upper Bound

Before being able to start searching for solutions to problems (1) and (2), we want to express the occupancy rates in such a way that we can use integer programming to find solutions. To this purpose, we introduce a number of methods that will help solving the problem. The methods explained in this chapter will be used to calculate the occupancy rates for finite $\mathcal{S} \subset \mathbb{Z}^2$, and to calculate upper bounds to $r(\mathcal{S}_\infty, a, b, c)$.

3.1 Characteristic Vectors

Consider a seating arrangement $\mathcal{A} \subseteq \mathcal{S} \subset \mathbb{Z}^2$. With $\mathcal{A}$, we can associate a characteristic vector $x \in \{0,1\}^{\mathcal{S}}$ with $x_s = 1$ if and only if $s \in \mathcal{A}$. In order to find $r(\mathcal{S}, a, b, c)$, we need to find an (a, b, c)-safe seating arrangement $\mathcal{A}$ of maximum size. This is equivalent to finding a characteristic vector x corresponding to a safe $\mathcal{A}$ with the most entries equal to 1. Throughout the rest of the report, such an x corresponding to a safe $\mathcal{A}$ will be called a safe characteristic vector. Then, maximising the size of safe $\mathcal{A}$ is equivalent to maximising the sum $\sum_{s \in \mathcal{S}_k} x_s = \mathbb{1} \cdot x$ for x safe. By definition, $\mathcal{A}$ is (a, b, c)-safe if $d(s, t) > c$ for all $s, t \in \mathcal{A}$, $s \neq t$. So, if $d(s, t) \leq c$ and $s \neq t$, we cannot have that $s \in \mathcal{A}$ and $t \in \mathcal{A}$. So, if $d(s, t) \leq c$ and $s \neq t$, we cannot have that $x_s = x_t = 1$. Therefore, x is safe if and only if

$$x_s + x_t \leq 1 \quad \forall s, t \in \mathcal{S} \text{ such that } d(s, t) \leq c \text{ and } s \neq t. \tag{3}$$

Thus, using the characteristic vector, we have that

$$q(\mathcal{S}, a, b, c) := \max \{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{S}, \mathcal{A} \text{ is } (a, b, c)\text{- safe}\} \tag{4}$$

$$= \max \left\{ \mathbb{1}x : x \in \{0,1\}^{\mathcal{S}}, (3) \right\}. \tag{5}$$

maximisation problem (5) is an integer linear program. Therefore, for any finite $\mathcal{S}$, we can now determine the optimal solution for $r(\mathcal{S}, a, b, c)$. However, solving (5) can be rather time-consuming, especially for large $\mathcal{S}$. Moreover, (5) provides no way to gain further information about the case where we have $\mathcal{S}_\infty$. So, we will analyse problems (1) and (2) more closely, using the characteristic vectors, in order to get a better understanding of how to find an upper bound to $r(\mathcal{S}_\infty, a, b, c)$.

3.2 Analysing the Objective Problems

3.2.1 LP-Relaxation and the Dual

In the above section, it was shown that equation (5) characterises $q(\mathcal{S}, a, b, c)$. Using this, we can rewrite $r(\mathcal{S}, a, b, c)$ as

$$r(\mathcal{S}, a, b, c) := \lim_{k \to \infty} \frac{q(\mathcal{S} \cap \mathcal{S}_k, a, b, c)}{|(\mathcal{S} \cap \mathcal{S}_k)|} = \lim_{k \to \infty} \frac{\max \left\{ \mathbb{1}x : x \in \{0,1\}^{\mathcal{S} \cap \mathcal{S}_k}, (3) \right\}}{|(\mathcal{S} \cap \mathcal{S}_k)|}.$$

By Theorem 1, we can acquire more information about this $r(\mathcal{S}, a, b, c)$ by considering the case where $\mathcal{S} = \mathcal{S}_\infty$. Therefore, we try to find bounds to $r(\mathcal{S}_\infty, a, b, c)$. In order to find an upper bound, we first consider the Linear Programming Relaxation (LP-Relaxation) of q. We have

$$q(\mathcal{S}, a, b, c) = \max \left\{ \mathbb{1}x : x \in \{0,1\}^{\mathcal{S}}, (3) \right\} \leq \max \left\{ \mathbb{1}x : x \in \mathbb{R}^{\mathcal{S}}, x \geq 0, (3) \right\}.$$

We can apply the Duality Theorem (Conforti et al., 2014) to this last expression to find an upper bound.

Duality Theorem: For any $m \times n$ matrix A and vectors $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$, we have

$$\max\{cx : Ax \leq b, x \in \mathbb{R}^n, x \geq 0\} = \min\{yb : yA \geq c, y \geq 0, y \in \mathbb{R}^m\},$$

provided that the maximisation problem or the minimisation problem, is both feasible and bounded.

In order to apply this theorem, we need to write constraint (3) in the form $Ax \leq \mathbb{1}$, where A is a matrix. In Section 3.2.2, we will explain how to do this. However, any matrix A that satisfies $Ax \leq 1$ for all safe x is sufficient to achieve an upper bound. Let A be a matrix such that $Ax \leq 1$ for all safe x on $\mathcal{S} \subseteq \mathbb{Z}^2$. Then A is an $m \times n$ matrix, where $n = |\mathcal{S}|$. Applying the Duality Theorem then yields

$$
\begin{aligned}
& q(\mathcal{S}, a, b, c) && (6)\\
\overset{\text{LP-Relaxation}}{\leq} & \max\left\{\mathbb{1}x : x \in \mathbb{R}^{\mathcal{S}}, x \geq 0, (3)\right\} && (7)\\
\leq & \max\left\{\mathbb{1}x : Ax \leq \mathbb{1}, x \in \mathbb{R}^{\mathcal{S}}, x \geq 0, \right\} && (8)\\
\overset{\text{Duality Theorem}}{=} & \min\left\{y\mathbb{1} : yA \geq \mathbb{1}, y \in \mathbb{R}^m, y \geq 0\right\} && (9)
\end{aligned}
$$

So, the dual of the LP-relaxation of $q(\mathcal{S}, a, b, c)$ provides a way to find upper bounds to $r(\mathcal{S}, a, b, c)$. Therefore the dual of the LP-relaxation of $q((\mathcal{S}_\infty \cap \mathcal{S}_k), a, b, c) = q(\mathcal{S}_k, a, b, c)$ provides a way to find an upper bound to $r(\mathcal{S}_\infty, a, b, c)$. Note that since $x \geq 0$, by constraint (3) we have for all $s \in \mathcal{S}$ and if there exists a $t \in \mathcal{S}$ such that $d(s,t) \leq c$ and $t \neq s$

$$x_s + x_t \leq 1 \Rightarrow x_s \leq 1.$$

Therefore, the constraint $x \leq 1$ is redundant in (8), and does not need to be considered.

3.2.2 The First Upper Bound

If we can write constraint 3 using a matrix A such that $Ax \leq \mathbb{1}$, we can apply the method as described in Section 3.2.1 to find an upper bound to $r(\mathcal{S}_\infty, a, b, c)$. We can do this in the following way. First of all, we define the following set using the distance function.

Definition 5: Let $w \in \mathbb{R}^2$. Then the set $\mathcal{B}_p(w) \subseteq \mathbb{R}^2$ is defined by

$$\mathcal{B}_p(w) := \{t \in \mathbb{R}^2 : d(w,t) \leq p\}.$$

Thus, the set $\mathcal{B}_p(w)$ is a ball around a point in $\mathbb{R}^2$ containing all points within a certain distance of its center. Now, let $\mathcal{S}$, a, b and c be fixed. Then the set $(\mathcal{B}_c(s) \cap \mathbb{Z}^2) \setminus \{s\}$ contains exactly all points t within distance c of point s such that $t \neq s$. We generate matrix A in the following way. For all $s \in \mathcal{S}$, we consider the set $(\mathcal{B}_c(s) \cap \mathbb{Z}^2) \setminus \{s\}$. For each element $t \in (\mathcal{B}_c(s) \cap \mathbb{Z}^2) \setminus \{s\}$, we add a row in matrix A containing a 1 at the position of t and a 1 at the position of s if $t \in \mathcal{S}$, and we add a row with just a 1 at the position of s if $t \notin \mathcal{S}$. Doing so for all $t \in (\mathcal{B}_c(s) \cap \mathbb{Z}^2) \setminus \{s\}$ and for all $s \in \mathcal{S}$, we have generated a matrix A that corresponds to constraint (3). Then, matrix A is an $m \times n$ matrix, where $n = |\mathcal{S}|$ and $m = \frac{|\mathcal{S}| \cdot |(\mathcal{B}_c(s) \cap \mathbb{Z}^2) \setminus \{s\}|}{2}$. Note that the sum of each column of matrix A equals $|\mathcal{B}_c(s) \cap \mathbb{Z}^2) \setminus \{s\}|$. Thus, the vector y where $y_i = \frac{1}{|(\mathcal{B}_c(s) \cap \mathbb{Z}^2) \setminus \{s\}|}$ for all $i \in \{1, ..., m\}$ is a feasible solution to the dual problem. Suppose that $(\mathcal{B}_c(s) \cap \mathbb{Z}^2) \setminus \{s\} \neq \emptyset$. Then the upper bound to $r(\mathcal{S}_\infty, a, b, c)$ that we found is

$$\lim_{k \to \infty} \frac{y\mathbb{1}}{|\mathcal{S}_k|} = \lim_{k \to \infty} \frac{|\mathcal{S}_k| \cdot |(\mathcal{B}_c(s) \cap \mathbb{Z}^2) \setminus \{s\}|}{2|\mathcal{S}_k|} \cdot \frac{1}{|(\mathcal{B}_c(s) \cap \mathbb{Z}^2) \setminus \{s\}|} = \frac{1}{2}.$$

This upper bound is not very good. Therefore, we need to find better way to apply the methods explained in Section 3.2.1 to find an upper bound to $r(\mathcal{S}_\infty, a, b, c)$.

3.2.3 Improving the Upper Bound

The quality of the LP-Relaxation determines the quality of the integer linear programming of the upper bound. Therefore, we will focus on improving the quality of the LP-Relaxation. First, we define the set Q as follows.

$$Q_{\mathcal{S},a,b,c} := \{x \in \{0,1\}^{\mathcal{S}} : (3)\}.$$

Then Q is the set containing all feasible solutions to maximisation problem $q(\mathcal{S},a,b,c)$. Constraint (3) can be seen as a list of inequalities that bounds a certain polyhedron in which all $x \in Q$ are located. We define this polyhedron P as

$$P_{\mathcal{S},a,b,c} := \mathrm{Conv}(Q_{\mathcal{S},a,b,c}).$$

Then, we can use set Q and polyhedron P to deduce the following. Suppose that $\mathcal{S}' \subseteq \mathcal{S}$. Then if $\mathcal{A}$ is (a,b,c)-safe on $\mathcal{S}$, $\mathcal{A}$ restricted to $\mathcal{S}'$ is also (a,b,c)-safe, on $\mathcal{S}'$. Namely, we have that $d(s,t) > c$ for all $s,t \in \mathcal{A} \subseteq \mathcal{S}$ such that $s \neq t$, so $d(s,t) > c$ for all $s,t \in \mathcal{A} \subseteq \mathcal{S}'$. Let $x \in \mathbb{R}^{\mathcal{S}}$ and $s \in \mathcal{S}$, then we define $x_{\mathcal{S}'} \in \mathbb{R}^{\mathcal{S}'}$ as

$$(x_{\mathcal{S}'})_s := x_s \ \forall\ s \in \mathcal{S}'.$$

Then we have

$$\begin{aligned} &\mathcal{S}' \subseteq \mathcal{S} \\ \Rightarrow\ &\{x_{\mathcal{S}'} : x \in Q_{\mathcal{S},a,b,c}\} \subseteq Q_{\mathcal{S}',a,b,c} \\ \\ \Rightarrow\ &\{x_{\mathcal{S}'} : x \in P_{\mathcal{S},a,b,c}\} \subseteq P_{\mathcal{S}',a,b,c}. \end{aligned} \tag{10}$$

So, if we can find an inequality that is true for all $P_{\mathcal{S}',a,b,c}$, then it is also true for all $\{x_{\mathcal{S}'} : x \in P_{\mathcal{S},a,b,c}\}$. Thus, we can use knowledge about subsets $\mathcal{S}' \subseteq \mathcal{S}$ in order to gain knowledge about the entire theatre. This is shown by the following theorem.

THEOREM 2: Let $\mathcal{S} \subset \mathbb{Z}^2$ finite, $a \in \mathbb{R}^{\mathcal{S}}$, $a \geq 0$ and suppose that a satisfies $ax \leq 1$ for all $x \in \{0,1\}^{\mathcal{S}}$ safe. Then

$$\frac{1}{\mathbb{1}a} \geq r(\mathcal{S}_\infty, a, b, c).$$

Proof: By symmetry, we can translate $\mathcal{S}$ over $\mathbb{Z}^2$. Let $l \in \mathbb{Z}^2$, then we can write such a translation as

$$\mathcal{S} + l := \{s \in \mathbb{Z}^2 : (s-l) \in \mathcal{S}\}.$$

Then, by symmetry, we have that $ax \leq 1$ on all these regions $\mathcal{S}+l$ for all $l \in \mathbb{Z}^2$. Note that by translation symmetry, for finite $k \in \mathbb{R}$, there exists a finite $L \subset \mathbb{Z}^2$ such that

$$\mathcal{S}_k \subseteq \bigcup_{l \in L} (\mathcal{S}+l).$$

We choose L such that $L_k = \{l \in \mathbb{Z}^2 : (\mathcal{S}+l) \cap \mathcal{S}_k \neq \emptyset\}$. Then, we generate a matrix A such that

$$A_{ls} = \begin{cases} a_{s-l} & \text{if } s \in \mathcal{S}+l \\ 0 & \text{otherwise.} \end{cases}$$

Then by (6), it follows that $Ax \leq \mathbb{1}$ for all safe $x \in \{0,1\}^{\mathcal{S}_k}$. Matrix A is an $L \times \mathcal{S}_k$ matrix. Then, by construction, A satisfies

$$\sum_{l \in L} A_{ls} = \sum_{l \in L} a_{s-l} = \mathbb{1}a.$$

Therefore, vector y such that

$$y_i = \frac{1}{\mathbb{1}a}$$

is a feasible solution to the dual problem:

$$\begin{aligned} &\sum_{l \in L} \frac{1}{\mathbb{1}a} A_{ls} \\ &= \frac{1}{\mathbb{1}a} \sum_{l \in L} A_{ls} \\ &= \frac{1}{\mathbb{1}a} \mathbb{1}a \\ &= 1. \end{aligned}$$

By the method explained in Section 3.2.1, we have

$$\begin{aligned} &r(s_\infty, a, b, c) \\ &\leq \lim_{k \to \infty} \frac{\mathbb{1}y}{|\mathcal{S}_k|} \\ &= \lim_{k \to \infty} \frac{|L|}{\mathbb{1}a|\mathcal{S}_k|} \\ &= \frac{1}{\mathbb{1}a} \lim_{k \to \infty} \frac{|L|}{|\mathcal{S}_k|}. \end{aligned}$$

By choice of L, we have that $|L| \geq |\mathcal{S}_k|$ for all $k \geq 0$. Therefore

$$\frac{|L|}{|\mathcal{S}_k|} \geq 1 \Rightarrow \lim_{k \to \infty} \frac{|L|}{|\mathcal{S}_k|} \geq 1.$$

Since $\mathcal{S}$ is finite, it follows that it is bounded. Therefore, there exists a $p \in \mathbb{R}$ such that $\mathcal{S} + l \subset B_p(\mathbf{0})$ for each $l \in L$. We fix such a p. Then we have

$$|L| \leq \left(\frac{k+2p}{b} + 1\right)\left(\frac{k+2p}{a} + 1\right) = \frac{k^2 + k(4p+a+b) + 4p^2 + 2ap + 2bp + ab}{ab}.$$

Therefore

$$\begin{aligned} &\lim_{k \to \infty} \frac{|L|}{|\mathcal{S}_k|} \\ &\leq \lim_{k \to \infty} \frac{\frac{k^2 + k(4p+a+b) + 4p^2 + 2ap + 2bp + ab}{ab}}{\frac{k^2}{ab}} \\ &= \lim_{k \to \infty} \frac{k^2 + k(4p+a+b) + 4p^2 + 2ap + 2bp + ab}{k^2} \\ &= \lim_{k \to \infty} \left(1 + \frac{(4p+a+b)}{k} + \frac{4p^2 + 2ap + 2bp + ab}{k^2}\right) = 1. \end{aligned}$$

So, we have that $1 \leq \lim_{k\to\infty} \frac{|L|}{|\mathcal{S}_k|} \leq 1$, and therefore $\lim_{k\to\infty} \frac{|L|}{|\mathcal{S}_k|} = 1$. From this, it follows that

$$\frac{1}{\mathbb{1}a} \geq r(\mathcal{S}_\infty, a, b, c). \quad \square$$

So, any a that satisfies $ax \leq 1$ on some finite subset $\mathcal{S} \subset \mathbb{Z}^2$ provides an upper bound to $r(\mathcal{S}_\infty, a, b, c)$. Such constraint vectors a on any finite $\mathcal{S}$ can be found using a cutting-plane method that will be explained in Section 3.4. In the cutting-plane method, we repeatedly need to generate safe x on $\mathcal{S}$. This can be done, as explained in Section 2.1, by solving maximisation problem (5)

$$\max\left\{\mathbb{1}x : x \in \{0,1\}^{\mathcal{S}}, (3)\right\}.$$

However, due to constraint (3) this method is quite slow and since we need to use it repeatedly, it slows down the cutting-plane method significantly. Therefore, we use some additional constraints. Before doing this, we give the proof for Theorem 1.

3.3 Proof for Theorem 1

The proof for Theorem 1 is given in this section. Theorem 1 states

Theorem 1: For all finite $\mathcal{S} \subset \mathbb{Z}^2$ and all $a, b, c \geq 0$ we have

$$r(\mathcal{S}, a, b, c) \geq r(\mathcal{S}_\infty, a, b, c).$$

Proof: Fix $\mathcal{S} \subset \mathbb{Z}^2$ finite, and $a, b, c \in \mathbb{R}$ such that $a, b, c \geq 0$. Suppose that $x \in \{0,1\}^{\mathcal{S}}$ is safe. Then by definition of q we have

$$\mathbb{1}x \leq q(\mathcal{S}, a, b, c).$$

Let $a = \frac{1}{q(\mathcal{S},a,b,c)}\mathbb{1}$, $a \in \mathbb{R}^{\mathcal{S}}$. Then

$$\mathbb{1}a = \mathbb{1}\frac{1}{q(\mathcal{S},a,b,c)}\mathbb{1} = \frac{|\mathcal{S}|}{q(\mathcal{S},a,b,c)}$$
$$\Rightarrow \frac{1}{\mathbb{1}a} = \frac{q(\mathcal{S},a,b,c)}{|\mathcal{S}|} = r(\mathcal{S},a,b,c).$$

Furthermore, we have

$$ax = \frac{1}{q(\mathcal{S},a,b,c)}\mathbb{1}x \leq \frac{q(\mathcal{S},a,b,c)}{q(\mathcal{S},a,b,c)} = 1.$$

So, $ax \leq 1$ for all $x \in \{0,1\}^{\mathcal{S}}$ safe. Therefore, we can apply Theorem 2, which yields

$$r(\mathcal{S},a,b,c) = \frac{1}{\mathbb{1}a} \overset{\text{Theorem 2}}{\geq} r(\mathcal{S}_\infty, a, b, c). \quad \square$$

3.4 Additional Constraints

3.4.1 The First Additional Constraint

Let some $\mathcal{S} \in \mathbb{Z}^2$, a, b and c be fixed. Consider a seat $s \in \mathcal{S}$, and a ball with radius $c/2$ around seat s. Then for all safe characteristic vectors x, we have that

$$\sum_{t \in \mathcal{B}_{c/2}(s)} x_t \leq 1 \forall s \in \mathcal{S}. \tag{11}$$

This is shown in the following theorem.

Theorem 3: Let $\mathcal{S} \subset \mathbb{Z}^2$, and a, b and c be fixed. Then all safe characteristic vectors x satisfy

$$\sum_{t \in (\mathcal{B}_{c/2}(w) \cap \mathcal{S})} x_t \leq 1 \forall w \in \mathbb{R}^2.$$

Proof: This can be seen in the following way. Suppose that $t \in (\mathcal{B}_{c/2}(w) \cap \mathcal{S})$, and that $x_t = 1$. By definition of the ball, we have that $d(t, w) \leq c/2$. Now consider any other $v \in (\mathcal{B}_{c/2}(w) \cap \mathcal{S})$, then $d(v, w) \leq c/2$. Using the triangle equality, it follows that

$$d(t, v) \leq d(t, w) + d(v, w) \leq \frac{c}{2} + \frac{c}{2} = c.$$

So, $d(t, v) \leq c$. Therefore, for any safe x on $\mathcal{S}$ we have that $x_t + x_v \leq 1$. Since we assumed $x_t = 1$, it follows that we need $x_v = 0$ for all $v \in (\mathcal{B}_{c/2}(w) \cap \mathcal{S})$ such that $v \neq t$. Thus, if a safe x on $\mathcal{S}$ contains a 1 on $(\mathcal{B}_{c/2}(w) \cap \mathcal{S})$, then all other entries of x on that ball must be 0. Therefore, it follows that for any safe x, $\sum_{t \in (\mathcal{B}_{c/2}(w) \cap \mathcal{S})} x_t \leq 1$. $\square$

Since this constraint holds for all safe x, we can add it to Q, and we obtain

$$Q_{\mathcal{S},a,b,c} = \{x \in \{0, 1\}^{\mathcal{S}} : (11), (3).\} \tag{12}$$

3.4.2 The Second Additional Constraint

In the first additional constraint, we calculate the value of characteristic vectors restricted to an area around a seat. However, in Theorem 3 it is not assumed that these balls have to be chosen around seats. The balls could be chosen around any element in $\mathbb{R}^2$, and the statement would still hold. Thus, in this second constraint we calculate the balls not just around seats in $\mathcal{S}$, but also around points halfway in between. We do not need to use more balls than this, as is shown by Theorem 5. The advantage of using extra balls is that they provide some extra correspondence between consecutive rows, especially for small c, in which case the balls around seats can be too small to contain seats from different rows. All safe characteristic vectors x satisfy

$$\sum_{t \in (\mathcal{B}_{c/2}(w) \cap \mathcal{S})} x_t \leq 1 \forall w \in (\frac{1}{2}\mathbb{Z})^2. \tag{13}$$

The proof of (13) is analogous to the proof of (11). Adding this extra constraint to Q yields

$$Q_{\mathcal{S},a,b,c} = \left\{x \in \{0, 1\}^{\mathcal{S}} : (11), (13), (3)\right\}. \tag{14}$$

If we want to find the contents of Q, or solve the corresponding maximisation problem $q(\mathcal{S}, a, b, c)$, we could in fact leave (3) and (11) out and only use (13), as we show in the following two theorems.

Theorem 4: Let $\mathcal{S} \subseteq \mathbb{Z}^2$, and a, b and c be fixed. Then we have

$$\begin{aligned} Q_{\mathcal{S},a,b,c} &= \left\{x \in \{0, 1\}^{\mathcal{S}} : x_s + x_t \leq 1 \ \forall\ s, t \in \mathcal{S} \text{ such that } d(s, t) \leq c,\ s \neq t\right\} \\ &= \left\{ x \in \{0, 1\}^{\mathcal{S}} : \sum_{t \in (B_{c/2}(w) \cap \mathcal{S})} x_t \leq 1 \forall w \in \mathbb{R}^2 \right\}. \end{aligned}$$

Proof: For ease on notation, we say

$$K := \left\{ x \in \{0,1\}^{\mathcal{S}} : \sum_{t \in (B_{c/2}(w) \cap \mathcal{S})} x_t \leq 1 \forall w \in \mathbb{R}^2 \right\}.$$

Then we need to show that $Q_{\mathcal{S},a,b,c} = K$. By Theorem 3, we have that $Q_{\mathcal{S},a,b,c} \subseteq K$. Therefore, it remains to be shown that $K \subseteq Q_{\mathcal{S},a,b,c}$. Suppose that $x \in K$. Then, it follows that

$$\sum_{t \in (B_{c/2}(w) \cap \mathcal{S})} x_t \leq 1 \forall w \in \mathbb{R}^2.$$

Let $s, t \in \mathcal{S}$ such that $d(s,t) \leq c$. Then for $w = (\frac{s_r - t_r}{2}, \frac{s_s - t_s}{2}) \in \mathbb{R}^2$ we have $d(w,t) \leq c/2$ and $d(w,s) \leq c/2$. Thus, $s, t \in (\mathcal{B}_{c/2}(w) \cap \mathcal{S})$. Therefore, $x_s + x_t \leq 1$. It follows that $K \subseteq Q_{\mathcal{S},a,b,c}$. Hence, $Q_{\mathcal{S},a,b,c} = K$. $\square$

THEOREM 5: Let $\mathcal{S} \subseteq \mathbb{Z}^2$, and a, b and c be fixed. Then we have

$$\left\{ x \in \{0,1\}^{\mathcal{S}} : \sum_{t \in (B_{c/2}(w) \cap \mathcal{S})} x_t \leq 1 \forall w \in \mathbb{R}^2 \right\} = \left\{ x \in \{0,1\}^{\mathcal{S}} : \sum_{t \in (B_{c/2}(w) \cap \mathcal{S})} x_t \leq 1 \forall w \in (1/2\mathbb{Z})^2 \right\}.$$

Proof: First of all, note that the zero vector is an element of both sets. For ease of notation, we say that

$$L := \left\{ x \in \{0,1\}^{\mathcal{S}} : \sum_{t \in (B_{c/2}(w) \cap \mathcal{S})} x_t \leq 1 \forall w \in (1/2\mathbb{Z})^2 \right\}.$$

We need to show that $K = L$. To do so, we first show that all elements of K are also an element of L, so $K \subseteq L$. Then, we show that all vectors that are not in K are also not in L, so $L \subseteq K$.

- Suppose that $x \in K$. Then x satisfies

  $$\sum_{t \in (B_{c/2}(w) \cap \mathcal{S})} x_t \leq 1 \forall w \in \mathbb{R}^2.$$

  Since $(1/2\mathbb{Z})^2 \subset \mathbb{R}^2$, it follows that

  $$\sum_{t \in (B_{c/2}(w) \cap \mathcal{S})} x_t \leq 1 \forall w \in (1/2\mathbb{Z})^2.$$

  Therefore $x \in L$. So, $K \subseteq L$.
- Now, suppose that $x \notin K$. Then there exist a $t, s \in \mathcal{S}$ such that $x_s = x_t = 1$ and $s, t \in B_{c/2}(w)$ for some $w \in \mathbb{R}^2$. We fix such a w. By the same argument as above, we have

  $$s, t \in (B_{c/2}((\frac{s_r - t_r}{2}, \frac{s_s - t_s}{2})) \cap \mathcal{S}).$$

  Thus, there exists a $w \in (\frac{1}{2}\mathbb{Z})^2$ such that $\sum_{t \in (B_{c/2}(w) \cap \mathcal{S})} x_t \geq 1$. Therefore, $x \notin L$, and so $L \subseteq K$. Hence $K = L$. $\square$

By Theorems 4 and 5, we have

$$Q_{\mathcal{S},a,b,c} = \{x \in \{0,1\}^{\mathcal{S}} : (13)\}.$$

Therefore, solutions to $q(\mathcal{S}, a, b, c)$ and $r(\mathcal{S}, a, b, c)$ can be found using only constraint (13).

3.5 Constraint on a Finite Region

Now, we are ready to explain how to find a constraint vector a that satisfies $ax \leq 1$ for all safe x on finite $\mathcal{S} \subset \mathbb{Z}^2$. Let an a, b and a forbidden distance $c \in \mathbb{R}$ be fixed. Since, as shown in Theorem 2, vector a will be used to find upper bounds to $r(\mathcal{S}_\infty, a, b, c)$, we try to find $\mathbb{1}a$ as large as possible. Thus, we have the following new optimisation problem (15) that we want to solve.

$$\max\{\mathbb{1}a : ax \leq 1 \ \forall x \in \{0,1\}^{\mathcal{S}} \text{ safe}, a \in [0,1]^{\mathcal{S}}\}. \tag{15}$$

We try to find a solution to problem (15) using the cutting-plane method explained in the following three steps. First of all, notice that any seating arrangement where only 1 seat is occupied is safe. Thus, to start our iterative process, we take vector a to be the vector of maximum size such that $ax \leq 1$ for all x where only 1 seat is occupied. Thus, we start with $a = \mathbb{1}$.

Step 1: Using the vector a that we found, we try to find a safe vector $x \in \{0,1\}^{\mathcal{S}}$ such that ax is as large as possible. Thus, we solve maximisation problem (16):

$$\max\{ax : x \in \{0,1\}^{\mathcal{S}}, x \text{ safe.}\}. \tag{16}$$

In Section 2.3, we explained the methods that we use to solve this problem.

Step 2: Let x^* denote the generated vector x. Since x^* is safe, we want to ensure that a satisfies that $ax^* \leq 1$. Thus, we try to find a new a using maximisation problem (17).

$$\max\{\mathbb{1}a : ax^* \leq 1, x \in \{0,1\}^{\mathcal{S}}, a \in [0,1]^{\mathcal{S}}\}. \tag{17}$$

Step 3: Using the newly found vector a, we repeat Step 1 and solve problem (16) again. Then, we add $ax^* \leq 1$ using the newly found safe vector x^* that has just been generated to the list of constraints of problem (17), and solve that again. This process is repeated until the solution of problem (16) is smaller than or equal to 1. Namely, if it is not possible anymore to generate a safe vector x such that $ax > 1$, we must have that $ax \leq 1$ for all safe x, which was the goal we tried to achieve. Therefore, the latest found a is a solution to problem (15).

3.6 The Upper Bound

Now that all methods needed to find an upper bound to $r(\mathcal{S}_\infty, a, b, c)$ have been explained, it remains to be explained which finite $\mathcal{S}$ we use to apply our methods to. Not every choice for $\mathcal{S}$ is equally suitable. For instance, for each forbidden distance c, $\mathcal{S}$ can be chosen such that $d(s,t) > c$ for all $s, t \in \mathcal{S}$. Then the vector $\mathbb{1}$ is a safe characteristic vector. Thus, a would have to satisfy $a\mathbb{1} \leq 1$. Thus, it follows that

$$\frac{1}{\mathbb{1}a} \geq 1.$$

1 is an upper bound to any occupancy rate, but it is a very poor one. Therefore, we need to be smart about the choice for $\mathcal{S}$ on which we try to find a. The best upper bound is achieved when $\mathbb{1}a$ is as large as possible, which in turn is larger if all safe x for which we need $ax \leq 1$ are smaller. Thus, a good $\mathcal{S}$ to choose is one with a small occupancy rate. In other words, we want to use an $\mathcal{S}$ that contains a cluster of seats, on which the distance constraints have large effect. Therefore, we choose $\mathcal{S} = (B_p(\mathbf{0}) \cap \mathbb{Z}^2)$, for some $p > 0$. So, for our upper bound we find a such that $ax \leq 1$ for all safe $x \in \{0,1\}^{(B_p(\mathbf{0}) \cap \mathbb{Z}^2)}$, and use $\frac{1}{\mathbb{1}a}$ as our upper bound.

4 Lower Bound

To find a lower bound to problem $r(\mathcal{S}_\infty, a, b, c)$, we first note that

$$\mathcal{A} \subseteq \mathcal{S}_\infty \ (a,b,c)\text{-safe} \ \Rightarrow r(\mathcal{S}_\infty, a, b, c) \geq \lim_{k\to\infty} \frac{|\mathcal{S}_k \cap \mathcal{A}|}{|\mathcal{S}_k|}. \tag{18}$$

So, the density of any seating arrangement $\mathcal{A} \subset \mathcal{S}_\infty$ that is safe is a lower bound. Therefore, the goal of this chapter is to develop a general method to find safe seating arrangements in $\mathcal{S}_\infty$ and to calculate the density of these safe arrangements. To do this, we assume that $\mathcal{A}$ follows a consistent pattern throughout the infinite theatre. In other words, we will try to find a safe $\mathcal{A}$, such that $\mathcal{A}$ forms a lattice spanned by two basis vectors within $\mathcal{S}_\infty$. In order to find the lower bound as close as possible to the upper bound, we want to find a safe lattice such that the density of that lattice is maximal.

4.1 Lattices and Density

Let $l_1, l_2 \in \mathbb{Z}^2$ be two vectors. Then lattice $\mathcal{L}_{l_1,l_2} \subseteq \mathcal{S}_\infty$ is the set

$$\mathcal{L}_{l_1,l_2} := \{s \in \mathcal{S}_\infty : s = \lambda l_1 + \mu l_2 \ \forall \ \lambda, \mu \in \mathbb{Z}\}. \tag{19}$$

Thus, a lattice contains all linear combinations of two basis vectors $l_1, l_2 \in \mathcal{S}_\infty$. In order to use these lattices as a lower bound, we need to determine the density of the seating arrangement associated to a certain lattice. To do so, we first note that, in the Euclidean space, the area of the parallelogram spanned by two vectors equals the absolute value of the determinant of the matrix that has these two vectors as columns. Thus, if P_{l_1,l_2} denotes the parallelogram spanned by l_1 and l_2, we have

$$\text{Area}(P_{l_1,l_2}) = \left|\det \begin{pmatrix} l_1 & l_2 \end{pmatrix}\right|. \tag{20}$$

By symmetry, this holds for each translation of P by $\lambda l_1 + \mu l_2$ for all $\lambda, \mu \in \mathbb{Z}$. Therefore, for any fixed $\lambda, \mu \in \mathbb{Z}$, a set $P^*_{l_1,l_2}(\lambda, \mu) \subseteq \mathbb{Z}^2$ of $|\det \begin{pmatrix} l_1 & l_2 \end{pmatrix}|$ points can be associated to point $(\lambda l_1 + \mu l_2)$, such that

$$\bigcup_{(\lambda,\mu)\in\mathbb{Z}^2} P^*_{l_1,l_2}(\lambda, \mu) = \mathbb{Z}^2 = \mathcal{S}_\infty. \tag{21}$$

As a result,

$$\frac{1}{\left|\det \begin{pmatrix} l_1 & l_2 \end{pmatrix}\right|}$$

gives the density of a lattice $\mathcal{L}_{l_1,l_2}$. Since this last fact does not depend on the distances within the grid, it also holds for the theatres considered in this report. An example of this method is given in Figure 7. Here, $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ are chosen as basis vectors. Thus, the absolute value of the determinant of the corresponding matrix is

$$\left|\det \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}\right| = |-7| = 7.$$

Thus, to each element of the corresponding lattice, a set P^* of 7 seats can be associated. One of these regions is depicted by the 7 blue dots. So, if we interpret the lattice as a seating arrangement, the corresponding density of that seating arrangement is $1/7$. An example of (21) is shown in

Figure 8. Moreover, all elements of the lattice are depicted by crosses, such that the fact that the density is 1/7 can more easily be seen.

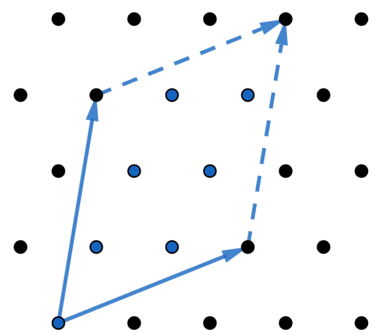

Figure 7: Parallelogram Spanned by $\begin{pmatrix}1\\2\end{pmatrix}$ and $\begin{pmatrix}3\\-1\end{pmatrix}$.

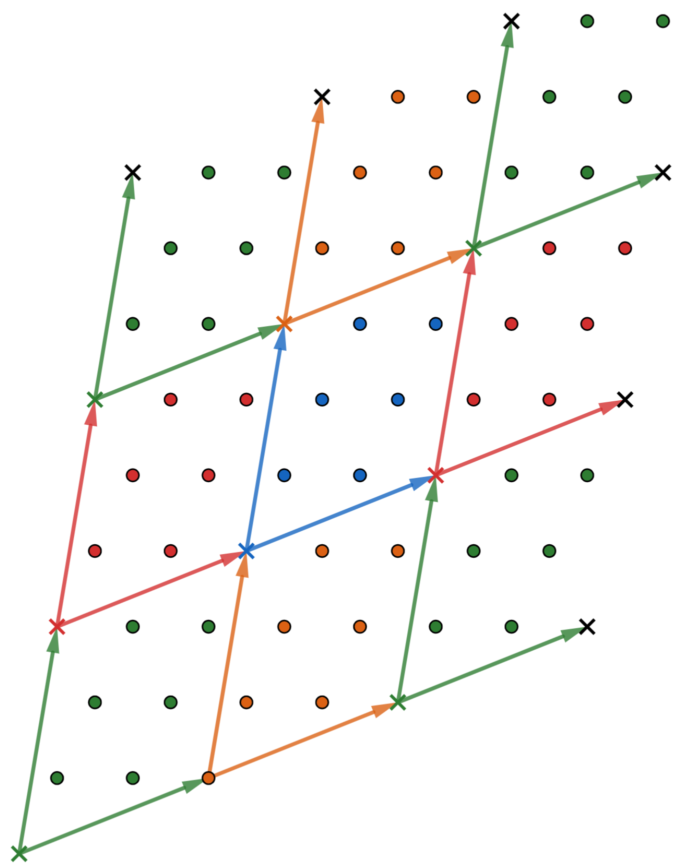

Figure 8: Union of Parallelograms.

4.2 Rewriting the Distance Function

Now that we have the methods to find the density of any lattice, we need to ensure that the lattices are associated to safe seating arrangements in order to find a lower bound to $r(\mathcal{S}_\infty, a, b, c)$. Before we do so, we explain a new way to write the distance function. Note that we can associate a seat's row and seat number to its coordinates in the Euclidean space. To do this, we have to use parameters a and b. Then, the vertical component of the coordinates of a seat s in the Euclidean space is given by $b \cdot s_r$, and the horizontal component is given by $a \cdot s_s + \frac{1}{2}a \cdot s_r$. Therefore, the coordinates of a seat s in the Euclidean space can be given by

$$\begin{pmatrix} b & 0 \\ \frac{1}{2}a & a \end{pmatrix} \cdot \begin{pmatrix} s_r \\ s_s \end{pmatrix},$$

where we call

$$\begin{pmatrix} b & 0 \\ \frac{1}{2}a & a \end{pmatrix}$$

the transition matrix. Using this, we rewrite the distance function, and give meaning to the inner product of some vectors $\mathbf{s}$ and $\mathbf{t}$. We have

$$\begin{aligned} d(s,t) = \|\mathbf{s} - \mathbf{t}\| &= \sqrt{(\begin{pmatrix} b & 0 \\ \frac{1}{2}a & a \end{pmatrix}(\mathbf{s}-\mathbf{t}))^T(\begin{pmatrix} b & 0 \\ \frac{1}{2}a & a \end{pmatrix}(\mathbf{s}-\mathbf{t}))} \\ &= \sqrt{(\mathbf{s}-\mathbf{t})^T(\begin{pmatrix} b & \frac{1}{2}a \\ 0 & a \end{pmatrix})(\begin{pmatrix} b & 0 \\ \frac{1}{2}a & a \end{pmatrix}(\mathbf{s}-\mathbf{t})} \\ &= \sqrt{(\mathbf{s}-\mathbf{t})^T \begin{pmatrix} b^2 + \frac{1}{4}a^2 & \frac{1}{2}a^2 \\ \frac{1}{2}a^2 & a^2 \end{pmatrix} (\mathbf{s}-\mathbf{t})}. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} \langle \mathbf{s}, \mathbf{t} \rangle = &\left(\begin{pmatrix} b & 0 \\ \frac{1}{2}a & a \end{pmatrix}\begin{pmatrix} s_r \\ s_s \end{pmatrix}\right)^T \left(\begin{pmatrix} b & 0 \\ \frac{1}{2}a & a \end{pmatrix}\begin{pmatrix} t_r \\ t_s \end{pmatrix}\right) \\ &\left(b^2 + \frac{1}{4}a^2\right) s_r t_r + \frac{1}{2}a^2 s_r t_s + \frac{1}{2}a^2 s_s t_r + a^2 s_s t_s. \end{aligned}$$

4.3 Safety of Lattices

A lattice is safe if the distance between each distinct pair of elements of the lattice is larger than c. Using the information of Section 4.2, we can state and proof the following lemma.

LEMMA 1: Let $s, t \in \mathcal{S}_\infty$. Suppose that $\|s\| \leq \|t\|$ and that $|\langle s, t \rangle| \leq \frac{1}{2}\|s\|^2$. Then

$$\min\{\|\lambda s + \mu t\| : \lambda, \mu \in \mathbb{Z}, (\lambda, \mu) \neq (0,0)\} = \|s\|.$$

Proof: Suppose that the statement does not hold. Then there exist $\lambda, \mu \in \mathbb{Z}$, $(\lambda, \mu) \neq (0,0)$, such that $\|s\| > \|\lambda s + \mu t\|$. Then

$$\begin{aligned} \|s\|^2 &> \|\lambda s + \mu t\|^2 \\ &= \lambda^2\|s\|^2 + 2\lambda\mu\langle s, t\rangle + \mu^2\|t\|^2 \\ &\geq \lambda^2\|s\|^2 - \lambda\mu\|s\|^2 + \mu^2\|s\|^2 \\ &= \|s\|^2(\lambda^2 - \lambda\mu + \mu^2). \end{aligned}$$

Thus

$$\|s\| > \|\lambda s + \mu t\| \iff \lambda^2 - \lambda\mu + \mu^2 < 1.$$

We consider four cases.

1. First, we consider the case $|\lambda| \geq |\mu|, \lambda \neq 0, \mu \neq 0$. In this case,
$$\lambda^2 - \lambda\mu + \mu^2 \geq \lambda^2 - \lambda^2 + \mu^2 = \mu^2 \geq 1.$$
2. Second, we consider the case $|\lambda| < |\mu|, \lambda \neq 0, \mu \neq 0$. In this case,
$$\lambda^2 - \lambda\mu + \mu^2 > \lambda^2 - \mu^2 + \mu^2 = \lambda^2 \geq 1.$$
3. Third, we consider the case where $\lambda = 0, \mu \neq 0$. Then,
$$\lambda^2 - \lambda\mu + \mu^2 = \mu^2 \geq 1.$$
4. Last, we consider the case where $\lambda \neq 0, \mu = 0$. Then,
$$\lambda^2 - \lambda\mu + \mu^2 = \lambda^2 \geq 1.$$

Thus, for all $(\lambda, \mu) \neq (0, 0)$, we have $\lambda^2 - \lambda\mu + \mu^2 \geq 1$. Therefore

$$\|s\|^2 > \|s\|^2(\lambda^2 - \lambda\mu + \mu^2) \geq \|s\|^2.$$

This is a contradiction. Therefore, no $\lambda, \mu \in \mathbb{Z}$, $(\lambda, \mu) \neq (0, 0)$, exist such that $\|s\| > \|\lambda s + \mu t\|$. Thus, it follows that $\|\lambda s + \mu t\| \geq \|s\|$ for such λ and μ. Notice that for $\lambda = 1$ and $\mu = 0$,

$$\|\lambda s + \mu t\| = \|s\|.$$

Therefore, if $\|s\| \leq \|t\|$ and $|\langle s, t\rangle| \leq \frac{1}{2}\|s\|^2$,

$$\min\{\|\lambda s + \mu t\| : \lambda, \mu \in \mathbb{Z}, (\lambda, \mu) \neq (0, 0)\} \leq \|s\|. \quad \square$$

Thus, if $\mathcal{L}_{l_1,l_2}$ is a lattice such that $\|l_2\| \leq \|l_1\| > c$ and $|\langle l_1, l_2\rangle| \geq \frac{1}{2}\|l_1\|^2$, it follows that for all $s, t \in \mathcal{L}_{l_1,l_2}$

$$\begin{aligned} d(s,t) &= d(\lambda_1 l_1 + \mu_1 l_2, \lambda_2 l_1 + \mu_2 l_2) \\ &= \|(\lambda_1 l_1 + \mu_1 l_2) - (\lambda_2 l_1 + \mu_2 l_2)\| \\ &= \|(\lambda_1 - \lambda_2) l_1 + (\mu_1 - \mu_2) l_2\| \\ &\overset{\text{Lemma 1}}{\geq} \|l_1\| > c. \end{aligned}$$

Therefore, $\mathcal{L}_{l_1,l_2}$ is safe.

4.4 The Lower Bound

To find a suitable lower bound to $r(\mathcal{S}_\infty, a, b, c)$ for any a, b and c, using the methods as explained in this chapter, we use an algorithm. The goal of this algorithm is to find a basis l_1, l_2 such that the lattice $\mathcal{L}_{l_1,l_2}$ is safe, and such that its density

$$\frac{1}{\left|\det\begin{pmatrix} l_1 & l_2 \end{pmatrix}\right|}$$

is maximal, or equivalently, such that $\left|\det\begin{pmatrix} l_1 & l_2 \end{pmatrix}\right|$ is minimal. The methods explained in this chapter allow us to find vectors that span safe lattices. However, we have no way to limit the area in which to look for such vectors. The following lemma forms the basis for a method to find such an area.

Lemma 2: Let $s, t \in \mathcal{S}_\infty$ be integer valued vectors. Suppose that $\|s\| \leq \|t\|$ and that $|\langle s, t\rangle| \leq \frac{1}{2}\|s\|^2$. Then

$$|\det(s,t)| \geq \frac{1}{2ab}\sqrt{3}\|s\|\|t\|.$$

Proof: We have

$$|\det(s,t)| = \left|\det\begin{pmatrix} s_r & t_r \\ s_s & t_s \end{pmatrix}\right| = |s_r t_s - t_r s_s|.$$

We want to apply the property of the cross product that for some $\mathbf{m}, \mathbf{n} \in \mathbb{R}^3$ we have $\mathbf{m} \times \mathbf{n} = \|\mathbf{m}\|\|\mathbf{n}\|\sin(\theta)$, where θ is the angle between vectors $\mathbf{m}$ and $\mathbf{n}$. However, this definition is applicable in the Euclidean space. Therefore, we have to use the Euclidean coordinates of s and t using the transition matrix. The vector corresponding to s in the Euclidean space is given by

$$\begin{pmatrix} bs_r \\ \frac{1}{2}as_r + as_s \end{pmatrix}$$

and the vector corresponding to t in the Euclidean space is given by

$$\begin{pmatrix} bt_r \\ \frac{1}{2}at_r + at_s \end{pmatrix}.$$

Let the Euclidean norm be denoted by $\|.\|_E$, and let the norm as given in Section 4.2 be denoted by $\|.\|$. Then

$$\begin{aligned}
\left\|\begin{pmatrix} bs_r \\ \frac{1}{2}as_r + as_s \\ 0 \end{pmatrix} \times \begin{pmatrix} bt_r \\ \frac{1}{2}at_r + at_s \\ 0 \end{pmatrix}\right\|_E &= \left\|\begin{pmatrix} 0 \\ 0 \\ \frac{1}{2}abs_r t_r + abs_r t_s - \frac{1}{2}abs_r t_r - abs_s t_r \end{pmatrix}\right\|_E \\
&= \left\|\begin{pmatrix} 0 \\ 0 \\ ab(s_r t_s - s_s t_r) \end{pmatrix}\right\|_E = ab|s_r t_s - t_r s_s| = ab|\det(s,t)|.
\end{aligned}$$

Then, by the definition of the cross-product,

$$\begin{aligned}
ab|\det(s,t)| &= \left\|\begin{pmatrix} bs_r \\ \frac{1}{2}as_r + as_s \\ 0 \end{pmatrix} \times \begin{pmatrix} bt_r \\ \frac{1}{2}at_r + at_s \\ 0 \end{pmatrix}\right\|_E \\
&= \left\|\begin{pmatrix} bs_r \\ \frac{1}{2}as_r + as_s \\ 0 \end{pmatrix}\right\|_E \left\|\begin{pmatrix} bt_r \\ \frac{1}{2}at_r + at_s \\ 0 \end{pmatrix}\right\|_E \sin(\theta) \\
&= \left\|\begin{pmatrix} bs_r \\ \frac{1}{2}as_r + as_s \end{pmatrix}\right\|_E \left\|\begin{pmatrix} bt_r \\ \frac{1}{2}at_r + at_s \end{pmatrix}\right\|_E \sin(\theta) \\
&= \|s\|\,\|t\|\sin(\theta)
\end{aligned}$$

where θ denotes the angle between the vectors $\begin{pmatrix} bs_r \\ \frac{1}{2}as_r + as_s \\ 0 \end{pmatrix}$ and $\begin{pmatrix} bt_r \\ \frac{1}{2}at_r + at_s \\ 0 \end{pmatrix}$. Let $\langle ., . \rangle_E$ be the Euclidean inner product, and $\langle ., . \rangle$ denote the inner product as given in Section 4.2.

By the properties of the inner product and by the assumptions follows

$$\begin{aligned}
\cos(\theta) &= \frac{\left\langle \begin{pmatrix} bs_r \\ \frac{1}{2}as_r + as_s \\ 0 \end{pmatrix}, \begin{pmatrix} bt_r \\ \frac{1}{2}at_r + at_s \\ 0 \end{pmatrix} \right\rangle_E}{\left\| \begin{pmatrix} bs_r \\ \frac{1}{2}as_r + as_s \\ 0 \end{pmatrix} \right\|_E \left\| \begin{pmatrix} bt_r \\ \frac{1}{2}at_r + at_s \\ 0 \end{pmatrix} \right\|_E} \\
&= \frac{\left\langle \begin{pmatrix} bs_r \\ \frac{1}{2}as_r + as_s \end{pmatrix}, \begin{pmatrix} bt_r \\ \frac{1}{2}at_r + at_s \end{pmatrix} \right\rangle_E}{\left\| \begin{pmatrix} bs_r \\ \frac{1}{2}as_r + as_s \end{pmatrix} \right\|_E \left\| \begin{pmatrix} bt_r \\ \frac{1}{2}at_r + at_s \end{pmatrix} \right\|_E} \\
&= \frac{\langle s, t \rangle}{\|s\| \|t\|} \\
&\geq \frac{-\frac{1}{2}\|s\|^2}{\|s\|^2} \\
&= -\frac{1}{2}.
\end{aligned}$$

Therefore,

$$-\frac{2}{3}\pi \leq \theta \leq \frac{2}{3}\pi. \tag{22}$$

By similar reasoning, we find

$$\begin{aligned}
\cos(\theta) &= \frac{\langle s, t \rangle}{\|s\| \|t\|} \\
&\leq \frac{\frac{1}{2}\|s\|^2}{\|s\|^2} \\
&= \frac{1}{2}.
\end{aligned}$$

So, we also have

$$\frac{1}{3}\pi \leq \theta \leq \frac{5}{3}\pi. \tag{23}$$

By (22) and (23),

$$\frac{1}{3}\pi \leq \theta \leq \frac{2}{3}\pi.$$

As a result,

$$\sin(\theta) \geq \frac{1}{2}\sqrt{3}.$$

Therefore,

$$\begin{aligned}
ab|\det(s, t)| &= \|s\| \|t\| \sin(\theta) \\
&\geq \|s\| \|t\| \frac{1}{2}\sqrt{3} \\
\Rightarrow |\det(s, t)| &\geq \|s\| \|t\| \frac{1}{2ab}\sqrt{3}. \quad \square
\end{aligned}$$

Thus, for any safe lattice $\mathcal{L}_{l_1,l_2}$, such that l_1 and l_2 satisfy the conditions of Lemma 1 and Lemma 2, we have

$$|\det(l_1,l_2)| \geq \frac{1}{2ab}\sqrt{3}\|l_1\|\|l_2\|.$$

Furthermore, the lattice $\overline{\mathcal{L}}$ spanned by $l_1 = \left(0, \left\lfloor \frac{c}{a} \right\rfloor + 1\right)$ and $l_2 = \left(2\left\lfloor \frac{c}{2b} \right\rfloor + 2, -\left\lfloor \frac{c}{2b} \right\rfloor - 1\right)$ satisfies

$$\left\|\left(0, \left\lfloor \frac{c}{a} \right\rfloor + 1\right)\right\| = \sqrt{\begin{pmatrix} 0 & \lfloor \frac{c}{a} \rfloor + 1 \end{pmatrix} \begin{pmatrix} b^2 + \frac{1}{4}a^2 & \frac{1}{2}a^2 \\ \frac{1}{2}a^2 & a^2 \end{pmatrix} \begin{pmatrix} 0 \\ \lfloor \frac{c}{a} \rfloor + 1 \end{pmatrix}} = \sqrt{a^2\left(\left\lfloor \frac{c}{a} \right\rfloor + 1\right)^2} > c,$$

$$\left\|\left(2\left\lfloor \frac{c}{2b} \right\rfloor + 2, -\left\lfloor \frac{c}{2b} \right\rfloor - 1\right)\right\| = \sqrt{\begin{pmatrix} 2\lfloor \frac{c}{2b} \rfloor + 2 & -\lfloor \frac{c}{2b} \rfloor - 1 \end{pmatrix} \begin{pmatrix} b^2 + \frac{1}{4}a^2 & \frac{1}{2}a^2 \\ \frac{1}{2}a^2 & a^2 \end{pmatrix} \begin{pmatrix} 2\lfloor \frac{c}{2b} \rfloor + 2 \\ -\lfloor \frac{c}{2b} \rfloor - 1 \end{pmatrix}}$$

$$= \sqrt{\begin{pmatrix} 2\lfloor \frac{c}{2b} \rfloor + 2 & -\lfloor \frac{c}{2b} \rfloor - 1 \end{pmatrix} \begin{pmatrix} b^2(2\lfloor \frac{c}{2b} \rfloor + 2) + \frac{1}{2}a^2(2\lfloor \frac{c}{2b} \rfloor + 2) + \frac{1}{2}a^2(-\lfloor \frac{c}{2b} \rfloor - 1) \\ a^2(\lfloor \frac{c}{2b} \rfloor + 1) + a^2(-\lfloor \frac{c}{2b} \rfloor - 1) \end{pmatrix}}$$

$$= \sqrt{\begin{pmatrix} 2\lfloor \frac{c}{2b} \rfloor + 2 & -\lfloor \frac{c}{2b} \rfloor - 1 \end{pmatrix} \begin{pmatrix} b^2(2\lfloor \frac{c}{2b} \rfloor + 2) \\ 0 \end{pmatrix}} = \sqrt{b^2(2\left\lfloor \frac{c}{2b} \right\rfloor + 2)^2} = 2b\left(\left\lfloor \frac{c}{2b} \right\rfloor + 1\right) > c$$

and

$$\begin{aligned} &\left|\left\langle \left(0, \left\lfloor \frac{c}{a} \right\rfloor + 1\right), \left(2\left\lfloor \frac{c}{2b} \right\rfloor + 2, -\left\lfloor \frac{c}{2b} \right\rfloor - 1\right)\right\rangle\right| \\ &= \left|\frac{1}{2}a^2\left(\left\lfloor \frac{c}{a} \right\rfloor + 1\right)\left(2\left\lfloor \frac{c}{2b} \right\rfloor + 2\right) + a^2\left(\left\lfloor \frac{c}{a} \right\rfloor + 1\right)\left(-\left\lfloor \frac{c}{2b} \right\rfloor - 1\right)\right| \\ &= \left|a^2\left(\left\lfloor \frac{c}{a} \right\rfloor + 1\right)\left(\left\lfloor \frac{c}{2b} \right\rfloor + 1\right) - a^2\left(\left\lfloor \frac{c}{a} \right\rfloor + 1\right)\left(\left\lfloor \frac{c}{2b} \right\rfloor + 1\right)\right| = 0. \end{aligned}$$

Therefore, using Lemma 1, $\overline{\mathcal{L}}$ is a safe lattice. Now, suppose that $\mathcal{L}_{l_1^*,l_2^*}$ is a safe lattice that gives an optimal lower bound. So, $|\det(l_1^*,l_2^*)|$ is minimal. Then

$$\begin{aligned} &\left|\det\begin{pmatrix} 0 & 2\lfloor \frac{c}{2b} \rfloor + 2 \\ \lfloor \frac{c}{a} \rfloor + 1 & -\lfloor \frac{c}{2b} \rfloor - 1 \end{pmatrix}\right| \geq |\det(l_1^*,l_2^*)| \geq \frac{1}{2ab}\sqrt{3}\|l_1^*\|\|l_2^*\| \\ &\Rightarrow \left(2\left\lfloor \frac{c}{2b} \right\rfloor + 2\right)\left(\left\lfloor \frac{c}{a} \right\rfloor + 1\right) \geq |\det(l_1^*,l_2^*)| \geq \frac{1}{2ab}\sqrt{3}\|l_1^*\|\|l_2^*\| \\ &\Rightarrow \left(\frac{c}{b} + 2\right)\left(\frac{c}{a} + 1\right) \geq |\det(l_1^*,l_2^*)| \geq \frac{1}{2ab}\sqrt{3}\|l_1^*\|\|l_2^*\| \\ &\Rightarrow \frac{2ab}{\sqrt{3}}\left(\frac{c}{b} + 2\right)\left(\frac{c}{a} + 1\right) \geq \frac{2ab\,|\det(l_1^*,l_2^*)|}{\sqrt{3}} \geq \|l_1^*\|\|l_2^*\|. \end{aligned}$$

Therefore, when looking for the lattice that provides an optimal lower bound, we are only interested in lattices $\mathcal{L}_{l_1,l_2}$ for which

$$\|l_1\|\|l_2\| \leq \frac{2ab}{\sqrt{3}}\left(\frac{c}{b} + 2\right)\left(\frac{c}{a} + 1\right).$$

By safety of $\mathcal{L}_{l_1,l_2}$, we have

$$\begin{aligned} \|l_1\|\|l_2\| &\geq c\|l_2\|, \\ \|l_1\|\|l_2\| &\geq c\|l_1\|. \end{aligned}$$

So, we are interested in lattices $\mathcal{L}_{l_1,l_2}$ for which

$$\|l_i\| \leq \frac{2ab}{\sqrt{3}c}\left(\frac{c}{b} + 2\right)\left(\frac{c}{a} + 1\right) \text{ for } i = 1, 2.$$

For ease, we define

$$m(a,b,c) := \frac{2ab}{\sqrt{3}c}\left(\frac{c}{b}+2\right)\left(\frac{c}{a}+1\right).$$

Then, in the search for a safe lattice with a minimal determinant it is sufficient to restrict to the area $\mathcal{B}_{m(a,b,c)}(\mathbf{0}) \cap \mathcal{S}_\infty$. We apply this in the algorithm used to find the lower bound. The algorithm consists of the following four steps.

Step 1: First of all, we consider the area $\mathcal{B}_{m(a,b,c)}(\mathbf{0}) \cap \mathcal{S}_\infty$. We assign a vector l_1 to a seat s in this area, such that $d(\mathbf{0}, s) > c$.

Step 2: We find a t in the area $\mathcal{B}_{m(a,b,c)}(\mathbf{0}) \cap \mathcal{S}_\infty$ such that the conditions of Lemma 1 are met, and assign vector l_2 to t. Then we calculate the determinant of those l_1 and l_2, and add it to a list.

Step 3: We repeat Step 2 until we have considered every $t \in \mathcal{B}_{m(a,b,c)}(\mathbf{0}) \cap \mathcal{S}_\infty$ such that the conditions of Lemma 1 are satisfied. Then, we repeat Step 1 until all seats $s \in \mathcal{B}_{m(a,b,c)}(\mathbf{0}) \cap \mathcal{S}_\infty$ such that $d(\mathbf{0}, s) > c$ are considered.

Step 4: By the previous three steps, a list is generated that consists of determinants of the vectors assigned to every possible pair of seats s and t in the area $\mathcal{B}_{m(a,b,c)}(\mathbf{0}) \cap \mathcal{S}_\infty$ such that s and t satisfy the conditions of Lemma 1. Thus, the inverse of each of these determinants is a lower bound to $r(\mathcal{S}_\infty, a, b, c)$. We select the minimum of this list, and use this number as our lower bound to $r(\mathcal{S}_\infty, a, b, c)$.

5 Results

In the previous two chapters, we discussed all methods that we need to calculate $r(\mathcal{S}, a, b, c)$ for finite $\mathcal{S}$, and the methods to find upper and lower bounds to $r(\mathcal{S}_\infty, a, b, c)$, which in turn is a lower bound to all occupancy rates using the same a, b and c by Theorem 1. In this chapter, we will show some of the results that we found using the methods as discussed.

5.1 Distinction between Solutions

At first glance, it would seem that there are an infinite number of strictly different solutions to both $r(\mathcal{S}, a, b, c)$ and $r(\mathcal{S}_\infty, a, b, c)$. After all, a, b and c are all real numbers larger than zero, and can therefore take an infinite number of values. In fact, the number of possible solutions is infinite, but, under a few assumptions, those with a certain relevance to real life are much more sparse than it would seem. To see this, we must note a few things. First of all, we consider a fixed a and b. Then, not all different c necessarily yield a different result. The restriction put on the optimal solutions that are investigated is that they must be safe. By Definition 1, an arrangement $\mathcal{A}$ is safe when $d(s,t) > c$ for all $s, t \in \mathcal{A}$ such that $s \neq t$. Thus, if for any $s \in \mathcal{A}$ and two $c_1, c_2 \in \mathbb{R}$ such that $c_1 \neq c_2$, we have

$$\{t \in \mathcal{S} : d(s,t) \leq c_1\} = \{t \in \mathcal{S} : d(s,t) \leq c_2\}, \tag{24}$$

then it follows that the safety constraint applies to the same set of seats for both c_1 and c_2, and so the optimal solution will be equal in both cases. We refer to a set such as the ones in (22) as the forbidden zone of seat s. In Figure 9, an example of this idea is visualised in a theatre with $a = 1$ and $b = 1$. The forbidden zone of a seat s consists of the seats within the circle. This is done for a forbidden distance of $c = 1.2$ in blue, and a forbidden distance of 1.7 in orange. Both forbidden zones contain the same set of seats. Therefore, the optimal solution in both cases will be equal.

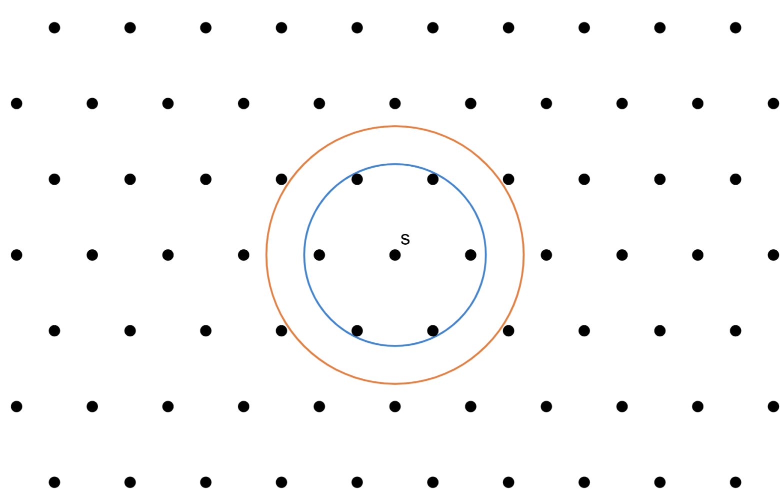

Figure 9: Forbidden Zone around seat s for $c = 1.2$ and $c = 1.7$.

The fact that for fixed a and b not all different c lead to different solutions greatly reduces the number of situations that need to be checked. In fact, if we consider only forbidden distances up to some finite number, then the number of different solutions for some fixed a and b are finite.

However, we wish to analyse solutions for varying a and b, not for some fixed situation. To this purpose, we can use the fact that not all different combinations of a and b yield a different situation. By scaling, a theatre in which we have $a = 1$, $b = 1$ and where we consider a forbidden distance of $c = 2$, will have the same solution as a theatre with $a = 2$, $b = 2$ and $c = 4$, since the forbidden zones consist of the same set of seats. Therefore, we are only interested in different ratios between a and b.

5.2 The Examined Situations

Of course, looking at varying ratios between a and b still leaves many different situations to analyse. In this report, we will show the results corresponding to a few examples of ratios of a and b between $a = 2b$ and $b = 2a$, since the measurements of a theatre typically lie within this region. We will consider the following four cases.

1. We take $a = 0.51$ and $b = 0.95$, since these are the distances in the Music Building in Eindhoven.
2. We take $a = 1$ and $b = 1$.
3. We take $a = 1$ and $b = \frac{1}{2}\sqrt{3}$, since in this case we have $d(\mathbf{0}, (0, 1)) = d(\mathbf{0}, (1, 0))$.
4. We take $a = 1$ and $b = \frac{1}{2}$.

Within these situations, we will consider forbidden distances such that the smallest forbidden zones are examined, up to some bound. This bound is chosen arbitrarily for each forbidden distance. We use this bound, because including too many larger distances would cause for too many results to be presented. Within this bound, we can present all solutions to $r(\mathcal{S}_\infty, a, b, c)$, as explained in Section 5.1. The solutions to $r(\mathcal{S}, a, b, c)$ also depend on the choice of the finite set $\mathcal{S}$. However, we will still present solutions for all c that yield different forbidden zones, and we will consider a number of different sized sets $\mathcal{S}$. Since the constraints have most effect on clusters of seats, we will consider $\mathcal{S} = B_p(\mathbf{0}) \cap \mathbb{Z}^2$, for various ridii p. By symmetry, these $\mathcal{S}$ could be translated to be centered around any seat $s \in \mathcal{S}_\infty$. For each of the four cases, we can easily determine all forbidden distances, up to the bound, that need to be considered in order to examine every possible forbidden zone in the following way. Observe that, if we were to increase the forbidden distance c starting from zero, the forbidden zone of any seat s would grow by some set of seats as soon as the forbidden distance reaches the distance between those seats and the centre seat s. For instance, the forbidden zone of seat $\mathbf{0}$ would include seats $(0, 1)$ and $(0, -1)$ as soon as $c = d(\mathbf{0}, (0, 1)) = a$. Thus, if we list all distances of seats $s \in B_p(\mathbf{0}) \cap \mathbb{Z}^2$ to seat $\mathbf{0}$, we can determine all c that yield distinct results by choosing one c between all distances in the list. Note that each grid is symmetrical along two lines, which is shown in Figure 10, where the blue lines are the lines of symmetry. Therefore, it suffices to list only the distances of seats in one quadrant of the set $B_p(\mathbf{0}) \cap \mathbb{Z}^2$. The quarter that we consider consists of the blue seats. For $a = 0.51$ and $b = 0.95$, this list of distances is given in Table 1.

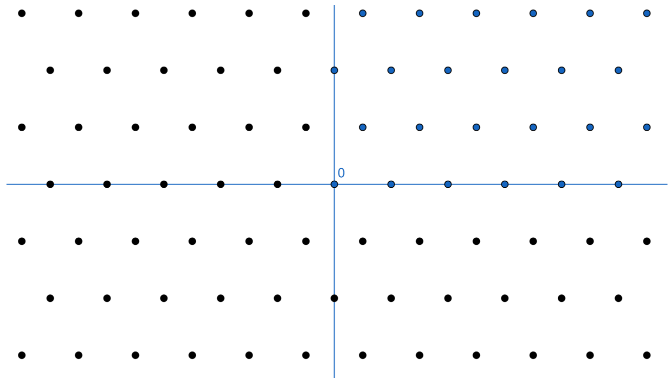

Figure 10: Symmetry within the grid.

Seat s	$d(\mathbf{0}, s)$
0	0
(0,1)	0.51
(1,0)	0.9863
(0,2)	1.02
(1,1)	1.2197
(0,3)	1.53
(1,2)	1.59
(2,-1)	1.9
(2,0)	1.9673
(1,3)	2.0221
(0,4)	2.04
(2,1)	2.1565
(2,2)	2.4394
(1,4)	2.4839
(0,5)	2.55
(2,3)	2.7878
(3,-1)	2.8614
(3,0)	2.9509
(1,5)	2.9615
(0,6)	3.06

(a) $a = 0.51$, $b = 0.95$

Seat s	$d(\mathbf{0}, s)$
0	0
(0,1)	1
(1,0)	1.118
(1,1)	1.8028
(0,2)	2
(2,-1)	2
(2,0)	2.2361
(1,2)	2.6926
(2,1)	2.8284
(0,3)	3
(3,-1)	3.0414
(3,0)	3.3541
(2,2)	3.6056
(1,3)	3.6401
(3,1)	3.9051
(0,4)	4
(4,-2)	4
(4,-1)	4.1231
(4,0)	4.4721
(2,3)	4.4721
(1,4)	4.6098
(3,2)	4.6098
(0,5)	5
(4,1)	5
(5,-2)	5.0249

(b) $a = 1$, $b = 1$

Seat s	$d(\mathbf{0}, s)$
0	0
(0,1)	1
(1,0)	1
(1,1)	1.7321
(2,-1)	1.7321
(0,2)	2
(2,0)	2
(1,2)	2.6458
(2,1)	2.6458
(3,-1)	2.6458
(0,3)	3
(3,0)	3
(2,2)	3.4641
(4,-2)	3.4641
(1,3)	3.6056
(3,1)	3.6056
(0,4)	4
(4,0)	4
(2,3)	4.3589
(3,2)	4.3589
(5,-2)	4.3589
(1,4)	4.5826
(4,1)	4.5826
(5,-1)	4.5826
(0,5)	5
(5,0)	5

(c) $a = 1$, $b = \frac{1}{2}\sqrt{3}$

Seat s	$d(\mathbf{0}, s)$
0	0
(1,0)	0.7071
(0.1)	1
(2,-1)	1
(2,0)	1.4142
(1,1)	1.5811
(3,-1)	1.5811
(0,2)	2
(4,-2)	2
(3,0)	2.1213
(2,1)	2.2361
(4,-1)	2.2361
(1,2)	2.5495
(5,-2)	2.5495
(4,0)	2.8284
(3,1)	2.9155
(5,-1)	2.9155
(0,3)	3
(6,-3)	3
(2,2)	3.1623
(6,-2)	3.1623
(1,3)	3.5355
(5,0)	3.5355
(7,-3)	3.5355
(4,1)	3.6056
(6,-1)	3.6056

(d) $a = 1$, $b = \frac{1}{2}$

Table 1: Distances between seats and **0** for various a and b.

5.3 Solutions to $r(\mathcal{S}, a, b, c)$

The solutions to $r(\mathcal{S}, a, b, c)$ that we found are presented in the following tables. In Table 2, the results for $a = 0.51$ and $b = 0.95$ are given. For $a = 1$ and $b = 1$, the results can be found in Table 3. The results for $a = 1$ and $b = \frac{1}{2}\sqrt{3}$ are listed in Table 4, and the results for $a = 1$ and $b = \frac{1}{2}$ are given in Table 5. In Figure 11, a visualisation of the optimal solution for a finite theatre is given.

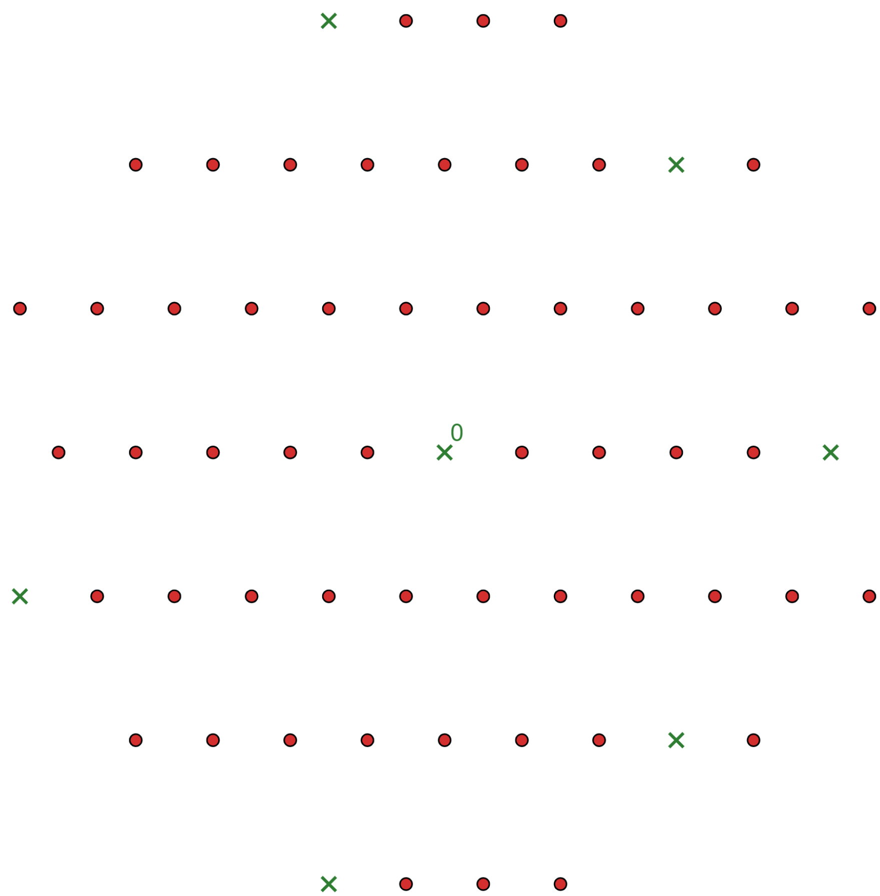

Figure 11: Optimal solution to $r(\mathcal{B}_p \cap \mathbb{Z}^2, a, b, c)$ for $p = 3$, $a = 0.51$, $b = 0.95$, and $c = 2.1$.

c	p	q($B_p(\mathbf{0}), a, b, c$)	r($B_p(\mathbf{0}), a, b, c$)	r($B_p(\mathbf{0}), a, b, c$) decimals
0.3	4	107	1	1
0.7	1	4	4/7	0.5714
0.7	2	14	14/25	0.56
1	2	10	10/25	0.4
1	3	22	22/61	0.3607
1.1	2	9	9/25	0.36
1.1	3	22	22/61	0.3607
1.3	2	7	7/25	0.28
1.3	3	14	14/61	0.2295
1.55	2	7	7/25	0.28
1.55	3	14	14/61	0.2295
1.6	2	5	5/25	0.2
1.6	3	11	11/61	0.1803
1.95	2	4	4/25	0.16
1.95	3	10	10/61	0.1639
2	2	4	4/25	0.16
2	3	9	9/61	0.1475
2	4	15	15/107	0.1402
2.03	3	8	8/61	0.1311
2.03	4	14	14/107	0.1308
2.1	3	7	7/61	0.1148
2.1	4	12	12/107	0.1121
2.2	3	7	7/61	0.1148
2.2	4	11	11/107	0.1028
2.45	3	7	7/61	0.1148
2.45	4	10	10/107	0.0935
2.5	3	7	7/61	0.1148
2.5	4	9	9/107	0.0841
2.6	3	7	7/61	0.1148
2.6	4	9	9/107	0.0841
2.8	3	5	5/61	0.0820
2.8	4	9	9/107	0.0841
2.9	3	5	5/61	0.0820
2.9	4	9	9/107	0.0841
2.951	3	5	5/61	0.0820
2.951	4	8	8/107	0.0748
3	3	5	5/61	0.0820
3	4	7	7/107	0.0654
3	5	11	11/167	0.0659
3.1	4	7	7/107	0.0654
3.1	5	10	10/167	0.0599
3.15	4	7	7/107	0.0654
3.15	5	10	10/167	0.0599
3.2	4	7	7/107	0.0654
3.2	5	9	9/167	0.0539

Table 2: Occupancy rates for finite $\mathcal{S}$ for $a = 0.51$ and $b = 0.95$.

c	p	q($B_p(\mathbf{0}), a, b, c$)	r($B_p(\mathbf{0}), a, b, c$)	r($B_p(\mathbf{0}), a, b, c$) decimals
0.7	1	3	1	1
1.1	1	2	2/3	0.6667
1.1	2	9	9/15	0.6
1.1	3	16	16/29	0.5517
1.5	2	7	7/15	0.4667
1.5	3	10	10/29	0.3448
1.9	2	5	5/15	0.3333
1.9	3	10	10/29	0.3448
2.1	2	4	4/15	0.2667
2.1	3	7	7/29	0.2414
2.1	4	11	11/53	0.2075
2.3	3	5	5/29	0.1724
2.3	4	9	9/53	0.1698
2.7	3	5	5/29	0.1724
2.7	4	9	9/53	0.1698
2.9	3	4	4/29	0.1379
2.9	4	7	7/53	0.1321
3	3	4	4/29	0.1379
3	4	7	7/53	0.1321
3	5	9	9/79	0.1139
3.1	4	7	7/53	0.1321
3.1	5	9	9/79	0.1139
3.5	4	7	7/53	0.1321
3.5	5	8	8/79	0.1013
3.61	4	6	6/53	0.1132
3.61	5	8	8/79	0.1013
3.7	4	5	5/53	0.0943
3.7	5	7	7/79	0.0886
3.95	4	5	5/53	0.0943
3.95	5	7	7/79	0.0886
4	4	5	5/53	0.0943
4	5	7	7/79	0.0886
4	6	9	9/111	0.0811
4.2	5	7	7/79	0.0886
4.2	6	9	9/111	0.0811
4.5	5	5	5/79	0.0633
4.5	6	7	7/111	0.0631
4.7	5	5	5/79	0.0633
4.7	6	7	7/111	0.0631
5	5	5	5/79	0.0633
5	6	7	7/111	0.0631
5	7	8	8/149	0.0537
5.03	6	7	7/111	0.0631
5.03	7	8	8/149	0.0537

Table 3: Occupancy rates for finite $\mathcal{S}$ for $a = 1$ and $b = 1$.

c	p	q($B_p(\mathbf{0}), a, b, c$)	r($B_p(\mathbf{0}), a, b, c$)	r($B_p(\mathbf{0}), a, b, c$) decimals
0.7	1	7	1	1
1.5	2	7	7/19	0.3684
1.5	3	13	13/37	0.3513
1.8	2	7	7/19	0.3684
1.8	3	10	10/37	0.2703
2.5	3	7	7/37	0.1892
2.5	4	9	9/61	0.1475
2.8	3	7	7/37	0.1892
2.8	4	7	7/61	0.1148
3.2	4	7	7/61	0.1148
3.2	5	8	8/91	0.0879
3.5	4	7	7/61	0.1148
3.5	5	8	8/91	0.0879
3.8	4	7	7/61	0.1148
3.8	5	7	7/91	0.0769
4.2	5	7	7/91	0.0769
4.2	6	8	8/127	0.0630
4.5	5	7	7/91	0.0769
4.5	6	7	7/127	0.0551
4.8	5	7	7/91	0.0769
4.8	6	7	7/127	0.0551
5	5	5	5/91	0.0549
5	6	7	7/127	0.0551
5	7	8	8/187	0.0428

Table 4: Occupancy rates for finite $\mathcal{S}$ for $a = 1$ and $b = \frac{1}{2}\sqrt{3}$.

c	p	q($B_p(\mathbf{0}), a, b, c$)	r($B_p(\mathbf{0}), a, b, c$)	r($B_p(\mathbf{0}), a, b, c$) decimals
0.7	1	9	1	1
0.8	1	5	5/9	0.5556
0.8	2	13	13/25	0.52
1.2	2	9	9/25	0.36
1.2	3	19	19/61	0.3115
1.5	2	6	6/25	0.24
1.5	3	13	13/61	0.2131
1.6	2	5	5/25	0.2
1.6	3	10	10/61	0.1639
2.1	3	9	9/61	0.1475
2.1	4	14	14/101	0.1386
2.2	3	9	9/61	0.1475
2.2	4	12	12/101	0.1188
2.3	3	7	7/61	0.1148
2.3	4	10	10/101	0.0990
2.6	3	7	7/61	0.1148
2.6	4	9	9/101	0.0891
2.9	3	7	7/61	0.1148
2.9	4	9	9/101	0.0891
2.95	3	5	5/61	0.0820
2.95	4	7	7/101	0.0693
3.1	4	7	7/101	0.0693
3.1	5	12	12/161	0.0743
3.2	4	7	7/101	0.0693
3.2	5	9	9/161	0.0559
3.6	4	7	7/101	0.0693
3.6	5	9	9/161	0.0559
3.7	4	5	5/101	0.0495
3.7	5	9	9/161	0.0559

Table 5: Occupancy rates for finite $\mathcal{S}$ for $a = 1$ and $b = \frac{1}{2}$.

5.4 Solutions to $r(\mathcal{S}_\infty, a, b, c)$

In Table 6, we show the upper and lower bounds to $r(\mathcal{S}_\infty, a, b, c)$ for various a, b and c. Furthermore, we show the p that is used as the radius for the balls used to find the upper bound. In Figure 12, a visualisation of an optimal solution to $r(\mathcal{S}_\infty, a, b, c)$ is given.

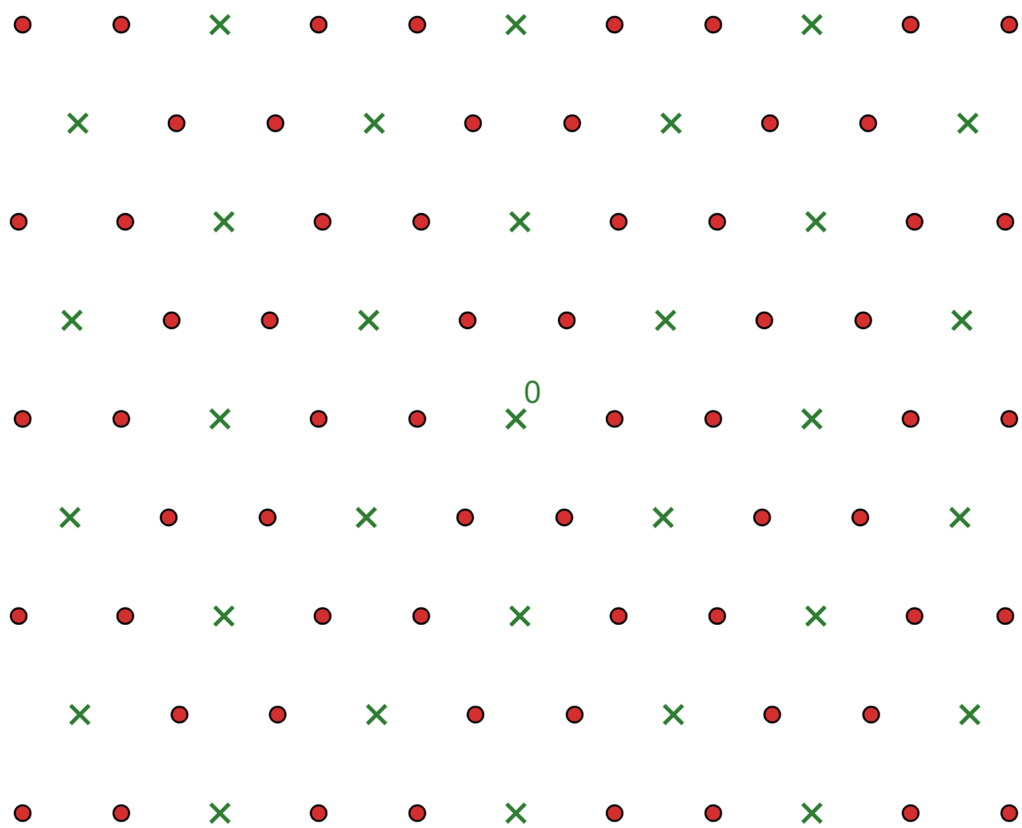

Figure 12: Optimal solution to $r(\mathcal{S}_\infty, a, b, c)$ for $a = b = 1$ and $c = 1.5$.

c	p	Upper Bound	Lower Bound
0.3	0.3	1	1
0.7	0.7	1/2	1/2
1	1	1/3	1/3
1.1	1.1	1/3	1/3
1.3	1.3	1/5	1/5
1.55	1.55	1/5	1/5
1.6	1.6	1/7	1/7
1.95	1.95	1/8	1/8
2	2	1/8	1/8
2.03	2.03	1/8	1/8
2.1	2.1	1/10	1/10
2.2	2.2	1/12	1/12
2.45	2.45	1/13	1/13
2.5	5	1/14.9984	1/15
2.6	2.6	1/15	1/15
2.8	2.8	1/16	1/16
2.9	2.9	1/17.7727	1/18
2.951	2.951	1/18	1/18
3	3	1/18	1/18
3.1	3.1	1/19.9231	1/20
3.15	3.15	1/20	1/20
3.2	3.2	1/21	1/21

(a) $a = 0.51$, $b = 0.95$

c	p	Upper Bound	Lower Bound
0.7	0.7	1	1
1.1	1.1	1/2	1/2
1.5	1.5	1/3	1/3
1.9	1.9	1/4	1/4
2.1	2.1	1/6	1/6
2.3	2.3	1/7	1/7
2.7	2.7	1/8	1/8
2.9	2.9	1/9	1/9
3	3	1/10	1/10
3.1	3.1	1/11.7143	1/12
3.5	3.5	1/12	1/12
3.61	3.61	1/13	1/13
3.7	3.7	1/15	1/15
3.95	3.95	1/16	1/16
4	4	1/17	1/17
4.2	4.2	1/19	1/19
4.5	4.5	1/20.8571	1/21
4.7	4.7	1/23.94	1/24
5	5	1/24	1/24
5.03	5.03	1/26	1/26

(b) $a = 1$, $b = 1$

c	p	Upper Bound	Lower Bound
0.7	0.7	1	1
1.5	1.5	1/3	1/3
1.8	1.8	1/4	1/4
2.5	2.5	1/7	1/7
2.8	2.8	1/9	1/9
3.2	3.2	1/12	1/12
3.5	3.5	1/13	1/13
3.8	3.8	1/16	1/16
4.2	4.2	1/19	1/19
4.5	4.5	1/21	1/21
4.8	4.8	1/25	1/25
5	5	1/27	1/27

(c) $a = 1$, $b = \frac{1}{2}\sqrt{3}$

c	p	Upper Bound	Lower Bound
0.7	0.7	1	1
0.8	0.8	1/2	1/2
1.2	1.2	1/4	1/4
1.5	1.5	1/5	1/5
1.6	1.6	1/8	1/8
2.1	2.1	1/9	1/9
2.2	2.2	1/10	1/10
2.3	2.3	1/12	1/12
2.6	2.6	1/15	1/15
2.9	2.9	1/15	1/15
2.95	2.95	1/18	1/18
3.1	3.1	1/19.75	1/20
3.2	3.2	1/22.931	1/23
3.6	3.6	1/24	1/24
3.7	3.7	1/27.8571	1/28

(d) $a = 1$, $b = \frac{1}{2}$

Table 6: Upper and lower bounds to $r(\mathcal{S}_\infty, a, b, c)$.

5.5 Interpretation of the Results

From the results it becomes apparent that the distance rules have a huge impact on the occupancy rates of theatres. For very small forbidden distances for which the forbidden zone consists of more than just one seat, an occupancy rate of 1/2 can be achieved, as we can see in Table 6. However, when we look further into the results, we see that the occupancy rate decreases very quickly for increasing forbidden distances. For instance, a theatre with the row and seat distance of the Music Building Eindhoven with a forbidden distance of 1.5 meters has an occupancy rate of around 28% for a small circular theatre with a radius of 2 meters, and the lower bound is merely 20%.

6 Discussion

Although the methods used in this report have been proven successful to solve the problem statement, there are still a number of things that could be improved, or are relevant for future research.

Firstly, a number of methods have a rather large computation time. Especially the cutting-plane method, explained in Section 3.4, can take several minutes and in some rare cases up to several hours to reach a solution, the total duration depending on the forbidden distance and the corresponding forbidden zone. When trying to find a solution for a limited amount of choices for a, b and c, this is manageable, but when trying to find solutions for a larger variety of selections for these parameters, this can become a problem, especially if a greater variety of larger forbidden distances are included. Thus, if one would be interested in calculating optimal seating arrangements for infinitely large theatres and large forbidden zones, it might be worthwhile to re-evaluate the methods to find the bounds, in order to save time. This could be relevant as well if one tries to find optimal seating arrangements for very large finite theatres.

Secondly, the upper and lower bounds to $r(\mathcal{S}_\infty, a, b, c)$ are not equal for all forbidden distances c. This might have several causes. For instance, it could be that there is an optimal solution that cannot be spanned by two vectors. In that case, the method for finding the lower bound is not good enough, since it relies on two vectors that span a grid. More likely, however, is that in some cases, the region $\mathcal{B}_p(\mathbf{0}) \cap \mathbb{Z}^2$ used to find vector a is not large enough to find the upper bound. In that case, we might find an upper bound equal to the lower bound if we use a larger p. How large p would have to be for which distance could be a separate research topic. In this paper, however, the previously discussed large computation times has prevented the calculation of a for larger p, perhaps such that all lower and upper bounds would have been equal.

Lastly, a number of suggestions that are not considered in current research, but might still be relevant of interesting for future research can be made. In the Netherlands, people of one household do not have to maintain distance from each other (*Corona en regels voor afstand houden — Coronavirus COVID-19 — Rijksoverheid.nl*, n.d.). This has not been taken into account in this paper, but would improve the relevance of the research. Another interesting idea that extends these exceptions to the distance rules and that has been included in previous research (Blom et al., 2020), and has proven to significantly improve the amount of people being able to attend a certain theatre performance, is to give the same show twice consecutively, for different audiences. This allows for a larger part of the theatre to be used.

All in all, there are several possibilities to improve or extend current research. Still, this paper provides satisfactory solutions to the problem statement, and the methods to find them. In doing so, a good basis is given that allows to find optimal seating arrangements for theatres of general size and shape.

7 Conclusion

In this report, we have investigated the occupancy rates of theatres of a general size or shape. Using several linear optimisation methods, we have been able to find these occupancy rates for both finite theatres, and, in most cases, for infinitely large theatres, which provide a lower bound for finite theatres. These methods are applicable for any combination of the parameters a, b and c, and any set of seats $\mathcal{S}$. In the results, the solutions for a number of examples are shown. We can see that the distance constraint has a huge impact on the occupancy rates of theatres, lowering the rate of seats that can be filled by at least 50% for an infinitely large theatre and a small forbidden distance, and even more for larger forbidden distances. To conclude, all the methods needed to find solutions to the objectives have been provided, and in most cases have proven successful to find results. In COVID times, these results are relevant and applicable for the improvement of occupancy in theatres while practicing social distancing.

References

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Appendix

This appendix shows the Sagemath scripts that were used to find the results.

```
c1 = 0.51
c2 = 0.95
c = 2.5
mf = 2
def d(s,t):
    return sqrt((c1*(s[1]-t[1]+1/2*(s[0]-t[0])))^2+(c2*(s[0]-t[0]))^2)

B = {(x,y) for x in range(-40,41) for y in range(-40,41)}
C = {(0.5*x,0.5*y) for x in range(-32,33) for y in range (-32,33)}
Czone = {s for s in C if d(s,[0,0]) <= c*mf}
S = {s for s in B if d(s,[0,0]) <= c*mf}
T = [s for s in S if d(s,[0,0])<= c/2]


print(S)
print(len(S))

#CALCULATING THE DISTANCE BETWEEN ANY SEAT AND SEAT 0
Distances = []
for s in S:
    z = d(s,[0,0])
    w = round(z,4)
    p = [w,s]
    Distances.append(tuple([p]))
Distances.sort()
print(Distances)

#SOLUTION FOR FINITE S AND INITIALIZING THE CUTTING-PLANE METHOD
print('Make MIP1')
print(len(S), len(T), round(len(S)/len(T),2))

MIP1 = MixedIntegerLinearProgram()
a = MIP1.new_variable(nonnegative=True)
for s in S:
    MIP1.add_constraint(a[s]<=1)
MIP1.set_objective(sum([a[s] for s in S]))


print('Objective Value: {}'.format(MIP1.solve()))
for i, v in sorted(MIP1.get_values(a).items()):
    print(f'a_{i} = {v}')

print('Make MIP2')
```

```
MIP2 = MixedIntegerLinearProgram()
x = MIP2.new_variable(binary=True)
#FIRST ADDITIONAL CONSTRAINT
for s in S:
    MIP2.add_constraint(sum(x[t] for t in S if d(s,t)<=c/2)<=1)
#SECOND ADDITIONAL CONSTRAINT
for s in Czone:
    MIP2.add_constraint(sum(x[t] for t in S if d(s,t)<=c/2)<=1)
#ORIGINAL CONSTRAINT
for s in S:
    for t in S:
        if t != s:
            if d(s,t) <= c:
                MIP2.add_constraint(x[s]+x[t]<=1)
MIP2.set_objective(sum(x[v] for v in S))
q=MIP2.solve()
print(len(S), round(len(S)/q,2),round(q,2))
print('Done')
```

```
print('Objective Value: {}'.format(MIP2.solve()))
for i, v in sorted(MIP2.get_values(x).items()):
     print(f'x_{i} = {v}')

#FINDING THE SOLUTION FOR INFINITE THEATRES
while True:
    p1 = MIP1.solve()
    a_sol = MIP1.get_values(a)
    MIP2.set_objective(sum([a_sol[n]*x[n] for n in S]))
    p2 = MIP2.solve()
    print(round(p1,4),round(p2,4))
    if p2<=1.0001:
        break
    x_sol=MIP2.get_values(x)
    print(sum(v for s, v in x_sol.items()))
    MIP1.add_constraint(sum([x_sol[s]*a[s] for s in S])<=1)

#VIEW THE FINAL CONSTRAINT VECTOR
for s in sorted(a_sol):
    if a_sol[s]>0.001:
        print(s, round(a_sol[s]*1,4))

#CODE FOR LOWER BOUND, BASED ON LEMMA 1 AND LEMMA 2
distance = 3.7
distanceseat = 1
distancerow = 1/2
```

```
l2 = (2*distancerow*distanceseat)/((sqrt(3))*distance)*(distance/distancerow+2)*
    (distance/distanceseat+1)
L = {s for s in B if d(s,[0,0]) <= l2}

def f(w,x,y,z):
    return abs(w*z-x*y)
def inp(w,x,y,z):
    return abs((distancerow^2+1/4*distanceseat^2)*w*y+1/2*distanceseat^2*w*z+1/2*
    distanceseat^2*x*y+distanceseat^2*x*z)

LegalDistances = []
LowerBound = []

for s in L:
    for t in L:
        d2 = d([0,0],s)
        d3 = d(t,[0,0])
        if d3>distance:
            if d2>=d3:
                if inp(s[0],s[1],t[0],t[1])<=1/2*(d3^2):
                    LegalDistances.append(tuple([s,t]))
                    LowerBound.extend([abs(s[0]*t[1]-t[0]*s[1])])

LB=min(LowerBound)
Basis={(s,t) for (s,t) in LegalDistances if abs(s[0]*t[1]-t[0]*s[1])<=LB}
print(Basis)
print(LB)
print(l2)
```