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AUTHOR(S):

KO, Eungil; LEE, Ji Eun; LEE, Mee-Jung

CITATION:

KO, Eungil ...[et al]. Properties of C-normal operators (Research on preserver problems on Banach algebras and related topics). 数理解析研究所講究録別冊 2023, B93: 117-124

ISSUE DATE:

2023-07

URL:

http://hdl.handle.net/2433/284874

RIMS Kôkyûroku Bessatsu
B93 (2023), 117–124

Properties of C-normal operators

By

Eungil Ko* Ji Eun Lee** and Mee-Jung Lee***

Abstract

We study various properties of C-normal operators, i.e., $T^*T = CTT^*C$ holds for a conjugation C on $\mathcal{H}$. Especially, we show that $T - \lambda I$ is C-normal for all $\lambda \in \mathbb{C}$ if and only if T is a complex symmetric operator with the conjugation C. In addition, we prove that if T is C-normal, then T is normal $\Leftrightarrow$ T is quasinormal $\Leftrightarrow$ T is hyponormal $\Leftrightarrow$ T is p-hyponormal for $0 < p \leq 1$. Finally, we investigate equivalent conditions so that Aluthge and Duggal transforms of C-normal operators to be C-normal operators.

§ 1. Introduction

This paper is mainly based on [10]. Let $\mathcal{H}$ be a separable complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. This paper will be appeared in other journal.

Definition 1.1. $C : \mathcal{H} \to \mathcal{H}$ is a conjugation operator on $\mathcal{H}$ if the following conditions hold:
(i) C is antilinear; $C(ax + by) = \bar{a}Cx + \bar{b}Cy$ for all $a, b \in \mathbb{C}$ and $x, y \in \mathcal{H}$.
(ii) C is isometric; $\langle Cx, Cy\rangle = \langle y, x\rangle$ for all $x, y \in \mathcal{H}$
(iii) C is involutive; $C^2 = I$.

By the polarization identity, the second condition (ii) is equivalent to $\|Cx\| = \|x\|$ for all $x \in \mathcal{H}$. Note that $(CTC)^k = CT^kC$ and $(CTC)^* = CT^*C$ for every positive

Received January 8, 2022. Revised March 6, 2022.
2020 Mathematics Subject Classification(s): 47A05, 47B15, 47B20.
Key Words: *C-normal operator, complex symmetric operator, operator transforms.*
*Department of Mathematics, Ewha Womans University, Seoul, 03760, Republic of Korea.
e-mail: `eiko@ewha.ac.kr`
**Department of Mathematics and Statistics, Sejong University, Seoul, 05006, Republic of Korea.
e-mail: `jieunlee7@sejoung.ac.kr`
***Kookmin University, Seoul, 02707, Republic of Korea, College of General Education.
e-mail: `meejung@ewhain.net`, `meejunglee@kookmin.ac.kr`

integer k, and $\|C\| = 1$. For a conjugation C, there is an orthonormal basis $\{e_n\}_{n=0}^{\infty}$ for $\mathcal{H}$ such that $Ce_n = e_n$ for all $n \geq 0$ Such a basis is C-real) (see [6]).

We give examples of conjugation operators for each spaces.

Example 1.2.

- $C(x_1, x_2, x_3, \cdots, x_n) = (\overline{x_1}, \overline{x_2}, \overline{x_3}, \cdots, \overline{x_n})$ on $\mathbb{C}^n$.
- $C(x_1, x_2, x_3, \cdots, x_n) = (\overline{x_n}, \overline{x_{n-1}}, \overline{x_{n-2}}, \cdots, \overline{x_1})$ on $\mathbb{C}^n$.
- $[Cf](x) = \overline{f(x)}$ on $\mathcal{L}^2(\mathcal{X}, \mu)$.
- $[Cf](x) = \overline{f(1-x)}$ on $L^2([0,1])$.
- $[Cf](x) = \overline{f(-x)}$ on $L^2(\mathbb{R}^n)$.
- $[Cf](z) = \overline{zf(z)}u(z) \in \mathcal{K}_u^2$ for all $f \in \mathcal{K}_u^2$ where u is inner function and $\mathcal{K}_u^2 = H^2 \ominus uH^2$ is Model space.

Definition 1.3. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be complex symmetric if there exists a conjugation C on $\mathcal{H}$ such that

$$T = CT^*C \tag{1.1}$$

where T^* is the adjoint of T. In this case, we say that T is a complex symmetric operator (CSO) with a conjugation C.

Example 1.4. The following operators are actually complex symmetric operators.

- All 2×2 complex matrix on $\mathbb{C}^2$
- Normal operators (i.e., $T^*T = TT^*$)
- Aluthge transforms of CSOs
- Algebraic operator of order 2 (i.e., $T^2 + aT + b = 0$)
- Truncated Toeplitz operators (i.e., $A_\varphi^u f = P_u(\varphi f), P_u : H^2 \to \mathcal{K}_u := H^2 \ominus uH^2$)
- The Volterra integration operator $Tf(x) = \int_0^x f(y)dy$ on $L^2([0,1])$ satisfies $T = CT^*C$ where $Cf(x) = \overline{f(1-x)}$ on $L^2([0,1])$.

Definition 1.5. For a conjugation C on $\mathcal{H}$, an operator $T \in \mathcal{L}(\mathcal{H})$ is *C-normal* if $T^*T = CTT^*C$ holds.

By the definition of C-normal operators that $C|T|^2C = |T^*|^2$ is equivalent to $C|T|C = |T^*|$. Note that T is C-normal if and only if T^* is C-normal. We denote by $\mathcal{N}_C(\mathcal{H})$ the set of all C-normal operators on $\mathcal{H}$.

We provide examples of C-normal operators which are not complex symmetric.

Example 1.6. Define a conjugation operator C on $\mathbb{C}^5$ as

$$C(x_1, x_2, x_3, x_4, x_5) = (\overline{x_5}, \overline{x_2}, \overline{x_4}, \overline{x_3}, \overline{x_1}).$$

Let $\{e_n\}_{n=1}^5$ be an orthonormal basis of $\mathbb{C}^5$ and let T have the form

$$T = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with respect to $\{e_n\}_{n=1}^5$. Then $|T| = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ and so

$$\begin{cases} C|T|Ce_1 = C|T|e_5 = Ce_5 = e_1, \\ C|T|Ce_2 = C|T|e_2 = Ce_2 = e_2, \\ C|T|Ce_3 = C|T|e_4 = C3e_4 = 3e_3, \\ C|T|Ce_4 = C|T|e_3 = Ce_3 = e_4, \\ C|T|Ce_5 = C|T|e_1 = 0 \cdot Ce_1 = 0 \cdot e_5, \end{cases}$$

So, $C|T|C = |T^*|$. Thus T is C-normal. We know from [15] that T is not complex symmetric. Moreover, it is clearly not normal.

For $\varphi \in L^\infty$, we say that the *Toeplitz operator* T_φ on the Hilbert Hardy space H^2 with symbol φ defined by

$$T_\varphi f = P(\varphi f)$$

for $f \in H^2$ where P is the orthogonal projection from L^2 onto H^2.

Example 1.7. Let C be a conjugation on H^2 given by $(Cf)(z) = \overline{f(\overline{z})}$. Assume that $\varphi(z) = \phi_-(z) + \widehat{\varphi}(0) + \phi_+(z)$ where $\phi_-(z) = \frac{-t\overline{z}}{1-it\overline{z}}$, $\phi_+(z) = \frac{itz}{1-itz}$, and $\widehat{\varphi}(0) = \frac{1+i}{2}$ for $-1 < t < 1$. Then T_φ is C-normal but it is not complex symmetric from Example 6.10 in [12].

§ 2. Main results

In this section, we study various properties of C-normal operators.

Theorem 2.1. *Let $T \in \mathcal{L}(\mathcal{H})$. Then the following statements hold.*
(i) *Every complex symmetric operator T with a conjugation C is C-normal. In particular, if T is normal, then T is C-normal.*
(ii) *$T - \lambda$ is C-normal for all $\lambda \in \mathbb{C}$ if and only if T is a complex symmetric operator with the conjugation C.*

Corollary 2.2. *Let $T = U_T|T|$ be the polar decomposition of T in $\mathcal{L}(\mathcal{H})$ where U_T is unitary. If T is quasinormal (i.e., T^*T and T commute), then T is C-normal.*

Theorem 2.3. *Let $\mathcal{N}_C(\mathcal{H})$ be the set of all C-normal operators on $\mathcal{H}$. Then the following statements hold.*
(i) *The class $\mathcal{N}_C(\mathcal{H})$ is norm closed in $\mathcal{L}(\mathcal{H})$.*
(ii) *$\mathcal{N}_C(\mathcal{H})$ is not translation invariant.*

Lemma 2.4. *(Hölder-McCarthy's inequality in [11])*
For any positive operator $T \in \mathcal{L}(\mathcal{H})$ and $x \in \mathcal{H}$, the following inequalities hold.
(i) $|\langle T^\gamma x, x\rangle| \le |\langle Tx, x\rangle|^\gamma \|x\|^{2(1-\gamma)}$ *if* $0 < \gamma \le 1$.
(ii) $|\langle T^\gamma x, x\rangle| \ge |\langle Tx, x\rangle|^\gamma \|x\|^{2(1-\gamma)}$ *if* $1 \le \gamma < \infty$.

Theorem 2.5. *Let $T = U_T|T|$ be the polar decomposition of T in $\mathcal{L}(\mathcal{H})$. If T is C-normal, then*

$$|\langle T^* x, y\rangle| \le (\|TCx\|\|y\|)^\alpha (\|TCU_{T^*}^* y\|\|x\|)^\beta$$

for every $x, y \in \mathcal{H}$ where $\alpha, \beta \ge 0$ with $\alpha + \beta = 1$.

Corollary 2.6. *If $T \in \mathcal{L}(\mathcal{H})$ is C-normal, then the following properties hold.*
(i) *T is bounded below if and only if T^* is bounded below.*
(ii) *T is injective if and only if T^* is injective.*

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *isometry* if $T^*T = I$ and *unitary* if $T^*T = TT^* = I$, respectively. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *hyponormal* if $T^*T - TT^* \ge 0$ and *p-hyponormal* operator if $(T^*T)^p \ge (TT^*)^p$, where $0 < p \le 1$, respectively. A closed subspace $\mathcal{M}$ is *nontrivial* if it is different from (0) and $\mathcal{H}$. A closed subspace $\mathcal{M} \subset \mathcal{H}$ is *invariant* for T if $T\mathcal{M} \subset \mathcal{M}$.

Theorem 2.7. *If $T \in \mathcal{L}(\mathcal{H})$ is C-normal, then the following statements are equivalent.*

(i) T *is normal.*
(ii) T *is quasinormal.*
(iii) T *is hyponormal.*
(iv) T *is p-hyponormal for* $0 < p \leq 1$.

Corollary 2.8. *Let* $T \in \mathcal{L}(\mathcal{H})$ *be* C*-normal. If* T *is p-hyponormal for* $0 < p \leq 1$ *where* $T \neq \lambda I$*, then* T *has a nontrivial invariant subspace.*

Theorem 2.9. *Assume that* $T \in \mathcal{L}(\mathcal{H})$ *is* C*-normal. Then*

$$T^* \text{ is an isometry} \iff T \text{ is unitary} \iff T \text{ is an isometry.}$$

Corollary 2.10. *If* $S \in \mathcal{L}(\mathcal{H})$ *is the unilateral shift defined by* $Se_n = e_{n+1}$ *where* $\{e_n\}$ *is an orthonormal basis for* $\mathcal{H}$*, then* S *is not* C*-normal.*

Recall that an operator T in $\mathcal{L}(\mathcal{H})$ has the unique polar decomposition $T = U|T|$, where $|T| = (T^*T)^{\frac{1}{2}}$ and U is the appropriate partial isometry satisfying $\ker U = \ker |T| = \ker T$ and $\ker U^* = \ker T^*$. We call the Aluthge transform $\widetilde{T}$ of $T \in \mathcal{L}(\mathcal{H})$ given by $|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ ([8]). It is known from [1] that the Aluthge transform of a complex symmetric operator is complex symmetric. The Aluthge transform of $T \in \mathcal{L}(\mathcal{H})$ does not preserve the C-normality.

Example 2.11. Define a conjugation operator C on $\mathbb{C}^5$ as $C(x_1, x_2, x_3, x_4, x_5) = (\overline{x_5}, \overline{x_2}, \overline{x_4}, \overline{x_3}, \overline{x_1})$. Let T have the form

$$T = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with respect to an orthonormal basis $\{e_n\}_{n=1}^5$ of $\mathbb{C}^5$. Then T is C-normal by Example 1.6. On the other hand, $\widetilde{T}$ is not C-normal. Indeed, the Aluthge transform of T is given by

$$\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since $|\widetilde{T}| = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&1&0&0\\ 0&0&0&\sqrt{3}&0\\ 0&0&0&0&\sqrt{3} \end{pmatrix}$ and $|\widetilde{T}^*| = \begin{pmatrix} 0&0&0&0&0\\ 0&1&0&0&0\\ 0&0&\sqrt{3}&0&0\\ 0&0&0&\sqrt{3}&0\\ 0&0&0&0&0 \end{pmatrix}$, it follows that

$$C|\widetilde{T}|C - |\widetilde{T}^*| = \begin{pmatrix} \sqrt{3}&0&0&0&0\\ 0&-1&0&0&0\\ 0&0&0&0&0\\ 0&0&0&1-\sqrt{3}&0\\ 0&0&0&0&0 \end{pmatrix} \neq 0.$$

Hence $\widetilde{T}$ is not C-normal.

Proposition 2.12. *Let $T = U_T|T| \in \mathcal{L}(\mathcal{H})$ be the polar decomposition of T and let C be a conjugation on $\mathcal{H}$. If T is C-normal, then the following statements hold.*
(i) *$T = CT^*CU_T^*$ on $\overline{\operatorname{ran} U_T^*}$. In particular, if T is a quasiaffinity, then $T^* = U_TCTC$.*
(ii) *$C|\widetilde{T}|C$ and $|(\widetilde{T})^*|$ are unitarily equivalent.*
(iii) *$\widetilde{T}$ is C-normal if and only if $|T^*| = J|T^*|J$ and*

$$|T^*|^{\frac{1}{4}}|T|^{\frac{1}{2}}|T^*|^{\frac{1}{4}} = |T|^{\frac{1}{4}}|T^*|^{\frac{1}{2}}|T|^{\frac{1}{4}}.$$

We say that $T \in \mathcal{L}(\mathcal{H})$ belongs to $\delta(\mathcal{H})$ if

$$U_T^2|T| = |T|U_T^2$$

where $T = U_T|T|$ is the polar decomposition of T. It is known that if T is quasinormal, i.e., T^*T and T commute, then $T \in \delta(\mathcal{H})$. But the converse does not hold.

Theorem 2.13. *Let $T = U_T|T| \in \mathcal{L}(\mathcal{H})$ be the polar decomposition of T and let T be C-normal. If $T \in \delta(\mathcal{H})$ is a quasiaffinity, then the following statements hold.*
(i) *$\widetilde{T}$ is normal, and hence $\widetilde{T}$ is C-normal.*
(ii) *The Duggal transform $\widetilde{T}^D := |T|U_T$ of T is C-normal.*

Corollary 2.14. *If $T \in \delta(\mathcal{H})$ is C-normal with $T \neq \mathbb{C}I$, then the following statements hold.*
(i) *T has a nontrivial invariant subspace.*
(ii) *There exists a positive integer K such that T^k has a nontrivial invariant subspace for every $k \geq K$.*

Proposition 2.15. *Let $T = U_T|T|$ be the polar decomposition of $T \in \mathcal{L}(\mathcal{H})$. Set $\widehat{T} = |T|^{\frac{1}{2}}V|T|^{\frac{1}{2}}$ where $V = CU_TC$. Let $T \in \mathcal{L}(\mathcal{H})$ be C-normal for some conjugation C and let $T = U_T|T|$ be the polar decomposition of T. Then $\widetilde{T}$ is C-normal if and only if $\widehat{T^*}$ is C-normal.*

Corollary 2.16. *Let $T \in \mathcal{L}(\mathcal{H})$ be C-normal for some conjugation C and let $T = U_T|T|$ be the polar decomposition of T. If U_T is complex symmetric with the conjugation C, i.e., $U_{T^*} = CU_TC$, then $\widetilde{T}$ is C-normal if and only if $\widehat{T^*}$ is C-normal.*

Example 2.17. Let T and C be given as in Example 1.6. Define a conjugation operator C on $\mathbb{C}^5$ as $C(x_1, x_2, x_3, x_4, x_5) = (\overline{x_5}, \overline{x_2}, \overline{x_4}, \overline{x_3}, \overline{x_1})$. Let T have the form

$$T = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with respect to an orthonormal basis $\{e_n\}_{n=1}^5$ of $\mathbb{C}^5$. Then T is C-normal and $\widetilde{T}$ is not C-normal. On the other hand, $\widehat{T^*}$ is not C-normal. Indeed, a direct computation shows that

$$V = CU_TC = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \text{ and } |T^*|^{\frac{1}{2}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus $\widehat{T^*} = |T^*|^{\frac{1}{2}} V |T^*|^{\frac{1}{2}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. Since

$$C|\widehat{T^*}|C - |(\widehat{T^*})^*| = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1-\sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{3} \end{pmatrix} \neq 0,$$

we conclude that $\widehat{T^*}$ is not C-normal.

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