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Surjective isometries on an algebra of analytic functions with C^n-boundary values

By

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Abstract

Let $\mathbb{D}$, $\overline{\mathbb{D}}$ and $\mathbb{T}$ be the open unit disk, closed unit disk and unit circle in $\mathbb{C}$. Let $A^n(\overline{\mathbb{D}})$ denote the algebra of all continuous functions f on $\overline{\mathbb{D}}$ which are analytic in $\mathbb{D}$ and whose restrictions $f|_{\mathbb{T}}$ to $\mathbb{T}$ are of class C^n. For each $f \in A^n(\overline{\mathbb{D}})$, the k-th derivative of $f|_{\mathbb{T}}$ as a function on $\mathbb{T}$ is denoted by $D^k(f)$. We characterize surjective, not necessarily linear, isometries on $A^n(\overline{\mathbb{D}})$ with respect to the norm $\|f\|_{\overline{\mathbb{D}}} + \sum_{k=1}^{n} \|D^k(f)\|_{\mathbb{T}}/k!$, where $\|\cdot\|_{\overline{\mathbb{D}}}$ and $\|\cdot\|_{\mathbb{T}}$ are the supremum norms on $\overline{\mathbb{D}}$ and $\mathbb{T}$, respectively.

§ 1. Introduction

A mapping $T : E_1 \to E_2$ between two normed spaces $(E_1, \|\cdot\|_1)$ and $(E_2, \|\cdot\|_2)$ is called an *isometry* if

$$\|T(f) - T(g)\|_2 = \|f - g\|_1$$

for every $f, g \in E_1$. We emphasize that we do not assume linearity for T. The characterization of isometries is a classical problem. Banach [1] characterized surjective, not necessarily linear, isometries on the Banach space $C_{\mathbb{R}}(K)$ of all continuous real-valued functions on a compact metric space K with the supremum norm. After that, characterizations of surjective linear isometries were given for various Banach spaces. For the space $C^1[0, 1]$ of all continuously differentiable functions on $[0, 1]$, Rao and Roy

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[14] determined the general form of surjective complex-linear isometries on $C^1[0,1]$ with respect to the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$, where $\|\cdot\|_\infty$ stands for the supremum norm. Novinger and Oberlin [13] consider the space S^p of all analytic functions on the open unit disk whose derivatives belong to the Hardy space H^p. They gave a characterization of complex-linear isometries on S^p $(1 \leq p < \infty)$ with respect to the norm $\|f\| = \|f\|_\infty + \|f'\|_{H^p}$. Jarosz investigated a class of unital semisimple commutative Banach algebras with the so-called *natural norm.* Jarosz [7] proved that every surjective unital complex-linear isometry with respect to the natural norm is actually an isometry with respect to the supremum norm. Note that the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$ becomes a natural norm on $C^1[0,1]$.

One of the most interesting results on study of isometries was proved by Mazur and Ulam. The Mazur-Ulam theorem [10] states that every surjective isometry between normed spaces must be (real) affine. Applying the Mazur-Ulam theorem, surjective, not necessarily linear, isometries were studied on various normed spaces by many researchers. Hatori and the second author [6] gave the characterization of surjective isometries between function algebras. Kawamura, Koshimizu and the second author [9] introduced a unified framework to treat several norms on $C^1[0,1]$, and gave the characterization of surjective isometries on $C^1[0,1]$ with respect to various norms including $\|f\| = \|f\|_\infty + \|f'\|_\infty$. Concerning such a framework, Kawamura [8] also considers the algebra $C^1(\mathbb{T})$ of all continuously differentiable functions on the unit circle $\mathbb{T}$, and gave the characterization of surjective isometries on $C^1(\mathbb{T})$ with respect to norms belonging to the framework. The second author and Niwa [11, 12] introduce the Novinger-Oberlin type space S_A of all analytic functions whose derivatives belong to the disk algebra. The space S_A admits several norms. They determined general forms of surjective isometries with respect to some norms, including $\|f\| = \|f\|_\infty + \|f'\|_\infty$.

§ 1.1. Notations and Main results

In this paper, let $\mathbb{N}$ and $\mathbb{N}_0$ be the sets of all positive integers and non-negative integers, respectively. For $m_1, m_2 \in \mathbb{N}_0$ with $m_1 \leq m_2$, we set $\mathbb{N}_{m_1}^{m_2} = \{k \in \mathbb{N}_0 : m_1 \leq k \leq m_2\}$.

For a compact Hausdorff space K, let $C(K)$ denote the Banach space of all complex-valued continuous functions on K, with the supremum norm

$$\|f\|_K = \sup_{x \in K} |f(x)| \qquad (f \in C(X)).$$

The constant functions on K taking the value only 0 and 1 are denoted by $\mathbf{0}$ and $\mathbf{1}$, respectively. Let $\mathbb{T}$ be the unit circle in the complex plane $\mathbb{C}$. For $n \in \mathbb{N}$, a function $f : \mathbb{T} \to \mathbb{C}$ is said to be *of class C^n* if the function F on $\mathbb{R}$ defined by $F(t) = f(e^{2\pi i t})$ is of class C^n in the usual sense. We denote by $C^n(\mathbb{T})$ the subalgebra of $C(\mathbb{T})$ consisting of all functions of class C^n. Let $\mathbb{D}$ be the open unit disk, and let $\overline{\mathbb{D}} = \mathbb{D} \cup \mathbb{T}$ be the closed unit disk. The *disk algebra* $A(\overline{\mathbb{D}})$ is the Banach algebra of all continuous functions on $\overline{\mathbb{D}}$ which are analytic in $\mathbb{D}$, with the supremum norm $\|\cdot\|_{\overline{\mathbb{D}}}$. Note that, by the maximum modulus principle, $\|f\|_{\overline{\mathbb{D}}} = \|f\|_{\mathbb{T}}$ for every $f \in A(\overline{\mathbb{D}})$.

Throughout this paper, we fix $n \in \mathbb{N}$. The main object of this paper is the algebra

$$A^n(\overline{\mathbb{D}}) = \{f \in A(\overline{\mathbb{D}}) : f|_{\mathbb{T}} \in C^n(\mathbb{T})\}.$$

For each $f \in A^n(\overline{\mathbb{D}})$ and $k \in \mathbb{N}_1^n$, the k-th derivative of $f|_{\mathbb{T}}$ at $e^{2\pi i t_0} \in \mathbb{T}$ is denoted by

$$D^k(f)(e^{2\pi i t_0}) = \left(\frac{1}{2\pi}\right)^k \frac{d^k}{dt^k}\bigg|_{t=t_0} f(e^{2\pi i t}).$$

Let $D^0(f) = f|_{\mathbb{T}}$. Since $f|_{\mathbb{T}}$ is a function of class C^n, the function $D^k(f) : \mathbb{T} \to \mathbb{C}$ is continuous on $\mathbb{T}$ for every $k \in \mathbb{N}_0^n$. Note that D^k satisfies the Leibniz rule

$$D^k(fg) = \sum_{j=0}^{k} \binom{k}{j} D^{k-j}(f) D^j(g)$$

for every $f, g \in A^n(\overline{\mathbb{D}})$. For each $f \in A^n(\overline{\mathbb{D}})$, set

$$\|f\|_\Sigma = \|f\|_{\overline{\mathbb{D}}} + \sum_{k=1}^{n} \frac{1}{k!} \|D^k(f)\|_{\mathbb{T}} = \sum_{k=0}^{n} \frac{1}{k!} \|D^k(f)\|_{\mathbb{T}}.$$

Then $(A^n(\overline{\mathbb{D}}), \|\cdot\|_\Sigma)$ is a unital commutative Banach algebra. The following theorem is the main result of this paper.

Theorem 1.1. *Suppose that $T : A^n(\overline{\mathbb{D}}) \to A^n(\overline{\mathbb{D}})$ is a surjective, not necessarily linear, isometry with respect to the norm $\|\cdot\|_\Sigma$. Then there exist constants $c, \lambda \in \mathbb{T}$ such that*

$$T(f)(z) = T(\mathbf{0})(z) + cf(\lambda z) \qquad (\forall f \in A^n(\overline{\mathbb{D}}), \forall z \in \overline{\mathbb{D}}), \quad \text{or}$$
$$T(f)(z) = T(\mathbf{0})(z) + \overline{cf(\overline{\lambda z})} \qquad (\forall f \in A^n(\overline{\mathbb{D}}), \forall z \in \overline{\mathbb{D}}).$$

Conversely, every mapping $T : A^n(\overline{\mathbb{D}}) \to A^n(\overline{\mathbb{D}})$ which is one of the above forms is a surjective isometry on $A^n(\overline{\mathbb{D}})$ with respect to the norm $\|\cdot\|_\Sigma$, where $T(\mathbf{0})$ is an arbitrary function in $A^n(\overline{\mathbb{D}})$.

§ 1.2. Some remarks

Note first that $(A^n(\overline{\mathbb{D}}), \|\cdot\|_\Sigma)$ is a unital semisimple commutative Banach algebra. Moreover, the norm $\|\cdot\|_\Sigma$ is a natural norm in the sense of Jarosz [7]. Hence it is relatively easy to determine the general form of surjective *complex-linear* isometry T on $A^n(\overline{\mathbb{D}})$ with $T(\mathbf{1}) = \mathbf{1}$ by the result of Jarosz [7, Theorem and Proposition 2]. On the other hand, our study is more complicated. In fact, we will investigate surjective isometry T on $A^n(\overline{\mathbb{D}})$, which need not be complex-linear nor unital, that is, $T(\mathbf{1}) = \mathbf{1}$ in Theorem 1.1.

The second author and Niwa [11] introduce the space S_A of all analytic functions f on $\mathbb{D}$ whose derivative f' is continuously extended to $\overline{\mathbb{D}}$, where f' is the usual derivative with respect to the complex variable. It is well-known that a holomorphic function $f : \mathbb{D} \to \mathbb{C}$ is continuously extended to $\overline{\mathbb{D}}$ with absolutely continuous boundary value if and only if the derivative f' belongs to the Hardy space H^1 (see [4, Theorem 3.11]). As a consequence of the fact, every function in S_A is continuously extended to $\overline{\mathbb{D}}$. The continuous extension of f will be denoted by $\hat{f}$. Now, for each $f \in S_A$, we set

$$\|f\|_{\Sigma,S_A} = \|\hat{f}\|_{\overline{\mathbb{D}}} + \|\widehat{f'}\|_{\overline{\mathbb{D}}}.$$

Then the space S_A becomes a unital commutative Banach algebra. The Banach algebra S_A is isometrically isomorphic to $A^1(\overline{\mathbb{D}})$. More precisely, we have the following proposition, which can be verified by the same argument as [4, Theorem 3.11].

Proposition 1.2. *A holomorphic function $f : \mathbb{D} \to \mathbb{C}$ is continuously extended to $\overline{\mathbb{D}}$ and its extension $\hat{f}$ belongs to $A^1(\overline{\mathbb{D}})$ if and only if f belongs to S_A. Moreover, if $f \in S_A$, then $\|\hat{f}\|_\Sigma = \|f\|_{\Sigma,S_A}$.*

In [12], a characterization of surjective, not necessarily linear, isometries on S_A with respect to the norm $\|\cdot\|_{\Sigma,S_A}$ was given. Hence Theorem 1.1 is considered as a generalization of the result.

§ 2. Preliminaries and embedding of $A^n(\overline{\mathbb{D}})$ into $C(X)$

§ 2.1. Polynomials

First, we consider each polynomial p as a function on $\overline{\mathbb{D}}$. It is obvious that $p \in A^n(\overline{\mathbb{D}})$. Let $p(z) = a_0 + \cdots + a_m z^m$. For $k \in \mathbb{N}$, let $p^{(k)}$ denote the k-th formal derivative of p, that is, $p^{(k)}(z) = k^{\underline{k}}\, a_k + \cdots + m^{\underline{k}}\, a_m z^{m-k}$, where $m^{\underline{k}}$ is the falling factorial $m(m-1)\cdots(m-k+1)$. Note that $D^k(p) = p^{(k)}|_{\mathbb{T}}$ does not hold. In fact, $D^k(\iota^j)(z) = i^k j^k z^j$, where $\iota^j(z) = z^j$. More generally, we see that $D^k(p)$ can be represented as

$$D^k(p)(z) = i^k \sum_{j=1}^{m} j^k a_j z^j. \tag{2.1}$$

On the other hand, the chain rule implies that $D^1(p)(z) = ip^{(1)}(z)$ and $D^2(p)(z) = -p^{(2)}(z)z^2 - p^{(1)}(z)z$. By induction, we see that $D^k(p)$ can also be represented as

$$D^k(p)(z) = \sum_{j=1}^{k} c_j p^{(j)}(z) z^j, \tag{2.2}$$

where $c_1, \dots, c_k$ are constants independent of the polynomial p.

For $m \in \mathbb{N}_0$, let $M_{m+1,n+1}(\mathbb{T})$ denote the set of all $(m+1) \times (n+1)$ matrices whose entries belong to $\mathbb{T}$.

Proposition 2.1. *Let $m \in \mathbb{N}_0$. Let $W = [w_{j,k}]_{j,k} \in M_{m+1,n+1}(\mathbb{T})$, and assume that $w_{0,0} \notin \{w_{1,0}, \dots, w_{m,0}\}$. Then there exists a polynomial p such that $p(w_{0,0}) \neq 0$ and $D^k(p)(w_{j,k}) = 0$ for every $(j,k) \neq (0,0)$, that is,*

$$\begin{bmatrix} p(w_{0,0}) & D^1(p)(w_{0,1}) & \cdots & D^n(p)(w_{0,n}) \\ p(w_{1,0}) & D^1(p)(w_{1,1}) & \cdots & D^n(p)(w_{1,n}) \\ \vdots & \vdots & \ddots & \vdots \\ p(w_{m,0}) & D^1(p)(w_{m,1}) & \cdots & D^n(p)(w_{m,n}) \end{bmatrix} = \begin{bmatrix} * & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{2.3}$$

Proof. Let $I_0 = \{(j,k) \in \mathbb{N}_0^m \times \mathbb{N}_0^n : w_{j,k} \neq w_{0,0}\}$, and let

$$q(z) = \prod_{(j,k) \in I_0} (z - w_{j,k})^{k+1}.$$

By definition, $q(w_{0,0}) \neq 0$. If $(j,k) \in I_0$, then the formal derivatives $q(z), q^{(1)}(z), \dots, q^{(k)}(z)$ have the factor $(z - w_{j,k})$, and thus, by equality (2.2), we have $D^k(q)(w_{j,k}) = 0$. Hence we obtain $q(w_{0,0}) \neq 0 = D^k(q)(w_{j,k})$ for every $(j,k) \in I_0$. If, in addition, $D^k(q)(w_{0,0}) = 0$ for every $k \in \mathbb{N}_1^n$, then q satisfies the condition (2.3). In this cases, q is the desired polynomial.

Now, assume that $D^k(q)(w_{0,0}) \neq 0$ for some $k \in \mathbb{N}_1^n$. Let $k_1 \in \mathbb{N}_1^n$ be the smallest $k \in \mathbb{N}_1^n$ such that $D^k(q)(w_{0,0}) \neq 0$. Then

$$D^1(q)(w_{0,0}) = \cdots = D^{k_1-1}(q)(w_{0,0}) = 0 \neq D^{k_1}(q)(w_{0,0}).$$

In particular, $D^k(q)(w_{0,0}) = 0$ for all $k \in \mathbb{N}_1^{k_1-1}$. Let $r(z) = q(z) - 2q(w_{0,0})$. Since $r(w_{0,0}) = -q(w_{0,0}) \neq 0$, we have $(qr)(w_{0,0}) \neq 0$. Moreover, if $(j,k) \in I_0$, then the Leibniz rule shows that $D^k(qr)(w_{j,k}) = 0$. Note that $D^k(r)(w_{0,0}) = D^k(q)(w_{0,0}) = 0$ for every $k \in \mathbb{N}_1^{k_1-1}$, and that $D^{k_1}(r)(w_{0,0}) = D^{k_1}(q)(w_{0,0}) \neq 0$. By the Leibniz rule, $D^k(qr)(w_{0,0}) = 0$ for every $k \in \mathbb{N}_1^{k_1-1}$. We also have

$$\begin{aligned} D^{k_1}(qr)(w_{0,0}) &= q(w_{0,0}) \cdot D^{k_1}(r_0)(w_{0,0}) + D^{k_1}(q)(w_{0,0}) \cdot r(w_{0,0}) \\ &= q(w_{0,0}) \cdot D^{k_1}(q)(w_{0,0}) - D^{k_1}(q)(w_{0,0}) \cdot q(w_{0,0}) = 0. \end{aligned}$$

Hence we obtain $D^k(qr)(w_{0,0}) = 0$ for all $k \in \mathbb{N}_1^{k_1}$. This shows that the polynomial qr has not only the same properties as q, but also $D^{k_1}(qr)(w_{0,0}) = 0$. Finally, applying the above argument repeatedly, at most finitely many times, we obtain a polynomial p satisfying condition (2.3) . The proof is completed. □

Proposition 2.2. *Let $m \in \mathbb{N}_0$, and let $k_0 \in \mathbb{N}_0^n$. Let $W = [w_{j,k}]_{j,k} \in M_{m+1,n+1}(\mathbb{T})$, and assume that $w_{0,k_0} \notin \{w_{1,k_0}, \dots, w_{m,k_0}\}$. Then there exists a polynomial p such that $D^{k_0}(p)(w_{0,k_0}) \neq 0$ and $D^k(p)(w_{j,k}) = 0$ for every $(j,k) \neq (0,k_0)$, that is,*

$$\begin{bmatrix} p(w_{0,0}) & \cdots & D^{k_0}(p)(w_{0,k_0}) & \cdots & D^n(p)(w_{0,n}) \\ p(w_{1,0}) & \cdots & D^{k_0}(p)(w_{1,k_0}) & \cdots & D^n(p)(w_{1,n}) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ p(w_{m,0}) & \cdots & D^{k_0}(p)(w_{m,k_0}) & \cdots & D^n(p)(w_{m,n}) \end{bmatrix} = \begin{bmatrix} 0 \cdots * \cdots 0 \\ 0 \cdots 0 \cdots 0 \\ \vdots \ddots \vdots \ddots \vdots \\ 0 \cdots 0 \cdots 0 \end{bmatrix}. \tag{2.4}$$

Proof. Let $I_1 = \{(j,k) \in \mathbb{N}_0^m \times \mathbb{N}_0^n : w_{j,k} \neq w_{0,k_0}\}$, and let $\{z_1, \dots, z_{m'}\}$ be an enumeration of $\{w_{j,k} : (j,k) \in I_1\}$. Applying Proposition 2.1 to the following $(m'+1) \times (n+1)$ matrix

$$W' = \begin{bmatrix} w_{0,k_0} & w_{0,k_0} & \cdots & w_{0,k_0} \\ z_1 & z_1 & \cdots & z_1 \\ \vdots & \vdots & \ddots & \vdots \\ z_{m'} & z_{m'} & \cdots & z_{m'} \end{bmatrix},$$

we see that there exists a polynomial q such that

$$\begin{cases} q(w_{0,k_0}) \neq 0 = D^l(q)(w_{0,k_0}) & (\forall l \in \mathbb{N}_1^n), \\ D^l(q)(w_{j,k}) = 0 & (\forall (j,k) \in I_1, \forall l \in \mathbb{N}_0^n). \end{cases}$$

Assume that we have constructed a polynomial r such that

$$D^{k_0}(r)(w_{0,k_0}) = 1 \neq 0 = D^l(r)(w_{0,k_0}) \tag{2.5}$$

for every $l \in \mathbb{N}_0^n \setminus \{k_0\}$. Set $p(z) = q(z)r(z)$. Since $q(w_{0,k_0}) \neq 0 = D^l(q)(w_{0,k_0})$ for every $l \in \mathbb{N}_1^n$, the Leibniz rule implies that $D^{k_0}(p)(w_{0,k_0}) \neq 0 = D^k(p)(w_{0,k_0})$ for every $k \in \mathbb{N}_0^n \setminus \{k_0\}$. Moreover, if $(j,k) \in I_1$, then $D^l(q)(w_{j,k}) = 0$ for every $l \in \mathbb{N}_0^n$, and thus the Leibniz rule implies that $D^k(p)(w_{j,k}) = 0$. Hence $p(z)$ satisfies the condition (2.4).

Now, it remains to construct a polynomial r satisfying the condition (2.5). It follows from equality (2.1) that a polynomial $r(z) = a_0 + a_1 z + \cdots + a_n z^n$ satisfies the condition (2.5) if and only if the coefficients of r satisfy the system of $n+1$ linear equations

$$\sum_{j=0}^{n} j^k w_{0,k_0}^j a_j = \begin{cases} i^{-k_0} & (k = k_0), \\ 0 & (\text{otherwise}). \end{cases}$$

The system of linear equations has a solution $(a_0, \dots, a_n) \in \mathbb{C}^{n+1}$. Indeed, the determinant

$$\begin{vmatrix} 1 & w_{0,k_0} & 2w_{0,k_0}^2 & \dots & nw_{0,k_0}^n \\ 0 & w_{0,k_0} & 2^2w_{0,k_0}^2 & \dots & n^2w_{0,k_0}^n \\ 0 & w_{0,k_0} & 2^3w_{0,k_0}^2 & \dots & n^3w_{0,k_0}^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & w_{0,k_0} & 2^nw_{0,k_0}^2 & \dots & n^nw_{0,k_0}^n \end{vmatrix} = w_{0,k_0}^{\frac{n(n+1)}{2}} \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 2 & \dots & n \\ 0 & 1 & 2^2 & \dots & n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 2^n & \dots & n^n \end{vmatrix}$$

is non-zero, because the right-hand side of the above equality is a determinant of a Vandermonde matrix whose columns are the geometric sequence with pairwise distinct common ratios. Hence we can find a polynomial r satisfying the condition (2.5). The proof is completed. □

Proposition 2.3. *Let $k_0 \in \mathbb{N}_0^n$, let $w_{k_0}, \dots, w_n \in \mathbb{T}$, and assume that $w_{k_0} \notin \{w_{k_0+1}, \dots, w_n\}$. For each $\varepsilon > 0$ and each neighborhood V of w_{k_0} in $\mathbb{T}$, there exists a polynomial p such that*

$$\begin{cases} \|D^l(p)\|_{\mathbb{T}} < \varepsilon & (l \in \mathbb{N}_0^{k_0-1}), \\ \|D^{k_0}(p)\|_{\mathbb{T}} = D^{k_0}(p)(w_{k_0}) = k_0!, \\ \|D^{k_0}(p)\|_{\mathbb{T}\setminus V} < \varepsilon, \\ |D^l(p)(w_l)| < \varepsilon & (l \in \mathbb{N}_{k_0+1}^n), \end{cases} \tag{2.6}$$

where $\|D^{k_0}(p)\|_{\mathbb{T}\setminus V}$ is the supremum of $|D^{k_0}(p)|$ on $\mathbb{T} \setminus V$.

Proof. For each $m \in \mathbb{N}$, consider the polynomial

$$p_m(z) = \frac{(-i)^{k_0}}{2^m} \sum_{j=0}^{m} \frac{1}{(m+j)^{k_0}} \binom{m}{j} (\overline{w_{k_0}} z)^{m+j}.$$

Let us show that the sequence $\{p_m\}_m$ has the following properties

$$\begin{cases} \|D^{l_1}(p_m)\|_{\mathbb{T}} \to 0 & (m \to \infty), \\ \|D^{k_0}(p_m)\|_{\mathbb{T}} = D^{k_0}(p_m)(w_{k_0}) = 1 & (\forall m \in \mathbb{N}), \\ \|D^{k_0}(p_m)\|_{\mathbb{T}\setminus V} \to 0 & (m \to \infty), \\ |D^{l_2}(p_m)(w_{l_2})| \to 0 & (m \to \infty) \end{cases} \tag{2.7}$$

for every neighborhood V of w_{k_0} in $\mathbb{T}$, $l_1 \in \mathbb{N}_0^{k_0-1}$ and $l_2 \in \mathbb{N}_{k_0+1}^n$.

First, by equality (2.1), we have

$$D^l(p_m)(z) = \frac{(-i)^{k_0-l}}{2^m} \sum_{j=0}^{m} \frac{1}{(m+j)^{k_0-l}} \binom{m}{j} (\overline{w_{k_0}} z)^{m+j}$$

for every $l \in \mathbb{N}_0^{k_0}$. In particular,

$$D^{k_0}(p_m)(z) = \left(\frac{\overline{w_{k_0}} z + (\overline{w_{k_0}} z)^2}{2} \right)^m .$$

For each $l \in \mathbb{N}_0^{k_0}$ and $w \in \mathbb{T}$,

$$|D^l(p_m)(w)| \leq \frac{1}{2^m} \sum_{j=0}^{m} \frac{1}{(m+j)^{k_0-l}} \binom{m}{j} \leq \frac{1}{2^m} \sum_{j=0}^{m} \frac{1}{m^{k_0-l}} \binom{m}{j} = \frac{1}{m^{k_0-l}},$$

and thus $\|D^l(p_m)\|_{\mathbb{T}} \leq 1/m^{k_0-l}$. This shows that $\|D^l(p_m)\|_{\mathbb{T}} \to 0$ as $m \to \infty$ for every $l \in \mathbb{N}_0^{k_0-1}$, and that $\|D^{k_0}(p_m)\|_{\mathbb{T}} \leq 1$ for every $m \in \mathbb{N}$. Since $D^{k_0}(p_m)(w_{k_0}) = 1$, we obtain $\|D^{k_0}(p_m)\|_{\mathbb{T}} = D^{k_0}(p_m)(w_{k_0}) = 1$. Let V be a neighborhood of w_{k_0} in $\mathbb{T}$. Since

$$\sup_{z \in \mathbb{T} \setminus V} \left| \frac{\overline{w_{k_0}} z + (\overline{w_{k_0}} z)^2}{2} \right| < 1,$$

we have $\|D^{k_0}(p_m)\|_{\mathbb{T} \setminus V} \to 0$ as $m \to \infty$.

Let us verify the rest of the property in (2.7). Let $l \in \mathbb{N}_1^{n-k_0}$. By equality (2.2),

$$D^{k_0+l}(p_m)(z) = i^l \sum_{j=1}^{l} c_j (D^{k_0}(p_m))^{(j)}(z) z^j,$$

where $c_1, \dots, c_l$ are constants independent of m. Thus, to show that $D^{k_0+l}(p_m)(z) \to 0$ as $m \to \infty$, it suffices to prove that $(D^{k_0}(p_m))^{(j)}(w_{k_0+l}) \to 0$ as $m \to \infty$ for every $j \in \mathbb{N}_1^l$. Fix $j_0 \in \mathbb{N}_1^l$. It is easy to see that for each positive integer m with $j_0 < m$, the j_0-th formal derivative of $D^{k_0}(p_m)$ can be written as

$$(D^{k_0}(p_m))^{(j_0)}(z) = \sum_{j=1}^{j_0} m^{\underline{j}} \, q_j(z) \left(\frac{\overline{w_{k_0}} z + (\overline{w_{k_0}} z)^2}{2} \right)^{m-j},$$

where $q_1, \dots, q_{j_0}$ are polynomials independent of m. By our hypothesis on w_{k_0}, we have $|\overline{w_{k_0}} w_{k_0+l} + (\overline{w_{k_0}} w_{k_0+l})^2|/2 < 1$, and thus

$$m^{\underline{j}} \, q_j(w_{k_0+l}) \left(\frac{\overline{w_{k_0}} w_{k_0+l} + (\overline{w_{k_0}} w_{k_0+l})^2}{2} \right)^{m-j} \to 0 \qquad (m \to \infty)$$

for every $j \in \mathbb{N}_0^{j_0}$, and thus $(D^{k_0}(p_m))^{(j_0)}(w_{k_0+l}) \to 0$ as $m \to \infty$, as desired.

Now, let $\varepsilon > 0$, and let V be a neighborhood of w_{k_0} in $\mathbb{T}$. Choose $m \in \mathbb{N}$ so large that

$$\|D^{l_1}(p_m)\|_{\mathbb{T}}, \ \|D^{k_0}(p_m)\|_{\mathbb{T} \setminus V}, \ |D^{l_2}(p_m)(w_{l_2})| < \frac{1}{k_0!} \varepsilon$$

for every $l_1 \in \mathbb{N}_0^{k_0-1}$ and $l_2 \in \mathbb{N}_{k_0+1}^n$. Then $p = k_0! p_m$ satisfies the condition (2.6). $\square$

§ 2.2. Embedding of $A^n(\overline{\mathbb{D}})$ into $C(X)$

Let $X = \mathbb{T}^{2n+1}$ be the compact Hausdorff space endowed with the product topology. We will write each element in X as $x = (\mathbf{w}, \boldsymbol{\zeta})$, where $\mathbf{w} = (w_0, \ldots, w_n) \in \mathbb{T}^{n+1}$ and $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_n) \in \mathbb{T}^n$. For simplicity of notation, we always assume $\zeta_0 = 1$. For each $f \in A^n(\overline{\mathbb{D}})$, define $\tilde{f} : X \to \mathbb{C}$ by

$$\tilde{f}(x) = f(w_0) + \sum_{k=1}^{n} \frac{1}{k!} D^k(f)(w_k)\zeta_k = \sum_{k=0}^{n} \frac{1}{k!} D^k(f)(w_k)\zeta_k \tag{2.8}$$

for every $x = (w_0, \ldots, w_n, \zeta_1, \ldots, \zeta_n) \in X$. It is obvious that $\tilde{f}$ is continuous on X. Note that $\tilde{\mathbf{1}}$ is the constant function on X taking the value only 1. In this notation, Proposition 2.2 is reformulated as follows:

Proposition 2.4. *Let $k_0 \in \mathbb{N}_0^n$, and let $\mathbf{w}_0, \ldots, \mathbf{w}_m \in \mathbb{T}^{n+1}$. For $j \in \mathbb{N}_0^m$, write $\mathbf{w}_j = (w_{j,0}, \ldots, w_{j,n})$. Assume that the k_0-th coordinate w_{0,k_0} of $\mathbf{w}_0$ is distinct from those of $\mathbf{w}_1, \ldots, \mathbf{w}_m$, namely, $w_{0,k_0} \notin \{w_{1,k_0}, \ldots, w_{m,k_0}\}$. Then there exists $f \in A^n(\overline{\mathbb{D}})$ such that*

$$\tilde{f}(\mathbf{w}_0, \boldsymbol{\zeta}) = \zeta_{k_0} \neq 0 = \tilde{f}(\mathbf{w}_1, \boldsymbol{\zeta}) = \cdots = \tilde{f}(\mathbf{w}_m, \boldsymbol{\zeta})$$

for all $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_n) \in \mathbb{T}^n$. In particular, if $\boldsymbol{\zeta}_0, \ldots, \boldsymbol{\zeta}_m \in \mathbb{T}^n$, and if $x_j = (\mathbf{w}_j, \boldsymbol{\zeta}_j)$, then f can be chosen so that $\tilde{f}(x_0) = 1 \neq 0 = \tilde{f}(x_1) = \cdots = \tilde{f}(x_m)$.

Proof. By Proposition 2.2, we can find $f \in A^n(\overline{\mathbb{D}})$ such that $D^{k_0}(f)(w_{0,k_0}) \neq 0$ and $D^k(f)(w_{j,k}) = 0$ for every $(j, k) \in (\mathbb{N}_0^m \times \mathbb{N}_0^n) \setminus \{(0, k_0)\}$. Multiplying a constant if needed, we may assume that $D^{k_0}(f)(w_{0,k_0}) = k_0!$. Then equality (2.8) shows that $\tilde{f}(\mathbf{w}_0, \boldsymbol{\zeta}) = \zeta_{k_0}$ and $\tilde{f}(\mathbf{w}_1, \boldsymbol{\zeta}) = \cdots = \tilde{f}(\mathbf{w}_m, \boldsymbol{\zeta}) = 0$ for all $\boldsymbol{\zeta} = (z_1, \ldots, \zeta_n)$.

Assume that $\boldsymbol{\zeta}_0, \ldots, \boldsymbol{\zeta}_m \in \mathbb{T}^n$, and that $x_j = (\mathbf{w}_j, \boldsymbol{\zeta}_j)$. Replacing f with the product of f and the complex conjugate of the k_0-th coordinate of $\boldsymbol{\zeta}_0$, we have $\tilde{f}(x_0) = 1 \neq 0 = \tilde{f}(x_1) = \cdots = \tilde{f}(x_m)$. □

Let $\widetilde{A^n} = \{\tilde{f} : f \in A^n(\overline{\mathbb{D}})\}$, and define $U : A^n(\overline{\mathbb{D}}) \to \widetilde{A^n}$ by

$$U(f) = \tilde{f} \qquad (f \in A^n(\overline{\mathbb{D}})). \tag{2.9}$$

Note that $\widetilde{A^n}$ is a complex linear subspace of $C(X)$, and hence $\widetilde{A^n}$ is a normed space with the supremum norm $\|\cdot\|_X$.

Lemma 2.5. *The mapping U, defined by (2.9), is a surjective complex-linear isometry from $(A^n(\overline{\mathbb{D}}), \|\cdot\|_\Sigma)$ onto $(\widetilde{A^n}, \|\cdot\|_X)$.*

Proof. By definition, it is obvious that U is surjective and complex-linear. To show that U is an isometry, fix $f \in A^n(\overline{\mathbb{D}})$. For each $x = (w_0, \dots, w_n, \zeta_1, \dots, \zeta_n) \in X$,

$$|\tilde{f}(x)| = \left|\sum_{k=0}^{n} \frac{1}{k!} D^k(f)(w_k)\zeta_k\right| \leq \sum_{k=0}^{n} \frac{1}{k!} |D^k(f)(w_k)| \leq \sum_{k=0}^{n} \frac{1}{k!} \|D^k(f)\|_{\mathbb{T}} = \|f\|_{\Sigma},$$

and thus we obtain $\|\tilde{f}\|_X \leq \|f\|_\Sigma$. On the other hand, for each $k \in \mathbb{N}_0^n$, choose $w_{0,k} \in \mathbb{T}$ so that $|D^k(f)(w_{0,k})| = \|D^k(f)\|_{\mathbb{T}}$. For each $k \in \mathbb{N}_0^n$, we set

$$\zeta_{0,k} = \frac{f(w_{0,0})}{|f(w_{0,0})|} \Big/ \frac{D^k(f)(w_{0,k})}{|D^k(f)(w_{0,k})|}$$

Here $f(w_{0,0})/|f(w_{0,0})|$ and $D^k(f)(w_{0,k})/|D^k(f)(w_{0,k})|$ read 1 if $f(w_{0,0}) = 0$ and $D^k(f)(w_{0,k}) = 0$, respectively. We also set $\zeta_{0,0} = 1$. Let $x_0 = (w_{0,1}, \dots, w_{0,n}, \zeta_{0,1}, \dots, \zeta_{0,n}) \in X$. Since $D^k(f)(w_{0,k})\zeta_{0,k}$ has the same argument as $f(w_{0,0})$, we have

$$\begin{aligned}\|f\|_\Sigma &= \sum_{k=0}^{n} \frac{1}{k!} \|D^k(f)\|_{\mathbb{T}} = \sum_{k=0}^{n} \frac{1}{k!} |D^k(f)(w_{0,k})| = \left|\sum_{k=0}^{n} \frac{1}{k!} D^k(f)(w_{0,k})\zeta_{0,k}\right| \\ &= |\tilde{f}(x_0)| \leq \|\tilde{f}\|_X.\end{aligned}$$

Therefore we obtain $\|\tilde{f}\|_X = \|f\|_\Sigma$, which proves that U is an isometry, as desired. □

Lemma 2.6. *The subspace $\widetilde{A^n}$ of $C(X)$ separates the points of X, that is, for each pair of distinct points $x_0, x_1 \in X$ there exists a function $\tilde{f} \in \widetilde{A^n}$ such that $\tilde{f}(x_0) \neq \tilde{f}(x_1)$.*

Proof. Let $x_0, x_1 \in X$ be distinct points, and write $x_j = (w_{j,0}, \dots, w_{j,n}, \zeta_{j,1}, \dots, \zeta_{j,n})$ for $j = 0, 1$. Assume that $w_{0,k_0} \neq w_{1,k_0}$ for some $k_0 \in \mathbb{N}_0^n$. By Proposition 2.4, there exists $f_0 \in A^n(\overline{\mathbb{D}})$ such that $\tilde{f}_0(x_0) \neq 0 = \tilde{f}_0(x_1)$.

Now, assume that $w_{0,k} = w_{1,k}$ for every $k \in \mathbb{N}_0^n$. Then $\zeta_{0,k_1} \neq \zeta_{1,k_1}$ for some $k_1 \in \mathbb{N}_1^n$. Set $\mathbf{w} = (w_{0,0}, \dots, w_{0,n}) = (w_{1,0}, \dots, w_{1,n})$. By Proposition 2.4, there exists $f_1 \in A^n(\overline{\mathbb{D}})$ such that $\tilde{f}_1(\mathbf{w}, \boldsymbol{\zeta}) = \zeta_{k_1}$ for all $\boldsymbol{\zeta} = (\zeta_1, \dots, \zeta_n)$. Hence we have $\tilde{f}_1(x_0) = \zeta_{0,k_1} \neq \zeta_{1,k_1} = \tilde{f}_1(x_1)$. The proof is completed. □

We have proved that $\widetilde{A^n}$ is a uniformly closed subspace of $C(X)$ which separates the points of X and contains the constant function $\tilde{\mathbf{1}}$. In the rest of this section, we consider the set $\operatorname{ext}((\widetilde{A^n})_1^*)$ of all extreme points of the unit ball $(\widetilde{A^n})_1^*$ of the dual space $(\widetilde{A^n})^*$ of $\widetilde{A^n}$.

For each $x \in X$, the *point evaluation* δ_x at x is a functional $\delta_x : \widetilde{A^n} \to \mathbb{C}$ defined by $\delta_x(\tilde{f}) = \tilde{f}(x)$ for every $\tilde{f} \in \widetilde{A^n}$. By the Arens-Kelley theorem (see [3, Lemma V.8.6]), every extreme point of the unit ball $(\widetilde{A^n})_1^*$ is of the form $\lambda\delta_x$ for some $x \in X$ and $\lambda \in \mathbb{T}$. Recall that the *Choquet boundary* of $\widetilde{A^n}$ is the set

$$\mathrm{Ch}(\widetilde{A^n}) = \{x \in X : \delta_x \in \mathrm{ext}((\widetilde{A^n})_1^*)\}.$$

Then the set $\mathrm{ext}((\widetilde{A^n})_1^*)$ can be written as

$$\mathrm{ext}((\widetilde{A^n})_1^*) = \{\lambda\delta_x : \lambda \in \mathbb{T},\ x \in \mathrm{Ch}(\widetilde{A^n})\}. \tag{2.10}$$

Let $x \in X$. Recall that a *representing measure* for δ_x is a positive regular Borel measure μ on X such that

$$\delta_x(\tilde{f}) = \int_X f d\mu$$

for every $\tilde{f} \in \widetilde{A^n}$. Since $\|\delta_x\| = \delta_x(\tilde{\mathbf{1}}) = 1$, we see that every representing measure μ must be a probability measure. Note also that there exists at least one representing measure for δ_x, namely, the Dirac measure concentrated at x. The Choquet boundary of $\widetilde{A^n}$ can be characterized in terms of representing measures.

Proposition 2.7. *Assume that each representing measure μ for δ_x is concentrated at x. Then δ_x is an extreme point of $(\widetilde{A^n})_1^*$, that is, $x \in \mathrm{Ch}(\widetilde{A^n})$.*

Proof. Assume that δ_x is written as $\delta_x = (1-t)\xi_1 + t\xi_2$ for some $\xi_1, \xi_2 \in (\widetilde{A^n})_1^*$ and $t \in (0,1)$. Then $|\xi_1(\tilde{\mathbf{1}})|, |\xi_2(\tilde{\mathbf{1}})| \leq 1$, and that $1 = \delta_x(\tilde{\mathbf{1}}) = (1-t)\xi_1(\tilde{\mathbf{1}}) + t\xi_2(\tilde{\mathbf{1}})$. Since 1 is an extreme point of the closed unit disk $\overline{\mathbb{D}}$, we have $\xi_1(\tilde{\mathbf{1}}) = \xi_2(\tilde{\mathbf{1}}) = 1$. It is well-known that the Dirac measure concentrated at x is the only representing measure for δ_x if and only if δ_x is an extreme point of the weak $*$-compact convex set $\{\xi \in (\widetilde{A^n})_1^* : \xi(\tilde{\mathbf{1}}) = 1\}$ (see [2, Theorem 2.2.8]). Hence $\delta_x = \xi_1 = \xi_2$, that is, δ_x is an extreme point of $(\widetilde{A^n})_1^*$. □

Consider the subset X_0 of $X = \mathbb{T}^{2n+1}$ consisting of all those points $(w_0, \ldots, w_n, \zeta_1, \ldots, \zeta_n)$ such that $w_0, \ldots, w_n$ are mutually distinct:

$$X_0 = \{(w_0, \ldots, w_n, \zeta_1, \ldots, \zeta_n) \in X : w_j \neq w_k \quad (j \neq k)\}. \tag{2.11}$$

It is clear that X_0 is dense in X. Let us show that every point in X_0 is an extreme point of the dual ball $(\widetilde{A^n})_1^*$. To see this, fix an arbitrary point $x_0 = (w_{0,0}, \ldots, w_{0,n}, \zeta_{0,1}, \ldots, \zeta_{0,n}) \in X_0$. For simplicity of notation, we set $\zeta_{0,0} = 1$. In view of Proposition 2.7, it suffices to show that any representing measure μ for δ_{x_0} is concentrated at x_0.

Lemma 2.8. *Any representing measure μ for δ_{x_0} is concentrated on the set $\{w_{0,0}\} \times \cdots \times \{w_{0,n}\} \times \mathbb{T}^n$.*

Proof. For each $k \in \mathbb{N}_0^n$, we set $X^{(k)} = \mathbb{T}^k \times \{w_{0,k}\} \times \cdots \times \{w_{0,n}\} \times \mathbb{T}^n$ and $X^{(n+1)} = X = \mathbb{T}^{2n+1}$. Let us show that each representing measure μ for δ_{x_0} is concentrated on $X^{(k_0)}$ for every $k_0 \in \mathbb{N}_0^n$ by induction. Fix an arbitrary representing measure μ for δ_{x_0}. If $k_0 = n+1$, then μ is concentrated on $X^{(n+1)} = \mathbb{T}^{2n+1}$ by definition. Assume that μ is concentrated on $X^{(k_0+1)}$ for $k_0 \in \mathbb{N}_0^n$; we will prove that it is concentrated on $X^{(k_0)}$.

Let W be an arbitrary open neighborhood of w_{0,k_0} in $\mathbb{T}$, and set

$$\begin{aligned} Q_W &= \mathbb{T}^{k_0} \times W \times \{w_{0,k_0+1}\} \times \cdots \times \{w_{0,n}\} \times \mathbb{T}^n, \quad \text{and} \\ Q_{W^c} &= \mathbb{T}^{k_0} \times W^c \times \{w_{0,k_0+1}\} \times \cdots \times \{w_{0,n}\} \times \mathbb{T}^n, \end{aligned}$$

where $W^c = \mathbb{T} \setminus W$. Note that Q_{W^c} is the complement of Q_W in $X^{(k_0+1)}$. Let us show that $\mu(Q_W) = 1$. To see this, choose ε with $0 < \varepsilon < 1/n$ arbitrarily. By Proposition 2.3, there exists $f_0 \in A^n(\overline{\mathbb{D}})$ such that

$$\begin{cases} \|D^l(f_0)\|_{\mathbb{T}} < \varepsilon & (l \in \mathbb{N}_0^{k_0-1}), \\ \|D^{k_0}(f_0)\|_{\mathbb{T}} = D^{k_0}(f_1)(w_{0,k_0}) = k_0!, \\ \|D^{k_0}(f_0)\|_{\mathbb{T}\setminus W} < \varepsilon, \\ |D^l(f_0)(w_{0,l})| < \varepsilon & (l \in \mathbb{N}_{k_0+1}^n). \end{cases}$$

It follows from equality (2.8) that $\|\tilde{f}_0\|_{X^{(k_0+1)}} < n\varepsilon + 1$ and $\|\tilde{f}_0\|_{Q_{W^c}} < (n+1)\varepsilon$. Also we have

$$\left| \sum_{k \in \mathbb{N}_0^n \setminus \{k_0\}} \frac{1}{k!} D^k(f_0)(w_{0,k})\zeta_{0,k} \right| < n\varepsilon < 1.$$

Since μ is a representing measure for δ_{x_0} concentrated on $X^{(k_0+1)}$, we have

$$\int_{X^{(k_0+1)}} \tilde{f}_0 d\mu = \delta_{x_0}(\tilde{f}_0) = \zeta_{0,k_0} + \sum_{k \in \mathbb{N}_0^n \setminus \{k_0\}} \frac{1}{k!} D^k(f_0)(w_{0,k})\zeta_{0,k},$$

and thus

$$\begin{aligned} 1 - n\varepsilon < |\delta_{x_0}(\tilde{f}_0)| &= \left| \int_{X^{(k_0+1)}} \tilde{f}_0 d\mu \right| \le \left| \int_{Q_W} \tilde{f}_0 d\mu \right| + \left| \int_{Q_{W^c}} \tilde{f}_0 d\mu \right| \\ &< (n\varepsilon + 1)\mu(Q_W) + (n+1)\varepsilon\mu(Q_{W^c}). \end{aligned}$$

Since ε is arbitrary, we have $1 \le \mu(Q_W) \le \mu(X) = 1$, that is, $\mu(Q_W) = 1$.

Now, let $\{W_n\}_n$ be a decreasing sequence of open neighborhoods of w_{0,k_0} in $\mathbb{T}$ whose intersection is precisely the singleton $\{w_{0,k_0}\}$. Then $\{Q_{W_n}\}$ is a decreasing sequence of sets of measure 1 with respect to μ and $\bigcap_{n=1}^{\infty} Q_{W_n} = X^{(k_0)}$. Therefore $\mu(X^{(k_0)}) = 1$, that is, μ is concentrated on $X^{(k_0)}$. Consequently, we have proved that μ is concentrated on $X^{(k)}$ for every $k \in \mathbb{N}_0^n$, in particular, it is concentrated on $X^{(0)} = \{w_{0,0}\} \times \cdots \times \{w_{0,n}\} \times \mathbb{T}^n$. □

Lemma 2.9. *Any representing measure μ for δ_{x_0} is concentrated at the point x_0.*

Proof. For simplicity, set $X' = \{w_{0,0}\}\times\cdots\times\{w_{0,n}\}\times\mathbb{T}^n$ and $\mathbf{w}_0 = (w_{0,0}, \ldots, w_{0,n})$. Fix $k_0 \in \mathbb{N}_1^n$. By Proposition 2.4, there exists $f_1 \in A^n(\overline{\mathbb{D}})$ such that $\tilde{f}_1(\mathbf{w}_0, \boldsymbol{\zeta}) = \zeta_{k_0}$ for every $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_n) \in \mathbb{T}^n$. Since, by lemma 2.8, the measure μ concentrated on $\{w_{0,0}\} \times \cdots \times \{w_n\} \times \mathbb{T}^n$, we have

$$\zeta_{0,k_0} = \delta_{x_0}(\tilde{f}_1) = \int_{X'} \tilde{f}_1 d\mu = \int_{X'} \zeta_{k_0} d\mu(\mathbf{w}, \boldsymbol{\zeta}),$$

and thus

$$\int_{X'} (1 - \overline{\zeta_{0,k_0}}\zeta_{k_0}) d\mu(\mathbf{w}, \boldsymbol{\zeta}) = 0.$$

Since μ is a positive measure, it follows that

$$\int_{X'} (1 - \mathrm{Re}[\overline{\zeta_{0,k_0}}\zeta_{k_0}]) d\mu(\mathbf{w}, \boldsymbol{\zeta}) = 0.$$

Hence the measure of the set $\{(\mathbf{w}, \boldsymbol{\zeta}) \in X' : \mathrm{Re}[\overline{\zeta_{0,k_0}}\zeta_{k_0}] \neq 1\}$ with respect to μ must be zero. This proves that μ is concentrated on $\{w_{0,0}\}\times\cdots\times\{w_{0,n}\}\times\mathbb{T}^{k_0-1}\times\{\zeta_{0,k_0}\}\times\mathbb{T}^{n-k_0}$. Since this holds for every $k_0 \in \mathbb{N}_1^n$, it follows that μ is concentrated at x_0. □

It now follows from Proposition 2.7 and Lemma 2.9 that X_0 is contained in the Choquet boundary $\mathrm{Ch}(\widetilde{A^n})$. Set

$$\mathcal{B} = \{\lambda\delta_x : \lambda \in \mathbb{T}, x \in X\}. \tag{2.12}$$

It is easy to see that $\mathcal{B}$ is a closed subset of the unit ball $(\widetilde{A^n})_1^*$ of the dual space of $\widetilde{A^n}$, and thus it is a compact Hausdorff space with respect to the relative weak $*$-topology. Let $\mathbb{T} \times X$ be the compact Hausdorff space endowed with the product topology. Define $\mathbf{h} : \mathbb{T} \times X \to \mathcal{B}$ by

$$\mathbf{h}(\lambda, x) = \lambda\delta_x \qquad ((\lambda, x) \in \mathbb{T} \times X). \tag{2.13}$$

The proof of the following lemma is the same argument as [12, Lemma 2.7].

Lemma 2.10. *The mapping $\mathbf{h} : \mathbb{T} \times X \to \mathcal{B}$ is a homeomorphism from $\mathbb{T} \times X$ onto $\mathcal{B}$. In particular,* $\mathbf{h}(\mathbb{T} \times \mathrm{Ch}(\widetilde{A^n})) = \mathrm{ext}((\widetilde{A^n})_1^*)$.

Proof. By definition, $\mathbf{h}$ is surjective. Since $\widetilde{A^n}$ separates the points of X, and since $\widetilde{A^n}$ contains the constant function $\tilde{\mathbf{1}}$, we see that $\mathbf{h}$ is injective.

To show that $\mathbf{h}$ is continuous, choose sequences $\{\lambda_n\}_n \subset \mathbb{T}$ and $\{x_k\}_k \subset X$ converging to $\lambda \in \mathbb{T}$ and $x \in X$, respectively. For each $\tilde{f} \in \widetilde{A^n}$,

$$\mathbf{h}(\lambda_k, x_k)(\tilde{f}) = \lambda_k \tilde{f}(x_k) \to \lambda \tilde{f}(x) = \mathbf{h}(\lambda, x)(\tilde{f}) \qquad (k \to \infty).$$

Thus the sequence $\{\mathbf{h}(\lambda_k, x_k)\}_k$ converges to $\mathbf{h}(\lambda, x)$ with respect to the relative weak $*$-topology, which proves the continuity of $\mathbf{h}$. Since $\mathbf{h}$ is a bijective continuous mapping from the compact space $\mathbb{T} \times X$ onto the Hausdorff space $\mathcal{B}$, it must be a homeomorphism from $\mathbb{T} \times X$ onto $\mathcal{B}$. In particular, equality (2.10) shows that $\mathbf{h}(\mathbb{T} \times \mathrm{Ch}(\widetilde{A^n})) = \mathrm{ext}((\widetilde{A^n})_1^*)$. □

§ 3. Surjective real-linear isometries on $(\widetilde{A^n}, \|\cdot\|_X)$

In this section, we will characterize the surjective real-linear isometries on the Banach space $(\widetilde{A^n}, \|\cdot\|_X)$. Throughout this section, fix a surjective real-linear isometry $S : \widetilde{A^n} \to \widetilde{A^n}$. Define $S_* : (\widetilde{A^n})^* \to (\widetilde{A^n})^*$ by

$$S_*(\xi)(\tilde{f}) = \mathrm{Re}[\xi(S(\tilde{f}))] - i\,\mathrm{Re}[\xi(S(i\tilde{f}))]$$

for every $\xi \in (\widetilde{A^n})^*$ and $\tilde{f} \in \widetilde{A^n}$. Note that S_* is a well-defined surjective real-linear isometry on $(\widetilde{A^n})^*$ with respect to the operator norm. In particular, we have $S_*(\mathrm{ext}((\widetilde{A^n})_1^*)) = \mathrm{ext}((\widetilde{A^n})_1^*)$. Proof of the next lemma is the same as that of [12, Lemma 2.8].

Lemma 3.1. *Let $\mathcal{B}$ be the compact Hausdorff space defined by* (2.12). *Then $S_*(\mathcal{B}) = \mathcal{B}$.*

Proof. Let $\mathbf{h} : \mathbb{T} \times X \to \mathcal{B}$ be the mapping defined by (2.13). Then $\mathbf{h}(\mathbb{T} \times \mathrm{Ch}(\widetilde{A^n})) = \mathrm{ext}((\widetilde{A^n})_1^*) = S_*(\mathrm{ext}((\widetilde{A^n})_1^*))$. Since $X_0 \subset \mathrm{Ch}(\widetilde{A^n}) \subset X$, where X_0 is the subset of $X = \mathbb{T}^{2n+1}$ defined by (2.11), we have

$$\begin{aligned} S_*(\mathbf{h}(\mathbb{T} \times X_0)) &\subset S_*(\mathbf{h}(\mathbb{T} \times \mathrm{Ch}(\widetilde{A^n}))) = S_*(\mathrm{ext}((\widetilde{A^n})_1^*)) \\ &= \mathbf{h}(\mathbb{T} \times \mathrm{Ch}(\widetilde{A^n})) \subset \mathbf{h}(\mathbb{T} \times X) = \mathcal{B}. \end{aligned}$$

Recall that the closure $\overline{X_0}$ of X_0 coincides with X, since X_0 is dense in X. It follows from Lemma 2.10 that

$$\mathcal{B} = \mathbf{h}(\mathbb{T} \times X) = \mathbf{h}(\mathbb{T} \times \overline{X_0}) = \overline{\mathbf{h}(\mathbb{T} \times X_0)},$$

where $\overline{\mathbf{h}(\mathbb{T} \times X_0)}$ is the closure of $\mathbf{h}(\mathbb{T} \times X_0)$ in $\mathcal{B}$ with respect to the relative weal $*$-topology. Since S_* is a surjective real-linear isometry on $(\widetilde{A^n})^*$ with respect to the operator norm, S_* is a homeomorphism with respect to the weak $*$-topology, and thus

$$S_*(\mathcal{B}) = S_*\left(\overline{\mathbf{h}(\mathbb{T} \times X_0)}\right) = \overline{S_*(\mathbf{h}(\mathbb{T} \times X_0))} \subset \mathcal{B}.$$

Hence $S_*(\mathcal{B}) \subset \mathcal{B}$. Applying the same argument to S_*^{-1}, we see that $S_*^{-1}(\mathcal{B}) \subset \mathcal{B}$. Thus $S_*(\mathcal{B}) = \mathcal{B}$. $\square$

Definition 3.2. Let $p_1 : \mathbb{T} \times X \to \mathbb{T}$ and $p_2 : \mathbb{T} \times X \to X$ be the canonical projections. Define $\alpha : \mathbb{T} \times X \to \mathbb{T}$ and $\Phi : \mathbb{T} \times X \to X$ by

$$\alpha = p_1 \circ \mathbf{h}^{-1} \circ S_* \circ \mathbf{h}, \quad \text{and} \quad \Phi = p_2 \circ \mathbf{h}^{-1} \circ S_* \circ \mathbf{h}.$$

Note that α and Φ are surjective continuous mappings. By definition, for each $(\lambda, x) \in \mathbb{T} \times X$, we have $(S_* \circ \mathbf{h})(\lambda, x) = \mathbf{h}(\alpha(\lambda, x), \Phi(\lambda, x))$, that is, $S_*(\lambda\delta_x) = \alpha(\lambda, x)\delta_{\Phi(\lambda,x)}$. Now for each $\lambda \in \mathbb{T}$, let $\alpha_\lambda(x) = \alpha(\lambda, x)$. Then

$$S_*(\lambda\delta_x) = \alpha_\lambda(x)\delta_{\Phi(\lambda,x)} \qquad (\forall(\lambda, x) \in \mathbb{T} \times X).$$

Lemma 3.3. *There exists $s_0 \in \{\pm 1\}$ such that $\alpha_i(x) = is_0\alpha_1(x)$ for all $x \in X$.*

Proof. First, let us show that for each $x \in X$, $\alpha_i(x) = i\alpha_1(x)$ or $\alpha_i(x) = -i\alpha_1(x)$. Fix $x \in X$. For $\lambda_0 = \frac{1+i}{\sqrt{2}} \in \mathbb{T}$, the real-linearity of S_* implies that

$$\sqrt{2}\alpha_{\lambda_0}(x)\delta_{\Phi(\lambda_0,x)} = S_*(\sqrt{2}\lambda_0\delta_x) = S_*(\delta_x) + S_*(i\delta_x) = \alpha_1(x)\delta_{\Phi(1,x)} + \alpha_i(x)\delta_{\Phi(i,x)}.$$

Hence we have $\sqrt{2}\alpha_{\lambda_0}(x)\delta_{\Phi(\lambda_0,x)} = \alpha_1(x)\delta_{\Phi(1,x)} + \alpha_i(x)\delta_{\Phi(i,x)}$. Evaluating this equality at $\tilde{\mathbf{1}}$, we obtain $\sqrt{2}\alpha_{\lambda_0}(x) = \alpha_1(x) + \alpha_i(x)$. Since $|\alpha_{\lambda_0}(x)| = 1$, we have

$$\sqrt{2} = |\alpha_1(x) + \alpha_i(x)| = |1 + \alpha_i(x)\overline{\alpha_1(x)}|,$$

and thus $\alpha_i(x)\overline{\alpha_1(x)} \in \{\pm i\}$. Therefore $\alpha_i(x) = i\alpha_1(x)$ or $\alpha_i(x) = -i\alpha_1(x)$.

Now, we set

$$K_+ = \{x \in X : \alpha_i(x) = i\alpha_1(x)\} \quad \text{and} \quad K_- = \{x \in X : \alpha_i(x) = -i\alpha_1(x)\}.$$

Then $K_+ \cup K_- = X$ and $K_+ \cap K_- = \emptyset$. The continuity of α_1 and α_i implies that K_+ and K_- are closed in X. Since $X = \mathbb{T}^{2n+1}$ is connected, $K_+ = X$ or $K_- = X$. This proves the existence of $s_0 \in \{\pm 1\}$ such that $\alpha_i(x) = is_0\alpha_1(x)$ for every $x \in X$. $\square$

Lemma 3.4. *For each $\lambda = r + it \in \mathbb{T}$ with $r, t \in \mathbb{R}$, and each $x \in X$,*

$$\lambda^{s_0}\tilde{f}(\Phi(\lambda, x)) = r\tilde{f}(\Phi(1, x)) + is_0t\tilde{f}(\Phi(i, x)) \tag{3.1}$$

for every $\tilde{f} \in \widetilde{A^n}$.

Proof. Let $\lambda = r + it \in \mathbb{T}$ with $r, t \in \mathbb{R}$, and let $x \in X$. Since S_* is real-linear,

$$\alpha_\lambda(x)\delta_{\Phi(\lambda,x)} = S_*(\lambda\delta_x) = rS_*(\delta_x) + tS_*(i\delta_x) = r\alpha_1(x)\delta_{\Phi(1,x)} + is_0t\alpha_1(x)\delta_{\Phi(i,x)},$$

and thus $\alpha_\lambda(x)\delta_{\Phi(\lambda,x)} = \alpha_1(x)(r\delta_{\Phi(1,x)} + is_0 t\delta_{\Phi(i,x)})$. Evaluating this equality at $\tilde{\mathbf{1}}$, we have $\alpha_\lambda(x) = \alpha_1(x)(r+is_0t)$. Since $\lambda \in \mathbb{T}$ and $s_0 \in \{\pm 1\}$, we have $\lambda^{s_0} = r+is_0t$. Hence $\alpha_\lambda(x) = \lambda^{s_0}\alpha_1(x)$. This implies that $\lambda^{s_0}\delta_{\Phi(\lambda,x)} = r\delta_{\Phi(1,x)} + is_0t\delta_{\Phi(i,x)}$. Therefore we obtain $\lambda^{s_0}\tilde{f}(\Phi(\lambda,x)) = r\tilde{f}(\Phi(1,x)) + is_0t\tilde{f}(\Phi(i,x))$ for every $\tilde{f} \in \widetilde{A^n}$. □

Definition 3.5. For $j \in \mathbb{N}_0^{2n}$, let $q_j : X = \mathbb{T}^{2n+1} \to \mathbb{T}$ be the j-th canonical projection. Define $\varphi_0, \dots, \varphi_n, \chi_1, \dots, \chi_n : \mathbb{T} \times X \to \mathbb{T}$ by

$$\varphi_k = q_k \circ \Phi \quad (k \in \mathbb{N}_0^n), \quad \text{and} \quad \chi_k = q_{n+k} \circ \Phi \quad (k \in \mathbb{N}_1^n),$$

that is, $\Phi(\lambda,x) = (\varphi_0(\lambda,x), \dots, \varphi_n(\lambda,x), \chi_1(\lambda,x), \dots, \chi_n(\lambda,x))$ for every $(\lambda,x) \in \mathbb{T}\times X$. For simplicity of notation, we set $\chi_0(\lambda,x) = 1$ for all $(\lambda,x) \in \mathbb{T} \times X$.

Note that the mappings $\varphi_0, \dots, \varphi_n, \chi_1, \dots, \chi_n$ are surjective continuous mappings for every $k \in \mathbb{N}_1^n$. For each $\lambda \in \mathbb{T}$ and $x \in X$, we set $\varphi_{k,\lambda}(x) = \varphi_k(\lambda,x)$ and $\chi_{k,\lambda}(x) = \chi_k(\lambda,x)$. Then $\Phi(\lambda,x) = (\varphi_{0,\lambda}(x), \dots, \varphi_{n,\lambda}(x), \chi_{1,\lambda}(x), \dots, \chi_{n,\lambda}(x))$, and thus equality (2.8) implies that

$$\tilde{f}(\Phi(\lambda,x)) = \sum_{k=0}^{n} \frac{1}{k!} D^k(f)(\varphi_{k,\lambda}(x))\chi_{k,\lambda}(x) \tag{3.2}$$

for every $f \in A^n(\overline{\mathbb{D}})$ and $(\lambda,x) \in \mathbb{T} \times X$.

Lemma 3.6. *Let $k \in \mathbb{N}_0^n$, and let $\lambda \in \mathbb{T}$. Then $\varphi_{k,\lambda}(x) = \varphi_{k,1}(x)$ for every $x \in X$.*

Proof. Fix $x \in X$. Let us show that $\varphi_{k,\lambda}(x) \in \{\varphi_{k,1}(x), \varphi_{k,i}(x)\}$ for every $k \in \mathbb{N}_0^n$ and every $\lambda \in \mathbb{T}$. Suppose, on the contrary, that $\varphi_{k_0,\lambda_0}(x) \notin \{\varphi_{k_0,1}(x), \varphi_{k_0,i}(x)\}$ for some $k_0 \in \mathbb{N}_0^n$ and $\lambda_0 \in \mathbb{T}$. By Proposition 2.4, there exists $f_0 \in A^n(\overline{\mathbb{D}})$ such that

$$\tilde{f}_0(\Phi(\lambda_0,x)) = 1 \neq 0 = \tilde{f}_0(\Phi(1,x)) = \tilde{f}_0(\Phi(i,x)).$$

Substituting these equalities into equality (3.1), we obtain $\lambda_0^{s_0} = 0$, which is a contradiction. Consequently, $\varphi_{k,\lambda}(x) \in \{\varphi_{k,1}(x), \varphi_{k,i}(x)\}$ for every $k \in \mathbb{N}_0^n$ and $\lambda \in \mathbb{T}$.

Now, we see that, for fixed $x \in X$ and $k \in \mathbb{N}_0^n$, the mapping $\lambda \mapsto \varphi_{k,\lambda}(x)$ is a continuous map from the connected space $\mathbb{T}$ onto $\{\varphi_{k,1}(x), \varphi_{k,i}(x)\}$. Hence $\{\varphi_{k,1}(x), \varphi_{k,i}(x)\}$ must be a singleton, that is, $\varphi_{k,\lambda}(x) = \varphi_{k,1}(x)$ for every $k \in \mathbb{N}_0^n$, $\lambda \in \mathbb{T}$ and $x \in X$. □

Lemma 3.7. *For each $k \in \mathbb{N}_1^n$, there exists $s_k \in \{\pm 1\}$ such that $\chi_{k,i}(x) = s_0 s_k \chi_{k,1}(x)$ for every $x \in X$.*

Proof. Fix $x \in X$ and $k \in \mathbb{N}_1^n$. Let us show that $\chi_{k,i}(x) = \chi_{k,1}(x)$ or $\chi_{k,i}(x) = -\chi_{k,1}(x)$. Let $\lambda_0 = \frac{1+i}{\sqrt{2}}$. By Lemma 3.6, $\Phi(\mu,x) = (\varphi_{0,1}(x), \dots, \varphi_{n,1}(x), \chi_{1,\mu}(x), \dots,$

$\chi_{n,\mu}(x))$ for $\mu = 1, i, \lambda_0$. Applying Proposition 2.4 with $\mathbf{w}_0 = (\varphi_{0,1}(x), \dots, \varphi_{n,1}(x))$, we can find $f \in A^n(\overline{\mathbb{D}})$ such that

$$\tilde{f}(\varphi_{0,1}(x), \dots, \varphi_{n,1}(x), \boldsymbol{\zeta}) = \zeta_k$$

for every $\boldsymbol{\zeta} = (\zeta_1, \dots, \zeta_n) \in \mathbb{T}^n$. In particular, we have $\tilde{f}_0(\Phi(\mu, x)) = \chi_{k,\mu}(x)$ for every $\mu = 1, i, \lambda_0$. Substituting these equalities to equality (3.1), we have $\sqrt{2}\lambda_0^{s_0}\chi_{k,\lambda_0}(x) = \chi_{k,1}(x) + is_0\chi_{k,i}(x)$. Since $\chi_{k,\lambda_0}(x) \in \mathbb{T}$, we obtain

$$\sqrt{2} = |\chi_{k,1}(x) + is_0\chi_{k,i}(x)| = |1 + is_0\chi_{k,i}(x)\overline{\chi_{k,1}(x)}|,$$

and thus $is_0\chi_{k,i}(x)\overline{\chi_{k,1}(x)} \in \{\pm i\}$. Hence $\chi_{k,i}(x) = s_0\chi_{k,1}(x)$ or $\chi_{k,i}(x) = -s_0\chi_{k,1}(x)$.

Now, we set

$$L_{k,+} = \{x \in X : \chi_{k,i}(x) = s_0\chi_{k,1}(x)\} \quad \text{and} \quad L_{k,-} = \{x \in X : \chi_{k,i}(x) = -s_0\chi_{k,1}(x)\}.$$

Then $L_{k,+} \cup L_{k,-} = X$ and $L_{k,+} \cap L_{k,-} = \emptyset$. The continuity of $\chi_{k,1}$ and $\chi_{k,i}$ implies that $L_{k,+}$ and $L_{k,-}$ are closed sets in the connected space $X = \mathbb{T}^{2n+1}$, and thus we obtain $L_{k,+} = X$ or $L_{k,-} = X$. This guarantees the existence of $s_k \in \{\pm 1\}$ such that $\chi_{k,i}(x) = s_0 s_k \chi_{k,1}(x)$ for every $x \in X$. □

In the rest of this paper, we use the following notation. If $a, b \in \mathbb{R}$ and $s \in \{\pm 1\}$, we denote $a + isb$ by $[a + ib]^s$, that is, for each $\lambda \in \mathbb{C}$, $[\lambda]^1 = \lambda$, and $[\lambda]^{-1} = \overline{\lambda}$. Clearly, $[\lambda\mu]^s = [\lambda]^s[\mu]^s$ for all $\lambda, \mu \in \mathbb{C}$. It is also clear that $[\lambda]^s = \lambda^s$ whenever $\lambda \in \mathbb{T}$.

Lemma 3.8. *For each $f \in A^n(\overline{\mathbb{D}})$ and $x \in X$,*

$$S(\tilde{f})(x) = \sum_{k=0}^{n} \frac{1}{k!}[\alpha_1(x)D^k(f)(\varphi_{k,1}(x))\chi_{k,1}(x)]^{s_k}. \tag{3.3}$$

Proof. Let $f \in A^n(\overline{\mathbb{D}})$, and let $x \in X$. By the definition of S_*, we have $\mathrm{Re}[S_*(\xi)(\tilde{f})] = \mathrm{Re}[\xi(S(\tilde{f}))]$ for every $\xi \in (\widetilde{A^n})^*$. Taking $\xi = \delta_x$ and $\xi = i\delta_x$, we derive that $\mathrm{Re}[S(\tilde{f})(x)] = \mathrm{Re}[S_*(\delta_x)(\tilde{f})]$ and $\mathrm{Im}[S(\tilde{f})(x)] = -\mathrm{Re}[S_*(i\delta_x)(\tilde{f})]$, respectively. Therefore

$$S(\tilde{f})(x) = \mathrm{Re}[S_*(\delta_x)(\tilde{f})] - i\,\mathrm{Re}[S_*(i\delta_x)(\tilde{f})]. \tag{3.4}$$

Recall that $S_*(\delta_x) = \alpha_1(x)\delta_{\Phi(1,x)}$ and $S_*(i\delta_x) = is_0\alpha_1(x)\delta_{\Phi(i,x)}$. Substituting these equalities into equality (3.4), we obtain

$$S(\tilde{f})(x) = \mathrm{Re}[\alpha_1\tilde{f}(\Phi(1,x))] + i\,\mathrm{Im}[s_0\alpha_1(x)\tilde{f}(\Phi(i,x))]. \tag{3.5}$$

It follows from Lemmas 3.6 and 3.7 that

$$\begin{aligned}\Phi(1,x) &= (\varphi_{0,1}(x),\dots,\varphi_{n,1}(x),\chi_{1,1}(x),\dots,\chi_{n,1}(x)) \quad \text{and}\\ \Phi(i,x) &= (\varphi_{0,1}(x),\dots,\varphi_{n,1}(x),s_0s_1\chi_{1,1}(x),\dots,s_0s_n\chi_{n,1}(x)).\end{aligned} \tag{3.6}$$

Keeping in mind that $s_0^2 = 1$, equalities (3.2), (3.5) and (3.6) imply that

$$\begin{aligned}S(\tilde{f})(x) &= \operatorname{Re}\left[\alpha_1(x)\sum_{k=0}^{n}\frac{1}{k!}D^k(f)(\varphi_{k,1}(x))\chi_{k,1}(x)\right]\\ &\qquad + i\operatorname{Im}\left[\alpha_1(x)\sum_{k=0}^{n}\frac{1}{k!}D^k(f)(\varphi_{k,1}(x))s_k\chi_{k,1}(x)\right]\\ &= \sum_{k=0}^{n}\frac{1}{k!}[\alpha_1(x)D^k(f)(\varphi_{k,1}(x))\chi_{k,1}(x)]^{s_k}.\end{aligned}$$

This completes the proof. □

For simplicity, we may write $\varphi_k(x) = \varphi_{k,1}(x)$ and $\chi_k(x) = \chi_{k,1}(x)$ for every $x \in X$. Then equality (3.3) is reduced to

$$S(\tilde{f})(x) = \sum_{k=0}^{n}\frac{1}{k!}[\alpha_1(x)D^k(f)(\varphi_k(x))\chi_k(x)]^{s_k} \tag{3.7}$$

for every $f \in A^n(\overline{\mathbb{D}})$ and $x \in X$.

§ 4. Proof of the main theorem

Let $T : A^n(\overline{\mathbb{D}}) \to A^n(\overline{\mathbb{D}})$ be a surjective, not necessarily linear, isometry on the Banach space $(A^n(\overline{\mathbb{D}}), \|\cdot\|_\Sigma)$. Define $T_0 : A^n(\overline{\mathbb{D}}) \to A^n(\overline{\mathbb{D}})$ by

$$T_0(f) = T(f) - T(\mathbf{0})$$

for every $f \in A^n(\overline{\mathbb{D}})$. By the Mazur-Ulam theorem (see [5, Theorem 1.3.5]), T_0 is a surjective real-linear isometry on $(A^n(\overline{\mathbb{D}}), \|\cdot\|_\Sigma)$. Let $S_0 : \widetilde{A^n} \to \widetilde{A^n}$ be defined by $U \circ T_0 \circ U^{-1}$, where U is defined by (2.9). Since U is a surjective complex-linear isometry from $A^n(\overline{\mathbb{D}})$ onto $\widetilde{A^n}$, S_0 is a surjective real-linear isometry on $\widetilde{A^n}$. Note that $S_0(\tilde{f}) = \widetilde{T_0(f)}$ for every $f \in A^n(\overline{\mathbb{D}})$. Replacing S by S_0 in equality (3.7), we obtain

$$\sum_{k=0}^{n}\frac{1}{k!}D^k(T_0(f))(w_k)\zeta_k = \sum_{k=0}^{n}\frac{1}{k!}[\alpha_1(x)D^k(f)(\varphi_k(x))\chi_k(x)]^{s_k} \tag{4.1}$$

for every $f \in A^n(\overline{\mathbb{D}})$ and $x = (w_0,\dots,w_n,\zeta_1,\dots,\zeta_n) \in X$. To prove the next lemma, we need the following elementary proposition.

Proposition 4.1. *Let $\lambda_0, \ldots, \lambda_n \in \mathbb{C}$, let $M \geq 0$, and assume that*

$$\left|\lambda_0 + \sum_{k=1}^{n} \lambda_k \zeta_k\right| = M \tag{4.2}$$

for every $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_n) \in \mathbb{T}^n$. Then there exists $k_0 \in \mathbb{N}_0^n$ such that $|\lambda_{k_0}| = M$ and $\lambda_k = 0$ for every $k \in \mathbb{N}_0^n \setminus \{k_0\}$.

Proof. If $M = 0$, then the proposition is clearly true. Assume that $M \neq 0$. Dividing (4.2) by M, we may assume that $M = 1$. Multiplying $\lambda_0, \ldots, \lambda_n$ by a suitable constant with modulus 1, we may also assume that λ_0 is non-negative. Note that at least one λ_k is non-zero. Assume $\lambda_{k_0} \neq 0$. Choose $\zeta_1, \ldots, \zeta_n \in \mathbb{T}$ so that $\lambda_k \zeta_k = |\lambda_k|$ for every $k \in \mathbb{N}_0^n$. By assumption, we have

$$\left| |\lambda_{k_0}| \pm \sum_{k \in \mathbb{N}_0^n \setminus \{k_0\}} |\lambda_k| \right| = \left| \lambda_0 + \lambda_{k_0}\zeta_{k_0} + \sum_{k \in \mathbb{N}_1^n \setminus \{k_0\}} \pm \lambda_k \zeta_k \right| = 1.$$

Set $\beta = \sum_{k \in \mathbb{N}_0^n \setminus \{k_0\}} |\lambda_k|$. Since $|\lambda_{k_0}|$ and β are non-negative numbers, the above equalities imply that $|\lambda_{k_0}| + \beta = 1$, and that either $\beta - |\lambda_{k_0}| = 1$ or $|\lambda_{k_0}| - \beta = 1$. If we had $\beta - |\lambda_{k_0}| = 1$, then, subtracting this equality from $|\lambda_{k_0}| + \beta = 1$, we would obtain $2|\lambda_0| = 0$, which contradicts $\lambda_0 \neq 0$. Hence we have $|\lambda_{k_0}| - \beta = 1$. Subtracting this equality from $|\lambda_{k_0}| + \beta = 1$, we obtain $\beta = 0$, which shows that $\lambda_k = 0$ for all $k \in \mathbb{N}_0^n \setminus \{k_0\}$. □

Lemma 4.2. *There exists a constant $c \in \mathbb{T}$ such that $\alpha_1(x) = c$ for all $x \in X$ and that $T_0(\mathbf{1}) = c^{s_0}$.*

Proof. Replacing f to the constant function $\mathbf{1}$ in equality (4.1), we have

$$\sum_{k=0}^{n} \frac{1}{k!} D^k(T_0(\mathbf{1}))(w_k)\zeta_k = [\alpha_1(x)]^{s_0} \tag{4.3}$$

for every $x = (\mathbf{w}, \boldsymbol{\zeta}) \in X$. If we had $T_0(\mathbf{1}) = \mathbf{0}$, then $D^k(T_0(\mathbf{1})) = \mathbf{0}$ for all $k \in \mathbb{N}_0^n$, and hence equality (4.3) would imply that $0 = [\alpha_1(x)]^{s_0}$, which contradicts $|[\alpha_1(x)]^{s_0}| = 1$. Thus there exists $w_{0,0} \in \mathbb{T}$ such that $T_0(\mathbf{1})(w_{0,0}) \neq 0$. By equality (4.3),

$$\left| T_0(\mathbf{1})(w_{0,0}) + \sum_{k=1}^{n} \frac{1}{k!} D^k(T_0(\mathbf{1}))(w_k)\zeta_k \right| = 1$$

for every $w_1, \ldots, w_n \in \mathbb{T}$ and $\zeta_1, \ldots, \zeta_n \in \mathbb{T}$. It follows from Proposition 4.1 that $D^k(T_0(\mathbf{1})) = \mathbf{0}$ for every $k \in \mathbb{N}_1^n$. Hence $T_0(\mathbf{1})$ is constant on $\mathbb{T}$, and equality (4.3) shows that $T_0(\mathbf{1})(w_0) = [\alpha_1(x)]^{s_0}$ for all $x = (w_0, \ldots, w_n, \zeta_1, \ldots, \zeta_n) \in X$. In particular, $\alpha_1 : X \to \mathbb{T}$ is constant. Let $c = \alpha_1(x)$. Then $c \in \mathbb{T}$ and $T_0(\mathbf{1})(w_0) = [c]^{s_0} = c^{s_0}$ for all $w_0 \in \mathbb{T}$. □

By Lemma 4.2, equality (4.1) is reduced to

$$\sum_{k=0}^{n} \frac{1}{k!} D^k(T_0(f))(w_k)\zeta_k = \sum_{k=0}^{n} \frac{1}{k!}[cD^k(f)(\varphi_k(x))\chi_k(x)]^{s_k} \tag{4.4}$$

for every $f \in A^n(\overline{\mathbb{D}})$ and every $x = (w_0, \ldots, w_n, \zeta_1, \ldots, \zeta_n) \in X$.

Lemma 4.3. *Let $k \in \mathbb{N}_0^n$, and let $(\mathbf{w}, \boldsymbol{\zeta}) \in X$, where $\mathbf{w} = (w_0, \ldots, w_n) \in \mathbb{T}^{n+1}$ and $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_n) \in \mathbb{T}^n$. Then the value $\varphi_k(\mathbf{w}, \boldsymbol{\zeta})$ is independent of $\boldsymbol{\zeta}$.*

Proof. Fix $k_0, k \in \mathbb{N}_0^n$. Let us prove that the value $\varphi_{k_0}(\mathbf{w}, \boldsymbol{\zeta})$ is independent of the k-th coordinate ζ_k of $\boldsymbol{\zeta}$. To see this, fix $\mathbf{w} = (w_0, \ldots, w_n) \in \mathbb{T}^{n+1}$ and $\zeta_l \in \mathbb{T}$ for $l \in \mathbb{N}_1^n \setminus \{k\}$.

For each triple $\zeta_{0,k}, \zeta_{1,k}, \zeta_{2,k} \in \mathbb{T}$, let $x_j = (\mathbf{w}, \zeta_1, \ldots, \zeta_{j,k}, \ldots, \zeta_n)$ for $j = 0, 1, 2$, and let $G_{k_0} = \{\varphi_{k_0}(x_0), \varphi_{k_0}(x_1), \varphi_{k_0}(x_2)\}$. First, let us show that G_{k_0} contains at most two points. Suppose, on the contrary, that $\varphi_{k_0}(x_0)$, $\varphi_{k_0}(x_1)$ and $\varphi_{k_0}(x_2)$ are mutually distinct. Then so are $\zeta_{0,k}$, $\zeta_{1,k}$ and $\zeta_{2,k}$. By Proposition 2.2, there exists $f_0 \in A^n(\overline{\mathbb{D}})$ such that $D^{k_0}(f_0)(\varphi_{k_0}(x_0)) \neq 0$ and that $D^l(f_0)(\varphi_l(x_j)) = 0$ for every $(j, l) \in \mathbb{N}_0^2 \times \mathbb{N}_0^n$. Multiplying f_0 by a suitable constant, we may assume that $D^{k_0}(f_0)(\varphi_{k_0}(x_0)) = k_0!$. By equality (4.4), we have

$$\frac{1}{k!} D^k(T_0(f_0))(w_k)\zeta_{j,k} + \sum_{l \in \mathbb{N}_0^n \setminus \{k\}} \frac{1}{l!} D^l(T_0(f_0))(w_l)\zeta_l = \begin{cases} [c\chi_{k_0}(x_0)]^{s_{k_0}} & (j = 0), \\ 0 & (j = 1, 2). \end{cases}$$

Since $\zeta_{1,k} \neq \zeta_{2,k}$, the above equalities imply that $D^k(T_0(f_0))(w_k) = 0$, and then

$$\sum_{l \in \mathbb{N}_0^n \setminus \{k\}} \frac{1}{l!} D^l(T_0(f_0))(w_l)\zeta_l = 0.$$

Hence $0 = [c\chi_{k_0}(x_0)]^{s_{k_0}}$, which is a contradiction. Therefore G_{k_0} contains at most two points.

Since φ_{k_0} is continuous on X, the mapping $\zeta_k \mapsto \varphi_{k_0}(\mathbf{w}, \zeta_1, \ldots, \zeta_k, \ldots, \zeta_n)$ is continuous on $\mathbb{T}$. Thus its image $H_{k_0} = \{\varphi_{k_0}(\mathbf{w}, \zeta_1, \ldots, \zeta_k, \ldots, \zeta_n) : \zeta_k \in \mathbb{T}\}$ is a connected set in $\mathbb{T}$. The previous paragraph implies that the above set contains at most two points, and thus the connectedness of H_{k_0} shows that the set H_{k_0} must be a singleton. This proves that the value $\varphi_{k_0}(\mathbf{w}, \boldsymbol{\zeta})$ is independent of the k-th coordinate ζ_k of $\boldsymbol{\zeta}$. Since this holds for every $k \in \mathbb{N}_1^n$, it follows therefore that the value $\varphi_{k_0}(\mathbf{w}, \boldsymbol{\zeta})$ is independent of $\boldsymbol{\zeta} \in \mathbb{T}^n$. □

By Lemma 4.3, we may write $\varphi_k(\mathbf{w}) = \varphi_k(\mathbf{w}, \boldsymbol{\zeta})$ for every $(\mathbf{w}, \boldsymbol{\zeta}) \in X$ and every $k \in \mathbb{N}_0^n$. Then we can rewrite equality (4.4) as

$$\sum_{k=0}^{n} \frac{1}{k!} D^k(T_0(f))(w_k)\zeta_k = \sum_{k=0}^{n} \frac{1}{k!} [cD^k(f)(\varphi_k(\mathbf{w}))\chi_k(\mathbf{w}, \boldsymbol{\zeta})]^{s_k} \tag{4.5}$$

for every $f \in A^n(\overline{\mathbb{D}})$ and $(\mathbf{w}, \boldsymbol{\zeta}) \in X$, where $\mathbf{w} = (w_0, \ldots, w_n) \in \mathbb{T}^{n+1}$ and $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_n) \in \mathbb{T}^n$.

Lemma 4.4. *Let $k_0, k \in \mathbb{N}_1^n$, and let $\mathbf{w} = (w_0, \ldots, w_n) \in \mathbb{T}^{n+1}$. Assume that $\chi_{k_0}(\mathbf{w}, \boldsymbol{\zeta})$ depends on the k-th coordinate ζ_k of $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_n) \in \mathbb{T}^n$. Then*

$$[\chi_{k_0}(\mathbf{w}, \boldsymbol{\zeta})]^{s_{k_0}} = [\chi_{k_0}(\mathbf{w}, 1, \ldots, 1)]^{s_{k_0}}\zeta_k$$

for every $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_n) \in \mathbb{T}^n$.

Proof. Assume that $\chi_{k_0}(\mathbf{w}, \boldsymbol{\zeta})$ depends on the k-th coordinate ζ_k of $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_n) \in \mathbb{T}^n$. Fix $\mathbf{w} = (w_0, \ldots, w_n) \in \mathbb{T}^{n+1}$. By Proposition 2.2, there exists $f_0 \in A^n(\overline{\mathbb{D}})$ such that $D^{k_0}(f_0)(\varphi_{k_0}(\mathbf{w})) \neq 0$ and $D^l(f_0)(\varphi_l(\mathbf{w})) = 0$ for every $l \in \mathbb{N}_0^n \setminus \{k_0\}$. Multiplying f_0 by a suitable constant, we may assume $D^{k_0}(f_0)(\varphi_{k_0}(\mathbf{w})) = c^{-1}k_0!$. By equality (4.5), we have

$$\sum_{l=0}^{n} \frac{1}{l!} D^l(T_0(f_0))(w_l)\zeta_l = [\chi_{k_0}(\mathbf{w}, \boldsymbol{\zeta})]^{s_{k_0}}$$

for all $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_n) \in \mathbb{T}^n$. By Proposition 4.1, there is a unique number $l_0 \in \mathbb{N}_0^n$ such that $D^{l_0}(T_0(f_0))(w_{l_0}) \neq 0$, and thus $(1/l_0!) \cdot D^{l_0}(T_0(f_0))(w_{l_0})\zeta_{l_0} = [\chi_{k_0}(\mathbf{w}, \boldsymbol{\zeta})]^{s_{k_0}}$ for all $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_n) \in \mathbb{T}^n$. Since $\chi_{k_0}(\mathbf{w}, \boldsymbol{\zeta})$ depends on ζ_k, the number l_0 must be k, that is,

$$\frac{1}{k!} D^k(T_0(f_0))(w_k)\zeta_k = [\chi_{k_0}(\mathbf{w}, \boldsymbol{\zeta})]^{s_{k_0}}$$

for all $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_n) \in \mathbb{T}^n$. This shows that the value $\chi_{k_0}(\mathbf{w}, \boldsymbol{\zeta})$ depends only on ζ_k. Thus we have

$$\frac{1}{k!} D^k(T_0(f_0))(w_k) = [\chi_{k_0}(\mathbf{w}, 1, \ldots, 1)]^{s_{k_0}}$$

Hence we obtain

$$\zeta_k = \frac{D^k(T_0(f_0))(w_k)\zeta_k}{D^k(T_0(f_0))(w_k)} = \frac{[\chi_{k_0}(\mathbf{w}, \boldsymbol{\zeta})]^{s_{k_0}}}{[\chi_{k_0}(\mathbf{w}, 1, \ldots, 1)]^{s_{k_0}}},$$

which implies that $[\chi_{k_0}(\mathbf{w}, \boldsymbol{\zeta})]^{s_{k_0}} = [\chi_{k_0}(\mathbf{w}, 1, \ldots, 1)]^{s_{k_0}}\zeta_k$ for all $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_n) \in \mathbb{T}^n$. □

Lemma 4.5. *Let $\mathbf{w} \in \mathbb{T}^{n+1}$. For each $k \in \mathbb{N}_0$, there exist a number $\sigma(k) \in \mathbb{N}_0^n$ with $\sigma(0) = 0$ and a constant $\gamma_k(\mathbf{w}) \in \mathbb{T}$ such that*

$$\gamma_k(\mathbf{w})\zeta_k = [c\chi_{\sigma(k)}(\mathbf{w}, \boldsymbol{\zeta})]^{s_{\sigma(k)}}$$

for every $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_n) \in \mathbb{T}^n$. Moreover, the mapping $\sigma : \mathbb{N}_0^n \to \mathbb{N}_0^n$ is bijective.

Proof. Fix $\mathbf{w} = (w_0, \ldots, w_n) \in \mathbb{T}^{n+1}$. Recall that we set $\zeta_0 = 1$ and $\chi_0(\mathbf{w}, \boldsymbol{\zeta}) = 1$ for all $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_n) \in \mathbb{T}^n$. Let $\sigma(0) = 0$ and $\gamma_0(\mathbf{w}) = [c]^{s_0}$. Then we have $\gamma_0(\mathbf{w}) \in \mathbb{T}$ and $\gamma_0(\mathbf{w})\zeta_0 = [c\chi_{\sigma(0)}(\mathbf{w}, \boldsymbol{\zeta})]^{s_{\sigma(0)}}$ for all $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_n) \in \mathbb{T}^n$.

Let $l \in \mathbb{N}_0^n$. We now assume that we have already constructed mutually distinct numbers $\sigma(0), \ldots, \sigma(l-1)$ with $\sigma(0) = 0$ and constants $\gamma_0(\mathbf{w}), \ldots, \gamma_{l-1}(\mathbf{w}) \in \mathbb{T}$ such that $\gamma_k(\mathbf{w})\zeta_k = [c\chi_{\sigma(k)}(\mathbf{w}, \boldsymbol{\zeta})]^{s_{\sigma(k)}}$ for all $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_n) \in \mathbb{T}^n$ and every $k \in \mathbb{N}_0^{l-1}$. Let us construct $\sigma(l)$ and $\gamma_l(\mathbf{w})$. By Proposition 2.4, there exists $g_l \in A^n(\overline{\mathbb{D}})$ such that $\tilde{g}_l(\mathbf{w}, \boldsymbol{\zeta}) = \zeta_l$ for all $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_n) \in \mathbb{T}^n$. The surjectiveity of T_0 guarantees the existence of $f_l \in A^n(\overline{\mathbb{D}})$ such that $g_l = T_0(f_l)$, and thus

$$\zeta_l = \widetilde{T_0(f_l)}(\mathbf{w}, \boldsymbol{\zeta}) = \sum_{k=0}^{n} \frac{1}{k!} D^k(T_0(f_l))(w_k)\zeta_k$$

for all $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_n) \in \mathbb{T}^n$. By equality (4.5) and the induction hypothesis, we have

$$\begin{aligned}\zeta_l &= \sum_{k=0}^{n} \frac{1}{k!} D^k(T_0(f_l))(w_k)\zeta_k = \sum_{k=0}^{n} \frac{1}{k!} [cD^k(f_l)(\varphi_k(\mathbf{w}))\chi_k(\mathbf{w}, \boldsymbol{\zeta})]^{s_k} \\ &= \sum_{k=0}^{l-1} \frac{1}{\sigma(k)} [D^{\sigma(k)}(f_l)(\varphi_{\sigma(k)}(\mathbf{w})]^{s_{\sigma(k)}} \gamma_k(\mathbf{w})\zeta_k \\ &\qquad + \sum_{k \neq \sigma(0), \ldots, \sigma(l-1)} \frac{1}{k!} [cD^k(f_l)(\varphi_k(\mathbf{w}))\chi_k(\mathbf{w}, \boldsymbol{\zeta})]^{s_k}\end{aligned}$$

for all $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_n) \in \mathbb{T}^n$. It follows that $\chi_k(\mathbf{w}, \boldsymbol{\zeta})$ depends on ζ_l for some $k \in \mathbb{N}_0^n \setminus \{\sigma(0), \ldots, \sigma(l-1)\}$. Choose $\sigma(l) \in \mathbb{N}_0^n \setminus \{\sigma(0), \ldots, \sigma(l-1)\}$ so that $\chi_{\sigma(l)}(\mathbf{w}, \boldsymbol{\zeta})$ depends on ζ_l. Then Lemma 4.4 implies that

$$[\chi_{\sigma(l)}(\mathbf{w}, 1, \ldots, 1)]^{s_{\sigma(l)}}\zeta_l = [\chi_{\sigma(l)}(\mathbf{w}, \boldsymbol{\zeta})]^{s_{\sigma(l)}}$$

for all $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_n) \in \mathbb{T}^n$. Let $\gamma_l(\mathbf{w}) = [c\chi_{\sigma(l)}(\mathbf{w}, 1, \ldots, 1)]^{s_{\sigma(l)}}$. Then we have $\gamma_l(\mathbf{w}) \in \mathbb{T}$ and $\gamma_l(\mathbf{w})\zeta_l = [c\chi_{\sigma(l)}(\mathbf{w}, \boldsymbol{\zeta})]^{s_{\sigma(l)}}$ for all $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_n) \in \mathbb{T}^n$.

Now, we have proved the first part of the lemma. By construction, the mapping $\sigma : \mathbb{N}_0^n \to \mathbb{N}_0^n$ is injective. Since $\mathbb{N}_0^n$ is a finite set, the mapping σ must be bijective. □

Lemma 4.6. *Let $f \in A^n(\overline{\mathbb{D}})$. Then*

$$T_0(f)(w_0) = [cf(\varphi_0(\mathbf{w}))]^{s_0} \tag{4.6}$$

and

$$\frac{1}{k!}D^k(T_0(f))(w_k) = \frac{1}{\sigma(k)!}[D^{\sigma(k)}(f)(\varphi_{\sigma(k)}(\mathbf{w}))]^{s_{\sigma(k)}}\gamma_k(\mathbf{w}) \tag{4.7}$$

for every $\mathbf{w} = (w_0, \dots, w_n) \in \mathbb{T}^{n+1}$. *In particular, the value* $\varphi_0(w_0, \dots, w_n)$ *is independent of* $w_1, \dots, w_n \in \mathbb{T}$.

Proof. Fix $f \in A^n(\overline{\mathbb{D}})$ and $\mathbf{w} = (w_0, \dots, w_n) \in \mathbb{T}^n$. By Lemma 4.5 and equality (4.5),

$$(T_0(f)(w_0) - [cf(\varphi_0(\mathbf{w}))]^{s_0}) + \sum_{k=1}^{n}\left(\frac{1}{k!}D^k(T_0(f))(w_k) - \frac{1}{\sigma(k)!}[D^{\sigma(k)}(f)(\varphi_{\sigma(k)}(\mathbf{w}))]^{s_{\sigma(k)}}\gamma_k(\mathbf{w})\right)\zeta_k = 0$$

for every $(\zeta_1, \dots, \zeta_n) \in \mathbb{T}^n$. Applying Proposition 4.1 with $M = 0$, we obtain equalities (4.6) and (4.7), as desired. □

By Lemma 4.6, we may write $\varphi(z) = \varphi_0(z, w_1, \dots, w_n)$. Then $\varphi : \mathbb{T} \to \mathbb{T}$ is a surjective continuous mapping. Moreover, equality (4.6) is now reduced to

$$T_0(f)(z) = [cf(\varphi(z))]^{s_0} \qquad (\forall f \in A^n(\overline{\mathbb{D}}), \forall z \in \mathbb{T}). \tag{4.8}$$

Proof of Theorem 1.1. Let $\iota \in A^n(\overline{\mathbb{D}})$ be the function defined by $\iota(z) = z$ for every $z \in \overline{\mathbb{D}}$. Let $\tau = c^{-s_0}T_0(\iota) \in A^n(\overline{\mathbb{D}})$. Then equality (4.8) shows that $c^{s_0}\tau(z) = T_0(\iota)(z) = [c\varphi(z)]^{s_0}$ for every $z \in \mathbb{T}$, and thus $\varphi(z) = [\tau(z)]^{s_0}$ for every $z \in \mathbb{T}$. Substituting this into equality (4.8), we have

$$T_0(f)(z) = [cf([\tau(z)]^{s_0})]^{s_0} \qquad (\forall f \in A^n(\overline{\mathbb{D}}), \forall z \in \mathbb{T}). \tag{4.9}$$

Since $\mathbb{T}$ is the Shilov boundary of the disk algebra $A(\overline{\mathbb{D}})$, equality (4.9) holds for every $z \in \overline{\mathbb{D}}$. Note that $\tau \in A^n(\overline{\mathbb{D}})$ and $|\tau(z)| = |\varphi(z)| = 1$ for every $z \in \mathbb{T}$. It follows from the maximum modulus principle that $\tau(\overline{\mathbb{D}}) \subset \overline{\mathbb{D}}$.

Since T_0^{-1} is also a surjective real-linear isometry on $(A^n(\overline{\mathbb{D}}), \|\cdot\|_\Sigma)$, applying the above argument to T_0^{-1}, there exist $c' \in \mathbb{T}$, $\rho \in A^n(\overline{\mathbb{D}})$ with $\rho(\overline{\mathbb{D}}) \subset \overline{\mathbb{D}}$, and $s_0' \in \{\pm 1\}$ such that

$$T_0^{-1}(g)(z) = [c'g([\rho(z)]^{s_0'})]^{s_0'} \qquad (\forall g \in A^n(\overline{\mathbb{D}}), \forall z \in \mathbb{T}). \tag{4.10}$$

Substituting $g = T_0(\mathbf{1})$ into equality (4.10), we have $1 = [c'T_0(\mathbf{1})([\rho(z)]^{s_0'})]^{s_0'}$. Since $T_0(\mathbf{1}) = c^{s_0}$, we obtain $1 = [c'c^{s_0}]^{s_0'}$. Substituting $g = T_0(\iota)$ into (4.10), we have

$$\begin{aligned} z = T_0^{-1}(T_0(\iota))(z) &= [c'T_0(\iota)([\rho(z)]^{s_0'})]^{s_0'} = [c'c^{s_0}\tau([\rho(z)]^{s_0'})]^{s_0'} \\ &= [c'c^{s_0}]^{s_0'}[\tau([\rho(z)]^{s_0'})]^{s_0'} = [\tau([\rho(z)]^{s_0'})]^{s_0'} \end{aligned}$$

for every $z \in \mathbb{T}$. This proves that $\overline{\mathbb{D}} \subset \tau(\overline{\mathbb{D}})$. Consequently $\tau(\overline{\mathbb{D}}) = \overline{\mathbb{D}}$.

Let us show that $\tau : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ is injective. Choose $z_1, z_2 \in \overline{\mathbb{D}}$, and assume that $\tau(z_1) = \tau(z_2)$. Let $g_0 = T_0^{-1}(\iota)$. Then

$$z_1 = T_0(g_0)(z_1) = [cg_0([\tau(z_1)]^{s_0})]^{s_0} = [cg_0([\tau(z_2)]^{s_0})]^{s_0} = T_0(g_0)(z_2) = z_2.$$

Hence τ is injective.

We have proved that τ is a continuous bijection on the compact Hausdorff space $\overline{\mathbb{D}}$, and thus it is a homeomorphism on $\overline{\mathbb{D}}$. Since φ maps $\mathbb{T}$ onto $\mathbb{T}$, so is τ. Hence $\tau|_{\mathbb{D}} : \mathbb{D} \to \mathbb{D}$ is a homeomorphism. It is well-known that such a function must be of the form

$$\tau(z) = \lambda \frac{z-a}{1-\overline{a}z} \qquad (z \in \mathbb{D})$$

for some $\lambda \in \mathbb{T}$ and $a \in \mathbb{D}$ (see [15, Theorem 12.6]).

Finally, let us show that $a = 0$. Note that τ is analytic in the open set containing $\overline{\mathbb{D}}$. Since $T_0(\iota)(z) = c^{s_0}\tau(z)$ for every $z \in \mathbb{T}$, the chain rule implies that

$$D^1(T_0(\iota))(z) = ic^{s_0}\tau'(z)z = ic^{s_0}\lambda z \frac{1-|a|^2}{(1-\overline{a}z)^2}$$

for every $z \in \mathbb{T}$, where τ' is the derivative as a function of one complex variable. Thus

$$1-|a|^2 = |D^1(T_0(\iota))(z)| \cdot |1-\overline{a}z|^2$$

for every $z \in \mathbb{T}$. On the other hand, by equality (4.7), we see that $|D^1(T_0(\iota))(w)| = \frac{1}{\sigma(1)!}$. Hence we have

$$\sigma(1)! \cdot (1-|a|^2) = |1-\overline{a}z|^2$$

for every $z \in \mathbb{T}$. By Proposition 4.1, we obtain $a = 0$.

Now we have $\tau(z) = \lambda z$ for every $z \in \overline{\mathbb{D}}$. Equality (4.9) is now reduced to

$$T_0(f)(z) = [cf([\lambda z]^{s_0})]^{s_0} \qquad (\forall f \in A^n(\overline{\mathbb{D}}), \forall z \in \overline{\mathbb{D}}).$$

Therefore we obtain

$$T(f)(z) = T(\mathbf{0})(z) + cf(\lambda z) \qquad (\forall f \in A^n(\overline{\mathbb{D}}), \forall z \in \overline{\mathbb{D}}), \quad \text{or}$$
$$T(f)(z) = T(\mathbf{0})(z) + \overline{cf(\overline{\lambda z})} \qquad (\forall f \in A^n(\overline{\mathbb{D}}), \forall z \in \overline{\mathbb{D}}),$$

as desired. □

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