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# Topological reflexivity of isometries on algebras of matrix-valued Lipschitz maps

By

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**Abstract**

Let $X$ and $Y$ be compact metric spaces and let $\mathcal{M}_n(\mathbb{C})$ be the Banach algebra of all $n \times n$ complex matrices. We prove that the set of all unital surjective linear isometries from $\mathrm{Lip}(X, \mathcal{M}_n(\mathbb{C}))$ to $\mathrm{Lip}(Y, \mathcal{M}_n(\mathbb{C}))$, whenever both spaces are endowed with the sum norm, is topologically reflexive.

## § 1. Introduction and statement of the result

Let $E$ and $F$ be Banach algebras, $\mathcal{B}(E, F)$ be the space of all linear continuous operators from $E$ to $F$ and $\mathcal{S}$ be a nonempty subset of $\mathcal{B}(E, F)$. We define the *algebraic reflexive closure* of $\mathcal{S}$ by

$$\mathrm{ref}_{\mathrm{alg}}(\mathcal{S}) = \{T \in \mathcal{B}(E, F) \colon \forall e \in E,\ \exists T_e \in \mathcal{S} \,|\, T_e(e) = T(e)\}$$

and the *topological reflexive closure* of $\mathcal{S}$ by

$$\mathrm{ref}_{\mathrm{top}}(\mathcal{S}) = \left\{T \in \mathcal{B}(E, F) \colon \forall e \in E,\ \exists \{T_{e,n}\}_{n\in\mathbb{N}} \subset \mathcal{S} \,|\, \lim_{n\to\infty} T_{e,n}(e) = T(e)\right\}.$$

The set $\mathcal{S}$ is said to be *algebraically reflexive* (respectively, *topologically reflexive*) if $\mathrm{ref}_{\mathrm{alg}}(\mathcal{S}) = \mathcal{S}$ (respectively, $\mathrm{ref}_{\mathrm{top}}(\mathcal{S}) = \mathcal{S}$). It is straightforward to verify that the

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topological reflexivity of $\mathcal{S}$ implies its algebraic reflexivity. The elements of $\mathrm{ref}_{\mathrm{alg}}(\mathcal{S})$ and $\mathrm{ref}_{\mathrm{top}}(\mathcal{S})$ are known as $\mathcal{S}$-local maps and approximate $\mathcal{S}$-local maps, respectively.

The study of these $\mathcal{S}$-local maps was addressed when $\mathcal{S}$ is the set of surjective linear isometries, the set of algebra automorphisms or the set of derivations of $E$ to $F$ (see the monograph [8] by Molnár).

In a recent paper [9], Oi proved that both the set of surjective linear isometries between spaces of complex-valued Lipschitz functions $\mathrm{Lip}(X)$ with the sum norm (see Theorem 3.1 in [9]) and the set of unital surjective linear isometries between spaces of matrix-valued Lipschitz maps $\mathrm{Lip}(X, \mathcal{M}_n(\mathbb{C}))$ with the sum norm (see Theorem 4.1 in [9]) are algebraically reflexive. In [5], we proved that the former set is topologically reflexive.

In this note, we prove that the latter set is also topologically reflexive. Two proofs of this result are presented: one –which is more direct– is based on Oi's second result, and the other on Oi's first result. We hope that these proofs with different viewpoints would give us some new ideas to deal with the local reflexivity problem of isometries on spaces of Lipschitz maps which take values in a non-commutative Banach algebra.

In [2, 4], some results were stated on the algebraic reflexivity of *-isomorphisms on $\mathrm{Lip}(X, \mathcal{B}(H))$, whenever $\mathcal{B}(H)$ is the C*-algebra of all bounded linear operators on a complex infinite-dimensional separable Hilbert space $H$, but, apparently, nothing is known on reflexivity of the isometry groups of $\mathrm{Lip}(X, \mathcal{B}(H))$.

Given $n \in \mathbb{N}$, the matrix algebra $\mathcal{M}_n(\mathbb{C})$ is the unital Banach algebra of all complex matrices of order $n$, with the operator norm:

$$\|A\|_{\mathcal{M}_n(\mathbb{C})} = \sup\left\{\|Ax\|_2 : x \in \mathbb{C}^n,\ \|x\|_2 \leq 1\right\} \qquad (A \in \mathcal{M}_n(\mathbb{C}))$$

where $\|\cdot\|_2$ is the Euclidean norm of $\mathbb{C}^n$. The identity of $\mathcal{M}_n(\mathbb{C})$ is the unit matrix $I_n$ of order $n$, and the group of invertible elements of $\mathcal{M}_n(\mathbb{C})$ is

$$\mathrm{Inv}(\mathcal{M}_n(\mathbb{C})) = \left\{A \in \mathcal{M}_n(\mathbb{C})\colon\ \det(A) \neq 0\right\}.$$

A result due to Schur [11] establishes that a map $\Phi\colon \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$ is a unital surjective linear isometry if and only if there exists a unitary matrix $V$ such that either $\Phi(A) = VAV^{-1}$ for all $A \in \mathcal{M}_n(\mathbb{C})$, or $\Phi(A) = VA^tV^{-1}$ for all $A \in \mathcal{M}_n(\mathbb{C})$.

Given a compact metric space $(X, d_X)$, the linear space

$$\mathrm{Lip}(X, \mathcal{M}_n(\mathbb{C})) = \left\{F\colon X \to \mathcal{M}_n(\mathbb{C}) \mid \sup_{x,y\in X,\, x\neq y} \frac{\|F(x) - F(y)\|_{\mathcal{M}_n(\mathbb{C})}}{d_X(x, y)} < \infty\right\}$$

is a unital complex Banach algebra with the sum norm:

$$\|F\|_\Sigma = \|F\|_\infty + \mathrm{Lip}(F) \qquad (F \in \mathrm{Lip}(X, \mathcal{M}_n(\mathbb{C}))),$$

where

$$\|F\|_\infty = \sup\left\{\|F(x)\|_{\mathcal{M}_n(\mathbb{C})} : x \in X\right\}$$

and

$$\mathrm{Lip}(F) = \sup\left\{\frac{\|F(x)-F(y)\|_{\mathcal{M}_n(\mathbb{C})}}{d_X(x,y)} : x, y \in X,\, x \neq y\right\}.$$

The constant map with value $I_n$ from $X$ to $\mathcal{M}_n(\mathbb{C})$ is the identity of the algebra $\mathrm{Lip}(X, \mathcal{M}_n(\mathbb{C}))$, and its set of invertible elements is

$$\mathrm{Inv}(\mathrm{Lip}(X, \mathcal{M}_n(\mathbb{C}))) = \{F \in \mathrm{Lip}(X, \mathcal{M}_n(\mathbb{C}))\colon\ \det(F(x)) \neq 0,\ \forall x \in X\}.$$

In particular, we denote by $\mathrm{Lip}(X)$ the Banach algebra of all complex-valued Lipschitz functions on $X$, equipped with the sum norm.

The algebras of matrix-valued Lipschitz maps $\mathrm{Lip}(X, \mathcal{M}_n(\mathbb{C}))$ have been investigated by some authors. Botelho and Jamison [3] characterized algebra homomorphisms and isomorphisms between such algebras. In two recent papers [9, 10], Oi studied unital surjective linear isometries between $\mathrm{Lip}(X, \mathcal{M}_n(\mathbb{C}))$ spaces.

Oi [10] proved that hermitian operators on $\mathrm{Lip}(X, \mathcal{M}_n(\mathbb{C}))$ are composition operators. Using this fact, she gave a complete description of unital surjective linear isometries between $\mathrm{Lip}(X, \mathcal{M}_n(\mathbb{C}))$ spaces, as follows. The third assertion of the following result is a reformulation of the second by applying Schur's theorem.

**Theorem 1.1.** *[10, Theorem 3.3] Let $X$ and $Y$ be compact metric spaces and let $T$ be a map from* $\mathrm{Lip}(X, \mathcal{M}_n(\mathbb{C}))$ *to* $\mathrm{Lip}(Y, \mathcal{M}_n(\mathbb{C}))$. *The following are equivalent:*

1. *$T$ is a unital surjective linear isometry.*
2. *There exist a unitary matrix $V \in \mathcal{M}_n(\mathbb{C})$ and a surjective isometry $\varphi\colon Y \to X$ such that either*

$$T(F)(y) = VF(\varphi(y))V^{-1} \qquad (F \in \mathrm{Lip}(X, \mathcal{M}_n(\mathbb{C})),\ y \in Y)$$

*or*

$$T(F)(y) = V(F(\varphi(y)))^t V^{-1} \qquad (F \in \mathrm{Lip}(X, \mathcal{M}_n(\mathbb{C})),\ y \in Y).$$

3. *There exist a unital surjective linear isometry $\Phi\colon \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$ and a surjective isometry $\varphi\colon Y \to X$ such that*

$$T(F)(y) = \Phi(F(\varphi(y))) \qquad (F \in \mathrm{Lip}(X, \mathcal{M}_n(\mathbb{C})),\ y \in Y).$$

□

To simplify, we introduce the following sets of maps:

$$\begin{aligned}
C(X,Y) &= \{f \text{ is a continuous map from } X \text{ to } Y\} \quad (X,Y, \text{ Hausdorff compact spaces}),\\
\mathrm{Iso}(X,Y) &= \{f \text{ is an isometry from } X \text{ onto } Y\} \quad (X,Y, \text{ compact metric spaces}),\\
\mathcal{B}(E,F) &= \{T \text{ is a continuous linear operator from } E \text{ to } F\} \quad (E,F, \text{ Banach spaces}),\\
\mathrm{Iso}(E,F) &= \{T \text{ is a linear isometry from } E \text{ onto } F\} \quad (E,F, \text{ Banach spaces}),\\
\mathrm{Iso}^u(E,F) &= \{T \text{ is a unital linear isometry from } E \text{ onto } F\} \quad (E,F, \text{ unital Banach algebras})
\end{aligned}$$

We write $\mathcal{B}(E)$, $\mathrm{Iso}(E)$ and $\mathrm{Iso}^u(E)$ instead of $\mathcal{B}(E,E)$, $\mathrm{Iso}(E,E)$ and $\mathrm{Iso}^u(E,E)$, respectively.

The main result of this note is the following.

**Theorem 1.2.** *Let $X$ and $Y$ be compact metric spaces. Then*

$$\mathrm{Iso}^u(\mathrm{Lip}(X,\mathcal{M}_n(\mathbb{C})),\mathrm{Lip}(Y,\mathcal{M}_n(\mathbb{C})))$$

*is topologically reflexive.*

## § 2. First proof of Theorem 1.2

This proof depends only on the analysis of the compactness of two sets of isometries which appear naturally in connection with approximate unital local isometries and on the algebraic reflexivity of $\mathrm{Iso}^u(\mathrm{Lip}(X,\mathcal{M}_n(\mathbb{C})),\mathrm{Lip}(Y,\mathcal{M}_n(\mathbb{C})))$.

*First proof of Theorem 1.2.* Let $T\in\mathrm{ref}_{\mathrm{top}}(\mathrm{Iso}^u(\mathrm{Lip}(X,\mathcal{M}_n(\mathbb{C})),\mathrm{Lip}(Y,\mathcal{M}_n(\mathbb{C}))))$. By Theorem 1.1, there exist two sequences $\{\Phi_{F,m}\}_{m\in\mathbb{N}}$ in $\mathrm{Iso}^u(\mathcal{M}_n(\mathbb{C}))$ and $\{\varphi_{F,m}\}_{m\in\mathbb{N}}$ in $\mathrm{Iso}(Y,X)$ for which

$$\lim_{m\to\infty}\left\|\Phi_{F,m}(F\circ\varphi_{F,m})-T(F)\right\|_{\Sigma}=0.$$

Firstly, notice that $\mathrm{Iso}^u(\mathcal{M}_n(\mathbb{C}))$ is compact in $(\mathcal{B}(\mathcal{M}_n(\mathbb{C})),\|\cdot\|_{\mathrm{op}})$ with the norm

$$\|T\|_{\mathrm{op}}=\sup\left\{\|T(A)\|_{\mathcal{M}_n(\mathbb{C})}: A\in\mathcal{M}_n(\mathbb{C}),\ \|A\|_{\mathcal{M}_n(\mathbb{C})}\leq 1\right\}\qquad(T\in\mathcal{B}(\mathcal{M}_n(\mathbb{C}))),$$

since $\mathrm{Iso}^u(\mathcal{M}_n(\mathbb{C}))$ is a closed and bounded subset of the finite-dimensional normed space $(\mathcal{B}(\mathcal{M}_n(\mathbb{C})),\|\cdot\|_{\mathrm{op}})$.

Secondly, observe that $\mathrm{Iso}(Y,X)$ is compact in $(C(Y,X),d^+)$ with the metric

$$d^+(f,g)=\sup\{d_X(f(y),g(y))\colon y\in Y\}\qquad(f,g\in C(Y,X)),$$

by the Arzelá–Ascoli theorem and the compactness of $X$ (see, for example, [6, Chapter XII]).

Therefore we can take two subsequences $\{\Phi_{F,m_k}\}_{k\in\mathbb{N}}$ and $\{\varphi_{F,m_k}\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty}\left\|\Phi_{F,m_k}-\Phi_F\right\|_{\mathrm{op}}=0\quad\text{and}\quad\lim_{k\to\infty}d^+(\varphi_{F,m_k},\varphi_F)=0,$$

for some $\Phi_F\in\mathrm{Iso}^u(\mathcal{M}_n(\mathbb{C}))$ and $\varphi_F\in\mathrm{Iso}(Y,X)$.

Pick $y\in Y$. For all $k\in\mathbb{N}$, an easy verification yields

$$\begin{aligned}
&\|\Phi_{F,m_k}(F(\varphi_{F,m_k}(y)))-\Phi_F(F(\varphi_F(y)))\|_{\mathcal{M}_n(\mathbb{C})}\\
&\leq\|\Phi_{F,m_k}(F(\varphi_{F,m_k}(y)))-\Phi_{F,m_k}(F(\varphi_F(y)))\|_{\mathcal{M}_n(\mathbb{C})}\\
&+\|\Phi_{F,m_k}(F(\varphi_F(y)))-\Phi_F(F(\varphi_F(y)))\|_{\mathcal{M}_n(\mathbb{C})}\\
&\leq\|F(\varphi_{F,m_k}(y))-F(\varphi_F(y))\|_{\mathcal{M}_n(\mathbb{C})}+\|\Phi_{F,m_k}-\Phi_F\|_{\mathrm{op}}\|F(\varphi_F(y))\|_{\mathcal{M}_n(\mathbb{C})}\\
&\leq\mathrm{Lip}(F)d_X(\varphi_{F,m_k}(y),\varphi_F(y))+\|\Phi_{F,m_k}-\Phi_F\|_{\mathrm{op}}\|F\|_\infty\\
&\leq\mathrm{Lip}(F)d^+(\varphi_{F,m_k},\varphi_F)+\|\Phi_{F,m_k}-\Phi_F\|_{\mathrm{op}}\|F\|_\infty.
\end{aligned}$$

On a hand, this implies that

$$\lim_{k\to\infty}\|\Phi_{F,m_k}(F(\varphi_{F,m_k}(y)))-\Phi_F(F(\varphi_F(y)))\|_{\mathcal{M}_n(\mathbb{C})}=0,$$

but, on the other hand, since

$$\|F(y)\|_{\mathcal{M}_n(\mathbb{C})} \leq \|F\|_\infty \leq \|F\|_\Sigma \qquad (F \in \mathrm{Lip}(Y, \mathcal{M}_n(\mathbb{C})),\ y \in Y)\,,$$

we have

$$\lim_{k\to\infty} \|\Phi_{F,m_k}(F(\varphi_{F,m_k}(y))) - T(F)(y)\|_{\mathcal{M}_n(\mathbb{C})} = 0.$$

Hence we deduce that

$$T(F)(y) = \Phi_F(F(\varphi_F(y))),$$

and since $y$ was arbitrary in $Y$, we get that

$$T(F) = \Phi_F(F \circ \varphi_F).$$

This tells us that $T \in \mathrm{ref_{alg}}(\mathrm{Iso}^u(\mathrm{Lip}(X, \mathcal{M}_n(\mathbb{C})), \mathrm{Lip}(Y, \mathcal{M}_n(\mathbb{C}))))$, and the algebraic reflexivity of $\mathrm{Iso}^u(\mathrm{Lip}(X, \mathcal{M}_n(\mathbb{C})), \mathrm{Lip}(Y, \mathcal{M}_n(\mathbb{C})))$ (see [9, Theorem 4.1]) implies that

$$T \in \mathrm{Iso}^u(\mathrm{Lip}(X, \mathcal{M}_n(\mathbb{C})), \mathrm{Lip}(Y, \mathcal{M}_n(\mathbb{C}))).$$

□

## § 3. Second proof of Theorem 1.2

We will give an alternative proof of Theorem 1.2 by using the algebraic reflexivity of the set $\mathrm{Iso}(\mathrm{Lip}(X), \mathrm{Lip}(Y))$. Another key tool is a variant by Aupetit [1] of the Gleason–Kahane–Żelazko theorem for unital surjective linear maps from a unital Banach algebra onto $\mathcal{M}_n(\mathbb{C})$ which preserve invertibility.

Throughout this section, we will use that if $f \in \mathrm{Lip}(X)$ and $A \in \mathcal{M}_n(\mathbb{C})$, then the map $f \otimes A$ from $X$ to $\mathcal{M}_n(\mathbb{C})$, defined by $(f \otimes A)(x) = f(x)A$, belongs to $\mathrm{Lip}(X, \mathcal{M}_n(\mathbb{C}))$ with $\|f \otimes A\|_\Sigma = \|f\|_\Sigma \|A\|_{\mathcal{M}_n(\mathbb{C})}$ since $\|f \otimes A\|_\infty = \|f\|_\infty \|A\|_{\mathcal{M}_n(\mathbb{C})}$ and $\mathrm{Lip}(f \otimes A) = \mathrm{Lip}(f) \|A\|_{\mathcal{M}_n(\mathbb{C})}$.

We denote by $1_X$ the constant function of value 1 on $X$. Note that $1_X \otimes I_n$ is the identity of the algebra $\mathrm{Lip}(X, \mathcal{M}_n(\mathbb{C}))$.

*Second proof of Theorem 1.2.* Let

$$T \in \mathrm{ref}_{\mathrm{top}}(\mathrm{Iso}^u(\mathrm{Lip}(X, \mathcal{M}_n(\mathbb{C})), \mathrm{Lip}(Y, \mathcal{M}_n(\mathbb{C})))).$$

We have divided this proof into several steps.

*Step* 1. $\|T(F)\|_\Sigma = \|F\|_\Sigma$ for all $F \in \mathrm{Lip}(X, \mathcal{M}_n(\mathbb{C}))$.

Let $F \in \mathrm{Lip}(X, \mathcal{M}_n(\mathbb{C}))$. By hypothesis, there is a sequence $\{T_{F,m}\}_{m\in\mathbb{N}}$ in

$$\mathrm{Iso}^u(\mathrm{Lip}(X, \mathcal{M}_n(\mathbb{C})), \mathrm{Lip}(Y, \mathcal{M}_n(\mathbb{C})))$$

such that

$$\lim_{m\to\infty} \|T_{F,m}(F) - T(F)\|_\Sigma = 0.$$

Since

$$\left| \|T_{F,m}(F)\|_\Sigma - \|T(F)\|_\Sigma \right| \leq \|T_{F,m}(F) - T(F)\|_\Sigma$$

and

$$\|T_{F,m}(F)\|_\Sigma = \|F\|_\Sigma$$

for all $m \in \mathbb{N}$, we infer that

$$\|T(F)\|_\Sigma = \|F\|_\Sigma .$$

*Step* 2. For each $F \in \mathrm{Lip}(X, \mathcal{M}_n(\mathbb{C}))$, there exist two sequences $\{\Phi_{F,m}\}_{m\in\mathbb{N}}$ in $\mathrm{Iso}^u(\mathcal{M}_n(\mathbb{C}))$ and $\{\varphi_{F,m}\}_{m\in\mathbb{N}}$ in $\mathrm{Iso}(Y, X)$ such that

$$\lim_{m\to\infty} \|\Phi_{F,m}(F \circ \varphi_{F,m}) - T(F)\|_\Sigma = 0.$$

Since $T \in \mathrm{ref}_{\mathrm{top}}(\mathrm{Iso}^u(\mathrm{Lip}(X, \mathcal{M}_n(\mathbb{C})), \mathrm{Lip}(Y, \mathcal{M}_n(\mathbb{C}))))$, Step 2 follows by applying Theorem 1.1.

*Step* 3. $\|T(F)\|_\infty = \|F\|_\infty$ and $\mathrm{Lip}(T(F)) = \mathrm{Lip}(F)$ for all $F \in \mathrm{Lip}(X, \mathcal{M}_n(\mathbb{C}))$.

Let $F \in \mathrm{Lip}(X, \mathcal{M}_n(\mathbb{C}))$ and take two sequences $\{\Phi_{F,m}\}_{m\in\mathbb{N}}$ and $\{\varphi_{F,m}\}_{m\in\mathbb{N}}$ as in Step 2. Since

$$\begin{aligned}\left|\|\Phi_{F,m}(F\circ\varphi_{F,m})\|_\infty - \|T(F)\|_\infty\right| &\le \|\Phi_{F,m}(F\circ\varphi_{F,m}) - T(F)\|_\infty \\ &\le \|\Phi_{F,m}(F\circ\varphi_{F,m}) - T(F)\|_\Sigma\end{aligned}$$

and

$$\begin{aligned}\|\Phi_{F,m}(F\circ\varphi_{F,m})\|_\infty &= \sup_{y\in Y}\|\Phi_{F,m}(F(\varphi_{F,m}(y)))\|_{\mathcal{M}_n(\mathbb{C})} \\ &= \sup_{y\in Y}\|F(\varphi_{F,m}(y))\|_{\mathcal{M}_n(\mathbb{C})} \\ &= \sup_{x\in X}\|F(x)\|_{\mathcal{M}_n(\mathbb{C})} = \|F\|_\infty\end{aligned}$$

for all $m \in \mathbb{N}$, we infer that $\|T(F)\|_\infty = \|F\|_\infty$. In view of Step 1, we have $\mathrm{Lip}(T(F)) = \mathrm{Lip}(F)$.

*Step* 4. $T(1_X \otimes I_n) = 1_Y \otimes I_n$.

By applying Step 3 yields

$$\begin{aligned}\|T(1_X\otimes I_n)\|_\infty = \|1_X\otimes I_n\|_\infty = \|1_X\|_\infty \|I_n\|_{\mathcal{M}_n(\mathbb{C})} = 1, \\ \mathrm{Lip}(T(1_X\otimes I_n)) = \mathrm{Lip}(1_X\otimes I_n) = \mathrm{Lip}(1_X)\|I_n\|_{\mathcal{M}_n(\mathbb{C})} = 0.\end{aligned}$$

This implies that there exists a constant $\lambda \in \mathbb{T}$ such that

$$T(1_X\otimes I_n) = \lambda(1_Y\otimes I_n).$$

By Step 2, there are $\{\Phi_{1_X\otimes I_n,m}\}_{m\in\mathbb{N}}$ in $\mathrm{Iso}^u(\mathcal{M}_n(\mathbb{C}))$ and $\{\varphi_{1_X\otimes I_n,m}\}_{m\in\mathbb{N}}$ in $\mathrm{Iso}(Y,X)$ such that

$$\lim_{m\to\infty}\|\Phi_{1_X\otimes I_n,m}((1_X\otimes I_n)\circ\varphi_{1_X\otimes I_n,m}) - T(1_X\otimes I_n)\|_\Sigma = 0.$$

Taking into account that

$$\Phi_{1_X\otimes I_n,m}((1_X\otimes I_n)\circ\varphi_{1_X\otimes I_n,m}) = 1_Y\otimes I_n$$

for all $m \in \mathbb{N}$, we obtain that

$$0 = \lim_{m\to\infty}\|(1_Y\otimes I_n) - \lambda(1_Y\otimes I_n)\|_\Sigma = \|(1-\lambda)(1_Y\otimes I_n)\|_\Sigma = |1-\lambda|,$$

hence $\lambda = 1$ and we conclude that $T(1_X\otimes I_n) = 1_Y\otimes I_n$.

*Step* 5. There exists a unital surjective linear isometry $\Phi\colon \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$ such that

$$\Phi(A) = T(1_X \otimes A)(y) \qquad (A \in \mathcal{M}_n(\mathbb{C})),$$

where $y$ is any point of $Y$.

For each $A \in \mathcal{M}_n(\mathbb{C})$, Step 2 gives sequences $\{\Phi_{1_X\otimes A,m}\}_{m\in\mathbb{N}}$ in $\mathrm{Iso}^u(\mathcal{M}_n(\mathbb{C}))$ and $\{\varphi_{1_X\otimes A,m}\}_{m\in\mathbb{N}}$ in $\mathrm{Iso}(Y, X)$ such that

$$\lim_{m\to\infty} \|\Phi_{1_X\otimes A,m}((1_X \otimes A) \circ \varphi_{1_X\otimes A,m}) - T(1_X \otimes A)\|_\Sigma = 0,$$

and this implies that

$$\lim_{m\to\infty} \|\Phi_{1_X\otimes A,m}(A) - T(1_X \otimes A)(y)\|_{\mathcal{M}_n(\mathbb{C})} = 0$$

for all $y \in Y$. Hence we can define the map $\Phi\colon \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$ by

$$\Phi(A) = T(1_X \otimes A)(y) \qquad (A \in \mathcal{M}_n(\mathbb{C})),$$

where $y$ is any point in $Y$. Since $T$ is linear, so is $\Phi$. By Step 4, given any $y \in Y$, we have

$$\Phi(I_n) = T(1_X \otimes I_n)(y) = (1_Y \otimes I_n)(y) = I_n.$$

Finally, we can take some $y_A \in Y$ such that $\|T(1_X \otimes A)(y_A)\|_{\mathcal{M}_n(\mathbb{C})} = \|T(1_X \otimes A)\|_\infty$. Using Step 3, we obtain that

$$\|\Phi(A)\|_{\mathcal{M}_n(\mathbb{C})} = \|T(1_X \otimes A)(y_A)\|_{\mathcal{M}_n(\mathbb{C})} = \|T(1_X \otimes A)\|_\infty = \|1_X \otimes A\|_\infty = \|A\|_{\mathcal{M}_n(\mathbb{C})}.$$

Hence $\Phi$ is a unital linear isometry of $\mathcal{M}_n(\mathbb{C})$, and since the vector space $\mathcal{M}_n(\mathbb{C})$ is finite dimensional, we conclude that $\Phi$ is surjective.

*Step* 6. For each $f \in \mathrm{Lip}(X)$, there exists a map $\varphi_{f\otimes I_n} \in \mathrm{Iso}(Y, X)$ such that

$$T(f \otimes I_n) = (f \circ \varphi_{f\otimes I_n}) \otimes I_n.$$

Let $f \in \mathrm{Lip}(X)$. We have

$$\lim_{m\to\infty} \|\Phi_{f\otimes I_n,m}((f \otimes I_n) \circ \varphi_{f\otimes I_n,m}) - T(f \otimes I_n)\|_\Sigma = 0,$$

with $\{\Phi_{f\otimes I_n,m}\}_{m\in\mathbb{N}}$ and $\{\varphi_{f\otimes I_n,m}\}_{m\in\mathbb{N}}$ being as in Step 2. Since

$$\Phi_{f\otimes I_n,m}((f \otimes I_n) \circ \varphi_{f\otimes I_n,m}) = (f \circ \varphi_{f\otimes I_n,m}) \otimes I_n,$$

for all $m \in \mathbb{N}$, we deduce that

$$\lim_{m\to\infty} \|(f \circ \varphi_{f\otimes I_n,m}) \otimes I_n - T(f \otimes I_n)\|_\Sigma = 0.$$

By the compactness of $X$ and the Arzelá-Ascoli theorem (see, for example, [6, Chapter XII]), the set $\mathrm{Iso}(Y,X)$, endowed with the topology induced by the metric

$$d^+(f,g) = \sup\{d(f(y),g(y))\colon y\in Y\} \qquad (f,g\in C(Y,X))$$

is compact. Hence we may assume that there exists a subsequence $\{\varphi_{f\otimes I_n,m_k}\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} d^+(\varphi_{f\otimes I_n,m_k},\varphi_{f\otimes I_n}) = 0$$

for some $\varphi_{f\otimes I_n}\in \mathrm{Iso}(Y,X)$. An easy verification shows that

$$\left\|(f\circ\varphi_{f\otimes I_n,m_k})\otimes I_n - (f\circ\varphi_{f\otimes I_n})\otimes I_n\right\|_\infty \le \mathrm{Lip}(f)d^+(\varphi_{f\otimes I_n,m_k},\varphi_{f\otimes I_n})$$

for all $k\in\mathbb{N}$, and on a hand we obtain

$$\lim_{k\to\infty}\left\|(f\circ\varphi_{f\otimes I_n,m_k})\otimes I_n - (f\circ\varphi_{f\otimes I_n})\otimes I_n\right\|_\infty = 0.$$

On the other hand, since $\|F\|_\infty \le \|F\|_\Sigma$ for all $F\in \mathrm{Lip}(Y,\mathcal{M}_n(\mathbb{C}))$, we also have

$$\lim_{k\to\infty}\left\|(f\circ\varphi_{f\otimes I_n,m_k})\otimes I_n - T(f\otimes I_n)\right\|_\infty = 0.$$

Hence we deduce that $T(f\otimes I_n) = (f\circ\varphi_{f\otimes I_n})\otimes I_n$.

*Step* 7. There exists a map $\varphi\in\mathrm{Iso}(Y,X)$ such that

$$T(f\otimes I_n) = (f\circ\varphi)\otimes I_n \qquad (f\in\mathrm{Lip}(X)).$$

Consider the map $\Psi\colon \mathrm{Lip}(X)\to\mathrm{Lip}(Y)$ defined by

$$\Psi(f) = f\circ\varphi_{f\otimes I_n},$$

where $\varphi_{f\otimes I_n}$ is the map of $\mathrm{Iso}(Y,X)$ given in Step 6. Observe that $\Psi(f)$ does not depend on $\varphi_{f\otimes I_n}$. Indeed, if $\phi_{f\otimes I_n}\in\mathrm{Iso}(Y,X)$ satisfies that

$$T(f\otimes I_n) = (f\circ\phi_{f\otimes I_n})\otimes I_n$$

as in Step 6, then $(f\circ\varphi_{f\otimes I_n})\otimes I_n = (f\circ\phi_{f\otimes I_n})\otimes I_n$ which implies that $f\circ\varphi_{f\otimes I_n} = f\circ\phi_{f\otimes I_n}$.

We next prove that $\Psi\colon\mathrm{Lip}(X)\to\mathrm{Lip}(Y)$ is a linear isometry. Clearly,

$$\|\Psi(f)\|_\Sigma = \|f\circ\varphi_{f\otimes I_n}\|_\Sigma = \|f\|_\Sigma \qquad (f\in\mathrm{Lip}(X)).$$

Pick $\alpha,\beta\in\mathbb{C}$ and $f,g\in\mathrm{Lip}(X)$. By Step 6, there are maps $\varphi_{(\alpha f+\beta g)\otimes I_n},\varphi_{f\otimes I_n},\varphi_{g\otimes I_n}\in \mathrm{Iso}(Y,X)$ such that

$$\begin{aligned} T((\alpha f+\beta g)\otimes I_n) &= ((\alpha f+\beta g)\circ\varphi_{(\alpha f+\beta g)\otimes I_n})\otimes I_n,\\ T(f\otimes I_n) &= (f\circ\varphi_{f\otimes I_n})\otimes I_n,\\ T(g\otimes I_n) &= (g\circ\varphi_{g\otimes I_n})\otimes I_n. \end{aligned}$$

Using the linearity of $T$, we obtain that

$$\begin{aligned}((\alpha f+\beta g)\circ\varphi_{(\alpha f+\beta g)\otimes I_n})\otimes I_n &= T((\alpha f+\beta g)\otimes I_n)\\ &= T(\alpha(f\otimes I_n)+\beta(g\otimes I_n))\\ &= \alpha T(f\otimes I_n)+\beta T(g\otimes I_n)\\ &= \alpha((f\circ\varphi_{f\otimes I_n})\otimes I_n)+\beta((g\circ\varphi_{g\otimes I_n})\otimes I_n)\\ &= [\alpha(f\circ\varphi_{f\otimes I_n})+\beta(g\circ\varphi_{g\otimes I_n})]\otimes I_n,\end{aligned}$$

and therefore

$$(\alpha f+\beta g)\circ\varphi_{(\alpha f+\beta g)\otimes I_n}=\alpha(f\circ\varphi_{f\otimes I_n})+\beta(g\circ\varphi_{g\otimes I_n})$$

that is,

$$\Psi(\alpha f+\beta g)=\alpha\Psi(f)+\beta\Psi(g),$$

and thus $\Psi$ is linear.

Since every surjective linear isometry from $(\mathrm{Lip}(X),\|\cdot\|_\Sigma)$ to $(\mathrm{Lip}(Y),\|\cdot\|_\Sigma)$ is a weighted composition operator of the form

$$f\mapsto\lambda(f\circ\varphi)\qquad(f\in\mathrm{Lip}(X)),$$

for some $\lambda\in\mathbb{T}$ and $\varphi\in\mathrm{Iso}(Y,X)$ (see Corollary 15 in [7]), we deduce that $\Psi$ belongs to $\mathrm{ref}_{\mathrm{alg}}(\mathrm{Iso}(\mathrm{Lip}(X),\mathrm{Lip}(Y)))$, and since $\mathrm{Iso}(\mathrm{Lip}(X),\mathrm{Lip}(Y))$ is algebraically reflexive by [9, Theorem 3.1], we obtain that $\Psi\in\mathrm{Iso}(\mathrm{Lip}(X),\mathrm{Lip}(Y))$. Hence there are some $\lambda\in\mathbb{T}$ and $\varphi\in\mathrm{Iso}(Y,X)$ such that

$$\Psi(f)=\lambda(f\circ\varphi)\qquad(f\in\mathrm{Lip}(X)).$$

Since $\Psi(1_X)=1_X\circ\varphi_{1_X\otimes I_n}=1_Y$ and $\Psi(1_X)=\lambda 1_Y$, we deduce that $\lambda=1$ and thus

$$\Psi(f)=f\circ\varphi\qquad(f\in\mathrm{Lip}(X)).$$

Finally, in view of Step 6, we get that

$$T(f\otimes I_n)=\Psi(f)\otimes I_n=(f\circ\varphi)\otimes I_n\qquad(f\in\mathrm{Lip}(X)).$$

*Step* 8. For each $y\in Y$, the map $T_y\colon\mathrm{Lip}(X,\mathcal{M}_n(\mathbb{C}))\to\mathcal{M}_n(\mathbb{C})$, given by

$$T_y(F)=T(F)(y)\qquad(F\in\mathrm{Lip}(X,\mathcal{M}_n(\mathbb{C}))),$$

is either an algebra homomorphism:

$$T_y(FG)=T_y(F)T_y(G)\qquad(F,G\in\mathrm{Lip}(X,\mathcal{M}_n(\mathbb{C}))),$$

or an algebra anti-homomorphism:

$$T_y(FG)=T_y(G)T_y(F)\qquad(F,G\in\mathrm{Lip}(X,\mathcal{M}_n(\mathbb{C}))).$$

Pick $y \in Y$. Since $T$ is linear, so is $T_y$. Notice that $T_y$ is sujective since if $B \in \mathcal{M}_n(\mathbb{C})$, Step 5 gives $A \in \mathcal{M}_n(\mathbb{C})$ such that $B = \Phi(A)$, that is, $B = T(1_X \otimes A)(y) = T_y(1_X \otimes A)$. Furthermore, by Step 4, we have

$$T_y(1_X \otimes I_n) = T(1_X \otimes I_n)(y) = (1_Y \otimes I_n)(y) = I_n.$$

According to a generalization of the Gleason–Kahane–Żelazko theorem established by Aupetit in [1, Theorem 1], to prove Step 8 it suffices to check that $T_y(F) \in \mathrm{Inv}(\mathcal{M}_n(\mathbb{C}))$ for all $F \in \mathrm{Inv}(\mathrm{Lip}(X, \mathcal{M}_n(\mathbb{C})))$.

Take such a function $F$ and Step 2 gives sequences $\{\Phi_{F,m}\}_{m\in\mathbb{N}}$ in $\mathrm{Iso}^u(\mathcal{M}_n(\mathbb{C}))$ and $\{\varphi_{F,m}\}_{m\in\mathbb{N}}$ in $\mathrm{Iso}(Y, X)$ for which

$$\lim_{m\to\infty} \|\Phi_{F,m}(F(\varphi_{F,m}(y))) - T_y(F)\|_{\mathcal{M}_n(\mathbb{C})} = 0.$$

An argument of compactness similar to that of the first proof of Theorem 1.2 shows that there exist $\Phi_F \in \mathrm{Iso}^u(\mathcal{M}_n(\mathbb{C}))$ and $\varphi_F \in \mathrm{Iso}(Y, X)$ such that $T_y(F) = \Phi_F(F(\varphi_F(y)))$. Since

$$\det(T_y(F)) = \det(\Phi_F(F(\varphi_F(y)))) = \det(F(\varphi_F(y))) \neq 0,$$

we deduce that $T_y(F) \in \mathrm{Inv}(\mathcal{M}_n(\mathbb{C}))$, as required.

*Step* 9. $T(f \otimes A) = \Phi((f \otimes A) \circ \varphi)$ for all $f \in \mathrm{Lip}(X)$ and $A \in \mathcal{M}_n(\mathbb{C})$.

Let $f \in \mathrm{Lip}(X)$ and $A \in \mathcal{M}_n(\mathbb{C})$. Using Steps 5, 7 and 8, for any $y \in Y$, we have

$$\begin{aligned} T(f \otimes A)(y) &= T_y(f \otimes A) = T_y((f \otimes I_n)(1_X \otimes A)) \\ &= \begin{cases} T_y(f \otimes I_n)T_y(1_X \otimes A) = T(f \otimes I_n)(y)T(1_X \otimes A)(y) = f(\varphi(y))I_n\Phi(A) \\ \text{or} \\ T_y(1_X \otimes A)T_y(f \otimes I_n) = T(1_X \otimes A)(y)T(f \otimes I_n)(y) = \Phi(A)f(\varphi(y))I_n \end{cases} \\ &= f(\varphi(y))\Phi(A) = \Phi(f(\varphi(y))A) \\ &= \Phi(((f \otimes A) \circ \varphi)(y)). \end{aligned}$$

*Step* 10. $T(F) = \Phi(F \circ \varphi)$ for all $F \in \mathrm{Lip}(X, \mathcal{M}_n(\mathbb{C}))$.

By Lemma 2.1 in [10], for each $F \in \mathrm{Lip}(X, \mathcal{M}_n(\mathbb{C}))$, there exist $m \in \mathbb{N}$, $f_1, \ldots, f_m \in \mathrm{Lip}(X)$ and $A_1, \ldots, A_m \in \mathcal{M}_n(\mathbb{C})$ such that $F = \sum_{k=1}^m f_k \otimes A_k$. By using Step 9, we have

$$\begin{aligned} T(F) &= T\left(\sum_{k=1}^m f_k \otimes A_k\right) = \sum_{k=1}^m T(f_k \otimes A_k) = \sum_{k=1}^m \Phi((f_k \otimes A_k) \circ \varphi) \\ &= \Phi\left(\sum_{k=1}^m ((f_k \otimes A_k) \circ \varphi)\right) = \Phi\left(\left(\sum_{k=1}^m f_k \otimes A_k\right) \circ \varphi\right) = \Phi(F \circ \varphi). \end{aligned}$$

By Theorem 1.1, $T$ belongs to $\mathrm{Iso}^u(\mathrm{Lip}(X, \mathcal{M}_n(\mathbb{C})), \mathrm{Lip}(Y, \mathcal{M}_n(\mathbb{C})))$. This completes this second proof of Theorem 1.2. $\square$

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