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Maximal regularity of the heat evolution equation on spatial local spaces and application to a singular limit problem of the Keller–Segel system

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Abstract

We consider the singular limit problem for the Cauchy problem of the (Patlak–) Keller–Segel system of parabolic-parabolic type. The problem is considered in the uniformly local Lebesgue spaces and the singular limit problem as the relaxation parameter τ goes to infinity, the solution to the Keller–Segel equation converges to a solution to the drift-diffusion system in the strong uniformly local topology. For the proof, we follow the former result due to Kurokiba–Ogawa [20–22] and we establish maximal regularity for the heat equation over the uniformly local Lebesgue and Morrey spaces which are non-UMD Banach spaces and apply it for the strong convergence of the singular limit problem in the scaling critical local spaces.

1 Keller–Segel system and drift-diffusion equation

We consider the Cauchy problem of the (parabolic–parabolic) (Patlak–)Keller–Segel system on n-dimensional Euclidean space $\mathbb{R}^n$:

$$\begin{cases} \partial_t u_\tau - \Delta u_\tau + \nabla \cdot (u_\tau \nabla \psi_\tau) = 0, & t > 0,\ x \in \mathbb{R}^n, \\ \dfrac{1}{\tau}\partial_t \psi_\tau - \Delta \psi_\tau + \lambda \psi_\tau = u_\tau, & t > 0,\ x \in \mathbb{R}^n, \\ \quad u_\tau(0,x) = u_0(x),\ \ \psi_\tau(0,x) = \psi_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $n \geq 3$, $\lambda > 0$, $\tau > 0$ and (u_0, ψ_0) are given initial data. The problem (1.1) was introduced by Patlak [33] and later on, Keller–Segel [13] rediscovered it for describing

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a model of chemotactic aggregation of microorganisms. In the chemotaxis model, $u_\tau = u_\tau(t,x)$ and $\psi_\tau = \psi_\tau(t,x)$ denote the unknown density of microorganisms and the distribution of the chemical substances, respectively. The parameter $\tau > 0$ is the relaxation time coefficient, and stands for the ratio of the relative speed of chemical substances. By passing $\tau \to \infty$, the limiting functions

$$\begin{cases} \lim_{\tau\to\infty} u_\tau(t,x) = u(t,x), \\ \lim_{\tau\to\infty} \psi_\tau(t,x) = \psi(t,x) \end{cases} \tag{1.2}$$

formally solve the Cauchy problem of the drift-diffusion equation:

$$\begin{cases} \partial_t u - \Delta u + \nabla\cdot(u\nabla\psi) = 0, & t>0,\ x\in\mathbb{R}^n, \\ -\Delta\psi + \lambda\psi = u, & t>0,\ x\in\mathbb{R}^n, \\ u(0,x) = u_0(x), & x\in\mathbb{R}^n, \end{cases} \tag{1.3}$$

and the problem (1.3) is often referred as the parabolic-elliptic Keller–Segel system. In this paper, we consider the singular limit problem (1.2) in uniformly local spaces.

Both of the Cauchy problems (1.1) and (1.3) show a (semi-)scaling invariant structure and the invariant scaling is given by the following: For $\mu > 0$,

$$\begin{cases} u_\mu(t,x) \equiv \mu^2 u(\mu^2 t, \mu x), \\ \psi_\mu(t,x) \equiv \psi(\mu^2 t, \mu x). \end{cases} \tag{1.4}$$

Under the above scaling, the systems are invariant if $\lambda = 0$. According to such a structure, the problem is typically considered as the scaling invariant spaces such as the Bochner–Lebesgue spaces $L^\theta(0,T;L^p(\mathbb{R}^n))$ with $2/\theta + n/p = 1$ for u and $L^\sigma(0,T;L^q(\mathbb{R}^n))$ with $2/\sigma + n/q = 0$ for ψ. Indeed, there are many results for the existence and the well-posedness of the problems (1.1) or (1.3) on the critical space (see Biler [1], Biler–Cannone–Guerra–Karch [4], Kurokiba–Ogawa [18, 19], Corrias–Perthame [6], Kozono–Sugiyama [14, 15], Iwabuchi [9], Iwabuchi–Nakamura [10], see for the bounded domain case Biler [2], Nagai [27], Nagai–Senba–Yoshida [28], and Senba–Suzuki [37]).

Among others, the singular limit problem (1.2) was considered by Raczyński [34] and Biler–Brandolese [3] in the pseudo-measure $\mathcal{PM}^s(\mathbb{R}^n) = \left\{f \in \mathcal{S}'(\mathbb{R}^n);\ \hat{f} \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \text{ and } \|f\|_{\mathcal{PM}^s} = \||\xi|^s\hat{f}\|_\infty < +\infty\right\}$ and the Lorentz spaces $L^{p,\infty}(\mathbb{R}^n)$, respectively. They showed the singular limit (1.2) for the two-dimension case under the condition of $u_0 \equiv 0$ and $\lambda = 0$. In [34], the same problem (1.2) was considered over the pseudo-measure space $\mathcal{PM}^0(\mathbb{R}^2)$ defined above. Lamarié-Rieusset [23] extended these results to the homogeneous Morrey space. Kurokiba–Ogawa [20–22] showed the singular limit problem (1.2) in the scaling critical Bochner–Lebesgue spaces for the large initial data and showed the appearance of the initial layer in the two and higher dimensional cases. Those results [20–22] cover the finite mass case, where the positive solution preserves the total mass, while the other results are not covering the finite mass case.

On the other hand, both the systems have the spatially non-local structure and it is interesting to consider the well-posedness of the problems in spatially local function classes. The second author showed that the Cauchy problem (1.3) is time locally well-posed in the uniformly local Lebesgue space in [38]. For $1 \le p < \infty$, let the uniformly local Lebesgue space is defined as follows:

$$L^p_{\mathrm{ul}}(\mathbb{R}^n) \equiv \left\{ f \in L^p_{\mathrm{loc}}(\mathbb{R}^n);\ \|f\|_{L^p_{\mathrm{ul}}} \equiv \sup_{x\in\mathbb{R}^n} \left(\int_{B_1(x)} |f(y)|^p\, dy \right)^{\frac{1}{p}} < +\infty \right\},$$

where $B_1(x)$ denotes the open ball in $\mathbb{R}^n$ with center x and radius 1. In the case of $\lambda > 0$, the solution ψ to the second equation in (1.3) is written by using the Bessel potential and it enables us to treat the Cauchy problem in such local function spaces. Analogous result is also obtained by Cygan–Karch–Krawczyk–Wakui [7], where they consider the stability of a constant solution. A natural question for the Keller–Segel system under such a setting is whether the singular limit problem (1.2) can be justified in such a locally uniform class of solutions. Namely such a singular limit problem also remains valid in the local uniform class that reflect a spatial structure of a solution to both problems.

Meanwhile the singular limit was established in the scaling invariant classes in [20–22] by applying the maximal regularity estimate for the Cauchy problem of the heat equation with $\lambda \ge 0$. Maximal regularity is a useful tool to see that the time local well-posedness of the problems (1.1) and (1.3) and it provides useful local estimates which are independent of the parameter $\tau > 0$. Hence it allowed us to show that the limit exists and it solves the limiting problem (1.3) in the critical Lebesgue space. Following basic method used in [20–22], we employ maximal regularity for the heat equation to show the singular limit problem (1.2).

In order to derive maximal regularity on the uniformly local Lebesgue space, we consider a slightly general setting, namely, the local (inhomogeneous) Morrey spaces $M^p_q(\mathbb{R}^n)$ and its real interpolation spaces, i.e., the local Besov–Morrey spaces $N^s_{p,q,\sigma}(\mathbb{R}^n)$ which are extensions of the uniformly local Lebesgue space, and extensively developed by Kozono–Yamazaki [16].

Definition (The (local) Morrey space). For $1 \le q \le p < \infty$, we define the (local) Morrey space $M^p_q(\mathbb{R}^n)$ with the norm

$$\|f\|_{M^p_q} \equiv \sup_{\substack{x\in\mathbb{R}^n,\\ 0<R\le 1}} |B_R|^{-\left(\frac{1}{q}-\frac{1}{p}\right)} \left(\int_{B_R(x)} |f(y)|^q\, dy \right)^{\frac{1}{q}}, \tag{1.5}$$

where $|B_R|$ is the Lebesgue measure of $B_R(x)$. We also introduce the completion of bounded uniformly continuous space $(BUC(\mathbb{R}^n))$ as

$$\mathcal{M}^p_q(\mathbb{R}^n) \equiv \overline{BUC(\mathbb{R}^n)}^{\|\cdot\|_{M^p_q}}.$$

The Sobolev spaces based on those Morrey spaces are analogously defined: For $1 \le p, q < \infty$, $s \in \mathbb{R}$,

$$M_q^{s,p}(\mathbb{R}^n) = \Big\{ f \in \mathcal{S}^*(\mathbb{R}^n);\ \langle\nabla\rangle^s f \in M_q^p(\mathbb{R}^n)\Big\},$$

where $\langle\nabla\rangle^s f = \mathcal{F}^{-1}\big[(1+|\xi|^2)^{s/2}\widehat{f}(\xi)\big]$ denotes the Bessel potential of order $s \in \mathbb{R}$ and $\mathcal{F}^{-1}$ stands for the inverse Fourier transform. The corresponding space $\mathcal{M}_q^{s,p}(\mathbb{R}^n)$ is also defined in a similar way. Since by the definition $M_p^p(\mathbb{R}^n) = L^p_{\mathrm{ul}}(\mathbb{R}^n)$ (and thus $\mathcal{M}_p^p(\mathbb{R}^n) = \mathcal{L}^p_{\mathrm{ul}}(\mathbb{R}^n) \equiv \overline{BUC(\mathbb{R}^n)}^{\|\cdot\|_{L^p_{\mathrm{ul}}}}$) for all $1 \le p \le \infty$, the local Morrey space is a generalization of the uniformly local Lebesgue space.

We note that the homogeneous Morrey class is defined by taking the supremum of R over $(0,\infty)$ which may denoted by $\dot{M}_q^p(\mathbb{R}^n)$ (and analogously $\dot{\mathcal{M}}_q^p(\mathbb{R}^n)$). We also remark that the local Morrey space $M_q^p(\mathbb{R}^n)$ is neither reflexive nor separable[1] and we may avoid difficulty to treat a non-C_0-semigroup of the heat evolution operator in $\mathcal{M}_q^p(\mathbb{R}^n)$. We then introduce a real interpolation space called as the Besov–Morrey space, which originally goes back to Netrusov [29] (cf. Kozono–Yamazaki [16], Nogayama–Sawano [30]). Let $\{\phi_j\}_{j\in\mathbb{Z}}$ be the Littlewood–Paley dyadic decomposition of unity. We set ψ by means of $\{\phi_j\}_{j\in\mathbb{Z}_-}$ as

$$\hat{\psi}(\xi) \equiv 1 - \sum_{j\le 0} \hat{\phi}_j(\xi)$$

and we often write $\phi_{\hat{0}} \equiv \psi$.

Definition (The Besov–Morrey and the Lizorkin–Triebel–Morrey spaces). For $1 \le q \le p < \infty$, $1 \le \sigma \le \infty$, $s \in \mathbb{R}$, we define the Besov–Morrey space $N^s_{p,q,\sigma}(\mathbb{R}^n)$ with the norm

$$\|f\|_{N^s_{p,q,\sigma}} \equiv \|\psi * f\|_{M^p_q} + \left\| \Big\{2^{sj}\|\phi_j * f\|_{M^p_q}\Big\}_{j\in\mathbb{N}} \right\|_{l^\sigma(\mathbb{N})}.$$

and the Lizorkin–Triebel–Morrey space $E^s_{p,q,\sigma}(\mathbb{R}^n)$ with the norm

$$\|f\|_{E^s_{p,q,\sigma}} \equiv \|\psi * f\|_{M^p_q} + \left\| \left\| \Big\{2^{sj}|\phi_j * f|\Big\}_{j\in\mathbb{N}} \right\|_{l^\sigma(\mathbb{N})} \right\|_{M^p_q},$$

where the convolution $f * g$ includes a correction of constant $(2\pi)^{-n/2}$.

For the Cauchy problem of the incompressible Navier–Stokes equations, Giga–Miyakawa [8] and Kato [12] showed the existence of a unique strong solution on the homogeneous Morrey space and Maekawa–Terasawa [24] constructed a mild solution on uniformly local Lebesgue spaces. Besides, Kozono–Yamazaki [16] introduced the Besov–Morrey space $N^s_{p,q,\sigma}$ by using the real interpolation theory and applied for the

[1] These are because of the same reason for the case of the uniformly local Lebesgue space.

well-posedness issue of the incompressible Navier–Stokes equations:

$$N^s_{p,q,\sigma} \equiv \left(\langle\nabla\rangle^{-s_1} M^p_q,\, \langle\nabla\rangle^{-s_2} M^p_q\right)_{\theta,\sigma},$$

where $s = (1-\theta)s_1 + \theta s_2$ and $0 \le \theta \le 1$. We should like to note that Mazzucato [25, 26] showed that $E^0_{p,q,2}(\mathbb{R}^n) \simeq M^p_q(\mathbb{R}^n)$ by the Littlewood–Paley theorem (see Proposition 2.3, below).

The systems (1.1) and (1.3) are invariant under the scaling transformation (1.4) and the invariant Bochner class (the Serrin class) is given by after (1.4). In this paper, we employ the corresponding invariant class and define the admissible exponents for the scaling critical spaces:

Definition (The admissible pair). Pairs of the exponents (θ, p) and (σ, r) are called the scaling invariant(Serrin) admissible if $\max\{r, \theta\} \le \sigma < \infty$, and

$$\begin{cases} u \in L^\theta(\mathbb{R}_+; M^p_q(\mathbb{R}^n)), & \dfrac{2}{\theta} + \dfrac{n}{p} = 2, \quad \dfrac{n}{2} < p \le \theta,\ 2 \le \theta < \infty, \\ \nabla\psi \in L^\sigma(\mathbb{R}_+; M^r_\alpha(\mathbb{R}^n)), & \dfrac{2}{\sigma} + \dfrac{n}{r} = 1, \quad 2 \le n < r \le \sigma < \infty. \end{cases} \tag{1.6}$$

We define mild solutions to (1.1) and (1.3):

Definition (The mild solution). Let $\tau > 0$, $1 \le p, r < \infty$, $1 \le q_1 \le p$, and $1 \le \alpha_1 \le r$. For initial data $(u_0, \nabla\psi_0) \in M^p_{q_1}(\mathbb{R}^n) \times M^r_{\alpha_1}(\mathbb{R}^n)$, (u_τ, ψ_τ) is a mild solution to (1.1) if the following integral equation is solved:

$$\begin{cases} u_\tau(t) = e^{t\Delta}u_0 - \displaystyle\int_0^t \nabla e^{(t-s)\Delta}\cdot(u_\tau(s)\nabla\psi_\tau(s))\,ds, \\ \psi_\tau(t) = e^{t\tau(\Delta-\lambda)}\psi_0 + \displaystyle\int_0^t e^{(t-s)\tau(\Delta-\lambda)}\tau u_\tau(s)\,ds \end{cases} \tag{1.7}$$

in $C(I; M^p_{q_1}(\mathbb{R}^n)) \times C(I; M^{1,r}_{\alpha_1}(\mathbb{R}^n))$.

Definition (The mild solution). Let $1 \le p, r < \infty$, $1 \le q_1 \le p$, and $1 \le \alpha_1 \le r$. For initial data $u_0 \in M^p_{q_1}(\mathbb{R}^n)$, (u, ψ) is a mild solution to (1.3) if the following integral equation is solved:

$$\begin{cases} u(t) = e^{t\Delta}u_0 - \displaystyle\int_0^t \nabla e^{(t-s)\Delta}\cdot(u(s)\nabla\psi(s))\,ds, \\ \psi(t) = (\lambda-\Delta)^{-1}u(t) = \displaystyle\int_0^\infty e^{s(\Delta-\lambda)}u(t)\,ds \end{cases}$$

in $C(I; M^p_{q_1}(\mathbb{R}^n)) \times C(I; M^{1,r}_{\alpha_1}(\mathbb{R}^n))$.

Our first result of this paper is the time local well-posedness for the Keller–Segel system (1.1) in the Besov–Morrey spaces:

Proposition 1.1 (The local well-posedness). *Let $n \ge 3$, $\tau, \lambda > 0$, $\tau \ge 1$ and (θ, p) and (σ, r) be admissible pair defined in (1.6) with $\theta < \sigma$. Suppose that $1 < q_0 \le n/2$,*

$q_0 \le q_1 \le p$, $1 < \alpha_0 \le n$, $\alpha_0 \le \alpha_2 \le r$ satisfy

$$\frac{2q_0}{n} = \frac{q_1}{p} = \frac{\alpha_0}{n} = \frac{\alpha_1}{r} \quad \text{and} \quad \frac{1}{q_0} + \frac{1}{\alpha_1} \le 1. \tag{1.8}$$

Assume that $(u_0, \psi_0) \in \mathcal{M}^{n/2}_{q_0}(\mathbb{R}^n) \cap N^{-2/\theta}_{p,q_1,\theta}(\mathbb{R}^n) \times \mathcal{M}^{1,n}_{\alpha_0}(\mathbb{R}^n) \cap N^{1-2/\sigma}_{r,\alpha_1,\sigma}(\mathbb{R}^n)$. Then there exists $T = T(u_0, \psi_0) > 0$ such that the unique mild solution (u_τ, ψ_τ) to (1.1) exists and satisfies

$$\begin{cases} u_\tau \in C([0,T); \mathcal{M}^{\frac{n}{2}}_{q_0}(\mathbb{R}^n)) \cap L^\theta((0,T); \mathcal{M}^p_{q_1}(\mathbb{R}^n)), \\ \psi_\tau \in C([0,T); \mathcal{M}^{1,n}_{\alpha_0}(\mathbb{R}^n)) \cap L^\sigma((0,T); \mathcal{M}^{1,r}_{\alpha_1}(\mathbb{R}^n)). \end{cases}$$

If the first condition in (1.8) is satisfied by $n/2 = q_0$, then the result in Proposition 1.1 implies the local well-posedness of the problem (1.1) in the uniformly local Lebesgue spaces, i.e., $(u_\tau, \nabla\psi_\tau) \in \mathcal{L}^{n/2}_{\mathrm{ul}}(\mathbb{R}^n) \times \mathcal{L}^n_{\mathrm{ul}}(\mathbb{R}^n)$. The assumption of the initial data in Proposition 1.1 is rather stringent than the one appeared in [38]. However, the most importantly, the existence time $T = T(u_0, \psi_0) > 0$ depends only on the initial data but not on the parameter $\tau \ge 1$.

Analogously we obtain the time local well-posedness for the Cauchy problem of to the limiting drift-diffusion system (1.3) in the same function class as above:

Proposition 1.2 (The local well-posedness). *Let $n \ge 3$, $\lambda > 0$, and (θ, p) and (σ, r) be admissible pair defined in (1.6) with $\theta \le \sigma$ and $1 < q_0 \le n/2$, $q_0 \le q_1 \le p$, $1 < \alpha_0 \le n$, $\alpha_0 \le \alpha_1 \le r$ satisfy (1.8). Assume that $u_0 \in \mathcal{M}^{n/2}_{q_0}(\mathbb{R}^n) \cap N^{-2/\theta}_{p,q_1,\theta}(\mathbb{R}^n)$. Then there exists $T = T(u_0) > 0$ such that the unique mild solution (u, ψ) to (1.3) exists and satisfies*

$$\begin{cases} u \in C([0,T); \mathcal{M}^{\frac{n}{2}}_{q_0}(\mathbb{R}^n)) \cap L^\theta((0,T); \mathcal{M}^p_{q_1}(\mathbb{R}^n)), \\ \psi \in C([0,T); \mathcal{M}^{1,n}_{\alpha_0}(\mathbb{R}^n)) \cap L^\sigma((0,T); \mathcal{M}^{1,r}_{\alpha_1}(\mathbb{R}^n)). \end{cases}$$

Proposition 1.2 is a distinct version of the time local well-posedness for the drift-diffusion system (1.3). Indeed, one can find a well-posedness result more general assumption the initial data (cf. [38]). The added regularity assumption on the data is required for applying maximal regularity.

The limiting process by $\tau \to \infty$ corresponds to observing the large time behavior of only the second component ψ of the system (1.1) as $t \to \infty$. In general, the decay of a solution as $t \to \infty$ or the stability of a stationary solution to the Cauchy problem of a partial differential equation is necessary to avoid the initial disturbance from the spatial infinity. For example, the decay of the solution to the heat equation is obtained by the initial data that prevents the initial disturbance at the spatial infinity. In our case, however, the initial data is taken from the uniformly local spaces and the initial turbulence from the spatial infinity is fully included. Hence the singular limit generally is not expected under such a setting. Nevertheless, we may show the singular limit problem (1.2) by the density of the initial class of $\mathcal{L}^p_{\mathrm{ul}}(\mathbb{R}^n)$, the use of the space-time integral norm, and the application to the Lebesgue dominated convergence theorem.

According to such observations, a presence of the positive parameter $\lambda > 0$ in (1.1) is essential because it implies exponential decay of the potential term ψ in time variable. Such a decay property enables us to show the strong convergence of the singular limit problem under the locally integrable function class.

We now state our main result.

Theorem 1.3 *Let $n \geq 3$, $\tau, \lambda > 0$, (θ, p) and (σ, r) be admissible pairs defined in (1.6) with $\theta < \sigma$. Suppose that $1 < q_0 \leq n/2$, $q_0 \leq q_1 \leq p$, $1 < \alpha_0 \leq n$, $\alpha_0 \leq \alpha_1 \leq r$ satisfy (1.8). Assume that $(u_0, \psi_0) \in \mathcal{M}^{n/2}_{q_0}(\mathbb{R}^n) \cap N^{-2/\theta}_{p,q_1,\theta}(\mathbb{R}^n) \times \mathcal{M}^{1,n}_{\alpha_0}(\mathbb{R}^n) \cap N^{1-2/\sigma}_{r,\alpha_1,\sigma}(\mathbb{R}^n)$. Let (u_τ, ψ_τ) be a unique mild solution to (1.1) in*

$$\left(C(I; \mathcal{M}^{\frac{n}{2}}_{q_0}(\mathbb{R}^n)) \cap L^\theta(I; \mathcal{M}^p_{q_1}(\mathbb{R}^n))\right) \times \left(C(I; \mathcal{M}^{1,n}_{\alpha_0}(\mathbb{R}^n)) \cap L^\sigma(I; \mathcal{M}^{1,r}_{\alpha_1}(\mathbb{R}^n))\right),$$

where (θ, p) and (σ, r) are admissible pairs defined in (1.6) and $I = (0, T)$ with $0 < T < \infty$. Then the following holds:

(1) For the same initial data u_0, there exists a unique mild solution (u, ψ) to (1.3) in

$$\left(C(I; \mathcal{M}^{\frac{n}{2}}_{q_0}(\mathbb{R}^n)) \cap L^\theta(I; \mathcal{M}^p_{q_1}(\mathbb{R}^n))\right) \times \left(C(I; \mathcal{M}^{1,n}_{\alpha_0}(\mathbb{R}^n)) \cap L^\sigma(I; \mathcal{M}^{1,r}_{\alpha_1}(\mathbb{R}^n))\right).$$

(2) For any admissible pairs (θ, p) and (σ, r) defined in (1.6) with $\theta < \sigma$, it holds that

$$\lim_{\tau \to \infty} \left(\|u_\tau - u\|_{L^\theta(I; M^p_{q_1})} + \|\nabla\psi_\tau - \nabla\psi\|_{L^\sigma(I; M^r_{\alpha_1})} \right) = 0. \quad (1.9)$$

(3) For any $0 < t_0 < T$, set $I_{t_0} \equiv (t_0, T)$. Then it holds that

$$\lim_{\tau \to \infty} \left(\|u_\tau - u\|_{L^\infty(I_{t_0}; M^{\frac{n}{2}}_{q_0})} + \|\nabla\psi_\tau(t) - \nabla\psi(t)\|_{L^\infty(I_{t_0}; M^n_{\alpha_0})} \right) = 0. \quad (1.10)$$

On the other hand, for some small $t_1 > 0$, let

$$\eta_\tau(t) \equiv \chi_{[0,\tau^{-1}t_1]}(t)(\psi_0 - (\lambda - \Delta)^{-1}u_0)$$

and $\chi_{[a,b]}(t)$ be the characteristic function on $[a, b]$. Then it holds that

$$\sup_{t \in [0,\tau^{-1}t_1]} \|u_\tau(t) - u(t)\|_{M^{\frac{n}{2}}_{q_0}} + \sup_{t \in [0,\tau^{-1}t_1]} \|\nabla\psi_\tau(t) - \nabla\psi(t) - \nabla\eta_\tau(t)\|_{M^n_{\alpha_0}} \to 0 \quad (1.11)$$

as $\tau \to \infty$, in other words, ψ_τ shows the initial layer $\psi_0 - (\lambda - \Delta)^{-1}u_0$ as $\tau \to \infty$.

As is stated in remark after Proposition 1.1, the above result for the singular limit problem also shows the corresponding result in the uniformly local Lebesgue space for $(u_\tau, \nabla\psi_\tau) \in \mathcal{L}^{n/2}_{\mathrm{ul}}(\mathbb{R}^n) \times \mathcal{L}^n_{\mathrm{ul}}(\mathbb{R}^n)$ due to the equivalence of the function classes.

The proof of the singular limit problem in Theorem 1.3 is based on maximal regularity for the Cauchy problem of the heat equation:

$$\begin{cases} \partial_t u - \mu \Delta u + \lambda u = f, & t > 0,\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n. \end{cases}$$

where $\lambda \geq 0$ and f and u_0 are given external force and initial data. The general theory of maximal regularity for the Cauchy problem of a parabolic equation is well established on function spaces satisfying the unconditional martingale differences (UMD). Since UMD Banach spaces are necessarily reflexive, maximal regularity in non-reflexive space requires distinct treatment. In particular, the uniformly local Lebesgue space is not reflexive by observing

$$L^p_{\mathrm{ul}}(\mathbb{R}^n) \simeq \ell^\infty\big(\mathbb{Z}^n; L^p(B_1(x_k))\big),$$

where $\ell^\infty(\mathbb{Z}^n)$ denotes a sequence space over the n-dimensional lattice point $x_k \in \mathbb{Z}^n$. Thus, maximal regularity for the heat equations on the uniformly local Lebesgue space requires independent argument. To show maximal regularity for the uniformly local Lebesgue space, we introduce the Besov–Morrey spaces and employ the real interpolation argument for proving maximal regularity (cf. [20, 31, 32]). After establishing maximal regularity we fully use the embedding relation between the Besov–Morrey space and the Lizorkin–Triebel–Morrey space (see Proposition 2.4) and the Littlewood-Paley theory obtained by Mazzucato [25, 26] to connect the Besov–Morrey space and the Morrey space. To this end, we use the smoothing properties of the heat evolution and the sub-suffixes of the Besov–Morrey spaces are fully improved (cf. Kozono–Yamazaki [16]) and this enables us to recover regularity of solution and convergence of the singular limit follows by an improved argument from [20] and [21]. Since $M^p_q(\mathbb{R}^n) = L^p_{\mathrm{ul}}(\mathbb{R}^n)$ for all $1 \leq p \leq q < \infty$, we complete the convergence of the singular limit in the scaling critical local spaces $M^p_q(\mathbb{R}^n)$ and hence $L^p_{\mathrm{ul}}(\mathbb{R}^n)$ as is seen below.

This paper is organized as follows. In the next section, we prepare properties of the Morrey and the Besov–Morrey spaces. In Sect. 3, we derive maximal regularity for the heat equation on the Besov–Morrey space. Section 4 is devoted to proving the well-posedness of the Cauchy problems of the parabolic-parabolic and the parabolic-elliptic Keller–Segel systems. In Sect. 5, we give proof of Theorem 1.3.

In the rest of paper, we use the following notation. Let $\hat{f}$ be the Fourier transformation of $f \in \mathcal{S}(\mathbb{R}^n)$:

$$\hat{f}(\xi) \equiv (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx.$$

$\mathbb{Z}^n$ denotes all the lattice point over $\mathbb{R}^n$. For $x \in \mathbb{R}^n$, $\langle x \rangle = (1+|x|^2)^{1/2}$ and $\langle \nabla \rangle^s f = (1-\Delta)^{s/2} f \equiv \mathcal{F}^{-1}\big[\langle \xi \rangle^s \widehat{f}(\xi)\big]$ is the Bessel potential of order $s \in \mathbb{R}$. For a various function space $X(\mathbb{R}^n)$ over $\mathbb{R}^n$, we abbreviate it as X such as $M^p_q = M^p_q(\mathbb{R}^n)$, $L^p_{\mathrm{ul}} = L^p_{\mathrm{ul}}(\mathbb{R}^n)$. The weak Lebesgue space for $1 \leq p < \infty$ is denoted by $L^p_{\mathrm{w}} = L^p_{\mathrm{w}}(\mathbb{R}^n)$.

Let B_λ be the Bessel potential defined by

$$B_\lambda(x) \equiv \frac{\lambda^{\frac{n}{2}-1}}{4\pi} \int_0^\infty e^{-\frac{\pi\lambda|x|^2}{s}} e^{-\frac{s}{4\pi}} s^{-\frac{n-2}{2}} \frac{ds}{s}. \tag{1.12}$$

For $f \in N^s_{p,q,\sigma}$, we use the simplified notation $\phi_{\hat{0}} * f \equiv \psi * f$ and the summation in the Besov–Morrey norm can be rewritten by

$$\begin{aligned}\|f\|_{N^s_{p,q,\sigma}} &= \|\psi * f\|_{M^p_q} + \left(\sum_{j=1}^\infty \left(2^{sj} \|\phi_j * f\|_{M^p_q} \right)^\sigma \right)^{\frac{1}{\sigma}} \\ &= \left(\sum_{j=\hat{0}}^\infty \left(2^{sj} \|\phi_j * f\|_{M^p_q} \right)^\sigma \right)^{\frac{1}{\sigma}}.\end{aligned}$$

2 Preliminaries

We first remark on the relation between the uniformly local Lebesgue space and the local Morrey spaces. Let $1 \le q \le p < \infty$. By the definition (1.5), we see that the following embeddings are continuous:

$$\begin{cases} L^q_{\mathrm{ul}} \supset M^p_q \supset L^p_{\mathrm{ul}}, & q < p, \\ M^p_q = L^q_{\mathrm{ul}}, & q \ge p. \end{cases}$$

Moreover, if $q_1 \le q_2$, then $M^p_{q_1} \supset M^p_{q_2}$ by the Hölder inequality. To see the equivalences $M^p_q = L^q_{\mathrm{ul}}$ for $q \ge p$, it follows from the definition (1.5) that

$$\begin{aligned}\|f\|_{M^p_q} &\equiv \sup_{\substack{x\in\mathbb{R}^n,\\ 0<R\le 1}} |B_R|^{-\left(\frac{1}{q}-\frac{1}{p}\right)} \left(\int_{B_R(x)} |f(y)|^q \, dy \right)^{\frac{1}{q}} \\ &\le |B_1|^{-\left(\frac{1}{q}-\frac{1}{p}\right)} \sup_{\substack{x\in\mathbb{R}^n,\\ 0<R\le 1}} \left(\int_{B_R(x)} |f(y)|^q \, dy \right)^{\frac{1}{q}} = C_n \|f\|_{L^q_{\mathrm{ul}}}.\end{aligned} \tag{2.1}$$

Note that (2.1) also implies $M^p_q \supset L^p_{\mathrm{ul}}$ if $q < p$ by regerding L^q_{ul} as L^p_{ul}. Conversely to see $M^p_q \subset L^q_{\mathrm{ul}}$ for $q \ge p$,

$$\begin{aligned}\|f\|_{L^q_{\mathrm{ul}}} &= C_n \sup_{x\in\mathbb{R}^n} |B_1|^{\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\int_{B_1(x)} |f(y)|^q\,dy\right)^{\frac{1}{q}} \\ &\le C_n \sup_{\substack{x\in\mathbb{R}^n,\\ 0<R\le 1}} |B_R|^{\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\int_{B_R(x)} |f(y)|^q\,dy\right)^{\frac{1}{q}} \\ &= \|f\|_{M^p_q}.\end{aligned} \tag{2.2}$$

The inequality (2.2) holds for the other case $q < p$ and hence the first embedding $L^q_{\mathrm{ul}} \supset M^p_q$ also holds.

Like in the uniformly local spaces, the Hölder type inequality also holds between the local Morrey spaces.

Proposition 2.1 (The Hölder type inequality). *Let $1 \le p_1, p_2 < \infty$, $1 \le q_j \le p_j$ for $j = 1, 2$. Suppose that for $1 < q \le r < \infty$,*

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p} \quad \text{and} \quad \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}.$$

Then for any $f \in M^{p_1}_{q_1}$ and $g \in M^{p_2}_{q_2}$, it holds that

$$\|fg\|_{M^p_q} \le \|f\|_{M^{p_1}_{q_1}} \|g\|_{M^{p_2}_{q_2}}. \tag{2.3}$$

The inequality (2.3) immediately follows from the Hölder inequality for the integration.

Proposition 2.2 (The Hausdorff-Young inequality). *Let $1 \le q \le p < \infty$. For any $f \in M^p_q(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$, it holds that*

$$\|f * g\|_{M^p_q} \le \|g\|_1 \|f\|_{M^p_q}. \tag{2.4}$$

Proof of Proposition 2.2 By Minkowski's inequality, we see that

$$\begin{aligned}&|B_R|^{-\left(\frac{1}{q}-\frac{1}{p}\right)} \left(\int_{B_R(x)} |(f*g)(y)|^q\,dy\right)^{\frac{1}{q}} \\ &= |B_R|^{-\left(\frac{1}{q}-\frac{1}{p}\right)} \left(\int_{B_R(x)} \left|\int_{\mathbb{R}^n} f(y-z)g(z)\,dz\right|^q dy\right)^{\frac{1}{q}} \\ &\le |B_R|^{-\left(\frac{1}{q}-\frac{1}{p}\right)} \int_{\mathbb{R}^n} \left(\int_{B_R(x)} |f(z)|^q |g(y-z)|^q\,dy\right)^{\frac{1}{q}} dz \\ &\le \|f\|_{M^p_q} \|g\|_1\end{aligned}$$

for any $x \in \mathbb{R}^n$ and $0 < R \le 1$. Thus, we obtain the inequality. □

The Littlewood–Paley theorem on the Morrey space was shown by Mazzucato [25, 26]:

Proposition 2.3 *([25, 26]). Let $1 < q \le p < \infty$ and $s \ge 0$. Then $E^0_{p,q,2} \simeq M^p_q$ (the norm equivalent), i.e.,*

$$\|f\|_{M^p_q} \simeq \|f\|_{E^0_{p,q,2}},$$
$$\|f\|_{M^{s,p}_q} \simeq \|f\|_{E^s_{p,q,2}}.$$

The embedding between Besov and Lizorkin–Triebel type Morrey spaces holds (Proposition 1.3 of Sawano [36]):

Proposition 2.4 *Let $1 \le q \le p \le \infty$, $1 \le \rho \le \infty$, and $s \in \mathbb{R}$. Then it holds that*

$$N^s_{p,q,\min\{q,\rho\}} \subset E^s_{p,q,\rho} \subset N^s_{p,q,\infty}.$$

The following potential estimate on Morrey spaces holds (see Taylor [39]):

Proposition 2.5 *Let $1 < p_0 < p_1 < \infty$ satisfy $1/p_0 - 1/p_1 \le 1/n$. Suppose that $1 < q_0 \le q_1 < \infty$ satisfy*

$$\begin{cases} \dfrac{q_0}{p_0} = \dfrac{q_1}{p_1} & \text{if } p_0 \le n, \\ \dfrac{q_0}{p_0} > \dfrac{q_1}{p_1} & \text{if } p_0 > n. \end{cases}$$

Let $B_\lambda(x)$ be the Bessel potential defined by (1.12). Then there exists a constant $C > 0$ such that for any $f \in M^{p_0}_{q_0}(\mathbb{R}^n)$,

$$\|\nabla B_\lambda * f\|_{M^{p_1}_{q_1}} \le C\|f\|_{M^{p_0}_{q_0}}.$$

The Sobolev embedding theorem was shown in Theorem 2.5 of Kozono–Yamazaki [16]:

Proposition 2.6 ([16]). *Let $1 \le q \le p < \infty$, $1 \le \sigma \le \infty$, and $s \in \mathbb{R}$. Then the following embedding holds:*

$$N^s_{p,q,\sigma} \subset B^{s-\frac{n}{p}}_{\infty,\sigma}.$$

Moreover, for $1 \le q_j \le p_j < \infty$ $(j = 0, 1)$ and $s_0 \le s_1$, it holds that

$$N^{s_1}_{p_1,q_1,\sigma} \subset N^{s_0}_{p_0,q_0,\sigma} \quad \text{with } \frac{1}{p_1} = \frac{1}{p_0} + \frac{s_1 - s_0}{n}, \quad \frac{1}{q_1} = \frac{1}{q_0} + \frac{p_0(s_1 - s_0)}{nq_0}. \tag{2.5}$$

We introduce a dissipative estimate or the heat evolution semigroup on the local Morrey spaces. Let $e^{t\Delta}f \equiv G_t * f$, where we set

$$G_t(x) \equiv (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}.$$

We derive the heat semigroup estimate (cf. Theorem 3.1 of [16] and see also [17]):

Proposition 2.7 *Let $1 \le q \le p < \infty$, $s_0 \le s_1$, and $1 \le \sigma \le \infty$. Then the following estimates hold:*

$$\left\| |\nabla|^{s_1} e^{t\Delta} f \right\|_{M_q^p} \le C t^{-\frac{s_1 - s_0}{2}} \left\| |\nabla|^{s_0} f \right\|_{M_q^p}, \tag{2.6}$$

$$\| e^{t\Delta} f \|_{N^{s_1}_{p,q,\sigma}} \le C(1 + t^{-\frac{s_1 - s_0}{2}}) \| f \|_{N^{s_0}_{p,q,\sigma}}. \tag{2.7}$$

Moreover, if $s_0 < s_1$, then it holds that

$$\| e^{t\Delta} f \|_{N^{s_1}_{p,q,1}} \le C \left(1 + t^{-\frac{s_1 - s_0}{2}} \right) \| f \|_{N^{s_0}_{p,q,\infty}}. \tag{2.8}$$

Proof of Proposition 2.7 In order to show (2.6), it suffices to consider the case of $s_1 = s$ and $s_0 = 0$ for $s > 0$. By (2.4), we have

$$\left\| |\nabla|^{s} e^{t\Delta} f \right\|_{M_q^p} \le \left\| |\nabla|^s G_t \right\|_1 \| f \|_{M_q^p}.$$

Since $\left\| |\nabla|^s G_t \right\|_1 \le C t^{-s/2}$, we obtain (2.6). By the definition of the norm of Besov–Morrey space and (2.6), we have

$$\| e^{t\Delta} f \|_{N^{s_1}_{p,q,\sigma}} = \left(\sum_{j \ge \hat{0}} 2^{s_1 j \sigma} \| e^{t\Delta} (\phi_j * f) \|^{\sigma}_{M_q^p} \right)^{\frac{1}{\sigma}} \le C(1 + t^{-\frac{s_1 - s_0}{2}}) \| f \|_{N^{s_0}_{p,q,\sigma}}.$$

For (2.8), we use the real interpolation theory. By (2.7), we see that

$$\begin{cases} \| e^{t\Delta} f \|_{N^{2s_1 - s_0}_{p,q,\infty}} \le C(1 + t^{-(s_1 - s_0)}) \| f \|_{N^{s_0}_{p,q,\infty}}, \\ \| e^{t\Delta} f \|_{N^{s_0}_{p,q,\infty}} \le C \| f \|_{N^{s_0}_{p,q,\infty}}. \end{cases} \tag{2.9}$$

We take $f \in N^{s_0}_{p,q_1,\infty}$ arbitrary and define the K-functor;

$$K(\lambda, f) \equiv \inf_{f = f_0 + f_1} \{ \| f_0 \|_{N^{s_0}_{p,q,\infty}} + \lambda \| f_1 \|_{N^{s_0}_{p,q,\infty}} \}. \tag{2.10}$$

By the above definition (2.10), for any $\varepsilon > 0$ and $\lambda > 0$, there exist $f_0, f_1 \in N^{s_0}_{p,q,\infty}$ such that

$$\| f_0 \|_{N^{s_0}_{p,q,\infty}} + \lambda \| f_1 \|_{N^{s_0}_{p,q,\infty}} \le (1 + \varepsilon) K(\lambda, f).$$

By (2.9), we see that $e^{t\Delta} f_0 \in N^{2s_1 - s_0}_{p,q,\infty}$ and $e^{t\Delta} f_1 \in N^{s_0}_{p,q,\infty}$. If we set $a(t) \equiv (1 + t^{-(s_1 - s_0)})$, then we have

$$\begin{aligned} \| e^{t\Delta} f_0 \|_{N^{2s_1 - s_0}_{p,q,\infty}} + a(t) \lambda \| e^{t\Delta} f_1 \|_{N^{s_0}_{p,q,\infty}} &\le C(1 + t^{-(s_1 - s_0)}) (\| f_0 \|_{N^{s_0}_{p,q,\infty}} + \lambda \| f_1 \|_{N^{s_0}_{p,q,\infty}}) \\ &\le C a(t) (1 + \varepsilon) K(\lambda, f). \end{aligned}$$

Since the real interpolation provides $(N^{2s_1-s_0}_{p,q,\infty}, N^{s_0}_{p,q,\infty})_{1/2,1} = N^{s_1}_{p,q,1}$, we obtain by changing the variable that

$$\begin{aligned}
&\|e^{t\Delta} f\|_{N^{s_1}_{p,q,1}}\\
&= \|e^{t\Delta} f\|_{(N^{2s_1-s_0}_{p,q,\infty}, N^{s_0}_{p,q,\infty})_{1/2,1}}\\
&\le \int_0^\infty \lambda^{-\frac{1}{2}} \left(\|e^{t\Delta} f_0\|_{N^{2s_1-s_0}_{p,q,\infty}} + \lambda \|e^{t\Delta} f_1\|_{N^{s_0}_{p,q,\infty}}\right) \frac{d\lambda}{\lambda}\\
&= \int_0^\infty (a(t)\rho)^{-\frac{1}{2}} \left(\|e^{t\Delta} f_0\|_{N^{2s_1-s_0}_{p,q,\infty}} + (a(t)\rho) \|e^{t\Delta} f_1\|_{N^{s_0}_{p,q,\infty}}\right) \frac{d\rho}{\rho}\\
&\le C(1+t^{-(s_1-s_0)})(1+\varepsilon) a(t)^{-\frac{1}{2}} \int_0^\infty \rho^{-\frac{1}{2}} K(\rho, f) \frac{d\rho}{\rho}\\
&\le C(1+\varepsilon)(1+t^{-(s_1-s_0)})^{\frac{1}{2}} \left(\int_1^\infty \rho^{-\frac{1}{2}} \|f\|_{N^{s_0}_{p,q,\infty}} \frac{d\rho}{\rho} + \int_0^1 \rho^{\frac{1}{2}} \|f\|_{N^{s_0}_{p,q,\infty}} \frac{d\rho}{\rho}\right)\\
&\le C(1+\varepsilon)(1+t^{-\frac{s_1-s_0}{2}}) \|f\|_{N^{s_0}_{p,q,\infty}}.
\end{aligned}$$

Since one can choose $\varepsilon > 0$ arbitrary, the inequality (2.8) holds. □

Concerning the heat semigroup, $\mathcal{M}^p_q(\mathbb{R}^n)$ is characterized by the following proposition:

Proposition 2.8 *Let* $1 \le q \le p < \infty$ *and assume that* $f \in M^p_q(\mathbb{R}^n)$*. Then the following statements are equivalent:*

(1) $f \in \mathcal{M}^p_q(\mathbb{R}^n)$.
(2) It holds that

$$\lim_{|y|\to 0} \|f(\cdot + y) - f\|_{M^p_q} = 0.$$

(3) It holds that

$$\lim_{t\to 0} \|e^{t\Delta} f - f\|_{M^p_q} = 0.$$

Proof of Proposition 2.8 We first suppose that $f \in \mathcal{M}^p_q(\mathbb{R}^n)$. For arbitrary fixed $\varepsilon > 0$, there exists a sequence $\{f_k\}_{k\in\mathbb{N}} \subset BUC(\mathbb{R}^n)$ and $K \in \mathbb{N}$ such that for any $k \ge K$,

$$\|f_k - f\|_{M^p_q} < \frac{\varepsilon}{3}.$$

By the triangle inequality, we have

$$\begin{aligned}
\|f(\cdot+y) - f\|_{M^p_q} &\le \|f(\cdot+y) - f_k(\cdot+y)\|_{M^p_q} + \|f_k(\cdot+y) - f_k\|_{M^p_q} + \|f_k - f\|_{M^p_q}\\
&\le \frac{2\varepsilon}{3} + \|f_k(\cdot+y) - f_k\|_{M^p_q}
\end{aligned}$$

for any $k \geq K$. Since $\{f_k\} \subset BUC(\mathbb{R}^n)$, there exists $\delta > 0$ such that

$$|f_k(z+y) - f_k(z)| < \frac{\varepsilon}{3|B_1|^{\frac{1}{p}}}$$

for all $y, z \in \mathbb{R}^n$ with $|y| \leq \delta$ and $k \in \mathbb{N}$. Thus, if $|y| \leq \delta$, then we have

$$\begin{aligned}|B_R|^{-\left(\frac{1}{q}-\frac{1}{p}\right)}\left(\int_{B_R(x)} |f_k(z+y) - f_k(z)|^q\, dz\right)^{\frac{1}{q}} &< |B_R|^{-\left(\frac{1}{q}-\frac{1}{p}\right)}|B_1|^{-\frac{1}{p}}|B_R|^{\frac{1}{q}}\frac{1}{3}\varepsilon \\ &= |B_R|^{\frac{1}{p}}|B_1|^{-\frac{1}{p}}\frac{1}{3}\varepsilon \leq \frac{1}{3}\varepsilon\end{aligned}$$

for any $x \in \mathbb{R}^n$, $0 < R \leq 1$, and $k \in \mathbb{N}$, which implies

$$\|f(\cdot + y) - f\|_{M^p_q} < \varepsilon$$

if $|y| < \delta$.

Secondary, we assume that (2) holds. By the representation of the heat semigroup, we have

$$\|e^{t\Delta} f - f\|_{M^p_q} \leq (4\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|z|^2}{4}} \|f(\cdot - \sqrt{t}z) - f\|_{M^p_q}\, dz$$

for any $t > 0$. It follows from the assumption (2) that

$$\|f(\cdot - \sqrt{t}z) - f\|_{M^p_q} \to 0 \quad \text{as } t \to 0.$$

On the other hand, we see taht

$$\|f(\cdot - \sqrt{t}z) - f\|_{M^p_q} \leq 2\|f\|_{M^p_q} \in L^1(\mathbb{R}^n; e^{-\frac{|\cdot|^2}{4}}\, dx).$$

By the Lebesgue dominated convergence theorem, we obtain

$$\|e^{t\Delta} f - f\|_{M^p_q} \to 0 \quad \text{as } t \to 0.$$

Lastly, we suppose that (3) holds. By the embedding $M^p_q(\mathbb{R}^n) \subset L^q_{\mathrm{ul}}(\mathbb{R}^n)$, we see that $e^{t\Delta} f \in BUC(\mathbb{R}^n)$ when $t > 0$ for any $f \in M^p_q(\mathbb{R}^n)$ (see Proposition 2.2 in [24]). This and the assumption (3) imply that $f \in \mathcal{M}^p_q(\mathbb{R}^n)$. □

The norm of the Morrey space M^p_q can be represented by the following equivalent norm:

$$\|f\|_{M^p_q} \simeq \sup_{\substack{j \in \mathbb{N}\cup\{0\},\\ k \in \mathbb{Z}^n}} 2^{nj\left(\frac{1}{q}-\frac{1}{p}\right)} \|f\|_{L^q(Q_j(k))},$$

where $Q_j(k)$ denotes an open cube in $\mathbb{R}^n$ whose side length is 2^{-j} and lower corner is $2^{-j}k$, that is, $Q_j(k) \equiv 2^{-j}k + 2^{-j}(0,1)^n$. By Rosenthal–Triebel [35] and Izumi–Sawano–Tanaka [11], the dual and the pre-dual spaces of the Morrey space are identified by the following way: For $1 < p < q < \infty$, we set $H^pL^q(\mathbb{R}^n)$ as all collection of $h \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$h = \sum_{\substack{j\in\mathbb{N}\cup\{0\},\\ k\in\mathbb{Z}^n}} h_{k,j} \quad \text{with supp}\, h_{k,j} \subset \overline{Q_j(k)}$$

satisfying

$$\sum_{\substack{j\in\mathbb{N}\cup\{0\},\\ k\in\mathbb{Z}^n}} 2^{-nj\left(\frac{1}{p}-\frac{1}{q}\right)} \|h_{j,k}\|_{L^q(Q_j(k))} < +\infty.$$

Furthermore, we define

$$\begin{aligned}
&\|h\|_{H^pL^q} \\
&\equiv \inf\Bigg\{ \sum_{\substack{j\in\mathbb{N}\cup\{0\},\\ k\in\mathbb{Z}^n}} 2^{-nj\left(\frac{1}{p}-\frac{1}{q}\right)} \|h_{j,k}\|_{L^q(Q_j(k))}; \\
&h = \sum_{\substack{j\in\mathbb{N}\cup\{0\},\\ k\in\mathbb{Z}^n}} h_{k,j} \ \text{ in } \mathcal{S}'(\mathbb{R}^n),\ \text{supp}\, h_{k,j} \subset \overline{Q_j(k)} \Bigg\}.
\end{aligned}$$

Proposition 2.9 *(The duality [11, 35]). Let* $1 < q < p < \infty$ *and* $1/p + 1/p' = 1/q + 1/q' = 1$. *Then*

(1) the dual space of $\mathcal{M}^p_q(\mathbb{R}^n)$ *satisfies*

$$\left(\mathcal{M}^p_q(\mathbb{R}^n)\right)^* = H^{p'}L^{q'}(\mathbb{R}^n).$$

(2) Conversely the dual space of $H^{p'}L^{q'}(\mathbb{R}^n)$ *is identified as*

$$\left(H^{p'}L^{q'}(\mathbb{R}^n)\right)^* = M^p_q(\mathbb{R}^n).$$

In particular, neither M^p_q *nor* $\mathcal{M}^p_q$ *is reflexive for all* $1 < q \le p < \infty$.

We introduce a new function space $\tilde{N}^s_{p,q,\sigma}$ which is a pre-dual of the space $N^{-s}_{p',q',\sigma}$ for $1/p + 1/p' = 1/q + 1/q' = 1$, $1 < p, q < \infty$ as follows:
Definition. For any $1 \le p, q, < \infty$, $s \in \mathbb{R}$ and $0 < \theta < 1$, let $\tilde{N}^s_{p,q,\sigma} = \tilde{N}^s_{p,q,\sigma}(\mathbb{R}^n)$ be the real interpolation space given by

$$\tilde{N}^s_{p,q,\sigma}(\mathbb{R}^n) \equiv \left((1-\Delta)^{-\frac{s_0}{2}} H^pL^q, (1-\Delta)^{-\frac{s_1}{2}} H^pL^q\right)_{\theta,\sigma},$$

where $s = (1-\theta)s_0 + \theta s_1$. Analogously

$$\tilde{N}^s_{p,q,0}(\mathbb{R}^n) \equiv \big((1-\Delta)^{-\frac{s_0}{2}} H^p L^q, (1-\Delta)^{-\frac{s_1}{2}} H^p L^q\big)_{\theta,0},$$

where

$$\|f\|_{(X,Y)_{\theta,0}} \equiv \sup_{j\in\mathbb{Z}} \sup_{2^j<\lambda\le 2^{j+1}} \lambda^{-\theta} \inf_{f=f_0+f_1} \big(\|f_0\|_X + \lambda\|f_1\|_Y\big)$$

with

$$\lim_{|j|\to\infty} \sup_{2^j<\lambda\le 2^{j+1}} \lambda^{-\theta} \inf_{f=f_0+f_1} \big(\|f_0\|_X + \lambda\|f_1\|_Y\big) = 0. \tag{2.11}$$

Proposition 2.10 *Let $1 \le p, q < \infty$, $s \in \mathbb{R}$ and $1 \le \sigma < \infty$. Then*

$$(\tilde{N}^s_{p,q,\sigma})^* \simeq N^{-s}_{p',q',\sigma'}. \tag{2.12}$$

If $\sigma = \infty$,

$$(\tilde{N}^s_{p,q,0})^* \simeq N^{-s}_{p',q',1}. \tag{2.13}$$

Proof of Proposition 2.10 For the first relation (2.12), by the duality result in Proposition 2.9, we see for any $1 \le p, q < \infty$, $s \in \mathbb{R}$ and $1 \le \sigma < \infty$ that

$$\begin{aligned}(\tilde{N}^s_{p,q,\sigma})^* &= \big(((1-\Delta)^{-\frac{s_1}{2}} H^p L^q)^*, ((1-\Delta)^{-\frac{s_2}{2}} H^p L^q)^*\big)_{\theta,\sigma'} \\ &= \big((1-\Delta)^{\frac{s_1}{2}} M^{p'}_{q'}, (1-\Delta)^{\frac{s_2}{2}} M^{p'}_{q'}\big)_{\theta,\sigma'} = N^{-s}_{p',q',\sigma'}.\end{aligned} \tag{2.14}$$

Noting the duality relation to the sequence space $(\ell_0)^* = \ell_1$, the relation (2.13) is also shown in a similar way, where $\ell_0 = \big\{\{a_k\}_{k\in\mathbb{N}};\ |a_k| \to 0 \ \text{ as } k \to \infty.\big\}$. □

3 Generalized maximal regularity

In this section, we consider maximal regularity for the Cauchy problem of the heat equation on Morrey spaces. By Proposition 2.9, we see that the (local) Morrey space is not reflexive and the general theory of UMD does not cover such a function space. We then employ the Besov–Morrey space $N^s_{p,q,\sigma}(\mathbb{R}^n)$ to derive maximal regularity for the heat equations on such a local function space:

Theorem 3.1 *Let $1 \le q \le p < \infty$, $1 \le \rho < \infty$, $\mu > 0$, $\lambda \ge 0$, and let $I = [0, T)$ for $0 < T \le \infty$ ($T < \infty$ if $\lambda = 0$). Given initial data $u_0 \in N^{2(1-1/\rho)}_{p,q,\rho}(\mathbb{R}^n)$ and the external force $f \in L^\rho(I; N^0_{p,q,\rho}(\mathbb{R}^n))$, suppose that u is the solution to the Cauchy problem of the heat equation*

$$\begin{cases} \partial_t u - \mu\Delta u + \lambda u = f, & t > 0,\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n. \end{cases}$$

Then there exists a constant $C > 0$ such that

$$\begin{aligned} \|\partial_t u\|_{L^\rho(I;N^0_{p,q,\rho})} + \mu\|\Delta u\|_{L^\rho(I;N^0_{p,q,\rho})} + \lambda\|u\|_{L^\rho(I;N^0_{p,q,\rho})} \\ \le C\big(\|u_0\|_{N^{2(1-1/\rho)}_{p,q,\rho}} + \|f\|_{L^\rho(I;N^0_{p,q,\rho})}\big). \end{aligned} \tag{3.1}$$

We note that the constant C appearing in the inequality (3.1) depends on T in the case of $\lambda = 0$. On the other hand, if $\lambda > 0$, then the constant C is independent of T.

The proof of Theorem 3.1 is decomposed into a homogeneous estimate and an inhomogeneous estimate.

Proposition 3.2 *Let $1 \le q \le p < \infty$, $1 \le \sigma \le \infty$, $1 \le \rho < \infty$, $\lambda \ge 0$, and let $I = [0,T)$ for $0 < T \le \infty$ ($T < \infty$ if $\lambda = 0$). Then there exists a constant $C > 0$ such that for any $u_0 \in N^{1-2/\rho}_{p,q,\rho}(\mathbb{R}^n)$, it holds that*

$$\|\nabla e^{t(\Delta-\lambda)} u_0\|_{L^\rho(I;N^0_{p,q,\sigma})} \le C\|u_0\|_{N^{1-2/\rho}_{p,q,\rho}}. \tag{3.2}$$

Proof of Proposition 3.2 By the embedding $l^1 \subset l^\sigma$ for $\sigma > 1$, it suffices to consider the case of $\sigma = 1$. By the definition of the norm of $N^0_{p,q,1}$, we see that

$$\|\nabla e^{t(\Delta-\lambda)} u_0\|_{N^0_{p,q,1}} = e^{-\lambda t}\sum_{j\ge\hat{0}} \|\nabla\phi_j * (e^{t\Delta}u_0)\|_{M^p_q}.$$

Using $\tilde{\phi}_j \equiv \phi_{j-1} + \phi_j + \phi_{j+1}$, then we have

$$\begin{aligned} \|\nabla e^{t(\Delta-\lambda)} u_0\|_{N^0_{p,q,1}} &= e^{-\lambda t}\sum_{j\ge\hat{0}} \|\nabla\tilde{\phi}_j * (\phi_j * (e^{t\Delta}u_0))\|_{M^p_q} \\ &= e^{-\lambda t}\sum_{j\ge\hat{0}} \|(\nabla\tilde{\phi}_j * G_t) * (\phi_j * u_0)\|_{M^p_q}. \end{aligned} \tag{3.3}$$

By changing the variable, we see that

$$\|\nabla\tilde{\phi}_j * G_t\|_1 \le C2^j e^{-2^{2j}t}. \tag{3.4}$$

It follows from (2.4), (3.3), and (3.4) that

$$\|\nabla e^{t(\Delta-\lambda)} u_0\|_{N^0_{p,q,1}} \le Ce^{-\lambda t}\left(\|\psi * u_0\|_{M^p_q} + \sum_{j\ge1} 2^j e^{-2^{2j}t}\|\phi_j * u_0\|_{M^p_q}\right). \tag{3.5}$$

We take $\alpha, \beta > 0$ satisfying

$$\alpha + \beta = 1 \quad \text{and} \quad \beta < \frac{2}{\rho}.$$

By the Hölder inequality with respect to j, we have

$$\begin{aligned}
&\sum_{j \geq 1} 2^j e^{-2^{2j}t} \|\phi_j * u_0\|_{M_q^p} \\
&\quad \leq \left(\sum_{j \geq 1} (2^{\alpha j} e^{-4^{j-1}t} \|\phi_j * u_0\|_{M_q^p})^\rho \right)^{\frac{1}{\rho}} \left(\sum_{j \geq 1} (2^{\beta j} e^{-3 \cdot 4^{j-1}t})^{\rho'} \right)^{\frac{1}{\rho'}} \\
&\quad \leq C t^{-\frac{\beta}{2}} \left(\sum_{j \geq 1} (2^{\alpha j} e^{-4^{j-1}t} \|\phi_j * u_0\|_{M_q^p})^\rho \right)^{\frac{1}{\rho}}.
\end{aligned}$$

Thus, it follows from (3.5) that

$$\begin{aligned}
\|\nabla e^{t(\Delta - \lambda)} u_0\|_{N^0_{p,q,1}}^\rho &\leq C e^{-\rho\lambda t} \left(\|\psi * u_0\|_{M_q^p} + \sum_{j \geq 1} 2^j e^{-2^{2j}t} \|\phi_j * u_0\|_{M_q^p} \right)^\rho \\
&\leq C 2^{\rho - 1} e^{-\rho\lambda t} \left(\|\psi * u_0\|_{M_q^p}^\rho + \left(\sum_{j \geq 1} 2^j e^{-2^{2j}t} \|\phi_j * u_0\|_{M_q^p} \right)^\rho \right) \\
&\leq C e^{-\rho\lambda t} \left(\|\psi * u_0\|_{M_q^p}^\rho + t^{-\frac{\beta\rho}{2}} \sum_{j \geq 1} (2^{\alpha j} e^{-4^{j-1}t} \|\phi_j * u_0\|_{M_q^p})^\rho \right).
\end{aligned}$$

By integration both sides with respect to t, we then have

$$\begin{aligned}
&\int_0^T \|\nabla e^{t(\Delta - \lambda)} u_0\|_{N^0_{p,q,1}}^\rho \, dt \\
&\leq C \Bigg(\int_0^T e^{-\rho\lambda t} \|\psi * u_0\|_{M_q^p}^\rho \, dt \\
&\quad + \int_0^\infty t^{-\frac{\beta\rho}{2}} \sum_{j \geq 1} \left(2^{\left(\frac{2}{\rho} - \beta\right)j} e^{-4^{j-1}t} 2^{j\left(1 - \frac{2}{\rho}\right)} \|\phi_j * u_0\|_{M_q^p} \right)^\rho dt \Bigg) \\
&= C \Bigg(\frac{1 - e^{-\rho\lambda T}}{\rho\lambda} \|\psi * u_0\|_{M_q^p}^\rho \\
&\quad + \sum_{j \geq 1} \left(2^{\left(\frac{2}{\rho} - \beta\right)j} 2^{j\left(1 - \frac{2}{\rho}\right)} \|\phi_j * u_0\|_{M_q^p} \right)^\rho \int_0^\infty t^{-\frac{\beta\rho}{2}} e^{-\rho \cdot 4^{j-1}t} \, dt \Bigg).
\end{aligned}$$

Since $\beta\rho/2 < 1$, we see that

$$\int_0^\infty t^{-\frac{\beta\rho}{2}} e^{-\rho \cdot 4^{j-1}t} \, dt = (\rho \cdot 4^{j-1})^{\frac{\beta\rho}{2} - 1} \int_0^\infty s^{-\frac{\beta\rho}{2}} e^{-s} \, ds \leq C 2^{(\beta\rho - 2)j}.$$

Therefore, we obtain

$$
\begin{aligned}
&\left(\int_0^T \|\nabla e^{t(\Delta-\lambda)}u_0\|_{N^0_{p,q,1}}^\rho \,dt\right)^{\frac{1}{\rho}} \\
&\le C_0\left(\|\psi * u_0\|_{M^p_q}^\rho + \sum_{j\ge 1}\left(2^{j\left(1-\frac{2}{\rho}\right)}\|\phi_j * u_0\|_{M^p_q}\right)^\rho\right)^{\frac{1}{\rho}} \\
&\le C_0\left(\|\psi * u_0\|_{M^p_q} + \left(\sum_{j\ge 1} 2^{j\left(1-\frac{2}{\rho}\right)}\|\phi_j * u_0\|_{M^p_q}^\rho\right)^{\frac{1}{\rho}}\right),
\end{aligned}
$$

which implies (3.2). If $\lambda > 0$, then the constant C_0 is independent of T. On the other hand, if $\lambda = 0$, then $C_0 = cT^{1/\rho}$ for some constant $c > 0$. □

We state the following slightly general form of maximal regularity for the inhomogeneous term:

Proposition 3.3 *Let $1 \le q \le p < \infty$, $1 \le \nu \le \sigma \le \rho \le \infty$, $\lambda \ge 0$, and let $I = [0, T)$ for $0 < T \le \infty$ ($T < \infty$ if $\lambda = 0$). Then there exists a constant $C > 0$ such that for any $f \in L^\nu(I; N^{-2/\rho+2/\nu}_{p,q,\sigma})$, it holds that*

$$
\left\|\int_0^t \Delta e^{(t-s)(\Delta-\lambda)} f(s)\,ds\right\|_{L^\rho(I;N^0_{p,q,\sigma})} \le C\|f\|_{L^\nu(I;N^{-\frac{2}{\rho}+\frac{2}{\nu}}_{p,q,\sigma})}. \tag{3.6}
$$

Proof of Proposition 3.3 By instituting $\tilde{\phi}_j$, (2.4) and Minkowski's inequality, we see that

$$
\begin{aligned}
&\left\|\int_0^t \Delta e^{(t-s)(\Delta-\lambda)} f(s)\,ds\right\|_{N^0_{p,q,\sigma}} \\
&\le \left\|\int_0^t \psi * (e^{(t-s)(\Delta-\lambda)} f(s))\,ds\right\|_{M^p_q} \\
&\quad + \left(\sum_{j\ge 1} 2^{2j\sigma}\left\|\int_0^t \phi_j * (e^{(t-s)\Delta} f(s))\,ds\right\|_{M^p_q}^\sigma\right)^{\frac{1}{\sigma}} \\
&\le C\int_0^t \|\psi * (e^{(t-s)(\Delta-\lambda)} f(s))\|_{M^p_q}\,ds \\
&\quad + \left\{\sum_{j\ge 1} 2^{2j\sigma}\left(\int_0^t \|\phi_j * (e^{(t-s)\Delta} f(s))\,ds\|_{M^p_q}\,ds\right)^\sigma\right\}^{\frac{1}{\sigma}}
\end{aligned}
$$

$$\leq C \int_0^t e^{-\lambda(t-s)} \|\psi * f(s)\|_{M_q^p} \, ds$$
$$+ \left\{ \sum_{j \geq 1} 2^{2j\sigma} \left(\int_0^t e^{-2^{2j}(t-s)} \|\phi_j * f(s)\|_{M_q^p} \, ds \right)^{\sigma} \right\}^{\frac{1}{\sigma}}.$$

For simplicity, we set

$$g_j \equiv 2^{2j} \int_0^t e^{-2^{2j}(t-s)} \|\phi_j * f(s)\|_{M_q^p} \, ds.$$

Since $\sigma \leq \rho$, it follows from Minkowski's inequality that

$$\big\| \|\{g_j\}_{j=1}^{\infty}\|_{l^\sigma} \big\|_{L^\rho(I)} \leq \big\| \|\{g_j\}_{j=1}^{\infty}\|_{L^\rho(I)} \big\|_{l^\sigma}.$$

Thus, we have

$$\left\| \int_0^t \Delta e^{(t-s)(\Delta-\lambda)} f(s) \, ds \right\|_{L^\rho(I; N^0_{p,q,\sigma})}$$
$$\leq C \left[\int_0^T \left(\int_0^t e^{-\lambda(t-s)} \|\psi * f(s)\|_{M_q^p} \, ds \right)^{\rho} dt \right]^{\frac{1}{\rho}}$$
$$+ C \left[\sum_{j \geq 1} 2^{2j\sigma} \left\{ \int_0^T \left(\int_0^t e^{-2^{2j}(t-s)} \|\phi_j * f\|_{M_q^p} \, ds \right)^{\rho} dt \right\}^{\frac{\sigma}{\rho}} \right]^{\frac{1}{\sigma}}.$$

By the Hausdorff–Young inequality with respect to t, we see (denoting $*_t$ the convolution by t-variable) that

$$\left(\int_0^T \left(\int_0^t e^{-\lambda(t-s)} \|\psi * f(s)\|_{M_q^p} \, ds \right)^{\rho} dt \right)^{\frac{1}{\rho}}$$
$$\leq \left\| e^{-\lambda t} *_t \|\psi * f(t)\|_{M_q^p} \right\|_{L^\rho(I)}$$
$$\leq C \left(\int_0^T e^{-\mu\lambda t} \, dt \right)^{\frac{1}{\mu}} \left(\int_0^T \|\psi * f(t)\|_{M_q^p}^{\nu} \, dt \right)^{\frac{1}{\nu}},$$
$$\left(\int_0^T \left(\int_0^t e^{-2^{2j}(t-s)} \|\phi_j * f(s)\|_{M_q^p} \, ds \right)^{\rho} dt \right)^{\frac{1}{\rho}}$$
$$\leq \left\| e^{-2^{2j}t} *_t \|\phi_j * f(t)\|_{M_q^p} \right\|_{L^\rho(I)}$$
$$\leq C \left(\int_0^T e^{-\mu 2^{2j} t} \, dt \right)^{\frac{1}{\mu}} \left(\int_0^T \|\phi_j * f(t)\|_{M_q^p}^{\nu} \, dt \right)^{\frac{1}{\nu}},$$

where $\mu \geq 1$ satisfies

$$1 + \frac{1}{\rho} = \frac{1}{\nu} + \frac{1}{\mu}.$$

Since

$$\int_0^T e^{-\mu 2^{2j} t}\, dt \leq \mu^{-1} 2^{-2j}$$

and $\nu \leq \sigma$, it follows from Minkowski's inequality that

$$\begin{aligned}
&\left\| \int_0^t \Delta e^{(t-s)(\Delta-\lambda)} f(s)\, ds \right\|_{L^\rho(I; N^0_{p,q,\sigma})} \\
&\leq C_0 \left(\int_0^T \|\psi * f(t)\|^\nu_{M^p_q}\, dt \right)^{\frac{1}{\nu}} \\
&\quad + C \left(\sum_{j \geq 1} 2^{2j\sigma} 2^{-\frac{2\sigma}{\mu} j} \left(\int_0^T \|\phi_j * f(t)\|^\nu_{M^p_q}\, dt \right)^{\frac{\sigma}{\nu}} \right)^{\frac{1}{\sigma}} \\
&\leq C_0 \left(\int_0^T \|\psi * f(t)\|^\nu_{M^p_q}\, dt \right)^{\frac{1}{\nu}} \\
&\quad + C \left(\int_0^T \left(\sum_{j \geq 1} 2^{2j\sigma\left(\frac{1}{\nu} - \frac{1}{\rho}\right)} \|\phi_j * f(t)\|^\sigma_{M^p_q} \right)^{\frac{\nu}{\sigma}} dt \right)^{\frac{1}{\nu}},
\end{aligned}$$

which implies (3.6).

If $\lambda > 0$, then the constant C_0 is independent of T. On the other hand, if $\lambda = 0$, then $C_0 = cT^{1/\mu}$ for some constant $c > 0$. □

As a corollary of Proposition 3.3 with $\nu = \sigma = \rho$, we obtain the maximal regularity for the inhomogeneous term:

Corollary 3.4 *Let $1 \leq q \leq p < \infty$ and $1 \leq \rho \leq \infty$, $\lambda \geq 0$, and let $I = [0, T)$ for $0 < T \leq \infty$ ($T < \infty$ if $\lambda = 0$). Then there exists a constant $C > 0$ such that for any $f \in L^\rho(I; N^0_{p,q,\rho})$, it holds that*

$$\left\| \int_0^t \Delta e^{(t-s)(\Delta-\lambda)} f(s)\, ds \right\|_{L^\rho(I; N^0_{p,q,\rho})} \leq C \|f\|_{L^\rho(I; N^0_{p,q,\rho})}.$$

Proof of Theorem 3.1 Combining Proposition 3.2 and Corollary 3.4, we conclude the estimate (3.1). □

By the refined dissipative estimates in Proposition 2.7, we introduce a version of maximal regularity for the inhomogeneous term like (3.6) as follows:

Proposition 3.5 *Let* $1 \le q \le p < \infty$, $1 < \nu < \rho \le \infty$, *and* $I = (0, T)$ *for* $0 < T < +\infty$. *Then there exists* $C = C(n, \mu, \rho, T) > 0$ *such that for any* $f \in L^\nu(I; N^{-1+2/\nu-2/\rho}_{p,q,\infty})$, *it holds that*

$$\left\| \int_0^t \nabla e^{(t-s)\Delta} f(s)\, ds \right\|_{L^\rho(I;N^0_{p,q,1})} \le C \|f\|_{L^\nu(I;N^{-1+\frac{2}{\nu}-\frac{2}{\rho}}_{p,q,\infty})}. \tag{3.7}$$

Proof of Proposition 3.5 We first treat the case $1 < \nu < \rho < \infty$. By (2.8), we have

$$\begin{aligned}\left\| \int_0^t \nabla e^{(t-s)\Delta} f(s)\, ds \right\|_{N^0_{p,q,1}} &\le \int_0^t \|e^{(t-s)\Delta} f(s)\|_{N^1_{p,q,1}}\, ds \\ &\le C \int_0^t \left(1 + (t-s)^{-1+\frac{1}{\nu}-\frac{1}{\rho}}\right) \|f(s)\|_{N^{-1+\frac{2}{\nu}-\frac{2}{\rho}}_{p,q,\infty}}\, ds.\end{aligned}$$

By the generalized Hausdorff–Young inequality, it holds that

$$\begin{aligned}&\left\| \int_0^t \nabla e^{(t-s)\Delta} f(s)\, ds \right\|_{L^\rho(I;N^0_{p,q,1})} \\ &\le C \left\| \int_0^\infty \chi_{(0,T)}(t-s)\left(1 + (t-s)^{-1+\frac{1}{\nu}-\frac{1}{\rho}}\right) \chi_{(0,t)}(s) \|f(s)\|_{N^{-1+\frac{2}{\nu}-\frac{2}{\rho}}_{p,q,\infty}}\, ds \right\|_{L^\rho(\mathbb{R}_+)} \\ &\le C \left\| 1 + t^{-1+\frac{1}{\nu}-\frac{1}{\rho}} \right\|_{L^\mu_{\mathrm{w}}(I)} \|f\|_{L^\nu(I;N^{-1+\frac{2}{\nu}-\frac{2}{\rho}}_{p,q,\infty})} \\ &\le C \|f\|_{L^\nu(I;N^{-1+\frac{2}{\nu}-\frac{2}{\rho}}_{p,q,\infty})},\end{aligned}$$

where L^μ_w denotes the weak L^μ norm and $\mu > 1$ satisfies

$$1 + \frac{1}{\rho} = \frac{1}{\mu} + \frac{1}{\nu}.$$

Since the interval I is bounded, we obtain (3.7) in the case of $1 < \nu < \rho < \infty$.

For the end-point case $1 < \nu < \rho = \infty$, we employ the duality argument: Since $0 < T < +\infty$, there exists $j_0 \in \mathbb{Z}$ such that $2^{j_0} \le T < 2^{j_0+1}$. For any $g \in C_0^\infty((0, T) \times \mathbb{R}^n)$, we see that

$$\begin{aligned}&\left| \int_0^T \left(\int_0^t \nabla e^{(t-s)\Delta} f(s)\, ds, g(t) \right)_{L^2} dt \right| \\ &\le \iint_{0<s<t<T} \left| \left(\nabla e^{(t-s)\Delta} f(s), g(t) \right)_{L^2} \right| ds\, dt \\ &\le \sum_{j \le j_0} \iint_{2^j \le t-s < 2^{j+1}} \left| \left(\nabla e^{(t-s)\Delta} f(s), g(t) \right)_{L^2} \right| ds\, dt.\end{aligned}$$

We set

$$T_j(f,g) \equiv \iint_{2^j \le t-s < 2^{j+1}} \left| \left(\nabla e^{(t-s)\Delta} f(s), g(t) \right)_{L^2} \right| ds\, dt$$

and

$$I_j(t) \equiv [t-2^{j+1}, t-2^j), \quad J_j(s) \equiv [s+2^j, s+2^{j+1}).$$

By the duality and (2.14) and the improved dissipative estimate (2.8), we have

$$\begin{aligned}
&T_j(f,g)\\
&\le \iint_{2^j \le t-s<2^{j+1}} \|\nabla e^{(t-s)\Delta} f(s)\|_{N^0_{p,q,1}} \|g(t)\|_{\tilde{N}^0_{p',q',\infty}} ds\, dt\\
&\le C \iint_{2^j \le t-s<2^{j+1}} \left(1+(t-s)^{-\left(1-\frac{1}{\nu}+\frac{1}{\rho}\right)}\right) \|f(s)\|_{N^{-1+\frac{2}{\nu}-\frac{2}{\rho}}_{p,q,\infty}} \|g(t)\|_{\tilde{N}^0_{p',q',\infty}} ds\, dt\\
&\le C\left(1+2^{-j\left(1-\frac{1}{\nu}+\frac{1}{\rho}\right)}\right) \iint_{2^j \le t-s<2^{j+1}} \|f(s)\|_{N^{-1+\frac{2}{\nu}-\frac{2}{\rho}}_{p,q,\infty}} \|g(t)\|_{\tilde{N}^0_{p',q',\infty}} ds\, dt.
\end{aligned}$$

In the case of $\rho = \infty$, it follows from the Hausdorff–Young inequality that

$$\begin{aligned}
&\sum_{j \le j_0} \iint_{2^j \le t-s<2^{j+1}} \left| \left(\nabla e^{(t-s)\Delta} f(s), g(t) \right)_{L^2} \right| ds\, dt\\
&\quad \le C \sum_{j\le j_0} \left(1+2^{-j\left(1-\frac{1}{\nu}\right)}\right) \iint_{2^j \le t-s<2^{j+1}} \|f(s)\|_{N^{-1+\frac{2}{\nu}}_{p,q,\infty}} \|g(t)\|_{\tilde{N}^0_{p',q',\infty}} ds\, dt\\
&\quad \le C \sum_{j\le j_0} \left(1+2^{-j\left(1-\frac{1}{\nu}\right)}\right) |I_j|^{\frac{1}{\nu'}} \left(\int_{I_j} \|f(s)\|^{\nu}_{N^{-1+\frac{2}{\nu}}_{p,q,\infty}} ds \right)^{\frac{1}{\nu}} \int_{I_j} \|g(t)\|_{\tilde{N}^0_{p',q',\infty}} dt\\
&\quad \le C \|f\|_{L^\nu(I; N^{-1+\frac{2}{\nu}-\frac{2}{\rho}}_{p,q,\infty})} \sum_{j \le j_0} (2^{\frac{j}{\nu'}}+1) \int_{I_j} \|g(t)\|_{\tilde{N}^0_{p',q',\infty}} dt.
\end{aligned}$$

Since

$$\begin{aligned}
\sum_{j\le j_0} (2^{\frac{j}{\nu'}}+1) \int_{I_j} \|g(t)\|_{\tilde{N}^0_{p',q',\infty}} dt &\le (2^{\frac{j_0}{\nu'}+1}+1) \int_I \|g(t)\|_{\tilde{N}^0_{p',q',\infty}} dt\\
&\le C_T \|g\|_{L^1(I;\tilde{N}^0_{p',q',\infty})}
\end{aligned}$$

and (2.11) holds for $g \in C_0^\infty((0,T)\times\mathbb{R}^n)$, we obtain

$$\left\langle \int_0^t \nabla e^{(t-s)\Delta} f(s)\, ds, g(t) \right\rangle_{L^2(I\times\mathbb{R}^n)} \le C_T \|f\|_{L^\nu(I;N^{-1+\frac{2}{\nu}}_{p,q,\infty})} \|g\|_{L^1(I;\tilde{N}^0_{p',q',0})}.$$

By the duality (2.13) in Proposition 2.10 we conclude

$$\left\| \int_0^t \nabla e^{(t-s)\Delta} f(s)\, ds \right\|_{L^\infty(I;N^0_{p,q,1})} \le C_T \|f\|_{L^\nu(I;N^{-1+\frac{2}{\nu}}_{p,q,\infty})}.$$

□

Since $N^s_{p,q,1} \subset M^{s,p}_q \subset N^s_{p,q,\infty}$ by Proposition 2.4, we immediately obtain the following estimate as a corollary of Proposition 3.5.

Corollary 3.6 *Let $1 \le q \le p < \infty$, $1 < \nu < \rho \le \infty$, and $I = (0,T)$. Then there exists $C = C(n,\mu,\rho,T) > 0$ such that for any $f \in L^\nu(I; M_q^{-1+2/\nu-2/\rho,p})$, it holds that*

$$\left\| \int_0^t \nabla e^{(t-s)\Delta} f(s)\, ds \right\|_{L^\rho(I;M^p_q)} \le C \|f\|_{L^\nu(I;M_q^{-1+\frac{2}{\nu}-\frac{2}{\rho},p})}. \tag{3.8}$$

4 Well-posedness of the Cauchy problems

In this section, we show the well-posedness of the Cauchy problem of the parabolic-parabolic Keller–Segel system (1.1). The proof of Proposition 1.2 for the parabolic-elliptic Keller–Segel system (1.3) is similar to the case for (1.1) and we do not show the case for (1.3) (cf. [38]).

Proof of Proposition 1.1 Let $1 < q_0 \le n/2$ and $1 < \alpha_0 \le n$ satisfy $2q_0 = \alpha_0$. Let (p,θ) and (r,σ) be admissible defined in (1.6) with $\theta < \sigma$. We further assume that the exponents $(q_0, q_1, \alpha_0, \alpha_1)$ are subject to the conditions (1.8).

For the initial data $(u_0, \nabla\psi_0) \in (M^{n/2}_{q_0}(\mathbb{R}^n) \cap N^{-2/\theta}_{p,q_1,\theta}(\mathbb{R}^n)) \times (M^n_{\alpha_0}(\mathbb{R}^n) \cap N^{-2/\sigma}_{r,\alpha_1,\sigma}(\mathbb{R}^n))$, let

$$\begin{cases} \Phi[u_\tau,\psi_\tau](t) = e^{t\Delta}u_0 - \displaystyle\int_0^t \nabla e^{(t-s)\Delta} \cdot (u_\tau(s)\nabla\psi_\tau(s))\, ds, \\ \Psi[u_\tau,\psi_\tau](t) = e^{\tau t(\Delta-\lambda)}\psi_0 + \displaystyle\int_0^{\tau t} e^{(\tau t - s)(\Delta-\lambda)} \Phi[u_\tau,\psi_\tau](\tau^{-1}s)\, ds. \end{cases}$$

We then introduce a metric space

$$\begin{aligned} X_M \equiv \Big\{ &(u,\psi) \in \Big(L^\infty(I; M^{\frac{n}{2}}_{q_0}) \cap L^\theta(I; M^p_{q_1})\Big) \times \Big(L^\infty(I; M^{1,n}_{\alpha_0}) \cap L^\sigma(I; M^{1,r}_{\alpha_1})\Big); \\ &\|u\|_{L^\infty(I;M^{\frac{n}{2}}_{q_0})} + \|\psi\|_{L^\infty(I;M^{1,n}_{\alpha_0})} \le M,\ \|u\|_{L^\theta(I;M^p_{q_1})} + \|\psi\|_{L^\sigma(I;M^{1,r}_{\alpha_1})} \le N \Big\}, \\ &d\big((u,\psi),(v,\phi)\big) \equiv \|u-v\|_{L^\theta(I;M^p_{q_1})} + \|\nabla(\psi-\phi)\|_{L^\sigma(I;M^r_{\alpha_1})}, \end{aligned}$$

where

$$M \equiv 4(\|u_0\|_{M^{\frac{n}{2}}_{q_0}} + \|\nabla\psi_0\|_{M^n_{\alpha_0}}) \tag{4.1}$$

and $N > 0$ which is chosen later. It is shown that the metric space (X_M, d) is complete. We show the proof of the completeness in Appendix. We now show that for any $(\Phi, \Psi) : (u_\tau, \psi_t) \to (\Phi[u_\tau, \psi_\tau], \Psi[u_\tau, \psi_\tau])$ is contraction in X_M. Then Banach–Cacciopolli fixed point theorem implies that there exists a unique solution $(u_\tau, \psi_\tau) \in X_M$ to the integral equation:

$$\begin{cases} u_\tau(t) = e^{t\Delta}u_0 - \int_0^t \nabla e^{(t-s)\Delta} \cdot (u_\tau(s)\nabla\psi_\tau(s))\, ds, \\ \psi_\tau(t) = e^{\tau t(\Delta-\lambda)}\psi_0 + \int_0^{\tau t} e^{(\tau t - s)(\Delta-\lambda)} u_\tau(\tau^{-1}s)\, ds, \end{cases} \tag{4.2}$$

which is equivalent to (1.7). By Proposition 2.2 and (4.1), we see that

$$\begin{aligned} \|e^{t\Delta}u_0\|_{L^\infty(I;M^{\frac{n}{2}}_{q_0})} &\le \|u_0\|_{M^{\frac{n}{2}}_{q_0}} \le \frac{1}{4}M, \\ \|\nabla e^{\tau t(\Delta-\lambda)}\psi_0\|_{L^\infty(I;M^n_{\alpha_0})} &\le \|\nabla\psi_0\|_{M^n_{\alpha_0}} \le \frac{1}{4}M. \end{aligned} \tag{4.3}$$

By the maximal regularity (3.7), we have

$$\begin{aligned} &\left\| \int_0^t \nabla e^{(t-s)\Delta} \cdot (u_\tau(s)\nabla\psi_\tau(s))\, ds \right\|_{L^\infty(I;M^{\frac{n}{2}}_{q_0})} \\ &\le \left\| \int_0^t \nabla e^{(t-s)\Delta} \cdot (u_\tau(s)\nabla\psi_\tau(s))\, ds \right\|_{L^\infty(I;N^0_{\frac{n}{2},q_0,1})} \\ &\le C\|u_\tau\nabla\psi_\tau\|_{L^\sigma(I;N^{-1+\frac{2}{\sigma}}_{\frac{n}{2},q_0,\infty})}. \end{aligned}$$

By the Sobolev embedding (2.5) (Proposition 2.6);

$$N^0_{\frac{nr}{n+2r},\frac{2rq_0}{n+2r},\infty} \subset N^{-1+\frac{2}{\sigma}}_{\frac{n}{2},q_0,\infty} \quad \text{with } \frac{n+2r}{nr} = \frac{2}{n} - \frac{1}{n}\left(-1+\frac{2}{\sigma}\right),$$

we have from Propositions 2.3, 2.4 and the Hölder inequality (Proposition 2.1) that

$$\begin{aligned} &\left\| \int_0^t \nabla e^{(t-s)\Delta} \cdot (u_\tau(s)\nabla\psi_\tau(s))\, ds \right\|_{L^\infty(I;M^{\frac{n}{2}}_{q_0})} \\ &\le C\|u_\tau\nabla\psi_\tau\|_{L^\sigma(I;N^0_{\frac{nr}{n+2r},\frac{2rq_0}{n+2r},\infty})} \\ &\le C\|u_\tau\nabla\psi_\tau\|_{L^\sigma(I;M^{\frac{nr}{n+2r}}_{\frac{2rq_0}{n+2r}})} \\ &\le C\|u_\tau\|_{L^\infty(I;M^{\frac{n}{2}}_{q_0})}\|\nabla\psi\|_{L^\sigma(I;M^r_{\alpha_1})} \le CMN. \end{aligned} \tag{4.4}$$

Combining (4.3) and (4.4), we see that

$$\|\Phi[u_\tau, \psi_\tau]\|_{L^\infty(I;M^{\frac{n}{2}}_{q0})} \le \frac{1}{4}M + CMN. \tag{4.5}$$

By the condition (1.8), the embedding $N^0_{p,q_1,1} \subset E^0_{p,q_1,2} \simeq M^p_{q_1}$ (with using Proposition 2.3) and maximal regularity (3.2), we have

$$\|e^{t\Delta}u_0\|_{L^\theta(I;M^p_{q_1})} \le \|e^{t\Delta}u_0\|_{L^\theta(I;N^0_{p,q_1,1})} \le C\|u_0\|_{N^{-2/\theta}_{p,q_1,\theta}},$$
$$\|e^{\tau t(\Delta-\lambda)}\psi_0\|_{L^\sigma(I;M^{1,r}_{\alpha_1})} \le \|e^{t(\Delta-\lambda)}\nabla\psi_0\|_{L^\sigma(I;N^0_{r,\alpha_1,1})} \le C\|\nabla\psi_0\|_{N^{-2/\sigma}_{r,\alpha_1,\sigma}}.$$

Then for $0 < \varepsilon_0 \le N/8$, we may choose T sufficiently small such that

$$\|e^{t\Delta}u_0\|_{L^\theta(I;M^p_{q_1})} \le \varepsilon_0 \quad \text{and} \quad \|\nabla e^{\tau t(\Delta-\lambda)}\psi_0\|_{L^\sigma(I;M^r_{\alpha_1})} \le \varepsilon_0, \tag{4.6}$$

where the choice of T does not depend on $\tau > 1$. By maximal regularity (3.7), the condition (1.8) and the Hölder inequality, we have

$$\begin{aligned}
&\left\|\int_0^t \nabla e^{(t-s)\Delta}\cdot(u_\tau(s)\nabla\psi_\tau(s))\,ds\right\|_{L^\theta(I;M^p_{q_1})}\\
&\le \left\|\int_0^t \nabla e^{(t-s)\Delta}\cdot(u_\tau(s)\nabla\psi_\tau(s))\,ds\right\|_{L^\theta(I;N^0_{p,q_1,1})}\\
&\le C\|u_\tau\nabla\psi_\tau\|_{L^{\frac{\theta\sigma}{\theta+\sigma}}(I;N^{-1+\frac{2}{\sigma}}_{p,q_1,\infty})}\\
&\le C\|u_\tau\nabla\psi_\tau\|_{L^{\frac{\theta\sigma}{\theta+\sigma}}(I;N^0_{\frac{pr}{p+r},\frac{q_1r}{p+r},\infty})}\\
&\le C\|u_\tau\nabla\psi_\tau\|_{L^{\frac{\theta\sigma}{\theta+\sigma}}(I;M^{\frac{pr}{p+r}}_{\frac{q_1r}{p+r}})}\\
&\le C\|u_\tau\|_{L^\theta(I;M^p_{q_1})}\|\nabla\psi_\tau\|_{L^\sigma(I;M^r_{\alpha_1})} \le CN^2,
\end{aligned} \tag{4.7}$$

where we use the Sobolev embedding;

$$N^0_{\frac{pr}{p+r},\frac{q_1r}{p+r},\infty} \subset N^{-1+\frac{2}{\sigma}}_{p,q_1,\infty} \quad \text{with} \ \frac{p+r}{pr} = \frac{1}{p} - \frac{1}{n}\left(-1+\frac{2}{\sigma}\right).$$

Thus, (4.6) and (4.7) imply

$$\|\Phi[u_\tau, \psi_\tau]\|_{L^\theta(I;M^p_{q_1})} \le \varepsilon_0 + CN^2 \le \frac{1}{4}N \tag{4.8}$$

if N is chosen as $N \le \min\{1/(8C), M/(4C)\}$. Similarly, it follows from (1.8) and (4.8) that

$$\begin{aligned}
&\left\| \nabla \int_0^{\tau t} e^{(\tau t-s)(\Delta-\lambda)} \Phi[u_\tau, \psi_\tau](\tau^{-1}s)\, ds \right\|_{L^\infty(I; M^n_{\alpha_0})} \\
&\le C \|\Phi[u_\tau, \psi_\tau](\tau^{-1}\cdot)|_{\cdot=\tau t}\|_{L^\theta(I; N^{-1+\frac{2}{\theta}}_{n,\alpha_0,\infty})} \\
&\le C \|\Phi[u_\tau, \psi_\tau]\|_{L^\theta(I; M^p_{q_1})} \le \frac{1}{4} CN, \qquad (4.9)\\
&\left\| \int_0^{\tau t} e^{(\tau t-s)(\Delta-\lambda)} \Phi[u_\tau, \psi_\tau](\tau^{-1}s)\, ds \right\|_{L^\infty(I; M^n_{\alpha_0})} \\
&\le C \|\Phi[u_\tau, \psi_\tau](\tau^{-1}\cdot)|_{\cdot=\tau t}\|_{L^\theta(I; N^{-2+\frac{2}{\theta}}_{n,\alpha_0,\infty})} \\
&\le C \|\Phi[u_\tau, \psi_\tau]\|_{L^\theta(I; N^{-1+\frac{2}{\theta}}_{n,\alpha_0,\infty})} \\
&\le C \|\Phi[u_\tau, \psi_\tau]\|_{L^\theta(I; M^p_{q_1})} \le \frac{1}{4} CN. \qquad (4.10)
\end{aligned}$$

Combining (4.3), (4.5), (4.9), and (4.10), we have

$$\|\Phi[u_\tau, \psi_\tau]\|_{L^\infty(I; M^{\frac{n}{2}}_{q_0})} + \|\nabla\Psi[u_\tau, \psi_\tau]\|_{L^\infty(I; M^{1,n}_{\alpha_0})} \le \frac{1}{2}M + CMN + \frac{1}{2}CN \le M. \qquad (4.11)$$

Since $\sigma > \theta$ and (1.8), it follows from (3.7) that

$$\begin{aligned}
&\left\| \nabla \int_0^{\tau t} e^{(\tau t-s)(\Delta-\lambda)} \Phi[u_\tau, \psi_\tau](\tau^{-1}s)\, ds \right\|_{L^\sigma(I; M^r_{\alpha_1})} \\
&\le C \|\Phi[u_\tau, \psi_\tau](\tau^{-1}\cdot)|_{\cdot=\tau t}\|_{L^\theta(I; N^{-1+\frac{2}{\theta}-\frac{2}{\sigma}}_{r,\alpha_1,\infty})} \\
&\le C \|\Phi[u_\tau, \psi_\tau]\|_{L^\theta(I; M^p_{\alpha_1 p/r})} \\
&\le C \|\Phi[u_\tau, \psi_\tau]\|_{L^\theta(I; M^p_{q_1})} \le \frac{1}{4} CN. \qquad (4.12)
\end{aligned}$$

Similarly, from (3.7) that

$$\begin{aligned}
&\left\| \int_0^{\tau t} e^{(\tau t-s)(\Delta-\lambda)} \Phi[u_\tau, \psi_\tau](\tau^{-1}s)\, ds \right\|_{L^\sigma(I; M^r_{\alpha_1})} \\
&\le C \|\Phi[u_\tau, \psi_\tau](\tau^{-1}\cdot)|_{\cdot=\tau t}\|_{L^\theta(I; N^{-2+\frac{2}{\theta}-\frac{2}{\sigma}}_{r,\alpha_1,\infty})} \\
&\le C \|\Phi[u_\tau, \psi_\tau]\|_{L^\theta(I; N^{-1+\frac{2}{\theta}-\frac{2}{\sigma}}_{r,\alpha_1,\infty})} \\
&\le C \|\Phi[u_\tau, \psi_\tau]\|_{L^\theta(I; M^p_{q_1})} \le \frac{1}{4} CN. \qquad (4.13)
\end{aligned}$$

Thus, by (4.6), (4.8), (4.12) and (4.13), we have

$$\|\Phi[u_\tau, \psi_\tau]\|_{L^\theta(I;M^p_{q_1})} + \|\Psi[u_\tau, \psi_\tau]\|_{L^\sigma(I;M^{1,r}_{\alpha_1})} \le \frac{1}{2}N + \varepsilon_0 + \frac{1}{2}CN \le N. \quad (4.14)$$

Combining (4.11) and (4.14), we see that $(\Phi[u_\tau, \psi_\tau], \Psi[u_\tau, \psi_\tau]) \in X_M$. Analogously from (4.7) and (4.12), we have

$$\begin{aligned}
&\|\Phi[u_\tau, \psi_\tau] - \Phi[v_\tau, \phi_\tau]\|_{L^\theta(I;M^p_{q_1})} \\
&= \left\| \int_0^t \nabla e^{(t-s)\Delta} \cdot (u_\tau(s)\nabla\psi_\tau(s) - v_\tau(s)\nabla\phi_\tau(s))\, ds \right\|_{L^\theta(I;M^p_{q_1})} \\
&\le C\|u_\tau\nabla\psi_\tau - v_\tau\nabla\phi_\tau\|_{L^{\frac{\theta\sigma}{\theta+\sigma}}(I;N^0_{\frac{pr}{p+r},\frac{q_1 r}{p+r},\infty})} \\
&\le C\|(u_\tau - v_\tau)\nabla\psi_\tau\|_{L^{\frac{\theta\sigma}{\theta+\sigma}}(I;N^0_{\frac{pr}{p+r},\frac{q_1 r}{p+r},\infty})} + C\|v_\tau(\nabla\psi_\tau - \nabla\phi_\tau)\|_{L^{\frac{\theta\sigma}{\theta+\sigma}}(I;N^0_{\frac{pr}{p+r},\frac{q_1 r}{p+r},\infty})} \\
&\le C\|u_\tau - v_\tau\|_{L^\theta(I;M^p_{q_1})}\|\nabla\psi_\tau\|_{L^\sigma(I;M^r_{\alpha_1})} + C\|v_\tau\|_{L^\theta(I;M^p_{q_1})}\|\nabla\psi_\tau - \nabla\phi_\tau\|_{L^\sigma(I;M^r_{\alpha_1})} \\
&\le CN(\|u_\tau - v_\tau\|_{L^\theta(I;M^p_{q_1})} + \|\nabla\psi_\tau - \nabla\phi_\tau\|_{L^\sigma(I;M^r_{\alpha_1})}) \le CNd((u_\tau, \psi_\tau), (v_\tau, \phi_\tau))
\end{aligned}$$

and similarly,

$$\begin{aligned}
&\|\nabla\Psi[u_\tau, \psi_\tau] - \nabla\Psi[v_\tau, \phi_\tau]\|_{L^\sigma(I;M^r_{\alpha_1})} \\
&= \left\| \int_0^{\tau t} \nabla e^{(\tau t-s)\Delta} \cdot (\Phi[u_\tau, \psi_\tau](\tau^{-1}s) - \Phi[v_\tau, \phi_\tau](\tau^{-1}s))\, ds \right\|_{L^\sigma(I;M^r_{\alpha_1})} \\
&\le C\|\Phi[u_\tau, \psi_\tau] - \Phi[v_\tau, \phi_\tau]\|_{L^\theta(I;M^p_{q_1})} \\
&\le CNd((u_\tau, \psi_\tau), (v_\tau, \phi_\tau))
\end{aligned}$$

for $(u_\tau, \psi_\tau), (v_\tau, \phi_\tau) \in X_T$. We choose N smaller as $CN \le 1/8$. Thus, we obtain

$$\begin{aligned}
&\|\Phi[u_\tau, \psi_\tau] - \Phi[v_\tau, \phi_\tau]\|_{L^\theta(I;M^p_{q_1})} + \|\nabla\Psi[u_\tau, \psi_\tau] - \nabla\Psi[v_\tau, \phi_\tau]\|_{L^\sigma(I;M^r_{\alpha_1})} \\
&\le \frac{1}{2}d((u_\tau, \psi_\tau), (v_\tau, \phi_\tau)),
\end{aligned}$$

which implies that (Φ, Ψ) is a contraction onto X_M. By the Banach fixed point theorem, there exists a unique fixed point $(u_\tau, \psi_\tau) \in X_M$ which solves (1.7).

We prove the continuous dependence on initial data. Let (u_τ, ψ_τ) and (v_τ, ϕ_τ) be solutions to (1.7) with initial data (u_0, ψ_0), (v_0, ϕ_0), respectively. Then, we see that

$$\begin{aligned}
&\|u_\tau - v_\tau\|_{L^\theta(I;M^p_{q_1})} \\
&\le \|e^{t\Delta}u_0 - e^{t\Delta}v_0\|_{L^\theta(I;M^p_{q_1})} \\
&\qquad + \left\| \int_0^t \nabla e^{(t-s)\Delta} \cdot (u_\tau(s)\nabla\psi_\tau(s) - v_\tau(s)\nabla\phi_\tau(s))\, ds \right\|_{L^\theta(I;M^p_{q_1})}
\end{aligned}$$

$$
\begin{aligned}
&\leq C\|u_0 - v_0\|_{N^0_{\frac{n}{2},q_0,\theta}} + CN(\|u_\tau - v_\tau\|_{L^\theta(I;M^p_{q_1})} + \|\nabla\psi_\tau - \nabla\phi_\tau\|_{L^\sigma(I;M^r_{\alpha_1})}),\\
&\|\nabla\psi_\tau - \nabla\phi_\tau\|_{L^\sigma(I;M^r_{\alpha_1})}\\
&\leq \|\nabla e^{\tau t(\Delta-\lambda)}\psi_0 - \nabla e^{\tau t(\Delta-\lambda)}\phi_0\|_{L^\sigma(I;M^r_{\alpha_1})}\\
&\quad + \left\|\nabla \int_0^{\tau t} e^{(\tau t - s)\Delta}(u_\tau(\tau^{-1}s) - v_\tau(\tau^{-1}s))\, ds\right\|_{L^\sigma(I;M^r_{\alpha_1})}\\
&\leq C\|\nabla\psi_0 - \nabla\phi_0\|_{N^0_{n,\alpha_0,\sigma}} + C\|u_\tau - v_\tau\|_{L^\theta(I;M^p_{q_1})},
\end{aligned}
$$

and hence, we have

$$
\begin{aligned}
&\|u_\tau - v_\tau\|_{L^\theta(I;M^p_{q_1})} + \|\nabla\psi_\tau - \nabla\phi_\tau\|_{L^\sigma(I;M^r_{\alpha_1})}\\
&\quad \leq C(\|u_0 - v_0\|_{N^0_{\frac{n}{2},q_0,\theta}} + \|\nabla\psi_0 - \nabla\phi_0\|_{N^0_{n,\alpha_0,\sigma}}).
\end{aligned} \tag{4.15}
$$

Similarly to the above argument, we see that

$$
\begin{aligned}
&\|u_\tau - v_\tau\|_{L^\infty(I;M^{\frac{n}{2}}_{q_0})}\\
&\leq \|e^{t\Delta}u_0 - e^{t\Delta}v_0\|_{L^\infty(I;M^{\frac{n}{2}}_{q_0})}\\
&\quad + \left\|\int_0^t \nabla e^{(t-s)\Delta}\cdot(u_\tau(s)\nabla\psi_\tau(s) - v_\tau(s)\nabla\phi_\tau(s))\, ds\right\|_{L^\infty(I;M^{\frac{n}{2}}_{q_0})}\\
&\leq \|u_0 - v_0\|_{M^{\frac{n}{2}}_{q_0}} + CN\|u_\tau - v_\tau\|_{L^\infty(I;M^{\frac{n}{2}}_{q_0})} + CM\|\nabla\psi_\tau - \nabla\phi_\tau\|_{L^\sigma(I;M^r_{\alpha_1})},\\
&\|\nabla\psi_\tau - \nabla\phi_\tau\|_{L^\infty(I;M^n_{\alpha_0})}\\
&\leq \|\nabla e^{\tau t(\Delta-\lambda)}\psi_0 - \nabla e^{\tau t(\Delta-\lambda)}\phi_0\|_{L^\infty(I;M^n_{\alpha_0})}\\
&\quad + \left\|\nabla \int_0^{\tau t} e^{(\tau t - s)\Delta}(u_\tau(\tau^{-1}s) - v_\tau(\tau^{-1}s))\, ds\right\|_{L^\infty(I;M^n_{\alpha_0})}\\
&\leq C\|\nabla\psi_0 - \nabla\phi_0\|_{M^n_{\alpha_0}} + C\|u_\tau - v_\tau\|_{L^\theta(I;M^p_{q_1})}.
\end{aligned}
$$

By (4.15), we obtain

$$
\begin{aligned}
&\|u_\tau - v_\tau\|_{L^\infty(I;M^{\frac{n}{2}}_{q_0})} + \|\nabla\psi_\tau - \nabla\phi_\tau\|_{L^\infty(I;M^n_{\alpha_0})}\\
&\leq C(\|u_0 - v_0\|_{M^{\frac{n}{2}}_{q_0}} + \|u_0 - v_0\|_{N^0_{\frac{n}{2},q_0,\theta}} + \|\nabla\psi_0 - \nabla\phi_0\|_{M^n_{\alpha_0}} + \|\nabla\psi_0 - \nabla\phi_0\|_{N^0_{n,\alpha_0,\sigma}}),
\end{aligned} \tag{4.16}
$$

which proves the continuous dependence on initial data. By (4.16), we see that $u_\tau(t) \in \mathcal{M}^{n/2}_{q_0}(\mathbb{R}^n)$ for any $t \geq 0$ and $u_\tau(t) \in \mathcal{M}^p_{q_1}(\mathbb{R}^n)$ almost everywhere $t > 0$.

We show that $(u_\tau, \nabla\psi_\tau) \in C([0,T); M^{n/2}_{q_0}(\mathbb{R}^n)) \times C([0,T); M^n_{\alpha_0}(\mathbb{R}^n))$. Let $0 < t < t+h < T$. We see that

$$\begin{aligned}
&\|u_\tau(t+h) - u_\tau(t)\|_{M^{\frac{n}{2}}_{q_0}} \\
&\quad \le \|e^{(t+h)\Delta}u_0 - e^{t\Delta}u_0\|_{M^{\frac{n}{2}}_{q_0}} + \left\| \int_t^{t+h} \nabla e^{(t+h-s)\Delta} \cdot (u_\tau(s)\nabla\psi_\tau(s))\,ds \right\|_{M^{\frac{n}{2}}_{q_0}} \\
&\quad + \left\| \int_0^t (\nabla e^{(t+h-s)\Delta} \cdot (u_\tau(s)\nabla\psi_\tau(s)) - \nabla e^{(t-s)\Delta} \cdot (u_\tau(s)\nabla\psi_\tau(s)))\,ds \right\|_{M^{\frac{n}{2}}_{q_0}}.
\end{aligned} \tag{4.17}$$

Since $e^{t\Delta}u_0 \in \mathcal{M}^{n/2}_{q_0}(\mathbb{R}^n)$, we have

$$\|e^{(t+h)\Delta}u_0 - e^{t\Delta}u_0\|_{M^{\frac{n}{2}}_{q_0}} \to 0 \quad \text{as } h \to 0.$$

By the estimate (3.8), we see that

$$\begin{aligned}
&\left\| \int_0^t (\nabla e^{(t+h-s)\Delta} \cdot (u_\tau(s)\nabla\psi_\tau(s)) - \nabla e^{(t-s)\Delta} \cdot (u_\tau(s)\nabla\psi_\tau(s)))\,ds \right\|_{M^{\frac{n}{2}}_{q_0}} \\
&\quad \le C\|e^{h\Delta}(u_\tau\nabla\psi_\tau) - u_\tau\nabla\psi_\tau\|_{L^\theta(I; M^{\frac{nr}{n+2r}}_{\frac{q_0\alpha_1}{q_0+\alpha_1}})}.
\end{aligned}$$

Again, since $u_\tau(t)\nabla\psi_\tau(t) \in \mathcal{M}^{nr/(n+2r)}_{q_0\alpha_1/(q_0+\alpha_1)}(\mathbb{R}^n)$, we have

$$\|e^{h\Delta}(u_\tau\nabla\psi_\tau) - u_\tau\nabla\psi_\tau\|_{M^{\frac{nr}{n+2r}}_{\frac{q_0\alpha_1}{q_0+\alpha_1}}} \to 0 \quad \text{as } h \to 0.$$

On the other hand, it follows from the Hölder inequality that

$$\begin{aligned}
\|e^{h\Delta}(u_\tau\nabla\psi_\tau) - u_\tau\nabla\psi_\tau\|_{L^\theta(I; M^{\frac{nr}{n+2r}}_{\frac{q_0\alpha_1}{q_0+\alpha_1}})} &\le 2\|u_\tau\nabla\psi_\tau\|_{L^\theta(I; M^{\frac{nr}{n+2r}}_{\frac{q_0\alpha_1}{q_0+\alpha_1}})} \\
&\le 2\|u_\tau\|_{L^\infty(I; M^{\frac{n}{2}}_{q_0})}\|\nabla\psi_\tau\|_{L^\theta(I; M^r_{\alpha_1})}.
\end{aligned}$$

Thus, the Lebesgue convergence theorem implies that

$$\left\| \int_0^t (\nabla e^{(t+h-s)\Delta} \cdot (u_\tau(s)\nabla\psi_\tau(s)) - \nabla e^{(t-s)\Delta} \cdot (u_\tau(s)\nabla\psi_\tau(s)))\,ds \right\|_{M^{\frac{n}{2}}_{q_0}} \to 0$$

as $h \to 0$.

On the second term of the right hand side in (4.17), we have

$$\left\| \int_t^{t+h} \nabla e^{(t+h-s)\Delta} \cdot (u_\tau(s) \nabla \psi_\tau(s))\, ds \right\|_{M^{\frac{n}{2}}_{q_0}} \le C \|\chi_{(t,t+h)} u_\tau \nabla \psi_\tau\|_{L^\theta(I; M^{\frac{nr}{n+2r}}_{\frac{q_0\alpha_1}{q_0+\alpha_1}})}$$
$$\to 0 \quad \text{as } h \to 0.$$

For $\nabla \psi_\tau$, we see that

$$\begin{aligned} &\|\nabla \psi_\tau(t+h) - \nabla \psi_\tau(t)\|_{M^n_{\alpha_0}} \\ &\quad \le \|\nabla e^{\tau(t+h)(\Delta-\lambda)} \psi_0 - \nabla e^{\tau t(\Delta-\lambda)} \psi_0\|_{M^n_{\alpha_0}} \\ &\qquad + \left\| \nabla \int_{\tau t}^{\tau(t+h)} e^{(\tau(t+h)-s)(\Delta-\lambda)} u_\tau(\tau^{-1}s)\, ds \right\|_{M^n_{\alpha_0}} \\ &\qquad + \left\| \nabla \int_0^{\tau t} (e^{(\tau(t+h)-s)(\Delta-\lambda)} u_\tau(\tau^{-1}s) - e^{(\tau t-s)(\Delta-\lambda)} u_\tau(\tau^{-1}s))\, ds \right\|_{M^n_{\alpha_0}}. \end{aligned} \tag{4.18}$$

Since $\nabla e^{\tau t(\Delta-\lambda)} \psi_0 \in \mathcal{M}^n_{\alpha_0}(\mathbb{R}^n)$, we have

$$\|\nabla e^{\tau(t+h)(\Delta-\lambda)} \psi_0 - \nabla e^{\tau t(\Delta-\lambda)} \psi_0\|_{M^n_{\alpha_0}} \to 0 \quad \text{as } h \to 0.$$

By (3.7), we see that

$$\begin{aligned} &\left\| \nabla \int_0^{\tau t} (e^{(\tau(t+h)-s)(\Delta-\lambda)} u_\tau(\tau^{-1}s) - e^{(\tau t-s)(\Delta-\lambda)} u_\tau(\tau^{-1}s))\, ds \right\|_{M^n_{\alpha_0}} \\ &\quad \le C \|e^{\tau h(\Delta-\lambda)} u_\tau - u_\tau\|_{L^\theta(I; M^p_{q_1})}. \end{aligned}$$

Again, since $u_\tau(t) \in \mathcal{M}^p_{q_1}(\mathbb{R}^n)$ almost everywhere $t > 0$, we have

$$\|e^{\tau h(\Delta-\lambda)} u_\tau(t) - u_\tau(t)\|_{M^p_{q_1}} \to 0 \quad \text{a.e. } t > 0$$

as $h \to 0$. On the other hand, it follows from the Hölder inequality that

$$\|e^{\tau h(\Delta-\lambda)} u_\tau - u_\tau\|_{L^\theta(I; M^p_{q_1})} \le C \|u_\tau\|_{L^\theta(I; M^p_{q_1})}.$$

Thus, the Lebesgue convergence theorem implies that

$$\left\| \nabla \int_0^{\tau t} (e^{(\tau(t+h)-s)(\Delta-\lambda)} u_\tau(\tau^{-1}s) - e^{(\tau t-s)(\Delta-\lambda)} u_\tau(\tau^{-1}s))\, ds \right\|_{M^n_{\alpha_0}} \to 0 \quad \text{as } h$$
$$\to 0.$$

On the second term of the right hand side in (4.18), we have

$$\begin{aligned}&\left\|\nabla\int_{\tau t}^{\tau(t+h)} e^{(\tau(t+h)-s)(\Delta-\lambda)}u_\tau(\tau^{-1}s)\,ds\right\|_{M^n_{\alpha_0}}\\&\le C\|\chi_{(\tau t,\tau(t+h))}u_\tau\|_{L^\theta(I;M^p_{q_1})}\to 0\quad\text{as } h\to 0.\end{aligned}$$

Furthermore, we have

$$\begin{aligned}&\|u_\tau(t)-u_0\|_{M^{\frac{n}{2}}_{q_0}}\\&\le\|e^{t\Delta}u_0-u_0\|_{M^{\frac{n}{2}}_{q_0}}+\left\|\int_0^t\nabla e^{(t-s)\Delta}\cdot(u_\tau(s)\nabla\psi_\tau(s))\,ds\right\|_{M^{\frac{n}{2}}_{q_0}},\\&\|\nabla\psi_\tau(t)-\nabla\psi_0\|_{M^n_{\alpha_0}}\\&\le\|\nabla e^{\tau t(\Delta-\lambda)}\psi_0-\nabla\psi_0\|_{M^n_{\alpha_0}}+\left\|\nabla\int_0^{\tau t}e^{(\tau t-s)(\Delta-\lambda)}u_\tau(\tau^{-1}s)\,ds\right\|_{M^n_{\alpha_0}}.\end{aligned}$$

Since $u_0\in\mathcal{M}^{n/2}_{q_0}(\mathbb{R}^n)$ and $\nabla\psi_0\in\mathcal{M}^n_{\alpha_0}(\mathbb{R}^n)$, the solution $(u(t),\nabla\psi(t))$ converges to $(u_0,\nabla\psi_0)$ in $M^{n/2}_{q_0}(\mathbb{R}^n)\times M^n_{\alpha_0}(\mathbb{R}^n)$ as $t\to 0$. □

5 Singular limit problem

Proof of Theorem 1.3 Let $n\ge 3,\lambda,\tau>0, 1<q_0\le n/2, 1<\alpha_0\le n$, and $I\equiv(0,T)$ for $0<T<\infty$. We take $(u_0,\nabla\psi_0)\in M^{n/2}_{q_0}(\mathbb{R}^n)\times M^n_{\alpha_0}(\mathbb{R}^n)$. Let (p,θ) and (r,σ) satisfy (1.6) and (q_0,q_1) and (α_0,α_2) satisfies (1.8). We recall that the solution to (1.1) solves the integral Eq. (4.2) and

$$\begin{cases}u(t)=e^{t\Delta}u_0+\displaystyle\int_{\mathbb{R}^n}\nabla e^{(t-s)\Delta}\cdot(u(s)\nabla\psi(s))\,ds,\\ \psi(t)=(\lambda-\Delta)^{-1}u(t)=\displaystyle\int_0^\infty e^{s(\Delta-\lambda)}u(t)\,ds.\end{cases}$$

By changing the variable, the potential term ψ_τ can be rewritten by

$$\psi_\tau(t)=e^{\tau t(\Delta-\lambda)}\psi_0+\int_0^{\tau t}e^{s(\Delta-\lambda)}u_\tau\left(t-\frac{s}{\tau}\right)ds,$$

and the difference of solutions to the first equation is written by

$$\begin{aligned}u_\tau(t)-u(t)=&\int_0^t\nabla e^{(t-s)\Delta}\cdot((u_\tau(s)-u(s))\nabla\psi_\tau(s))\,ds\\&+\int_0^t\nabla e^{(t-s)\Delta}\cdot(u(s)(\nabla\psi_\tau(s)-\nabla\psi(s)))\,ds.\end{aligned}\tag{5.1}$$

and

$$\begin{aligned}
&\nabla\psi_\tau(t) - \nabla\psi(t)\\
&\quad = \nabla e^{\tau t(\Delta-\lambda)}\psi_0 + \int_0^{\tau t} \nabla e^{s(\Delta-\lambda)} \left(u_\tau\left(t-\frac{s}{\tau}\right) - u\left(t-\frac{s}{\tau}\right)\right) ds \\
&\qquad + \int_0^{\tau t} \nabla e^{s(\Delta-\lambda)} \left(u\left(t-\frac{s}{\tau}\right) - u(t)\right) ds - \int_{\tau t}^{\infty} \nabla e^{s(\Delta-\lambda)} u(t)\, ds.
\end{aligned} \tag{5.2}$$

We estimate the difference (5.1) of solutions to the first equation. Taking the norm of $L^\theta(I; M^p_{q_1})$, we have

$$\begin{aligned}
\|u_\tau - u\|_{L^\theta(I;M^p_{q_1})} \le &\left\| \int_0^t \nabla e^{(t-s)\Delta} \cdot ((u_\tau(s) - u(s))\nabla\psi_\tau(s))\, ds \right\|_{L^\theta(I;M^p_{q_1})} \\
&+ \left\| \int_0^t \nabla e^{(t-s)\Delta} \cdot (u(s)(\nabla\psi_\tau(s) - \nabla\psi(s)))\, ds \right\|_{L^\theta(I;M^p_{q_1})}.
\end{aligned}$$

It suffices to consider only the first term of the right hand side. By the maximal regularity (3.7), we see that

$$\begin{aligned}
&\left\| \int_0^t \nabla e^{(t-s)\Delta} \cdot ((u_\tau(s) - u(s))\nabla\psi_\tau(s))\, ds \right\|_{L^\theta(I;M^p_{q_1})} \\
&\quad \le C \left\| \int_0^t \nabla e^{(t-s)\Delta} ((u_\tau(s) - u(s))\nabla\psi_\tau(s))\, ds \right\|_{L^\theta(I;N^0_{p,q_1,1})} \\
&\quad \le C \left\| (u_\tau(s) - u(s))\nabla\psi_\tau(s) \right\|_{L^{\frac{\theta\sigma}{\theta+\sigma}}(I;N^{-1+\frac{2}{\sigma}}_{p,q_1,\infty})}.
\end{aligned}$$

The Sobolev embedding

$$N^0_{\frac{pr}{p+r},\frac{q_1 r}{p+r},\infty} \subset N^{-1+\frac{2}{\sigma}}_{p,q_1,\infty} \quad \text{with} \quad \frac{p+r}{pr} = \frac{1}{p} - \frac{1}{n}\left(-1+\frac{2}{\sigma}\right)$$

and the Hölder inequality give

$$\begin{aligned}
&\|(u_\tau(s) - u(s))\nabla\psi_\tau(s)\|_{L^{\frac{\theta\sigma}{\theta+\sigma}}(I;N^{-1+\frac{2}{\sigma}}_{p,q_1,\infty})} \\
&\quad \le C \left\|(u_\tau(s) - u(s))\nabla\psi_\tau(s)\right\|_{L^{\frac{\theta\sigma}{\theta+\sigma}}(I;N^0_{\frac{pr}{p+r},\frac{q_1 r}{p+r},\infty})} \\
&\quad \le C \left\|(u_\tau(s) - u(s))\nabla\psi_\tau(s)\right\|_{L^{\frac{\theta\sigma}{\theta+\sigma}}(I;M^{\frac{pr}{p+r}}_{\frac{q_1 r}{p+r}})} \\
&\quad \le C\|u_\tau - u\|_{L^\theta(I;M^p_{q_1})} \|\nabla\psi_\tau\|_{L^\sigma(I;M^r_{\alpha_1})}.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}&\left\|\int_0^t \nabla e^{(t-s)\Delta}\cdot((u_\tau(s)-u(s))\nabla\psi_\tau(s))\,ds\right\|_{L^\theta(I;M^p_{q_1})}\\&\quad\le C\|u_\tau-u\|_{L^\theta(I;M^p_{q_1})}\|\nabla\psi_\tau\|_{L^\sigma(I;M^r_{\alpha_1})}.\end{aligned}$$

Similarly to the above argument, we have

$$\begin{aligned}&\left\|\int_0^t \nabla e^{(t-s)\Delta}\cdot(u(s)(\nabla\psi_\tau(s)-\nabla\psi(s)))\,ds\right\|_{L^\theta(I;M^p_{q_1})}\\&\quad\le C\|u\|_{L^\theta(I;M^p_{q_1})}\|\nabla\psi_\tau-\nabla\psi\|_{L^\sigma(I;M^r_{\alpha_1})},\end{aligned}$$

and hence, we obtain

$$\|u_\tau-u\|_{L^\theta(I;M^p_{q_1})}\le CM(\|u_\tau-u\|_{L^\theta(I;M^p_{q_1})}+\|\nabla\psi_\tau-\nabla\psi\|_{L^\sigma(I;M^r_{\alpha_1})}), \tag{5.3}$$

where $M\equiv\max\{\|\nabla\psi_\tau\|_{L^\sigma(I;M^r_{\alpha_1})},\|u\|_{L^\theta(I;M^p_{q_1})}\}$, which is independent of $\tau>0$.

We decompose (5.2) as follows:

$$\begin{aligned}&\|\nabla\psi_\tau-\nabla\psi\|_{L^\sigma(I;M^r_{\alpha_1})}\\&\quad\le\|\nabla e^{\tau t(\Delta-\lambda)}\psi_0\|_{L^\sigma(I;M^r_{\alpha_1})}\\&\qquad+\left\|\int_0^{\tau t}\nabla e^{s(\Delta-\lambda)}\left(u_\tau\left(t-\frac{s}{\tau}\right)-u\left(t-\frac{s}{\tau}\right)\right)ds\right\|_{L^\sigma(I;M^r_{\alpha_1})}\\&\qquad+\left\|\int_0^{\tau t}\nabla e^{s(\Delta-\lambda)}\left(u\left(t-\frac{s}{\tau}\right)-u(t)\right)ds\right\|_{L^\sigma(I;M^r_{\alpha_1})}\\&\qquad+\left\|\int_{\tau t}^{\infty}\nabla e^{s(\Delta-\lambda)}u(t)\,ds\right\|_{L^\sigma(I;M^r_{\alpha_1})}\\&\quad\equiv I_0+I_1+I_2+I_3.\end{aligned}$$

For any $\varepsilon>0$, taking $\tau>0$ sufficiently large, then it follows from Proposition 3.2 that

$$I_0=\left(\int_0^T\|\nabla e^{\tau t(\Delta-\lambda)}\psi_0\|^\sigma_{M^r_{\alpha_1}}\,dt\right)^{\frac{1}{\sigma}}\le\tau^{-\frac{1}{\sigma}}\left(\int_0^\infty\|\nabla e^{s(\Delta-\lambda)}\psi_0\|^\sigma_{M^r_{\alpha_1}}\,ds\right)^{\frac{1}{\sigma}}<\varepsilon. \tag{5.4}$$

By maximal regularity (3.7) with $\sigma > \theta$ and Proposition 3.5, it follows

$$\begin{aligned}
I_1 &= \left\| \int_0^{\tau t} \nabla e^{(\tau t - s)(\Delta - \lambda)} \left(u_\tau \left(\frac{s}{\tau} \right) - u \left(\frac{s}{\tau} \right) \right) ds \right\|_{L^\sigma(I; M^r_{\alpha_1})} \\
&\le C \left\| \int_0^{\tau t} \nabla e^{(\tau t - s)(\Delta - \lambda)} \left(u_\tau \left(\frac{s}{\tau} \right) - u \left(\frac{s}{\tau} \right) \right) ds \right\|_{L^\sigma(I; N^0_{r,\alpha_1,1})} \\
&\le C \left\| \left(u_\tau \left(\frac{s}{\tau} \right) - u \left(\frac{s}{\tau} \right) \right) \Big|_{s=\tau t} \right\|_{L^\theta(I; N^{-1+\frac{2}{\theta}-\frac{2}{\sigma}}_{r,\alpha_1,\infty})} \\
&\le C \|u_\tau - u\|_{L^\theta(I; N^0_{p,\frac{\alpha_1 p}{r},\infty})} \\
&\le C \|u_\tau - u\|_{L^\theta(I; M^p_{q_1})}.
\end{aligned} \tag{5.5}$$

For I_2, we have

$$\begin{aligned}
I_2 &= \left\| \int_0^{\tau t} \nabla e^{s(\Delta - \lambda)} \left(u \left(t - \frac{s}{\tau} \right) - u(t) \right) ds \right\|_{L^\sigma(I; M^r_{\alpha_1})} \\
&\le \left\| \int_0^{\tau t} \nabla e^{s(\Delta - \lambda)} \left(u \left(t - \frac{s}{\tau} \right) - u(t) \right) ds \right. \\
&\qquad \left. - \int_0^t \nabla (\lambda - \Delta)^{-1} e^{\tau s(\Delta - \lambda)} \Delta u(t-s)\, ds\, ds \right\|_{L^\sigma(I; M^r_{\alpha_1})} \\
&\quad + \left\| \int_0^t \nabla e^{\tau s(\Delta - \lambda)} (\lambda - \Delta)^{-1} \Delta u(t-s)\, ds \right\|_{L^\sigma(I; M^r_{\alpha_1})} \\
&\equiv \tilde{I}_2 + \left\| \int_0^t \nabla e^{\tau s(\Delta - \lambda)} (\lambda - \Delta)^{-1} \Delta u(t-s)\, ds \right\|_{L^\sigma(I; M^r_{\alpha_1})}.
\end{aligned}$$

It follows from (3.8) that for any $\varepsilon > 0$, taking $\tau > 0$ sufficiently large,

$$\begin{aligned}
&\left\| \int_0^t \nabla e^{\tau s(\Delta - \lambda)} (\lambda - \Delta)^{-1} \Delta u(t-s)\, ds \right\|_{L^\sigma(I; M^r_{\alpha_1})} \\
&\quad \le C \left\| \int_0^t \nabla e^{\tau(t-s)(\Delta - \lambda)} u(s)\, ds \right\|_{L^\sigma(I; M^r_{\alpha_1})} \\
&\quad \le C \tau^{-1} \|u\|_{L^\theta(I; M^p_{q_1})} < \varepsilon.
\end{aligned} \tag{5.6}$$

By the Sobolev inequality, we see that

$$\begin{aligned}
\tilde{I}_2 \le C &\left\| \int_0^{\tau t} (\lambda - \Delta) e^{s(\Delta - \lambda)} \left(u \left(t - \frac{s}{\tau} \right) - u(t) \right) ds \right. \\
&\left. - \int_0^t e^{\tau s(\Delta - \lambda)} \Delta u(s)\, ds\, ds \right\|_{L^\sigma(I; M^{\frac{nr}{n+r}}_{\frac{n\alpha_1}{n+r}})}.
\end{aligned}$$

By the mean value theorem and Fubini's theorem, we have

$$\begin{aligned}
&\int_0^{\tau t} (\lambda - \Delta) e^{s(\Delta-\lambda)} \left(\int_0^s \frac{\partial}{\partial \rho} u\left(t - \frac{\rho}{\tau}\right) d\rho \right) ds \\
&= \int_0^{\tau t} \left(\int_\rho^{\tau t} (\lambda - \Delta) e^{s(\Delta-\lambda)} ds \right) \frac{\partial}{\partial \rho} u\left(t - \frac{\rho}{\tau}\right) d\rho \\
&= \int_0^{\tau t} \left(e^{\rho(\Delta-\lambda)} - e^{\tau t(\Delta-\lambda)} \right) \frac{\partial}{\partial \rho} u\left(t - \frac{\rho}{\tau}\right) d\rho \\
&= \int_0^{\tau t} e^{\rho(\Delta-\lambda)} \frac{\partial}{\partial \rho} u\left(t - \frac{\rho}{\tau}\right) d\rho - e^{\tau t(\Delta-\lambda)} \left[u\left(t - \frac{\rho}{\tau}\right) \right]_{\rho=0}^{\rho=\tau t}.
\end{aligned}$$

By the triangle inequality, we decompose $\tilde{I}_2$ as follows:

$$\begin{aligned}
\tilde{I}_2 \le C &\left\| \int_0^{\tau t} e^{\rho(\Delta-\lambda)} \frac{\partial}{\partial \rho} u\left(t - \frac{\rho}{\tau}\right) d\rho - \int_0^t e^{\tau s(\Delta-\lambda)} \Delta u\,(t-s)\, ds \right\|_{L^\sigma(I; M^{\frac{nr}{n+r}}_{\frac{n\alpha_1}{n+r}})} \\
&+ C \left\| e^{\tau t(\Delta-\lambda)} u_0 \right\|_{L^\sigma(I; M^{\frac{nr}{n+r}}_{\frac{n\alpha_1}{n+r}})} + C \left\| e^{\tau t(\Delta-\lambda)} u(t) \right\|_{L^\sigma(I; M^{\frac{nr}{n+r}}_{\frac{n\alpha_1}{n+r}})} \equiv I_{2,1} + I_{2,2} + I_{2,3}.
\end{aligned}$$

For $I_{2,1}$, changing the variable and using the first equation of (1.1), we have

$$\begin{aligned}
I_{2,1} &= C \left\| \int_0^t e^{\tau s(\Delta-\lambda)} \frac{\partial}{\partial s} u(t-s)\, ds - \int_0^t e^{\tau s(\Delta-\lambda)} \Delta u\,(t-s)\, ds \right\|_{L^\sigma(I; M^{\frac{nr}{n+r}}_{\frac{n\alpha_1}{n+r}})} \\
&= C \left\| \int_0^t e^{\tau s(\Delta-\lambda)} \nabla \cdot (u(t-s) \nabla \psi(t-s))\, ds \right\|_{L^\sigma(I; M^{\frac{nr}{n+r}}_{\frac{n\alpha_1}{n+r}})}.
\end{aligned}$$

By maximal regularity (3.7), for any $\varepsilon > 0$, taking $\tau > 0$ sufficiently large, we then see that

$$\begin{aligned}
I_{2,1} &= C \left\| \int_0^t e^{\tau s(\Delta-\lambda)} \nabla \cdot (u(t-s) \nabla \psi(t-s))\, ds \right\|_{L^\sigma(I; M^{\frac{nr}{n+r}}_{\frac{n\alpha_1}{n+r}})} \\
&\le C \tau^{-1} \| u \nabla \psi \|_{L^{\frac{\theta\sigma}{\theta+\sigma}}(I; N^{-1+\frac{2}{\theta}}_{\frac{nr}{n+r}, \frac{n\alpha_1}{n+r}, \infty})} \\
&\le C \tau^{-1} \| u \nabla \psi \|_{L^{\frac{\theta\sigma}{\theta+\sigma}}(I; M^{\frac{pr}{p+r}}_{\frac{q_1\alpha_1}{q_1+\alpha_1}})} \\
&\le C \tau^{-1} \| u \|_{L^\theta(I; M^p_{q_1})} \| \nabla \psi \|_{L^\sigma(I; M^r_{\alpha_1})} < \varepsilon.
\end{aligned}$$

Thus, we obtain

$$I_{2,1} < \varepsilon. \tag{5.7}$$

For $I_{2,2}$ and $I_{2,3}$, $u_0, u(t) \in M^{\frac{n}{2}}_{q_0}$ satisfy

$$\|e^{\tau t(\Delta-\lambda)}u_0\|_{M^{\frac{nr}{n+r}}_{\frac{n\alpha_1}{n+r}}} \leq Ce^{-\tau t\lambda}\left(1+(\tau t)^{-\frac{n}{2}\left(\frac{1}{n}-\frac{1}{r}\right)}\right)\|u_0\|_{M^{\frac{n}{2}}_{q_0}},$$
$$\|e^{\tau t(\Delta-\lambda)}u(t)\|_{M^{\frac{nr}{n+r}}_{\frac{n\alpha_1}{n+r}}} \leq Ce^{-\tau t\lambda}\left(1+(\tau t)^{-\frac{n}{2}\left(\frac{1}{n}-\frac{1}{r}\right)}\right)\|u(t)\|_{M^{\frac{n}{2}}_{q_0}}.$$

By the maximal regularity (3.2) and the heat estimate, we have

$$\|e^{t(\Delta-\lambda)}u_0\|_{L^\sigma(I;M^{\frac{nr}{n+r}}_{\frac{n\alpha_1}{n+r}})} \leq C\|u_0\|_{N^{-\frac{2}{\sigma}}_{\frac{nr}{n+r},\frac{n\alpha_1}{n+r},\sigma}} \leq C\|u_0\|_{N^0_{\frac{n}{2},q_1,\sigma}} \leq C\|u_0\|_{N^0_{\frac{n}{2},q_1,\theta}},$$
$$\|e^{t(\Delta-\lambda)}u(t)\|_{L^\sigma(I;M^{\frac{nr}{n+r}}_{\frac{n\alpha_1}{n+r}})} \leq C\|u\|_{L^\sigma(I;M^{\frac{nr}{n+r}}_{\frac{n\alpha_1}{n+r}})}.$$

By the Hölder inequality, we see that

$$\|u\|_{L^\sigma(I;M^{\frac{nr}{n+r}}_{\frac{n\alpha_1}{n+r}})} = \left\|\|u\|_{M^{\frac{nr}{n+r}}_{\frac{n\alpha_1}{n+r}}}\right\|_{L^\sigma(I)} \leq \left\|\|u\|^{1-\mu}_{M^{\frac{n}{2}}_{\tilde{q}_1}}\|u\|^{\mu}_{M^p_{q_1}}\right\|_{L^\sigma(I)}$$
$$\leq \|u\|^{1-\mu}_{L^\infty(I;M^{\frac{n}{2}}_{\tilde{q}_1})}\|u\|^{\mu}_{L^\theta(I;M^p_{q_1})},$$

where

$$\mu \equiv \frac{p(r-n)}{r(2p-n)}, \quad \text{i.e., } \mu\sigma = \theta,$$
$$\frac{n+r}{n\alpha_1} = \frac{\mu}{q_1} + \frac{1-\mu}{\tilde{q}_1}, \quad \text{i.e., } \tilde{q}_1 = \frac{nq_1}{2p} = q_0.$$

For any $\tau > 1$, we see that

$$\|e^{\tau t(\Delta-\lambda)}u_0\|^\sigma_{M^{\frac{nr}{n+r}}_{\frac{n\alpha_1}{n+r}}} \leq C\|e^{t(\Delta-\lambda)}u_0\|^\sigma_{M^{\frac{nr}{n+r}}_{\frac{n\alpha_1}{n+r}}} \in L^1(I),$$
$$\|e^{\tau t(\Delta-\lambda)}u(t)\|^\sigma_{M^{\frac{nr}{n+r}}_{\frac{n\alpha_1}{n+r}}} \leq C\|e^{t(\Delta-\lambda)}u(t)\|^\sigma_{M^{\frac{nr}{n+r}}_{\frac{n\alpha_1}{n+r}}} \in L^1(I).$$

By the Lebesgue dominated convergence theorem, for any $\varepsilon > 0$, taking $\tau > 0$ sufficiently large, we then have

$$I_{2,2} + I_{2,3} < \varepsilon. \tag{5.8}$$

Therefore, by (5.6), (5.7), and (5.8), we obtain

$$I_2 < \varepsilon. \tag{5.9}$$

For I_3, it follows from Proposition 2.5 that

$$\begin{aligned} I_3 &= \left\| \int_{\tau t}^{\infty} \nabla e^{s(\Delta-\lambda)}\, ds\, u(t) \right\|_{L^\sigma(I;M^r_{\alpha_1})} = \left\| \nabla(\lambda-\Delta)^{-1} e^{\tau t(\Delta-\lambda)} u(t) \right\|_{L^\sigma(I;M^r_{\alpha_1})} \\ &\le C \left\| e^{\tau t(\Delta-\lambda)} u(t) \right\|_{L^\sigma(I;M^{\frac{nr}{n+r}}_{\frac{n\alpha_1}{n+r}})} \\ &\le C \left(\int_0^T \| e^{\tau t(\Delta-\lambda)} u(t) \|^\sigma_{M^{\frac{nr}{n+r}}_{\frac{n\alpha_1}{n+r}}} dt \right)^{\frac{1}{\sigma}} \le C \left(\int_0^T \| u(t) \|^\sigma_{M^{\frac{nr}{n+r}}_{\frac{n\alpha_1}{n+r}}} dt \right)^{\frac{1}{\sigma}}. \end{aligned}$$

Similarly to the above argument, for any $\varepsilon > 0$, taking $\tau > 0$ sufficiently large, we then have

$$I_3 < \varepsilon. \tag{5.10}$$

Summing up these estimates (5.4), (5.5), (5.9), and (5.10), for any $\varepsilon > 0$, taking $\tau > 0$ sufficiently large, we have

$$\|\nabla\psi_\tau - \nabla\psi\|_{L^\theta(I;M^r_{\alpha_1})} \le C\|u_\tau - u\|_{L^\theta(I;M^p_{q_1})} + \varepsilon, \tag{5.11}$$

and hence, it follows from (5.3) and (5.11) that

$$\|u_\tau - u\|_{L^\theta(I;M^p_{q_1})} \le CM(\|u_\tau - u\|_{L^\theta(I;M^p_{q_1})} + \varepsilon).$$

Since $u \in L^\theta(I; M^p_{q_1})$ and $\nabla\psi \in L^\sigma(I; M^r_{\alpha_1})$, one can take a small constant M for T small enough. For a small constant M, taking τ sufficiently large, we have

$$\|u_\tau - u\|_{L^\theta(I;M^p_{q_1})} + \|\nabla\psi_t - \nabla\psi\|_{L^\theta(I;M^r_{\alpha_1})} \le \varepsilon. \tag{5.12}$$

Repeating the same argument, we obtain (1.9).

By maximal regularity (3.7), the Sobolev inequality and the Hölder inequality, we have

$$\begin{aligned} \|u_\tau - u\|_{L^\infty(I;M^{\frac{n}{2}}_{q_0})} &= \left\| \int_0^t \nabla e^{(t-s)\Delta} \cdot (u_\tau(s)\nabla\psi_\tau(s) - u(s)\nabla\psi(s))\, ds \right\|_{L^\infty(I;M^{\frac{n}{2}}_{q_0})} \\ &\le C\|u_\tau\nabla\psi_\tau - u\nabla\psi\|_{L^{\frac{\theta\sigma}{\theta+\sigma}}(I;N^{-1+\frac{2}{\theta}+\frac{2}{\sigma}}_{\frac{n}{2},q_0,\infty})} \\ &\le C\|u_\tau\nabla\psi_\tau - u\nabla\psi\|_{L^{\frac{\theta\sigma}{\theta+\sigma}}(I;M^{\frac{pr}{p+r}}_{\frac{2prq_0}{n(p+r)}})} \\ &\le C\|u_\tau - u\|_{L^\theta(I;M^p_{q_1})}\|\nabla\psi_\tau\|_{L^\sigma(I;M^r_{\alpha_1})} \\ &\quad + C\|u\|_{L^\theta(I;M^p_{q_1})}\|\nabla\psi_\tau - \nabla\psi\|_{L^\sigma(I;M^r_{\alpha_1})}. \end{aligned} \tag{5.13}$$

For any $0 < t_0 < T$, we set $I_{t_0} \equiv (t_0, T)$. By the similar argument from (5.4), (5.5), (5.9), and (5.10), we see for large $\tau > 0$ that

$$\begin{aligned}
&\|\nabla\psi_\tau - \nabla\psi\|_{L^\infty(I_{t_0}; M^n_{\alpha_0})} \\
&\le \|\nabla e^{\tau t(\Delta-\lambda)}\psi_0\|_{L^\infty(I_{t_0}; M^n_{\alpha_0})} \\
&\quad + \left\| \int_0^{\tau t} \nabla e^{s(\Delta-\lambda)} \left(u_\tau\left(t - \frac{s}{\tau}\right) - u\left(t - \frac{s}{\tau}\right)\right) ds \right\|_{L^\infty(I_{t_0}; M^n_{\alpha_0})} \\
&\quad + \left\| \int_0^{\tau t} \nabla e^{s(\Delta-\lambda)} \left(u\left(t - \frac{s}{\tau}\right) - u(t)\right) ds \right\|_{L^\infty(I_{t_0}; M^n_{\alpha_0})} \\
&\quad + \left\| \int_{\tau t}^{\infty} \nabla e^{s(\Delta-\lambda)} u(t)\, ds \right\|_{L^\infty(I_{t_0}; M^n_{\alpha_0})} \\
&\le \|\nabla e^{\tau t_0(\Delta-\lambda)}\psi_0\|_{M^n_{\alpha_0}} + C\|u_\tau - u\|_{L^\theta(I_{t_0}; M^p_{q_1})} + 2\varepsilon < 4\varepsilon. \qquad (5.14)
\end{aligned}$$

On the other hand, for some small $t_1 > 0$, let

$$\eta_\tau(t) \equiv \chi_{[0,\tau^{-1}t_1]}(t)(\psi_0 - (\lambda - \Delta)^{-1}u_0).$$

Since $\nabla\psi_0 \in \mathcal{M}^n_{\alpha_0}(\mathbb{R}^n)$, we choose $t_1 > 0$ small enough so that

$$\begin{aligned}
&\|\nabla\psi_\tau - \nabla\psi - \nabla\eta_\tau\|_{L^\infty((0,\tau^{-1}t_1); M^n_{\alpha_0})} \\
&\le \|\nabla e^{\tau t(\Delta-\lambda)}\psi_0 - \nabla\psi_0\|_{L^\infty((0,\tau^{-1}t_1); M^n_{\alpha_0})} \\
&\quad + \left\| \int_0^{\tau t} \nabla e^{(\tau t - s)(\Delta-\lambda)} u_\tau(\tau^{-1}s)\, ds \right\|_{L^\infty((0,\tau^{-1}t_1); M^n_{\alpha_0})} \\
&\quad + \left\| \nabla(\lambda - \Delta)^{-1}u_0 - \nabla(\lambda - \Delta)^{-1}u \right\|_{L^\infty((0,\tau^{-1}t_1); M^n_{\alpha_0})} \\
&\le \|(e^{t_1(\Delta-\lambda)} - I)\nabla\psi_0\|_{M^n_{\alpha_0}} + C\|u_\tau\|_{L^\theta((0,\tau^{-1}t_1); M^p_{q_1})} \\
&\quad + C\|u - u_0\|_{L^\infty((0,\tau^{-1}t_1); M^{\frac{n}{2}}_{q_0})} < 3\varepsilon. \qquad (5.15)
\end{aligned}$$

The last inequality follows from the strong continuity of $u_\tau(t)$ in $M^{n/2}_{q_0}(\mathbb{R}^n)$ and the uniformly estimate for $u_\tau \in L^\theta(I; M^p_{q_1})$. Therefore, by passing $\tau \to \infty$ in (5.13), (5.14), and (5.15), we conclude from (5.12) that the convergence (1.10) and (1.11) hold. □

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Declarations

Conflict of interest On behalf of all authors, the corresponding author Takeshi Suguro states that there is no conflict of interest.

Appendix A. Proof of the completeness

Let $1 < q_0 \le n/2$ and $1 < \alpha_0 \le n$ satisfy $2q_0 = \alpha_0$. Let (p, θ) and (r, σ) be admissible as defined in (1.6) with $\theta < \sigma$. We further assume that the exponents $(q_0, q_1, \alpha_0, \alpha_1)$ is subject to the conditions (1.8). For $M, N > 0$ and $I = (0, T)$, we set

$$\begin{aligned} X_M \equiv \Big\{ &(u, \psi) \in \big(L^\infty(I; M_{q_0}^{\frac{n}{2}}) \cap L^\theta(I; M_{q_1}^p)\big) \times \big(L^\infty(I; M_{\alpha_0}^{1,n}) \cap L^\sigma(I; M_{\alpha_1}^{1,r})\big); \\ &\|u\|_{L^\infty(I; M_{q_0}^{\frac{n}{2}})} + \|\psi\|_{L^\infty(I; M_{\alpha_0}^{1,n})} \le M,\ \|u\|_{L^\theta(I; M_{q_1}^p)} + \|\psi\|_{L^\sigma(I; M_{\alpha_1}^{1,r})} \le N \Big\}, \\ d\big((u, \psi), (v, \phi)\big) &\equiv \|u - v\|_{L^\theta(I; M_{q_1}^p)} + \|\nabla(\psi - \phi)\|_{L^\sigma(I; M_{\alpha_1}^r)}. \end{aligned}$$

We prove that the metric space (X_M, d) is complete.

If we set

$$Y \equiv L^\theta(I; M_{q_1}^p) \times L^\sigma(I; M_{\alpha_1}^{1,r}),$$

then (Y, d) is a complete space. We show that X_M is a closed subspace of Y. Let $\{(u_k, \psi_k)\}_{k\in\mathbb{N}} \subset X_M$ and $(u, \psi) \in Y$ satisfy

$$d\big((u_k, \psi_k), (u, \psi)\big) \to 0 \quad \text{as } k \to \infty. \tag{A.1}$$

By Proposition 2.9, we see that

$$M_{\alpha_0}^n = (H^{n'} L^{\alpha_0'})^*.$$

Since $1 < \alpha_0 \le n < \infty$, $H^{n'} L^{\alpha_0'}$ is separable. Thus, by the Banach–Alaoglu theorem (see Brezis [5]), there exist a subsequence $\{(u_{k_j}, \psi_{k_j})\}_{j\in\mathbb{N}} \subset \{(u_k, \psi_k)\}$ and $(\tilde{u}, \tilde{\psi}) \in X_M$ such that

$$(u_{k_j}, \psi_{k_j}) \rightharpoonup (\tilde{u}, \tilde{\psi}) \quad \text{weak-}* \text{ in } L^\infty(I; M_{q_0}^{\frac{n}{2}}) \times L^\infty(I; M_{\alpha_0}^{1,n}),$$

$$(u_{k_j}, \psi_{k_j}) \rightharpoonup (\tilde{u}, \tilde{\psi}) \quad \text{weak-}* \text{ in } L^\theta(I; M^p_{q_1}) \times L^\sigma(I; M^{1,r}_{\alpha_1}).$$

On the other hand, it follows from (A.1) that

$$(u_k, \psi_k) \to (u, \psi) \quad \text{in } \mathcal{D}'(I \times \mathbb{R}^n).$$

Thus, it holds that $\tilde{u} = u$ and $\tilde{\psi} = \psi$ because of the uniqueness of the convergence limit. Hence, we see that

$$(u_{k_j}, \psi_{k_j}) \rightharpoonup (u, \psi) \quad \text{weak-}* \text{ in } L^\infty(I; M^{\frac{n}{2}}_{q_0}) \times L^\infty(I; M^{1,n}_{\alpha_0}),$$
$$(u_{k_j}, \psi_{k_j}) \rightharpoonup (u, \psi) \quad \text{weak-}* \text{ in } L^\theta(I; M^p_{q_1}) \times L^\sigma(I; M^{1,r}_{\alpha_1}).$$

By the weak lower semicontinuity of norms, we have

$$\begin{aligned}
&\|u\|_{L^\infty(I; M^{\frac{n}{2}}_{q_0})} + \|\psi\|_{L^\infty(I; M^{1,n}_{\alpha_0})} \\
&\quad \le \liminf_{k\to\infty} \|u_k\|_{L^\infty(I; M^{\frac{n}{2}}_{q_0})} + \liminf_{k\to\infty} \|\psi_k\|_{L^\infty(I; M^{1,n}_{\alpha_0})} \le M, \\
&\|u\|_{L^\theta(I; M^p_{q_1})} + \|\psi\|_{L^\sigma(I; M^{1,r}_{\alpha_1})} \\
&\quad \le \liminf_{k\to\infty} \|u_k\|_{L^\theta(I; M^p_{q_1})} + \liminf_{k\to\infty} \|\psi_k\|_{L^\sigma(I; M^{1,r}_{\alpha_1})} \le N,
\end{aligned}$$

which implies that $(u, \psi) \in X_M$. Therefore, (X_M, d) is complete.

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