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Möbius gyrovector spaces and functional analysis

By

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Abstract

This is a survey and résumés of previously published articles, including some announcement of new results. We discuss some aspects of the Einstein and Möbius gyrovector spaces from a viewpoint of elementary functional analysis.

§ 1. Introduction

Theory of gyrogroups and gyrovector spaces was initiated by A.A. Ungar in the late 1980s. For standard definitions and results of gyrocommutative gyrogroups and gyrovector spaces, one can refer to [16]. Ungar also defined the Einstein gyrovector space, Möbius gyrovector space, Proper Velocity gyrovector space and he showed that they are isomorphic each other as real inner product gyrovector spaces. Although gyro operations are generally not commutative, associative or distributive, there exist rich symmetrical structures in the real inner product gyrovector spaces due to Ungar.

Abe and Hatori [3] introduced the notion of generalized gyrovector spaces (GGVs), which is a generalization of the notion of real inner product gyrovector spaces. Abe [1] introduced the notion of normed gyrolinear spaces, which is a further generalization of the notion of GGVs. Hatori [10] showed various substructures of positive invertible elements of a unital C*-algebra are actually GGVs. These are more complicated objects from our point of view, but they are expected to be developed in future research. Ferreira and Suksumran [7] introduced the notion of real inner product gyrogroups, which is a generalization of well-known gyrogroups in the literature.

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In recent years, various notions in the Möbius gyrovector spaces have been established as counterparts to those in Hilbert spaces, such as orthogonal gyrodecomposition with respect to closed gyrovector subspaces, orthogonal gyroexpansion with respect to orthogonal bases and a weight sequence, Cauchy-Schwarz type inequalities, and continuous quasi gyrolinear mappings (cf.[4], [18]–[23]). In this article, from a perspective of basic theory of functional analysis, we present a survey and résumés of previously published papers, including some announcement of results in [26]. There is a large overlap with [28] and the author would like to beg understanding.

§ 2. Real inner product gyrovector spaces

Let us recall a minimal set of definitions and theorems related to the Einstein and Möbius gyrovector spaces whose career is a real inner product space. In this section, we denote any real inner product space by $(\mathbb{V}, \cdot)$. For any $s > 0$, we denote the open ball of radius s, centered at the origin of the whole space $\mathbb{V}$ by

$$\mathbb{V}_s = \{\boldsymbol{a} \in \mathbb{V};\ ||\boldsymbol{a}|| < s\},$$

where $||\boldsymbol{a}|| = (\boldsymbol{a}\cdot\boldsymbol{a})^{\frac{1}{2}}$.

Definition 2.1 (cf.[16]). The Einstein addition $\oplus_{\mathrm{E}}$, the Möbius addition $\oplus_{\mathrm{M}}$ and gyro scalar multiplication $\otimes$ are given by the equations

$$\boldsymbol{a} \oplus_{\mathrm{E}} \boldsymbol{b} = \frac{1}{1 + \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{s^2}} \left\{ \boldsymbol{a} + \frac{1}{\gamma_{\boldsymbol{a}}}\boldsymbol{b} + \frac{1}{s^2}\frac{\gamma_{\boldsymbol{a}}}{1+\gamma_{\boldsymbol{a}}}(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{a} \right\}$$

$$\boldsymbol{a} \oplus_{\mathrm{M}} \boldsymbol{b} = \frac{\left(1 + \frac{2}{s^2}\boldsymbol{a}\cdot\boldsymbol{b} + \frac{1}{s^2}||\boldsymbol{b}||^2\right)\boldsymbol{a} + \left(1 - \frac{1}{s^2}||\boldsymbol{a}||^2\right)\boldsymbol{b}}{1 + \frac{2}{s^2}\boldsymbol{a}\cdot\boldsymbol{b} + \frac{1}{s^4}||\boldsymbol{a}||^2||\boldsymbol{b}||^2}$$

$$r \otimes \boldsymbol{a} = s\,\tanh\left(r\tanh^{-1}\frac{||\boldsymbol{a}||}{s}\right)\frac{\boldsymbol{a}}{||\boldsymbol{a}||} \qquad (\text{if } \boldsymbol{a} \neq \boldsymbol{0})$$

$$r \otimes \boldsymbol{0} = \boldsymbol{0}$$

for any $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{V}_s$ and $r \in \mathbb{R}$, where $\gamma_{\boldsymbol{a}} = \dfrac{1}{\sqrt{1 - \frac{||\boldsymbol{a}||^2}{s^2}}}$. Please note that the parameter s does not appear in these notations of the left hand sides, however, it is just omitted.

Definition 2.2 (cf.[16]). For any $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{V}_s$, we use the notation for the Einstein subtraction

$$\boldsymbol{a} \ominus_{\mathrm{E}} \boldsymbol{b} = \boldsymbol{a} \oplus_{\mathrm{E}} (-\boldsymbol{b})$$

as in vector spaces. The Einstein distance function h_{E} is defined by the equation

$$h_{\mathrm{E}}(\boldsymbol{a},\boldsymbol{b}) = \tanh^{-1}\frac{||\boldsymbol{b}\ominus_{\mathrm{E}}\boldsymbol{a}||}{s}.$$

Furthermore, we use notations $\ominus_{\mathrm{M}}$ and h_{M} in the Möbius gyrovector spaces similarly.

Theorem 2.3 (cf.[16] See also [6], [14], [17], [11])**.**

(1) $(\mathbb{V}_s,\oplus_{\mathrm{E}})$, $(\mathbb{V}_s,\oplus_{\mathrm{M}})$ *are gyrocommutative gyrogroups.*

(2) $(\mathbb{V}_s,h_{\mathrm{E}})$, $(\mathbb{V}_s,h_{\mathrm{M}})$ *are metric spaces. If* $\mathbb{V}$ *is a Hilbert space, then* $(\mathbb{V}_s,h_{\mathrm{E}})$, $(\mathbb{V}_s,h_{\mathrm{M}})$ *are also complete as metric spaces.*

(3) $(\mathbb{V}_s,\oplus_{\mathrm{E}},\otimes)$, $(\mathbb{V}_s,\oplus_{\mathrm{M}},\otimes)$ *are real inner product gyrovector spaces.*

Ungar also defined the Proper Velocity gyrovector space, and he showed that those 3 spaces are isomorphic each other as real inner product gyrovector spaces. The isomorphism due to Ungar between the Einstein and Möbius gyrovector spaces is described in the following theorem.

Theorem 2.4 (cf.[16])**.** *The identity*

$$2\otimes(\boldsymbol{a}\oplus_{\mathrm{M}}\boldsymbol{b}) = (2\otimes\boldsymbol{a})\oplus_{\mathrm{E}}(2\otimes\boldsymbol{b})$$

holds for any $\boldsymbol{a},\boldsymbol{b}\in\mathbb{V}_s$*. The mapping*

$$\boldsymbol{a}\mapsto 2\otimes\boldsymbol{a}$$

is an isomorphism from $(\mathbb{V}_s,\oplus_{\mathrm{M}},\otimes)$ *onto* $(\mathbb{V}_s,\oplus_{\mathrm{E}},\otimes)$ *as real inner product gyrovector spaces.*

In the rest of this section, we state some results in [24]. Let $\mathbb{U},\mathbb{V}$ be two (real or complex) inner product spaces, let $s,s'>0$, and let T be a bounded linear operator from $\mathbb{U}$ into $\mathbb{V}$. If $||T||\leq\frac{s'}{s}$, then it is obvious

$$||Tx||\leq||T||||x||<s'$$

for any $x\in\mathbb{U}_s$. Thus the restriction of T maps $\mathbb{U}_s$ into $\mathbb{V}_{s'}$, and such restrictions form one of the most fundamental class of mappings between two balls. Although the restrictions of bounded linear operators do not preserve gyro operations in general, they can be considered as the most natural counterpart to bounded linear operators between Hilbert spaces. (Next, the class of quasi gyrolinear mappings investigated in [22], [23], [27] might be considered. Continuous maps that preserve a gyro addition are known to be special in a sense. cf. [12, Theorem 1], [5, Theorem 6], [22, Theorem 11]). Assume $s=s'$ for simplicity.

Theorem 2.5. *Let $\mathbb{U}, \mathbb{V}$ be real inner product spaces and let $T : \mathbb{U} \to \mathbb{V}$ be a bounded real linear operator with $||T|| \leq 1$. For any $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{U}$ and $s > \max\{||\boldsymbol{a}||, ||\boldsymbol{b}||\}$, the following inequality holds:*

$$h_{\mathrm{M}}(T\boldsymbol{a}, T\boldsymbol{b}) \leq ||T|| h_{\mathrm{M}}(\boldsymbol{a}, \boldsymbol{b}).$$

That is

$$||T\boldsymbol{a} \ominus_{\mathrm{M}} T\boldsymbol{b}|| \leq ||T|| \otimes ||\boldsymbol{a} \ominus_{\mathrm{M}} \boldsymbol{b}||$$

or

$$\sqrt{\frac{||T\boldsymbol{a}||^2 - 2T\boldsymbol{a}\cdot T\boldsymbol{b} + ||T\boldsymbol{b}||^2}{1 - \frac{2}{s^2} T\boldsymbol{a}\cdot T\boldsymbol{b} + \frac{1}{s^4}||T\boldsymbol{a}||^2||T\boldsymbol{b}||^2}} \leq ||T|| \otimes \sqrt{\frac{||\boldsymbol{a}||^2 - 2\boldsymbol{a}\cdot\boldsymbol{b} + ||\boldsymbol{b}||^2}{1 - \frac{2}{s^2}\boldsymbol{a}\cdot\boldsymbol{b} + \frac{1}{s^4}||\boldsymbol{a}||^2||\boldsymbol{b}||^2}}.$$

The equality holds if and only if one of the following conditions is satisfied:

(i) $\boldsymbol{a} = \boldsymbol{b}$

(ii) $T = 0$

(iii) $||T\boldsymbol{a}|| = ||\boldsymbol{a}||$ *and* $||T\boldsymbol{b}|| = ||\boldsymbol{b}||$.

Remark.

- The inequality $||T\boldsymbol{a} \ominus_{\mathrm{M}} T\boldsymbol{b}|| \leq ||T(\boldsymbol{a} \ominus_{\mathrm{M}} \boldsymbol{b})||$ does not hold in general.
- By letting $s \to \infty$, the following norm inequality can be recaptured

$$||T\boldsymbol{a} - T\boldsymbol{b}|| \leq ||T|| ||\boldsymbol{a} - \boldsymbol{b}||.$$

Theorem 2.6. *Let $\mathbb{U}, \mathbb{V}$ be real inner product spaces and let $T : \mathbb{U} \to \mathbb{V}$ be a bounded real linear operator with $||T|| \leq 1$. For any $s > 0$, the following identity holds:*

$$\sup_{||\boldsymbol{a}||, ||\boldsymbol{b}|| < s,\, \boldsymbol{a} \neq \boldsymbol{b}} \frac{h_{\mathrm{M}}(T\boldsymbol{a}, T\boldsymbol{b})}{h_{\mathrm{M}}(\boldsymbol{a}, \boldsymbol{b})} = ||T||.$$

§ 3. Complex inner product gyrovector spaces

In this section, we denote any complex inner product space by $(\mathcal{H}, \langle \cdot, \cdot \rangle)$. For any $s > 0$, we denote the open s-ball of $\mathcal{H}$ centered at the origin by

$$\mathcal{H}_s = \{v \in \mathcal{H};\, ||v|| < s\},$$

where $||v|| = \langle v, v \rangle^{\frac{1}{2}}$.

Let us recall the notion of the Schwarz-Pick system due to L.A. Harris.

Definition 3.1 ([9]. See also [8]). Call any system which assigns a pseudometric to each domain in every normed linear space a Schwarz-Pick system if the following conditions hold:

(i) The pseudometric assigned to the open unit disc $\mathbb{D}$ in the complex plain is the Poincaré metric.

(ii) If ρ_1 and ρ_2 are the pseudometrics assigned to domains $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively, then $\rho_2(h(x), h(y)) \leq \rho_1(x, y)$ for all holomorphic $h : \mathcal{D}_1 \to \mathcal{D}_2$ and all $x, y \in \mathcal{D}_1$.

Theorem 3.2 (cf.[8, Chapter 2, Section 15]). *All Schwarz-Pick systems assign the same metric ρ to the open unit ball $\mathcal{H}_1$ of any complex Hilbert space $\mathcal{H}$. Moreover, we know that*

$$\rho(u, v) = \tanh^{-1} (1 - \sigma(u, v))^{\frac{1}{2}} ,$$

where

$$\sigma(u, v) = \frac{(1 - ||u||^2)(1 - ||v||^2)}{|1 - \langle u, v\rangle|^2}.$$

for all $u, v \in \mathcal{H}_1$.

Next, let us recall some fundamental definitions and results related to the Einstein addition due to Ungar, where the carrier is a complex inner product space.

Definition 3.3 ([15]). Let $\mathcal{H}$ be a complex inner product space. The abstract complex relativistic velocity addition is given by the equation

$$u \oplus_{\mathrm{E}} v = \frac{1}{1 + \frac{\langle v,u\rangle}{s^2}} \left\{ u + \frac{1}{\gamma_u} v + \frac{1}{s^2} \frac{\gamma_u}{1 + \gamma_u} \langle v, u\rangle u \right\} \tag{3.1}$$

for any $u, v \in \mathcal{H}_s$, where $\gamma_v = \dfrac{1}{\sqrt{1 - \frac{||v||^2}{s^2}}}$.

Remark.

- Although we use the same notation $\oplus_{\mathrm{E}}$ as in the case where the carrier is a real inner product space, there will be no confusion. Because the notations for vectors and spaces are different as $\boldsymbol{a} \in \mathbb{V}$ and $u \in \mathcal{H}$ etc.

- If $\mathcal{H}$ is the one dimensional complex plain $\mathbb{C}$ with the standard inner product, then $(\mathcal{H}_1, \oplus_{\mathrm{E}})$ is nothing but the classical Poincaré disc $(\mathbb{D}, \oplus)$.

Theorem 3.4 ([15]). *Let $\mathcal{H}$ be a complex inner product space. $(\mathcal{H}_s, \oplus_{\mathrm{E}})$ is a gyrocommutative gyrogroup.*

Definition 3.5. We use the notations

$$u \ominus_{\mathrm{E}} v = u \oplus_{\mathrm{E}} (-v)$$

$$h_{\mathrm{E}}(u,v) = \tanh^{-1} \frac{||v \ominus_{\mathrm{E}} u||}{s}.$$

for any $u, v \in \mathcal{H}_s$.

Assume $s = 1$ for a while, for the sake of simplicity.

Theorem 3.6 ([15, (3.8)]). *Let $\mathcal{H}$ be a complex inner product space. The following identity holds*

$$||u \ominus_{\mathrm{E}} v||^2 = \frac{||u||^2 - 2\mathrm{Re}\langle u,v\rangle + ||v||^2 + |\langle u,v\rangle|^2 - ||u||^2||v||^2}{1 - 2\mathrm{Re}\langle u,v\rangle + |\langle u,v\rangle|^2} = 1 - \sigma(u,v)$$

for any $u, v \in \mathcal{H}_1$. Therefore, $h_{\mathrm{E}} = \rho$ holds.

Equation [15, (3.8)] is expressed as

$$\gamma_{u\oplus v} = \gamma_u \gamma_v \left| 1 + \frac{\langle v,u\rangle}{s^2} \right|$$

using the gamma factors for general $s > 0$. We give a proof here for the convenience, which is just a straightforward calculation of inner product.

Proof.

$$|1 + \langle v,u\rangle|^2 \, ||u \oplus_{\mathrm{E}} v||^2$$

$$= \left\langle u + \frac{1}{\gamma_u} v + \frac{\gamma_u}{1+\gamma_u} \langle v,u\rangle u, \; u + \frac{1}{\gamma_u} v + \frac{\gamma_u}{1+\gamma_u} \langle v,u\rangle u \right\rangle$$

$$= ||u||^2 + \sqrt{1-||u||^2}\, \langle u,v\rangle + \frac{\frac{1}{\sqrt{1-||u||^2}}}{1+\frac{1}{\sqrt{1-||u||^2}}} \langle u,v\rangle \, ||u||^2$$

$$+ \sqrt{1-||u||^2}\, \langle v,u\rangle + (1-||u||^2)||v||^2 + \frac{1}{1+\frac{1}{\sqrt{1-||u||^2}}} |\langle u,v\rangle|^2$$

$$+ \frac{\frac{1}{\sqrt{1-||u||^2}}}{1+\frac{1}{\sqrt{1-||u||^2}}} \langle v,u\rangle \, ||u||^2 + \frac{1}{1+\frac{1}{\sqrt{1-||u||^2}}} |\langle u,v\rangle|^2 + \left(\frac{\frac{1}{\sqrt{1-||u||^2}}}{1+\frac{1}{\sqrt{1-||u||^2}}} \right)^2 |\langle u,v\rangle|^2 ||u||^2$$

$$
\begin{aligned}
&= ||u||^2 + \sqrt{1-||u||^2}\,\langle u,v\rangle + \frac{1-\sqrt{1-||u||^2}}{||u||^2}\,\langle u,v\rangle\,||u||^2 \\
&\quad + \sqrt{1-||u||^2}\,\langle v,u\rangle + (1-||u||^2)||v||^2 + \frac{\sqrt{1-||u||^2}(1-\sqrt{1-||u||^2})}{||u||^2}\,|\langle u,v\rangle|^2 \\
&\quad + \frac{1-\sqrt{1-||u||^2}}{||u||^2}\,\langle v,u\rangle\,||u||^2 + \frac{\sqrt{1-||u||^2}(1-\sqrt{1-||u||^2})}{||u||^2}\,|\langle u,v\rangle|^2 \\
&\qquad + \left(\frac{1-\sqrt{1-||u||^2}}{||u||^2}\right)^2 |\,\langle u,v\rangle\,|^2||u||^2 \\
&= ||u||^2 + \langle u,v\rangle + \langle v,u\rangle + (1-||u||^2)||v||^2 + 2\frac{\sqrt{1-||u||^2}(1-\sqrt{1-||u||^2})}{||u||^2}\,|\langle u,v\rangle|^2 \\
&\quad + \frac{1-2\sqrt{1-||u||^2}+(1-||u||^2)}{||u||^4}|\,\langle u,v\rangle\,|^2||u||^2 \\
&= ||u||^2 + 2\mathrm{Re}\,\langle u,v\rangle + ||v||^2 + |\langle u,v\rangle|^2 - ||u||^2||v||^2.
\end{aligned}
$$

□

As far as concerning the hyperbolic metric h_{E}, the Lipschitz continuity of the restriction of contractive complex linear operators to the open unit ball of a complex Hilbert space is just a specific consequence of a generalization of Schwarz lemma related to holomorphic functions.

Corollary 3.7. *Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces. The following inequality holds*

$$h_{\mathrm{E}}(f(u), f(v)) \leq h_{\mathrm{E}}(u,v)$$

for any holomorphic maps $f : \mathcal{H}_1 \to \mathcal{K}_1$ and any $u, v \in \mathcal{H}_1$. In particular,

$$h_{\mathrm{E}}(Tu, Tv) \leq h_{\mathrm{E}}(u,v)$$

for any complex linear operator $T : \mathcal{H} \to \mathcal{K}$ with $||T|| \leq 1$ and any $u, v \in \mathcal{H}_1$.

In the case that $\mathcal{H}$ and $\mathcal{K}$ are finite dimensional, the corollary above is nothing but [13, Theorem 8.1.4]. ¿From a viewpoint of theory of holomorphic mappings, the complex Einstein addition $\oplus_{\mathrm{E}}$ due to Ungar can be regarded as the best binary operation on the open unit ball of a complex Hilbert space.

One can consider possibly various gyro additions on open balls centered at the origin of a complex inner product space. We state some related results of [25] below.

Definition 3.8. Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a complex inner product space. It is well-known as an elementary fact that one can regard the complex linear space $\mathcal{H}$ as a real linear space and define

$$\langle u, v \rangle_R = \mathrm{Re}\, \langle u, v \rangle$$

for any $u, v \in \mathcal{H}$, then $(\mathcal{H}, \langle \cdot, \cdot \rangle_R)$ is a real inner product space. Since $\langle u, u \rangle_R = \langle u, u \rangle$, the norm of vectors and the open balls centered at the origin are identical in both spaces. For any $s > 0$, the Einstein and Möbius additions are defined on $\mathcal{H}_s$, because $\mathcal{H}_s$ is an open s-ball of a real inner product space in this way. They are given by the equations

$$(3.2)\qquad u \oplus_{\mathrm{RE}} v = \frac{1}{1 + \frac{1}{s^2}\langle u, v \rangle_R} \left\{ u + \frac{1}{\gamma_u} v + \frac{1}{s^2} \frac{\gamma_u}{1 + \gamma_u} \langle u, v \rangle_R u \right\}$$

$$= \frac{1}{1 + \frac{1}{s^2}\mathrm{Re}\langle u, v \rangle} \left\{ u + \frac{1}{\gamma_u} v + \frac{1}{s^2} \frac{\gamma_u}{1 + \gamma_u} \mathrm{Re}\langle u, v \rangle u \right\}$$

$$(3.3)\qquad u \oplus_{\mathrm{RM}} v = \frac{\left(1 + \frac{2}{s^2}\langle u, v \rangle_R + \frac{1}{s^2}||v||^2\right) u + \left(1 - \frac{1}{s^2}||u||^2\right) v}{1 + \frac{2}{s^2}\langle u, v \rangle_R + \frac{1}{s^4}||u||^2||v||^2}$$

$$= \frac{\left(1 + \frac{2}{s^2}\mathrm{Re}\langle u, v \rangle + \frac{1}{s^2}||v||^2\right) u + \left(1 - \frac{1}{s^2}||u||^2\right) v}{1 + \frac{2}{s^2}\mathrm{Re}\langle u, v \rangle + \frac{1}{s^4}||u||^2||v||^2}$$

for any $u, v \in \mathcal{H}_s$.

Remark.

- In [24] and [25], we used the notations $\oplus_{\mathrm{E}}, \oplus_{\mathrm{M}}$ or $\oplus_s$, instead of the notations $\oplus_{\mathrm{RE}}$ and $\oplus_{\mathrm{RM}}$ as mentioned above.
- If $\mathcal{H}$ is the one dimensional complex plain $\mathbb{C}$ with the standard inner product, then $(\mathcal{H}_1, \oplus_{\mathrm{RM}})$ is nothing but the classical Poincaré disc $(\mathbb{D}, \oplus)$.

Definition 3.9. We define the notations $\ominus_{\mathrm{RE}}$, h_{RE} from $\oplus_{\mathrm{RE}}$ (resp. $\ominus_{\mathrm{RM}}$, h_{RM} from $\oplus_{\mathrm{RM}}$) in the same manner that we defined the notations $\ominus_{\mathrm{E}}$, h_{E} from $\oplus_{\mathrm{E}}$.

The following theorem is directly derived from the results by Ungar and the fact that a complex inner product space is naturally a real inner product space. Although the author gave the proofs of [25, Theorem 2.1, Theorem 5.1] by calculation, they are not necessary.

Theorem 3.10. *Let $\mathcal{H}$ be a complex inner product space and let $s > 0$. Then*

(1) $(\mathcal{H}_s, \oplus_{\mathrm{RE}})$ and $(\mathcal{H}_s, \oplus_{\mathrm{RM}})$ are gyrocommutative gyrogroups.

(2) The following identities hold

$$||u \ominus_{\mathrm{RE}} v||^2 = \frac{||u||^2 - 2\mathrm{Re}\langle u, v\rangle + ||v||^2 + (\mathrm{Re}\langle u, v\rangle)^2 - ||u||^2||v||^2}{1 - \frac{2}{s^2}\mathrm{Re}\langle u, v\rangle + \frac{1}{s^4}(\mathrm{Re}\langle u, v\rangle)^2}$$

$$||u \ominus_{\mathrm{RM}} v||^2 = \frac{||u||^2 - 2\mathrm{Re}\langle u, v\rangle + ||v||^2}{1 - \frac{2}{s^2}\mathrm{Re}\langle u, v\rangle + \frac{1}{s^4}||u||^2||v||^2}$$

for any $u, v \in \mathcal{H}_s$.

(3) $(\mathcal{H}_s, h_{\mathrm{RE}})$ *and* $(\mathcal{H}_s, h_{\mathrm{RM}})$ *are metric spaces. If* $\mathcal{H}$ *is a Hilbert space, then* $(\mathcal{H}_s, h_{\mathrm{RE}})$ *and* $(\mathcal{H}_s, h_{\mathrm{RM}})$ *are also complete as metric spaces.*

(4) The following identity holds

$$2 \otimes (u \oplus_{\mathrm{RM}} v) = (2 \otimes u) \oplus_{\mathrm{RE}} (2 \otimes v)$$

for any $u, v \in \mathcal{H}_s$.

The following gyro scalar multiplication by complex numbers can be naturally introduced as an extension of the scalar multiplication by real numbers due to Ungar.

Definition 3.11. Let $\mathcal{H}$ be a complex inner product space and let $s > 0$. For any complex number α with its polar form $\alpha = |\alpha|e^{i\theta}$, we define a scalar multiplication $\otimes$ by the equation

$$\alpha \otimes v = e^{i\theta} s \tanh\left(|\alpha| \tanh^{-1}\frac{||v||}{s}\right)\frac{v}{||v||}$$

for any nonzero vector $v \in \mathcal{H}_s$ and $\alpha \otimes 0 = 0$.

Theorem 3.12. *Let* $\mathcal{H}$ *be a complex inner product space and let* $s > 0$. *Then the triplet* $(\mathcal{H}_s, \oplus_{\mathrm{RM}}, \otimes)$ *possesses the following properties:*

(i) $\mathrm{Re}\,\langle \mathrm{gyr}[u, v]w_1, \mathrm{gyr}[u, v]w_2\rangle = \mathrm{Re}\langle w_1, w_2\rangle$

(ii) $1 \otimes v = v$

(iii) $(r_1 + r_2) \otimes v = r_1 \otimes v \oplus_{\mathrm{RM}} r_2 \otimes v$

(iv) $(\alpha\beta) \otimes v = \alpha \otimes (\beta \otimes v)$

(v) $\mathrm{gyr}[u, v](r \otimes w) = r \otimes \mathrm{gyr}[u, v]w$

(vi) $\mathrm{gyr}[r_1 \otimes v, r_2 \otimes v] = id.$

(vii) $||\alpha \otimes v|| = |\alpha| \otimes ||v||$

(viii) $||u \oplus_{\mathrm{RM}} v|| \leq ||u|| \oplus ||v||$

for any real numbers $r_1, r_2, r \in \mathbb{R}$, *complex numbers* $\alpha, \beta \in \mathbb{C}$ *and elements* $u, v, w, w_1, w_2 \in \mathcal{H}_s$. *Here, id. is the identity map on* $\mathcal{H}_s$. *The triplet* $(\mathcal{H}_s, \oplus_{\mathrm{RE}}, \otimes)$ *also possesses the similar properties.*

Remark. In this remark, let $(\mathbb{D}, \oplus)$ be the classical Poincaré disc. Let $\otimes : \mathbb{R} \times \mathbb{D} \to \mathbb{D}$ be the multiplication of real numbers due to Ungar. In [2], Dr. Abe showed that it is impossible to extend $\otimes$ to a map $\otimes_{\mathbb{C}} : \mathbb{C} \times \mathbb{D} \to \mathbb{D}$ so that the extension satisfies the following identities

$$(\alpha + \beta) \otimes_{\mathbb{C}} a = (\alpha \otimes_{\mathbb{C}} a) \oplus (\beta \otimes_{\mathbb{C}} a)$$

$$(\alpha\beta) \otimes_{\mathbb{C}} a = \alpha \otimes_{\mathbb{C}} (\beta \otimes_{\mathbb{C}} a)$$

for any complex numbers $\alpha, \beta \in \mathbb{C}$ and element $a \in \mathbb{D}$. As a personal opinion of the author of the present article, this result is significant in the context of relationship of linearity and gyro structure, and it should be published in an appropriate journal.

A binary operation on the open balls of a complex inner product space satisfying a set of fundamental properties is uniquely determined to be the Möbius addition $\oplus_{\mathrm{RM}}$ given by equation (3.3).

Requirements 3.13. Let $\mathcal{H}$ be a complex inner product space and let $s > 0$. For any binary operation $\oplus : \mathcal{H}_s \times \mathcal{H}_s \to \mathcal{H}_s$, consider the following conditions (R1)–(R3):

(R1) $u \oplus v = \dfrac{u + v}{1 + \frac{\lambda}{s^2}||u||^2}$ whenever $v = \lambda u$ for some $\lambda \in \mathbb{C}$

(R2) $u \oplus v = \dfrac{\left(1 + \frac{1}{s^2}||v||^2\right) u + \left(1 - \frac{1}{s^2}||u||^2\right) v}{1 + \frac{1}{s^4}||u||^2||v||^2}$ whenever $u \perp v$

(R3) $u \oplus (v \oplus w) = (u \oplus v) \oplus w$ whenever $u \perp w$ and $v \perp w$.

Theorem 3.14. *Let* $\mathcal{H}$ *be a complex inner product space and let* $s > 0$. *Then*

(1) The Möbius addition $\oplus_{\mathrm{RM}}$ *given by equation* (3.3) *satisfies the requirements* (R1)–(R3).

(2) Conversely, suppose that a binary operation $\oplus : \mathcal{H}_s \times \mathcal{H}_s \to \mathcal{H}_s$ *satisfies the requirements* (R1)–(R3). *Then,* $\oplus$ *coincides with the Möbius addition* $\oplus_{\mathrm{RM}}$ *given by equation* (3.3).

Next, we state some results in [24].

Theorem 3.15. *Let $\mathcal{H}, \mathcal{K}$ be complex inner product spaces, let u, v be elements in $\mathcal{H}$ and let T be a bounded linear operator from $\mathcal{H}$ into $\mathcal{K}$. If $s > \max\{||u||, ||v||\}$ and $||T|| \leq 1$, then the following inequality holds:*

$$h_{\mathrm{RM}}(Tu, Tv) \leq ||T|| h_{\mathrm{RM}}(u, v).$$

That is

$$||Tu \ominus_{\mathrm{RM}} Tv|| \leq ||T|| \otimes ||u \ominus_{\mathrm{RM}} v||$$

or

$$\sqrt{\frac{||Tu||^2 - 2\mathrm{Re}\langle Tu, Tv\rangle + ||Tv||^2}{1 - \frac{2}{s^2}\mathrm{Re}\langle Tu, Tv\rangle + \frac{1}{s^4}||Tu||^2||Tv||^2}} \leq ||T|| \otimes \sqrt{\frac{||u||^2 - 2\mathrm{Re}\langle u, v\rangle + ||v||^2}{1 - \frac{2}{s^2}\mathrm{Re}\langle u, v\rangle + \frac{1}{s^4}||u||^2||v||^2}}.$$

The equality holds if and only if one of the following conditions is satisfied:

(i) $u = v$

(ii) $T = 0$

(iii) $||Tu|| = ||u||$ *and* $||Tv|| = ||v||$.

Remark.

- The inequality $||Tu \ominus_{\mathrm{RM}} Tv|| \leq ||T(u \ominus_{\mathrm{RM}} v)||$ does not hold in general.
- By letting $s \to \infty$, the following norm inequality can be recaptured

$$||Tu - Tv|| \leq ||T|| ||u - v||.$$

Theorem 3.16. *Let $\mathcal{H}, \mathcal{K}$ be complex inner product spaces and let $T : \mathcal{H} \to \mathcal{K}$ be a bounded linear operator with $||T|| \leq 1$. For any $s > 0$, the following identity holds:*

$$\sup_{||u||, ||v|| < s,\, u \neq v} \frac{h_{\mathrm{RM}}(Tu, Tv)}{h_{\mathrm{RM}}(u, v)} = ||T||.$$

In the rest of this section, we announce some results in [26]. Assume $s = 1$ for simplicity. It is easy to extend the result to general $s > 0$.

Definition 3.17. Let $\mathcal{H}$ be a complex inner product space. We define a binary operation $\oplus_{\mathrm{M}}$ on the open unit ball $\mathcal{H}_1$ by the equation

$$u \oplus_{\mathrm{M}} v = \frac{|c| - (1 - ||u||^2)(1 - ||v||^2)}{\sqrt{|c|^2 - (1 - ||u||^2)^2(1 - ||v||^2)^2}} \cdot \frac{c}{|c|} \cdot \frac{(1 + 2\langle v, u\rangle + ||v||^2)u + (1 - ||u||^2)v}{\frac{1}{2}\sqrt{|c|^2 - (1 - ||u||^2)^2(1 - ||v||^2)^2}}$$

for any $u, v \in \mathcal{H}_1$, where $c = (1 + ||u||^2)(1 + ||v||^2) + 4\langle u, v\rangle$.

Theorem 3.18. *Let $\mathcal{H}$ be a complex inner product space. Then the following identity holds*

$$2 \otimes (u \oplus_{\mathrm{M}} v) = (2 \otimes u) \oplus_{\mathrm{E}} (2 \otimes v) \tag{3.4}$$

for any $u, v \in \mathcal{H}_1$.

Definition 3.19. For the simplicity of notations, we put

$$a = ||u||^2, \qquad b = ||v||^2 \qquad \text{and} \qquad z = \langle u, v \rangle$$

for any $u, v \in \mathcal{H}_1$.

Lemma 3.20. *The following identities hold*

$$2 \otimes u = \frac{2}{1+a} u, \qquad \gamma_{2\otimes u} = \frac{1+a}{1-a} \qquad \textit{and} \qquad \frac{1}{2} \otimes w = \frac{1 - \sqrt{1 - ||w||^2}}{||w||^2} w$$

for any $u, w \in \mathcal{H}_1$.

Lemma 3.21. *The following identity holds*

$$\begin{aligned} ||(1 + 2\overline{z} + b)u + (1-a)v||^2 &= (a+b)(1+ab) + 2\mathrm{Re}z(1+a)(1+b) + 4|z|^2 \\ &= \frac{1}{4} \left\{ |(1+a)(1+b) + 4z|^2 - (1-a)^2(1-b)^2 \right\} \end{aligned}$$

for any $u, v \in \mathcal{H}_1$.

Lemma 3.22. *If we put*

$$w = \frac{2}{(1+a)(1+b) + 4\overline{z}} \left\{ (1 + 2\overline{z} + b)u + (1-a)v \right\},$$

then the following identity holds

$$|(1+a)(1+b) + 4z|^2 \, (1 - ||w||^2) = (1-a)^2(1-b)^2.$$

Proof of Theorem 3.18. By using formula (3.1) and Lemma 3.20, a straightforward calculation shows

$$\begin{aligned} &(2 \otimes u) \oplus_{\mathrm{E}} (2 \otimes v) \\ &= \frac{1}{1 + \langle 2 \otimes v, 2 \otimes u \rangle} \left\{ 2 \otimes u + \frac{1}{\gamma_{2\otimes u}} (2 \otimes v) + \frac{\gamma_{2\otimes u}}{1 + \gamma_{2\otimes u}} \langle 2 \otimes v, 2 \otimes u \rangle (2 \otimes u) \right\} \\ &= \frac{1}{1 + \left\langle \frac{2}{1+b} v, \frac{2}{1+a} u \right\rangle} \left\{ \frac{2}{1+a} u + \frac{1}{\frac{1+a}{1-a}} \frac{2}{1+b} v + \frac{\frac{1+a}{1-a}}{1 + \frac{1+a}{1-a}} \left\langle \frac{2}{1+b} v, \frac{2}{1+a} u \right\rangle \frac{2}{1+a} u \right\} \\ &= \frac{2}{(1+a)(1+b) + 4\overline{z}} \left\{ (1 + 2\overline{z} + b)u + (1-a)v \right\}. \end{aligned}$$

Therefore, we can obtain

$$\frac{1}{2} \otimes \{(2 \otimes u) \oplus_{\mathrm{E}} (2 \otimes v)\} = \frac{1}{2} \otimes w = \frac{1 - \sqrt{1 - ||w||^2}}{||w||^2} w$$

$$= \frac{1 - \frac{(1-a)(1-b)}{|(1+a)(1+b)+4z|}}{\frac{4}{|(1+a)(1+b)+4z|^2} \left|\left|(1 + 2\overline{z} + b)u + (1 - a)v\right|\right|^2}$$

$$\cdot \frac{2}{(1 + a)(1 + b) + 4\overline{z}} \{(1 + 2\overline{z} + b)u + (1 - a)v\}$$

$$= \frac{|(1 + a)(1 + b) + 4z| - (1 - a)(1 - b)}{2\,||(1 + 2\overline{z} + b)u + (1 - a)v||} \cdot \frac{|(1 + a)(1 + b) + 4z|}{(1 + a)(1 + b) + 4\overline{z}} \cdot \frac{(1 + 2\overline{z} + b)u + (1 - a)v}{||(1 + 2\overline{z} + b)u + (1 - a)v||}$$

$$= \frac{|c| - (1 - ||u||^2)(1 - ||v||^2)}{\sqrt{|c|^2 - (1 - ||u||^2)^2(1 - ||v||^2)^2}} \cdot \frac{c}{|c|} \cdot \frac{(1 + 2\,\langle v, u\rangle + ||v||^2)u + (1 - ||u||^2)v}{\frac{1}{2}\sqrt{|c|^2 - (1 - ||u||^2)^2(1 - ||v||^2)^2}}$$

$$= u \oplus_{\mathrm{M}} v$$

as desired. □

Thus, via identity (3.4), the binary operation $\oplus_{\mathrm{M}}$ corresponds to the complex Einstein addition $\oplus_{\mathrm{E}}$ defined by equation (3.1) due to Ungar. So we might be able to call $\oplus_{\mathrm{M}}$ *the complex Möbius addition*. Entire work with details including the following theorem will be published elsewhere.

Theorem 3.23. *Let $\mathcal{H}$ be a complex inner product space and let $s > 0$. Then*

(1) $(\mathcal{H}_s, \oplus_{\mathrm{M}})$ is a gyrocommutative gyrogroup.

(2) $(\mathcal{H}_s, h_{\mathrm{M}})$ is a metric space. If $\mathcal{H}$ is a Hilbert space, then $(\mathcal{H}_s, \oplus_{\mathrm{M}})$ is also complete as metric space.

(3) The triplet $(\mathcal{H}_s, \oplus_{\mathrm{M}}, \otimes)$ satisfies the properties of "inner product gyrovector spaces".

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