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1. Prologue: Kolmogorov's theory and Onsager conjecture

One of the important assumptions Kolmogorov has made implicitly in his theory of isotropic turbulence [49, 50, 51] was that the energy dissipation rate, say ϵ, converges to a strictly positive finite value as the viscosity of fluids tends to zero, which is now called the dissipative anomaly. Kolmogorov claimed that some statistical quantities of turbulent fluctuations can be derived by a dimensional analysis with using the ensemble average of the dissipation rate $\langle\epsilon\rangle$. Suppose that the ensemble average of the pth moment of the velocity increment in the radial direction $\boldsymbol{r}$ between two points $\boldsymbol{x}$ and $\boldsymbol{x}+\boldsymbol{r}$ at time t, $(\Delta_r u) = (\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r},t) - \boldsymbol{u}(\boldsymbol{x},t))\cdot \boldsymbol{r}/|\boldsymbol{r}|$, is steady, homogeneous and isotropic, i.e., it depends only on the distance $|\boldsymbol{r}|$ between the two points. Since the energy dissipation rate and the pth moment $(\Delta u)^p$ have the dimensions $[\text{Length}]^2[\text{Time}]^{-3}$ and $[\text{Length}]^{2p}[\text{Time}]^{-2p}$ respectively, we have the following *Kolmogorov's scaling law* by equating the dimensions:

$$\langle(\Delta_r u)^p\rangle = C_p\langle\epsilon\rangle^{\frac{p}{3}}|\boldsymbol{r}|^{\frac{p}{3}}, \tag{1.1}$$

in which C_p denotes a non-dimensional constant. Historically, the scaling laws for the special cases $p=2$ and $p=3$ were derived in [49, 50], which have been generalized for general p in [51].

In the meantime, Onsager [65] and Weizsäker [75] have derived the statistical law of the energy spectra for turbulent velocity fields. Let $\widehat{\boldsymbol{u}}(\boldsymbol{k},t)$ denote the Fourier transform of the velocity of the wavenumber $\boldsymbol{k}\in\mathbb{R}^3$ satisfying the periodic boundary condition. The energy spectrum integrated over the spectral radius $k=|\boldsymbol{k}|$ is represented by $E(k)=\langle E(k,t)\rangle$. Then a dimensional analysis shows that there exists a real constant C_K such that the following holds.

$$E(k)\sim C_K\langle\epsilon\rangle^{\frac{2}{3}}k^{-\frac{5}{3}}. \tag{1.2}$$

This is known as the 5/3-*law of the energy spectra* in the turbulence theory [32]. We note that it is equivalent to the Kolmogorov's scaling law (1.1) for $p=2$. The statistical law of the energy spectrum has been verified through many laboratory experiments and numerical simulations of the Navier-Stokes equations, according to which the law (1.2) holds well for the intermediate range of wavenumbers, called the *inertial range* [32]. Regarding the scaling law (1.1) for $p=3$, recent laboratory experiments [1, 74] and numerical simulations [44, 46, 74] indicate the relation is correct, although some subdominant small corrections are required. In addition,

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as discussed later, since the scaling law for $p = 3$ is derived rigorously from the Navier-Stokes equations, it is considered to be a valid estimate. On the contrary, for higher p, data obtained in laboratory and numerical experiments suggest that a deviation from (1.1) exists. The deviation from the exponent $p/3$ in (1.1) is called the anomalous scaling of turbulence or *intermittency*. This phenomenon is a famous open problem in nonlinear physics, but we don't go into details about this topic in this article.

Let us now consider the Kolmogorov's scaling law based on the Navier-Stokes equations describing the motion of incompressible fluids on the flat torus $\mathbb{T}^3$ with the periodic boundary condition, which is given by

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \nu \triangle \boldsymbol{u} - \nabla p + \boldsymbol{f}, \quad \text{div}\ \boldsymbol{u} = 0, \quad \boldsymbol{x} \in \mathbb{T}^3,\ t > 0. \tag{1.3}$$

Here, p is the pressure, ν is the kinetic viscous coefficient and $\boldsymbol{f}$ represents an external forcing. Suppose that solutions of the Navier-Stokes equation without the forcing ($\boldsymbol{f} = \boldsymbol{0}$) are smooth. Then the equation of the energy balance per unit mass is derived by taking the inner product between (1.3) and $\boldsymbol{u}$ and the integration by parts under the incompressibility condition and the periodic boundary condition as follows.

$$\epsilon \equiv \partial_t \int \frac{|\boldsymbol{u}|^2}{2} d\boldsymbol{x} = -\nu \int |\nabla \boldsymbol{u}|^2 \, d\boldsymbol{x}, \tag{1.4}$$

Suppose further that solutions of the Navier-Stokes equations are samples of turbulent flows and their ensemble average of the energy dissipation rate is equivalent to $\langle \epsilon \rangle$. Then it follows from (1.4) that $\langle \epsilon \rangle \to 0$ as $\nu \to 0$ as long as the solutions are sufficiently smooth, which contradicts the Kolmogorov's assumption that $\langle \epsilon \rangle$ tends to a strictly positive constant in the inviscid limit. To resolve this contradiction, the following statement on the regularity of solutions of the Navier-Stokes equations has been pointed out first by Onsager [66] without any rigorous proof, and later re-discovered by Eyink [27], which is now known as *Onsager's conjecture.*

> Non differentiable velocity fields may violate the classical energy balance equation (1.4), yielding an anomalous energy dissipation. If the velocity fields satisfy $|\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{u}(\boldsymbol{x})| < C|\boldsymbol{r}|^\alpha$ with $\alpha > 1/3$, then the energy dissipation never occurs.

The constant C is independent of $\boldsymbol{x}$. See the review by Eyink and Sreenivasan [30] for a history about this statement . It claims that the critical exponent of Hölder regularity of velocity fields is $\alpha = 1/3$. In other words, turbulent vector fields having the dissipative anomaly should belong to the Hölder space with the exponent at least less than $1/3$. Let us remark that it has been positively verified by some laboratory experiments and numerical computations that turbulent flows for sufficiently small ν satisfy the dissipative anomaly [32, 45].

Onsager's conjecture is mathematically formulated in terms of the regularity of solutions of the Navier-Stokes equations:

> Suppose that weak solutions of the Navier-Stokes equations are Hölder continuous with the exponent greater than $1/3$. Then the energy dissipation rate for the weak solutions vanishes in the inviscid limit. Namely, $\langle \epsilon \rangle \to 0$ as $\nu \to 0$.

Similarly we can consider the Euler equations, which is equivalent to the Navier-Stokes equations without viscous and forcing terms,

$$\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla p, \quad \mathrm{div}\ \boldsymbol{u} = 0, \quad \boldsymbol{x}\in\mathbb{T}^3, \quad t>0. \tag{1.5}$$

Onsager's conjecture for weak solutions of the Euler equations is stated as follows.

> Suppose that weak solutions of the Euler equations are Hölder continuous with the exponent greater than 1/3. Then the energy is conserved.

It is well-known that the Euler equations (1.5) admits various unphysical weak solutions such as L^2 integrable weak solutions with compact support in space-time [71, 72]. We need to find physically admissible inviscid flows satisfying Onsager's conjecture. Eyink [27] has constructed a non-classical solution of the Euler equations that conserves the energy. The solution has the Hölder continuity of exponent 1/2, whose regularity is stronger than what Onsager has suggested. Later, Constantin, E and Titi [15] have proven the following sharp result on the regularity of weak solutions of the Euler equations with the energy conservation.

Theorem 1.1. *Let $\boldsymbol{u} = (u_1, u_2, u_3) \in L^3([0,T]; B^{\alpha}_{3\infty}(\mathbb{T}^3)) \cap C([0,T]; L^2(\mathbb{T}^3))$ be a weak solution of the 3D incompressible Euler's equations. If $\alpha > 1/3$, then we have $\|\boldsymbol{u}(\cdot,t)\|_{L^2} = \|\boldsymbol{u}(\cdot,0)\|_{L^2}$ for $t\in[0,T)$.*

See [15] for the definition of the weak solution in detail. Since any function $\boldsymbol{u}$ in the Besov space $B^{\alpha}_{3\infty}(\mathbb{T}^3)$ satisfies

$$\|\boldsymbol{u}(\cdot+\boldsymbol{r},t) - \boldsymbol{u}(\cdot,t)\|_{L^3} \le C(t)|\boldsymbol{r}|^{\alpha} \tag{1.6}$$

for a certain function $C(t)$, this theorem is regarded as a mathematically rigorous statement of Onsager's conjecture in terms of Besov space. We remark that there is a critical result on Onsager's conjecture for a Besov space with the exponent $\alpha = 1/3$, provided by Cheskidov et. al [13]. Onsager's conjecture is also proven on weak solutions of the Euler equations on bounded domains [2].

Onsager's conjecture is the statement on the energy conservation in terms of the Hölder exponent of weak solutions of the Euler equations. On the other hand, in order to characterize function spaces of turbulent vector fields realizing Kolmogorov's statistical laws, it is required to confirm whether or not the Hölder exponent $\alpha = 1/3$ is the critical value for the existence of weak solutions of the Euler equations yielding the anomalous energy dissipation. The problem is called the *backward Onsager's conjecture*, which has been resolved positively by Buckmaster et al. [9].

Theorem 1.2. *Let $e : [0,T] \to \mathbb{R}$ be a strictly positive smooth function. Then for any $0 < \alpha < 1/3$, there is a weak solution $\boldsymbol{u} \in C^{\alpha}(\mathbb{T}^3 \times [0,T])$ of the Euler equations, which satisfies*

$$\int_{\mathbb{T}^3} |\boldsymbol{u}(\boldsymbol{x},t)|^2 d\boldsymbol{x} = e(t).$$

This theorem indicates that for *any* given smooth positive function $e(t)$ on $[0,T]$, there exists a weak solution of the Euler equations in the space of Hölder continuous functions with an exponent less than 1/3, whose corresponding energy variation is equivalent to $e(t)$. Hence, we can obtain any kinds of weak solutions with dissipating, growing or oscillating energy profiles. To obtain physically admissible weak solutions, they introduce the following "strict subsolution" of the Euler equations.

Definition 1.3. *A smooth strict subsolution of (1.5) on* $\mathbb{T}^3 \times [0,T]$ *is a smooth triplet* $(\bar{\boldsymbol{u}}, \bar{p}, \bar{R})$ *with* $\bar{R}$ *a symmetric 2-tensor, such that*

$$\partial_t \bar{\boldsymbol{u}} + div(\bar{\boldsymbol{u}} \otimes \bar{\boldsymbol{u}}) = -\nabla \bar{p} + div\, \bar{R}, \quad div\, \bar{\boldsymbol{u}} = 0, \tag{1.7}$$

and $\bar{R}(x,t)$ *is positive definite for all* (x,t).

They then prove the existence of a sequence of C^α solutions for any $\alpha < 1/3$ approximating any smooth strict subsoltuion.

Theorem 1.4. *Let* $(\bar{\boldsymbol{u}}, \bar{p}, \bar{R})$ *be a smooth strict subsolution of the Euler equations on* $\mathbb{T}^3 \times [0,T]$ *and let* $\alpha < 1/3$*. Then there exists a sequence* $(\boldsymbol{u}_k, p_k)$ *of weak solutions of (1.5) such that* $\bar{\boldsymbol{u}} \in C^\alpha(\mathbb{T}^3 \times [0,T])$ *with* $\boldsymbol{u}_k \rightharpoonup \bar{\boldsymbol{u}}$ *and* $\boldsymbol{u}_k \otimes \boldsymbol{u}_k \rightharpoonup \bar{\boldsymbol{u}} \otimes \bar{\boldsymbol{u}} + \bar{R}$ *in* L^∞ *weak-*$*$ *topology uniformly in time. Furthermore, for all* $t \in [0,T]$,

$$\int_{\mathbb{T}^3} |\boldsymbol{u}_k(\boldsymbol{x},t)|^2 d\boldsymbol{x} = \int_{\mathbb{T}^3} (|\bar{\boldsymbol{u}}(\boldsymbol{x},t)|^2 + tr\bar{R}(\boldsymbol{x},t)) d\boldsymbol{x}.$$

Chronologically, DeLellis and Székelyhidi Jr.[19, 20, 21] first constructed weak solutions of the Euler equations with Hölder continuity $0 < \alpha < 1/10$, which is a substantial contribution to mathematical investigations of Onsager's conjecture. The Hölder exponent was improved to $\alpha = 1/5$ in various manner [7, 42]. Isett [43] has then successfully obtained weak solutions belonging to $C_t C_x^\alpha$ with $0 < \alpha < 1/3$, but they are defined on a compact support in $\mathbb{R} \times \mathbb{T}^3$, which are somewhat unphysical. Theorem 1.2 provides weak continuous solutions with smooth energy variations, which are physically admissible.

We call weak solutions of the fluid equations with dissipating the inviscid conserved quantity "*dissipative weak solutions*". A strict subsolution given by Theorem 1.4 can be regarded as one of dissipative weak solutions of the Euler equations. The next step is clarifying how those dissipative weak solutions of fluid equations satisfy the statistical laws of turbulence. Kolmogorov [49] derived the statistical law of the third moment of the velocity increment from the Navier-Stokes equations under the assumption of the isotropy and the homogeneity as

$$\left\langle (\Delta_r u)^3 \right\rangle = -\frac{4}{5} \langle \epsilon \rangle |\boldsymbol{r}|. \tag{1.8}$$

This is called *Kolmogorov's four-fifth law*, which is perhaps the only exact relation that can be derived from the Navier-Stokes equations directly. It also insists that the constant in the Kolmogorov's scaling law (1.1) for $p = 3$ is exactly $C_3 = -4/5$.

Duchon and Robert [24] and Eyink [29] have made an important progress in the understanding of the relation between weak solutions of the Euler equations with the anomalous energy dissipation and Kolmogorov's scaling law. Let φ be any infinitely differentiable function with compact support on $\mathbb{R}^3$, which is spherically symmetric and non-negative and $\int_{\mathbb{R}^3} \varphi(\boldsymbol{x}) d\boldsymbol{x} = 1$. With $\varphi^\varepsilon(\boldsymbol{\xi}) = (1/\varepsilon^3)\varphi(\boldsymbol{\xi}/\varepsilon)$ for $\varepsilon > 0$, they introduce the following quantity $D_\varepsilon[\boldsymbol{u}](\boldsymbol{x},t)$ for a given velocity field $\boldsymbol{u}(\boldsymbol{x},t)$,

$$D_\varepsilon[\boldsymbol{u}](\boldsymbol{x},t) = \frac{1}{4} \int \nabla \varphi^\varepsilon(\boldsymbol{\xi}) \cdot \delta \boldsymbol{u}(\boldsymbol{x},\boldsymbol{\xi},t) |\delta \boldsymbol{u}(\boldsymbol{x},\boldsymbol{\xi},t)|^2 d\boldsymbol{\xi},$$

where $\delta \boldsymbol{u}(\boldsymbol{x},\boldsymbol{\xi},t) = \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{\xi},t) - \boldsymbol{u}(\boldsymbol{x},t)$ denotes the velocity increment in the $\boldsymbol{\xi}$-direction. Then the following proposition claims that $D_\varepsilon[\boldsymbol{u}]$ works as a dissipative term of the energy for weak solutions of the unforced Navier-Stokes equations.

Proposition 1.5. *Let $\boldsymbol{u}^\nu \in L^2([0,T);H^1(\mathbb{T}^3)) \cap L^\infty([0,T);L^2(\mathbb{T}^3))$ be weak solutions of the Navier-Stokes equations. Then, as $\varepsilon \to 0$, the function $D_\varepsilon[\boldsymbol{u}]$ converges to $D_{NS}[\boldsymbol{u}^\nu]$, which is independent of φ, in the sense of distributions on $(0,T)\times\mathbb{T}^3$. Furthermore, it satisfies the following the energy budget equality.*

$$\partial_t\left(\frac{|\boldsymbol{u}^\nu|^2}{2}\right)+\partial_i\left(u_i^\nu\left(\frac{|\boldsymbol{u}^\nu|^2}{2}+p\right)\right)-\nu\partial_i^2\left(\frac{|\boldsymbol{u}^\nu|^2}{2}\right)+\nu|\nabla\boldsymbol{u}^\nu|^2+D_{NS}[\boldsymbol{u}^\nu]=0. \tag{1.9}$$

The limit $D_{NS}[\boldsymbol{u}^\nu]$ is called the defect distribution and the weak solution is called *dissipative*, if the defect term satisfies $D[\boldsymbol{u}^\nu]\geq 0$ in the sense of distributions. We also remark that Leray-Hopf weak solutions [55, 56] are dissipative weak solutions in this sense. They showed the similar result on dissipative weak solutions of the Euler solutions.

Proposition 1.6. *Let $\boldsymbol{u}\in L^3([0,T);L^3(\mathbb{T}^3))$ be a weak solution of the Euler equation. Then the term $D_\varepsilon[\boldsymbol{u}]$ converges, in the sense of distributions, to $D[\boldsymbol{u}]$, not depending on φ, and the following local energy equation holds:*

$$\partial_t\left(\frac{|\boldsymbol{u}|^2}{2}\right)+div\left(\boldsymbol{u}\left(\frac{|\boldsymbol{u}|^2}{2}+p\right)\right)+D[\boldsymbol{u}]=0.$$

In relation to Onsager's conjecture, we obtain the following sufficient condition that the inviscid velocity field $\boldsymbol{u}$ satisfying $D[\boldsymbol{u}]=0$.

Proposition 1.7. *Let a weak solution of the Euler equations $\boldsymbol{u}\in L^3([0,T);L^3(\mathbb{T}^3))$ satisfy*

$$\int_{\mathbb{T}^3}|\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r},t)-\boldsymbol{u}(\boldsymbol{x},t)|^3d\boldsymbol{x}\leq C(t)|\boldsymbol{r}|\sigma(|\boldsymbol{r}|), \tag{1.10}$$

where $\sigma(a)$ tends to 0 with $a\to 0$, and $C(t)$ satisfies $\int_0^T C(t)dt<+\infty$. Then $D[\boldsymbol{u}]=0$.

The weak solution $\boldsymbol{u}$ is smoother than the Hölder continuity with exponent $1/3$ owing to the existence of the function $\sigma(a)$ in the sufficient condition (1.10).

Dissipative weak solutions with $D[\boldsymbol{u}]>0$ are of a great significance in relation with Kolmogorov's theory of turbulence, since it recovers some statistical laws of turbulence [24]. Let us now introduce the function $S[\boldsymbol{u},\boldsymbol{l}](\boldsymbol{x},t)\in L^1(\mathbb{T}^3\times[0,T])$ as

$$\begin{aligned} S[\boldsymbol{u},\boldsymbol{l}](\boldsymbol{x},t) &:= \frac{3}{4\pi|\boldsymbol{l}|^3}\int_{|\boldsymbol{\xi}|=|\boldsymbol{l}|}\delta u_L(\boldsymbol{x},t\,;\xi)\,|\delta_{\boldsymbol{\xi}}\boldsymbol{u}(\boldsymbol{x},t)|^2\,\mathcal{H}^2(d\,\widehat{\boldsymbol{\xi}}) \\ &= \frac{3}{4\pi|\boldsymbol{l}|}\int_{\mathbb{S}^2}\delta u_L(\boldsymbol{x},t\,;|\boldsymbol{l}|\widehat{\boldsymbol{\omega}})\,|\delta_{\boldsymbol{\xi}}\boldsymbol{u}(\boldsymbol{x},t)|^2\,\mathcal{H}^2(d\,\widehat{\boldsymbol{\omega}}\,), \end{aligned}$$

where

$$\delta u_L(\boldsymbol{x},t\,;\boldsymbol{l}):=(\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{l},t)-\boldsymbol{u}(\boldsymbol{x},t))\cdot\boldsymbol{l},$$

and $\mathcal{H}^2$ denotes the 2-dimensional Haar measure on the surface of sphere $\mathbb{S}^2$. Duchon and Robert [24] have shown that $S[\boldsymbol{u},\boldsymbol{l}]$ converges as $l=|\boldsymbol{l}|\to 0$ and the limit satisfies the following equality in the sense of distributions:

$$S[\boldsymbol{u}]:=\lim_{l\to 0}S[\boldsymbol{u},\boldsymbol{l}]=-\frac{4}{3}D[\boldsymbol{u}]. \tag{1.11}$$

On the other hand, applying the Kolmogorov's theory, we obtain the following statistical laws, known as *Kármán-Howarth-Monin* 4/3*-law* [32, 47, 63].

$$\left\langle\delta u_L|\delta\boldsymbol{u}|^2\right\rangle\sim-\frac{4}{3}\langle\epsilon\rangle|\boldsymbol{l}|. \tag{1.12}$$

The equality (1.11) for the dissipative weak solutions with $D[u] > 0$ corresponds to the statistical law (1.12), when we regard the defect term as the energy dissipation rate, i.e., $D[\boldsymbol{u}] \sim \epsilon > 0$. Additionally, Eyink [29] has shown that the quantity,

$$S_L[\boldsymbol{u}, \boldsymbol{l}](\boldsymbol{x}, t) = \frac{1}{4\pi|\boldsymbol{l}|} \int_{\mathbb{S}^2} \left(\delta u_L(\boldsymbol{x}, t; |\boldsymbol{l}|\hat{\boldsymbol{\omega}})\right)^3 \mathcal{H}^2(d\hat{\boldsymbol{\omega}})$$

also converges as $l \to 0$ in the sense of distributions and it satisfies

$$S_L[\boldsymbol{u}] = -\frac{4}{5} D[\boldsymbol{u}],$$

when $S_L[\boldsymbol{u}] = \lim_{|\boldsymbol{l}|\to 0} S_L(\boldsymbol{u}, \boldsymbol{l})$ exists. This is equivalent to the Kolmogorov's four-fifth law (1.8) in terms of dissipative weak solutions.

As mentioned in this section, a great progress in mathematical investigations of Onsager's conjecture has been made in relation to the isotropic turbulence in the past decades. We also see dissipative weak solutions to the Euler equations would play an important role in the understanding of turbulent phenomena. In the meantime, those studies mainly focus on the regularity of weak solutions of the Navier-Stokes equations and the Euler equations satisfying the dissipation anomaly, which is the assumption of the Kolmogorov's theory of turbulence. Regarding the statistical laws, the Kolmogorov's scaling law for $p = 3$ is just described in terms of dissipative weak solutions, but it is still uncertain whether or not the dissipative weak solutions satisfy the 5/3-law of the energy spectra (1.2). Moreover, more importantly, it is uncertain how dissipative weak solutions look like as fluid flows and how they generate the statistical laws as dynamical systems. In this sense, theoretical understanding on turbulence has just started, and more mathematical investigations are required from various points of view. In this regard, we here introduce our recent works of hydrodynamic model equations, characterizing turbulent flow phenomena mathematically in terms of the theory of dynamical systems.

2. Finite-time collapse of point vortices with anomalous enstrophy dissipation

It is not an easy task to describe dissipative weak solutions of the Navier-Stokes equations and the Euler equations in three-dimensional spaces as dynamical systems, since the existence and the uniqueness of global weak solutions have not yet been established. To make a mathematical argument rigorous, we consider two-dimensional incompressible flows in the unbounded plane $\mathbb{R}^2$ for simplicity. In 2D turbulence, it is also pointed out that there appears an inertial range in the energy spectra in the inviscid limit corresponding to the anomalous dissipation of the enstrophy, which is L^2-norm of the vorticity and an inviscid conserved quantity [3, 52, 57]. Suppose that 2D turbulent flows in the inviscid limit are subject to the Euler equations whose smooth solutions conserve the enstrophy. Then, it is expected that weak solutions of the Euler equations with the anomalous enstrophy dissipation would play a certain role in the understanding of the statistical laws of 2D turbulence as analogous to that of 3D turbulence mentioned in Section 1. Moreover, since there are many mathematical results on 2D Euler equations, the existence of weak solutions can be established globally in time, and we thereby characterize the dynamic properties of weak solutions with the anomalous enstrophy dissipation rigorously. In [70], we consider the Euler-α equations, which is a dispersive regularization of the Euler equations with a scale parameter α, for an initial data consisting of point wise δ-distributions, called "point vortices". It has been shown that a variational part of the enstrophy dissipates anomalously at the singular time when the three point vortices collapse as $\alpha \to 0$. Later, the result still holds true for more general dispersively regularized Euler equations [36, 37], which will be explained in detail in what follows.

The 2D Euler equations have a unique weak solution if the initial data belongs to $L^1(\mathbb{R}^2)\cap L^p(\mathbb{R}^2)$ with $1 < p \leq \infty$. See, e.g., [23, 33, 60, 76]. Eyink [28] has shown that the enstrophy never dissipates for weak solutions in L^p with $p > 2$. Hence, it is required to deal with the initial vorticity distributions having a weaker regularity such as the space of Radon measures $\mathcal{M}(\mathbb{R}^2)$ on $\mathbb{R}^2$. The existence of a global-in-time weak solution has been established when the initial data $\omega_0 \in \mathcal{M}(\mathbb{R}^2)$ with a distinguished sign induces the velocity field belonging to $L^2_{loc}(\mathbb{R}^2)$ [22, 59]. Inspired by the numerical computation of three point vortices [70], we want to consider point-vortex distributions as the initial data. However, unfortunately, since the velocity field induced by the point vortices is no longer the element of $L^2_{loc}(\mathbb{R}^2)$, it is difficult to construct weak solutions of the Euler equations. To overcome this contradictory situation, let us first construct a unique global weak solution of a regularized Euler equation with a scaling parameter ε, and we then take the limit $\varepsilon \to 0$ to obtain singular inviscid and incompressible flows.

The dispersively regularized Euler equations [31, 39, 40, 41] considered here are derived as follows. For a given incompressible vector field $\boldsymbol{v}$, we introduce a new velocity field $\boldsymbol{u}^\varepsilon$ as

$$\boldsymbol{u}^\varepsilon(\boldsymbol{x}) = (h^\varepsilon * \boldsymbol{v})(\boldsymbol{x}) = \int_{\mathbb{R}^2} h^\varepsilon(\boldsymbol{x}-\boldsymbol{y})\,\boldsymbol{v}(\boldsymbol{y})d\boldsymbol{y}, \tag{2.1}$$

where h^ε is defined from a smooth scalar function $h(\boldsymbol{x})$ on $\mathbb{R}^2$ as follows.

$$h^\varepsilon(\boldsymbol{x}) = \frac{1}{\varepsilon^2} h\left(\frac{\boldsymbol{x}}{\varepsilon}\right). \tag{2.2}$$

Since $\boldsymbol{u}^\varepsilon$ is smoother than $\boldsymbol{v}$, it is a regularized velocity. From the velocity fields $\boldsymbol{v}$ and $\boldsymbol{u}^\varepsilon$, we introduce the vorticity and its regularization as $q = \operatorname{curl}\boldsymbol{v}$ and $\omega^\varepsilon = \operatorname{curl}\boldsymbol{u}^\varepsilon$, respectively. Note that $\operatorname{div}\boldsymbol{u}^\varepsilon = 0$ and $\omega^\varepsilon = h^\varepsilon * q$ are satisfied. Then we consider the following equations for $(\boldsymbol{u}^\varepsilon, \boldsymbol{v})$ on $\mathbb{R}^2$:

$$\partial_t \boldsymbol{v} + (\boldsymbol{u}^\varepsilon\cdot\nabla)\boldsymbol{v} - (\nabla\boldsymbol{v})^T\cdot\boldsymbol{u}^\varepsilon - \nabla\Pi = 0, \qquad \operatorname{div}\boldsymbol{u}^\varepsilon = \operatorname{div}\boldsymbol{v} = 0, \tag{2.3}$$

where Π denotes a (generalized) pressure term. The equations are equivalent to the Euler equations when ε is exactly 0. We remark that they are derived from an application of Hamilton's principle to Hamiltonian

$$\mathcal{H} = \frac{1}{2}\int_{\mathbb{R}^2} \boldsymbol{v}(\boldsymbol{x})\cdot\boldsymbol{u}^\varepsilon(\boldsymbol{x})d\boldsymbol{x}$$

subject to the divergence-free condition. Applying the operator $\nabla\times$ to the equations (2.3), we obtain the equations for the regularized vorticity.

$$\partial_t q + (\boldsymbol{u}^\varepsilon\cdot\nabla)q = 0, \qquad \boldsymbol{u}^\varepsilon = \boldsymbol{K}^\varepsilon * q, \qquad \boldsymbol{K}^\varepsilon = \boldsymbol{K} * h^\varepsilon. \tag{2.4}$$

The function $\boldsymbol{K}^\varepsilon$ is defined by $\boldsymbol{K}^\varepsilon = \nabla^\perp G^\varepsilon$, where G^ε is the solution of Poisson's equation $-\triangle G^\varepsilon = h^\varepsilon$. We further assume that the regularization function h is radially symmetric, i.e., $h_r(|\boldsymbol{x}|) = h(\boldsymbol{x})$, and

$$\int_{\mathbb{R}^2} h(\boldsymbol{x})d\boldsymbol{x} = 2\pi\int_0^\infty r h_r(r)dr = 1. \tag{2.5}$$

Then, G^ε also becomes radially symmetric, say $G^\varepsilon(\boldsymbol{x}) = G_r^\varepsilon(|\boldsymbol{x}|)$, which satisfies

$$G^\varepsilon(\boldsymbol{x}) = G^1\left(\frac{\boldsymbol{x}}{\varepsilon}\right) - \frac{1}{2\pi}\log\varepsilon, \qquad \boldsymbol{K}^\varepsilon(\boldsymbol{x}) = \frac{\boldsymbol{x}^\perp}{\varepsilon|\boldsymbol{x}|}\frac{\mathrm{d}G_r^1}{\mathrm{d}r}\left(\frac{|\boldsymbol{x}|}{\varepsilon}\right). \tag{2.6}$$

Gotoda [38] has shown the existence of a unique global weak solution of (2.4) for initial vorticity distributions belonging to the space of Radon measures $\mathcal{M}(\mathbb{R}^2)$. Suppose that the regularization function $h \in C^1(\mathbb{R}^2 \setminus \{0\})$ vanishes as $|\boldsymbol{x}| \to \infty$, and it satisfies the following conditions.

$$\chi_1 h \in L^1(\mathbb{R}^2), \quad \nabla h \in L^1(\mathbb{R}^2), \quad \chi_{\log}^- h \in L^\infty(\mathbb{R}^2), \quad \chi_1^- \nabla h \in L^\infty(\mathbb{R}^2), \tag{2.7}$$

in which $\chi_\alpha(\boldsymbol{x}) = |\boldsymbol{x}|^\alpha$ on $\boldsymbol{x} \in \mathbb{R}^2$ and

$$\chi_{\log}^-(\boldsymbol{x}) = \begin{cases} (1-\log|\boldsymbol{x}|)^{-1} & , |\boldsymbol{x}| \le 1, \\ 0 & , |\boldsymbol{x}| > 1, \end{cases} \qquad \chi_\alpha^-(\boldsymbol{x}) = \begin{cases} |\boldsymbol{x}|^\alpha & , |\boldsymbol{x}| \le 1, \\ 0 & , |\boldsymbol{x}| > 1. \end{cases}$$

Moreover, let $\boldsymbol{\eta}^\varepsilon$ denote the orbit of a particle advected by the regularized velocity fields $\boldsymbol{u}^\varepsilon$, called Lagrangian flow map, which is the solution of the following initial value problem.

$$\partial_t \boldsymbol{\eta}^\varepsilon(\boldsymbol{x}, t) = \boldsymbol{u}^\varepsilon\left(\boldsymbol{\eta}^\varepsilon(\boldsymbol{x}, t), t\right), \qquad \boldsymbol{\eta}^\varepsilon(\boldsymbol{x}, 0) = \boldsymbol{x}. \tag{2.8}$$

For any initial vorticity distribution $q_0 \in \mathcal{M}(\mathbb{R}^2)$, there exists a unique global solution of (2.4) such that $\boldsymbol{\eta}^\varepsilon \in C^1(\mathbb{R}; \mathscr{G})$, $\boldsymbol{u}^\varepsilon \in C(\mathbb{R}; C(\mathbb{R}^2; \mathbb{R}^2))$ and $q \in C(\mathbb{R}; \mathcal{M}(\mathbb{R}^2))$, where $\mathscr{G}$ denotes the group of homeomorphisms on $\mathbb{R}^2$ that preserve the Lebesgue measure.

The dispersive Euler equations has a unique global weak solution for any initial vortex distribution in the space of Randon measures $\mathcal{M}(\mathbb{R}^2)$, which gives rise to a different situation from the Euler equations. Let us suppose, in particular, that the initial vortex distribution is represented by a linear combination of δ-measures, called ε-point vortices, i.e.,

$$q_0(\boldsymbol{x}) = \sum_{n=1}^{N} \Gamma_n \delta(\boldsymbol{x} - \boldsymbol{x}_n^0), \tag{2.9}$$

where $\boldsymbol{x}_n^0 = (x_n^0, y_n^0) \in \mathbb{R}^2$ for $n = 1, \ldots N$ denotes the supports of the δ-measures. The coefficient $\Gamma_n \in \mathbb{R}$ is referred to as the *strength* of the nth ε-point vortex at $\boldsymbol{x}_n^0$. It corresponds to the circulation around the points that is conserved along the orbit $\boldsymbol{\eta}^\varepsilon$. For the initial data, the unique global solution of (2.4) is then expressed by

$$q(\boldsymbol{x}, t) = \sum_{n=1}^{N} \Gamma_n \delta(\boldsymbol{x} - \boldsymbol{\eta}^\varepsilon(\boldsymbol{x}_n^0, t)). \tag{2.10}$$

Since the Lagrangian flow map $\boldsymbol{\eta}^\varepsilon$ is defined globally in time, the orbits of the ε-point vortices never collide with each other in finite time. Furthermore, the evolution of the ε-point vortices at $\boldsymbol{x}_n^\varepsilon(t) = \boldsymbol{\eta}^\varepsilon(\boldsymbol{x}_n^0, t) = (x_n^\varepsilon(t), y_n^\varepsilon(t))$ is governed by

(2.11)

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}_n^\varepsilon(t) = \boldsymbol{u}^\varepsilon\left(\boldsymbol{x}_n^\varepsilon(t), t\right) = -\frac{1}{2\pi}\sum_{m \ne n}^{N} \Gamma_m \frac{(\boldsymbol{x}_n^\varepsilon - \boldsymbol{x}_m^\varepsilon)^\perp}{(l_{mn}^\varepsilon)^2} P_K\left(\frac{l_{mn}^\varepsilon}{\varepsilon}\right), \quad n = 1, \ldots, N,$$

in which $l^\varepsilon_{mn}(t) = |\boldsymbol{x}^\varepsilon_n(t) - \boldsymbol{x}^\varepsilon_m(t)|$ is the distance between the two ε-point vortices at $\boldsymbol{x}^\varepsilon_n$ and $\boldsymbol{x}^\varepsilon_m$. The function $P_K(r)$ in the equation is defined by

$$P_K(r) = -2\pi r \frac{\mathrm{d}G^1_r}{\mathrm{d}r},$$

where $G^1_r(|\boldsymbol{x}|) = G^1(\boldsymbol{x})$ is the radially symmetric solution of Poisson's equation $-\triangle G^\varepsilon = h^\varepsilon$ for the regularization function $h^\varepsilon_r(|\boldsymbol{x}|) = h^\varepsilon(\boldsymbol{x})$ with $\varepsilon = 1$. The evolution equation (2.11) is called the ε-point vortex (ε-PV) system. The equation (2.11) is formulated as a Hamiltonian dynamical system with the Hamiltonian

$$\mathscr{H}^\varepsilon = -\frac{1}{2\pi}\sum_{n=1}^{N}\sum_{m=n+1}^{N}\Gamma_n\Gamma_m\left[\log l^\varepsilon_{mn} + H_G\left(\frac{l^\varepsilon_{mn}}{\varepsilon}\right)\right], \tag{2.12}$$

in which $H_G(r) = -\log r - 2\pi G^1_r(r)$. The ε-PV system admits the first integrals $(\mathscr{H}^\varepsilon, (Q^\varepsilon)^2 + (P^\varepsilon)^2, I^\varepsilon)$ that are in involution with each other, which are given by

$$Q^\varepsilon + iP^\varepsilon = \sum_{n=1}^{N} x^\varepsilon_n + iy^\varepsilon_n, \qquad I^\varepsilon = \sum_{n=1}^{N}\Gamma_n\left[(x^\varepsilon_n)^2 + (y^\varepsilon_n)^2\right].$$

Hence, the ε-PV system for $N \le 3$ is integrable for any vortex strengths. The following conserved quantity, defined by these first integrals, plays an important role in the analysis stated in this section.

$$M^\varepsilon = \sum_{n\neq m}^{N}\Gamma_n\Gamma_m(l^\varepsilon_{mn})^2 = 2(\Gamma I^\varepsilon - (Q^\varepsilon)^2 - (P^\varepsilon)^2). \tag{2.13}$$

Here, $\Gamma = \sum_{m=1}^{N}\Gamma_m$.

Since the unique global weak solution of the dispersive Euler equations (2.4) is given by (2.10), the variations of the energy and the enstrophy, denoted by $\mathscr{Z}^\varepsilon(t)$ and $E^\varepsilon(t)$ respectively, are expressed in terms of the solution of the ε-PV system based on Novikov's argument [69]. See [34] for the detailed derivations.

(2.14)

$$\mathscr{Z}^\varepsilon(t) := \frac{1}{2\pi\varepsilon^2}\sum_{n=1}^{N}\sum_{m=n+1}^{N}\Gamma_n\Gamma_m\int_0^\infty s\left|2\pi\widehat{h}(s)\right|^2 J_0\left(s\frac{l^\varepsilon_{mn}(t)}{\varepsilon}\right)ds,$$

$$E^\varepsilon(t) := -\frac{1}{2\pi}\sum_{n=1}^{N}\sum_{m=n+1}^{N}\Gamma_n\Gamma_m\left[\log l^\varepsilon_{mn}(t) + \int_0^\infty \frac{1}{s}\left(1 - \left|2\pi\widehat{h}(s)\right|^2\right) J_0\left(s\frac{l^\varepsilon_{mn}(t)}{\varepsilon}\right)ds\right],$$

in which J_0 is the Bessel function of the first kind, $\widehat{h}$ denotes the Fourier transform of h. The second term in the energy variation is non-singular since $2\pi\widehat{h}(0) = 1$ owing to (2.5). Let us remark that the energy and the enstrophy themselves are not well-defined from the solution of the ε-PV system as $\varepsilon \to 0$, since each of them contains a divergent term in the limit.

In what follows, we pay attention to the evolution of the triple ε-point vortices of $N = 3$. We first introduce the following new scaled variables with the parameter ε.

$$\boldsymbol{X}_n(t) = \frac{1}{\varepsilon}\boldsymbol{x}^\varepsilon_n(\varepsilon^2 t + t^*), \qquad L_{mn}(t) = \frac{1}{\varepsilon}l^\varepsilon_{mn}(\varepsilon^2 t + t^*), \tag{2.15}$$

where $m, n = \{1, 2, 3\}$ ($m \neq n$), and $t^* \in \mathbb{R}$ is an arbitrary real, which is determined later. Then the evolution equation for the new variables $\boldsymbol{X}_n(t)$ is derived from (2.4) as follows.

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{X}_n = -\frac{1}{2\pi}\sum_{m\neq n}^{3}\Gamma_m\frac{(\boldsymbol{X}_n-\boldsymbol{X}_m)^{\perp}}{L_{mn}^2}P_K\left(L_{mn}\right), \qquad \boldsymbol{X}_n(0) = \frac{\boldsymbol{x}_n^{\varepsilon}(t^*)}{\varepsilon}. \tag{2.16}$$

This is formally equivalent to (2.4) with $\varepsilon = 1$ and it also defines a Hamiltonian dynamical system with the Hamiltonian,

$$\mathscr{H} = \mathscr{H}^1 = -\frac{1}{2\pi}\left[\Gamma_2\Gamma_3 H_P\left(L_{23}^2\right) + \Gamma_3\Gamma_1 H_P\left(L_{31}^2\right) + \Gamma_1\Gamma_2 H_P\left(L_{12}^2\right)\right]. \tag{2.17}$$

Here, $H_P(r) = \log\sqrt{r} + H_G\left(\sqrt{r}\right)$. It follows from (2.13) that the system has the following invariant.

$$M = M^1 = \Gamma_2\Gamma_3 L_{23}^2 + \Gamma_3\Gamma_1 L_{31}^2 + \Gamma_1\Gamma_2 L_{12}^2. \tag{2.18}$$

The evolution equations of the distance L_{mn} is obtained from (2.16) as follows.

$$\frac{\mathrm{d}}{\mathrm{d}t}L_{mn}^2 = \frac{2}{\pi}\Gamma_k A\left[\frac{1}{L_{nk}^2}P_K\left(L_{nk}\right) - \frac{1}{L_{km}^2}P_K\left(L_{km}\right)\right]. \tag{2.19}$$

Here, A is the signed area of the triangle formed by the three ε point vortices at $\boldsymbol{X}_1(t)$, $\boldsymbol{X}_2(t)$ and $\boldsymbol{X}_3(t)$. It is important to note that the evolution equation (2.16) does not contain the parameter ε except for the initial data. Hence, the solution of the original equation (2.11) is constructed from those of (2.16) and (2.19) via

$$\boldsymbol{x}_n^{\varepsilon}(t) = \varepsilon\boldsymbol{X}_n\left(\frac{t-t^*}{\varepsilon^2}\right), \qquad l_{mn}^{\varepsilon}(t) = \varepsilon L_{mn}\left(\frac{t-t^*}{\varepsilon^2}\right). \tag{2.20}$$

In terms of the scaled variables, the enstrophy variation $\mathscr{Z}^{\varepsilon}(t)$ is expressed as

$$\mathscr{Z}^{\varepsilon}(t) = -\frac{1}{\varepsilon^2}\mathscr{Z}_0\left(\frac{t-t^*}{\varepsilon^2}\right), \qquad \mathscr{Z}_0(\tau) = -\frac{1}{2\pi}\sum_{n=1}^{N}\sum_{m=n+1}^{N}\Gamma_n\Gamma_m Z_{mn}(\tau), \tag{2.21}$$

in which

$$Z_{mn}(\tau) = \int_0^{\infty} s\left|2\pi\widehat{h}(s)\right|^2 J_0\left(sL_{mn}(\tau)\right)ds. \tag{2.22}$$

Since $\mathscr{H}^{\varepsilon}$ remains a constant along the solution of (2.16), the energy variation $E^{\varepsilon}(t)$ is rewritten as

$$E^{\varepsilon}(t) = \mathscr{H}^{\varepsilon} + E_0\left(\frac{t-t^*}{\varepsilon^2}\right), \qquad E_0(\tau) = -\frac{1}{2\pi}\sum_{n=1}^{N}\sum_{m=n+1}^{N}\Gamma_n\Gamma_m E_{mn}(\tau),$$

in which E_{mn} is given by

$$E_{mn}(\tau) = -H_G\left(L_{mn}(\tau)\right) + \int_0^{\infty}\frac{1}{s}\left(1 - \left|2\pi\widehat{h}(s)\right|^2\right)J_0\left(sL_{mn}(\tau)\right)ds.$$

The energy dissipation rate $\mathscr{D}_E^{\varepsilon}(t)$ is obtained by differentiating the energy $E^{\varepsilon}(t)$ with respect to t.

We are concerned with whether or not there exists a singular weak solution of the dispersive Euler equation with the anomalous enstrophy dissipation in the limit of $\varepsilon \to 0$. As a candidate of such singular weak solutions, we consider a well-known solution where three point vortices collapses self-similarly in finite time [48]. A necessary condition for the existence of the triple collapse is given by

$$\frac{1}{\Gamma_1} + \frac{1}{\Gamma_2} + \frac{1}{\Gamma_3} = 0, \tag{2.23}$$

which we also assume here. Note that this condition (2.23) says that Γ_3 is determined from Γ_1 and Γ_2. To state the main result, let us introduce the function $\psi(r)$ and the constants $k_\pm$.

(2.24)

$$\psi(r) = \left(\frac{1}{1+r}\right)^{1/\Gamma_1} \left(\frac{r}{1+r}\right)^{1/\Gamma_2}, \quad k_\pm = \left(\frac{\Gamma_1 + \Gamma_2 \pm \sqrt{\Gamma_1^2 + \Gamma_1\Gamma_2 + \Gamma_2^2}}{\Gamma_2}\right)^2.$$

We also use the constant $k_0 = \operatorname*{argmin}_{k \in \{k_-, k_+\}} \psi(\Gamma_1/\Gamma_2 k)$. Then the following theorem holds.

Theorem 2.1. *Let $h \in C^1(\mathbb{R}^2 \backslash \{0\})$ be a positive radial function satisfying $\mathrm{d}h_r/\mathrm{d}r < 0$, (2.5), (2.7), $\chi_1 \nabla h \in L^1(\mathbb{R}^2)$, $\chi_{3+\eta} h \in L^\infty(\mathbb{R}^2)$ $(\eta > 0)$, and let $\mathscr{H}_c$ be the constant satisfying*

$$\frac{\Gamma_1^2\Gamma_2^2}{4\pi(\Gamma_1+\Gamma_2)} \log\left(\psi\left(\frac{\Gamma_1}{\Gamma_2}k_0\right)\left[\psi\left(\frac{\Gamma_1}{\Gamma_2}\right)\right]^{-1}\right) < \mathscr{H}_c < 0. \tag{2.25}$$

Suppose (2.23) and that for any initial condition with $\mathscr{H}^\varepsilon = \mathscr{H}_c$, the solution of (2.16) does not converge to a relative equilibrium as either of $t \to \pm\infty$. Then, there exists a constant t^* such that $\varepsilon \to 0$ as $l_{mn}^\varepsilon(t^*) \to 0$, and

$$\lim_{\varepsilon \to 0} \mathscr{Z}^\varepsilon = -z_0 \delta(\cdot - t^*), \qquad \lim_{\varepsilon\to 0} \mathscr{D}_E^\varepsilon = 0$$

holds in the sense of distributions. Here, the constant z_0 is given by

$$z_0 = \int_{-\infty}^{\infty} \mathscr{Z}_0(\tau) d\tau. \tag{2.26}$$

This theorem asserts that, in the limit of $\varepsilon \to 0$, for any initial data with (2.25), the distance of the three ε point vortices tends to zero. Hence, at the critical time t^* when the collapse of three point vortices occurs, the enstrophy variation tends to a δ-measure with the weight $-z_0$ in the sense of distributions. In other words, the total enstrophy variation converges, in the sense of distributions, to the Heaviside function $\mathcal{H}$.

$$\int_{-\infty}^{t} \mathscr{Z}^\varepsilon(\tau) d\tau \longrightarrow -z_0 \mathcal{H}(t - t^*). \tag{2.27}$$

This indicates that if $z_0 > 0$ the enstrophy *dissipates* discontinuously at the collapse time t^*. Since the ε-PV Hamiltonian system is time-reversible, the enstrophy dissipation still occurs when the time is reversed. This means the emergence of the irreversibility of time direction in the conservative dynamical system as $\varepsilon \to 0$.

With the function defined from (2.22),

$$Z(r) = \int_0^\infty s \left|2\pi\widehat{h}(s)\right|^2 J_0\left(s\sqrt{r}\right) ds, \tag{2.28}$$

a sufficient condition for $z_0 > 0$ is given as follows.

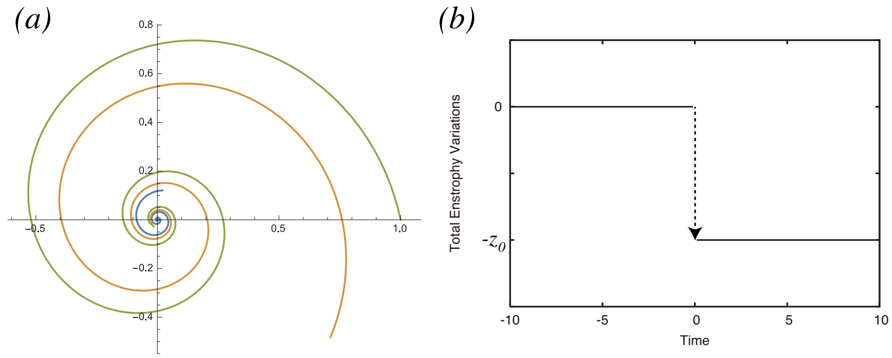

FIGURE 1. (a) The orbit of a self-similar triple collapse. (b) A schematic of the anomalous enstrophy dissipation, showing the total enstrophy variation drops discontinuously by $-z_0$ at the collapsing time $t = 0$. Our results prove the existence of this singular phenomena.

Corollary 2.2. *Suppose that $Z(r)$ is monotonically decreasing and concave. Then, for any initial configuration satisfying the assumptions of Theorem 2.1 and $M \geq 0$, we have $z_0 > 0$. For the case of $M < 0$, if the functions $Z(r)$ and $H_P(r)$ satisfy the additional condition*

$$Z''(r)H_P'(r) - Z'(r)H_P''(r) > 0, \tag{2.29}$$

then we have $z_0 > 0$.

Theorem 2.1 indicates the existence of the collapse time when the enstrophy dissipates anomalously, but it is still uncertain how the three vortex points behaves dynamically. The following theorem answers the question.

Theorem 2.3. *Under the same assumptions of Theorem 2.1, in the limit of $\varepsilon \to 0$, the orbit of three point vortices defined by (2.11) converges to the self-similar singular orbits collapsing to a point for $t < t^*$ and expanding to infinity for $t^* < t$ with the same value of the Hamiltonian in the three PV system.*

Hence, in the $\varepsilon \to 0$ limit, the orbit of the three vortex points is self-similarly collapsing at $t = t^*$ to a point as shown in Figure 1(a). The total enstrophy variation dissipates discontinuously as in Figure 1(b) at the event of the collapse.

According to Corollary 2.2, the sufficient condition for the existence of the anomalous enstrophy dissipation via the self-similar triple collapse is reduced to the condition (2.29) described by the regularization function h. We provide some examples of h that are of significance from the application points of view. The first example is a Gaussian regularization function $h(r) = \mathrm{e}^{-r^2}/\pi$, which is used in a numerical method for the Euler equations, called the *vortex method* [16]. The second example is given by $h_b(r) = 1/(\sqrt{\pi}(r^2+1))^2$, which also appears in a numerical method for the Euler equation, called the *vortex-blob method* [54]. The last example is found in the Euler-α equations, which is a model of turbulent flows. Then the regularization function is $h_\alpha(r) = K_0(r)$, where K_0 denotes the modified Bessel function of the first kind. These regularization functions satisfy the condition (2.29). This indicates that the anomalous enstrophy dissipation via the triple collapse of point vortices is less dependent on the choice of the regularization function.

Although the self-similar collapse of three point vortices with the anomalous enstrophy dissipation is a special solution, this is the first example of weak solutions of the 2D incompressible flow equations whose dynamical behavior is clarified with a mathematical rigor. Physically, it would be interesting to investigate the relation between the vortex collapse with the vortex merger that is regarded to be one of the fundamental mechanisms generating in 2D free decaying turbulence. Numerical study of the regularized point vortices showed that the collapse of four point vortices is also a trigger of the anomalous enstrophy dissipation [35]. Hence, we expect the existence of collapsing solution of more than three point vortices dissipating the enstrophy. However, it would be difficult to construct such singular solutions with the method using here, since the ε-PV system is no longer integrable for $N \geq 4$. Another mathematical approach is required.

3. One-dimensional hydrodynamic PDE model for the cascade of conserved quantity in turbulent flows

As mentioned in Section 1, many flow experiments and numerical simulations have supported the emergence of the 5/3-law of the energy spectra (1.2) in the inertial range. Up to this point, we focus on constructing weak solutions of the hydrodynamic equations dissipating of an inviscid conserved quantity, whose existence is required for the generation of the inertial range. In this section, changing the view point, we consider how the inertial range is formed dynamically in terms of such singular solutions of hydrodynamic equations. Let us imagine, for instance, that a turbulent flow generated by a big fan in a wind tunnel attains the scaling law (1.2). Then, a large scale (a low wavenumber) energy input by the fan cascades towards small scales (high wavenumbers) where the energy dissipates strongly. The energy input in the large scale and the energy dissipation in small scales are balanced as a whole. Otherwise, the total energy keeps increasing or decreasing. Roughly speaking, the inertial ranges connect these two scales. In physics of turbulence, it is considered that the large scale energy input is transferred to small scale dissipation range at a constant rate in the inertial range without losing energy. That is to say, the downstream energy flux is locally "balanced" in the inertial range, which is called the energy cascade. As the viscosity tends to zero, the inertial range expands, while the energy for smooth flows should be conserved in the inviscid limit. This gives rise to a contradiction. Hence, we expect turbulent flows in the inviscid limit become singular. This anomalous cascade of the inviscid conserved quantity is a phenomenon not limited to 3D turbulence. Indeed, as we mentioned in Section 2, there emerges the inertial range of energy spectra corresponding to the cascade of the enstrophy in 2D turbulence. Hence, the cascade of inviscid conserved quantities is a common feature shared with turbulent flow phenomena.

Since it is mathematically difficult to deal with the Euler equations and the Navier-Stokes equations, one way to tackle this problem is to investigate phenomenological and qualitative models with reduced degrees of freedom by choosing a few selected aspects of turbulence. The famous models are Burgers' equation [5] and the shell models [6], which are certainly more amenable to analytical and numerical studies. In this section, we propose another model family of turbulent flows based on the Constantin-Lax-Majda (CML) model [14], from which we gain some insights into the anomalous cascade phenomena in terms of singular solutions of hydrodynamic equations.

The one-dimensional hydrodynamic equation is constructed by observing some analytic properties of the 3D Navier-Stokes equations. First, applying $\nabla\times$ to (1.3) yields the following equations for the vorticity $\boldsymbol{\omega}(\boldsymbol{x},t) = \nabla\times\boldsymbol{u}(\boldsymbol{x},t)$.

$$\partial_t\boldsymbol{\omega} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u} = \nu\triangle\boldsymbol{\omega} + \boldsymbol{F}, \qquad \boldsymbol{u} = \mathcal{D}(\boldsymbol{\omega}), \tag{3.1}$$

where $\boldsymbol{F} = \nabla\times\boldsymbol{f}$ is an external force, and $\mathcal{D}$ denotes the Biot-Savart operator recovering the velocity field from the vorticity field. The nonlinear terms $(\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega}$ and $(\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u}$ in the first equation of (3.1) are called the advection term and the vortex stretching term. Naively speaking, since the large scale energy input and the dissipation at small scales are represented by $\boldsymbol{F}$ and $\nu\triangle\boldsymbol{\omega}$ respectively, the cascade phenomenon in the inertial range between these scales is generated by a subtle balance between these nonlinear terms. To model this situation, we consider the following one-dimensional partial differential equation for a scalar "model vorticity" $\omega(x,t)$ and a "model velocity" $u(x,t)$ as follows.

$$\partial_t\omega + au\omega_x - \omega u_x = \nu\omega_{xx} + f, \qquad u_x = H(\omega). \tag{3.2}$$

Here, H represents the Hilbert transform and $a\in\mathbb{R}$ is a parameter. The correspondence between the first equation of (3.1) and that of (3.2) is clear. A remarkable similarity can be found in the second equation where the vorticity and the velocity is connected. That is to say, by differentiating the Biot-Savart integrable, $\boldsymbol{u} = \mathcal{D}(\boldsymbol{\omega})$ in the 3D vorticity equation (3.1), we find that the derivative of the velocity is given by the convolution of the vorticity with the singular kernel $1/|\boldsymbol{x}|^3$ whose singularity is the same as the spatial dimension 3. Similarly, the second equation $u_x = H(\omega)$ in (3.2) is equating the derivative of the velocity and the convolution of the vorticity with the singular kernel $1/|x|$ whose singularity is the spatial dimension 1.

The idea of modeling the Biot-Savart integral with the Hilbert transform was originally proposed by Constantin, Lax and Majda [14]. They derived the celebrated Constantin-Lax-Majda (CLM) model for the 3D Euler equations, which is equivalent to (3.2) with $a=0$, $\nu=0$, $f=0$. They obtain an analytic solution of this equation that blows up in finite time. Soon later, adding the viscous term $\nu\omega_{xx}$ to the CLM equations, Schochet [73] obtained an analytic solution that blows up in finite time. However, the critical time of this solution is earlier than that of the inviscid solution, which is unphysical. To understand the balance between the vortex stretching term with the viscous diffusion term, the following generalized CLM equation with the hypo-viscous term was considered subject to the periodic boundary condition in [67, 68].

$$\partial_t\omega - \omega u_x = -\nu(-\partial_{xx})^{\frac{\alpha}{2}}\omega, \qquad u_x = H(\omega), \tag{3.3}$$

where $\alpha\in\mathbb{R}$ is a parameter. For any $\alpha\geq 0$, there exists ν^* depending on $\|\omega_0\|_{L^2}$ such that the following theorem holds.

Theorem 3.1. *For every $0<\nu<\nu^*$, the solution of (3.3) blows up in L^2 in finite time. Namely, there exists a time $T^*(\nu)$ such that $\|\omega(\cdot,t)\|_{L^2}\to\infty$ as $t\to T^*(\nu)$.*

This theorem indicates that however large the viscous dissipation rate sets in the system, the solution of (3.3) blows up for sufficiently small viscous coefficients. In other words, the viscous dissipation cannot control the rapid growth of the vorticity due to the vortex stretching term. This is in contrast to the fact that the solution of the Navier-Stokes with the hypo-viscosity of $\alpha > 5/2$ exists globally in time.

To improve the model, DeGregorio [17] added an advection term, which is a missing term in the CLM equation. This is equivalent to (3.2) with $\nu = 0$, $f = 0$ and $a = 1$. Furthermore, the model has been generalized by introducing a free parameter $a \in \mathbb{R}$ in the advection term to observe the balance between the nonlinear terms. We thus obtain the generalized Constantin-Lax-Majda-DeGregorio (gCLMG) equation. Under the periodic boundary condition, the existence of unique local solution of this equation for the initial data $\omega_0 \in H^1(S^1)/\mathbb{R}$ is proved in [64].

Theorem 3.2. *For any $\omega_0 \in H^1(S^1)/\mathbb{R}$, there exists $T > 0$ depending only on a and $\|\partial_x \omega_0\|_{L^2}$ such that a unique solution of the (3.2) with $\nu = 0$, $f = 0$ and $a = 1$ exists locally in time, i.e., $\omega \in C^0([0,T]; H^1(S^1)/\mathbb{R}) \cap C^1([0,T]; L^2(S^1)/\mathbb{R})$.*

In addition, a sufficient condition for the existence of global solutions is provided.

Theorem 3.3. *Suppose that the solution of (3.2) with $\nu = 0$, $f = 0$, $a \in \mathbb{R}$ exists for $\omega_0 \in H^1(S^1)/\mathbb{R}$ in $[0,T)$ and it satisfies*

$$\int_0^T \|H\omega(\cdot,t)\|_{L^\infty}\, dt < \infty.$$

Then the solution exists in $0 \le t \le T + \delta$ for some $\delta > 0$.

This theorem is an analogue of Beale-Kato-Majda criterion for the global existence of solutions of the 3D Euler equations [4]. In this regard, gCLMG equation is a valid mathematical model of the 3D Euler equation. Although this criterion has not yet been proved, we can confirm numerically the existence of global-in-time solution by using the criterion in [64], according to which it is expected that there exists an $a_c \in (0,1)$ such that the solution blows up in finite time for $a < a_c$ ($a \le a_c$) and exists globally in time for $a \ge a_c$ ($a > a_c$). It claims that the original equation given by DeGregorio [17] has a unique solution globally in time, but it is interesting to note that numerical simulations in [64] and mathematical studies [12, 26] suggest more subtle issues on the existence of blowing-up/growing-up solutions for $a = 1$. The existence of a blowing-up solution has been shown for $a < 0$ in [11] and for sufficiently small $|a|$ in [25].

In the meantime, for negative $a < -1$, the model equation (3.2) without the viscous term and the forcing term, i.e., $\nu = 0$ and $f = 0$, admits a conserved quantity [64].

Proposition 3.4. *Let us consider the equation (3.2) with $\nu = 0$, $f = 0$ and the parameter $-\infty < a \le -1$. Then, for any $p = -a$, its solution for the initial data ω_0 satisfies $\|\omega(\cdot,t)\|_{L^p} = \|\omega_0\|_{L^p}$.*

Since there exists no inviscid conserved quantity for $a > 0$, we confirm numerically whether the one-dimensional model (3.2) for $a \le -1$ becomes a model for the cascade of the inviscid conserved quantity subject to a random forcing, which has been considered in [61, 62]. Let us remark that although the Galilean invariance is lost in this model for general a, we focus on the formal resemblance of the mathematical structures between this model equation and Navier-Stokes equations to understand the cascade phenomenon in terms of dynamical system.

To achieve a statistically steady state, the forcing f is set to be random whose Fourier coefficient $\widehat{f}$ is nonzero only for the wave numbers $k = \pm 1$ and whose real

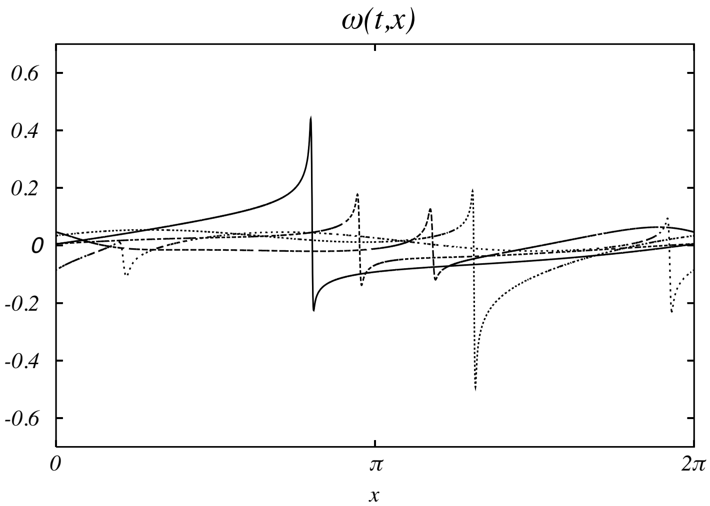

FIGURE 2. Snapshots of numerical solutions $\omega(x,t)$ of the gGLMG equation (3.2) with the parameters $a = -2$ and $\nu = \nu_1 := 2.5 \times 10^{-5}$ and a large-scale Gaussian random forcing. They are illustrated in one figure.

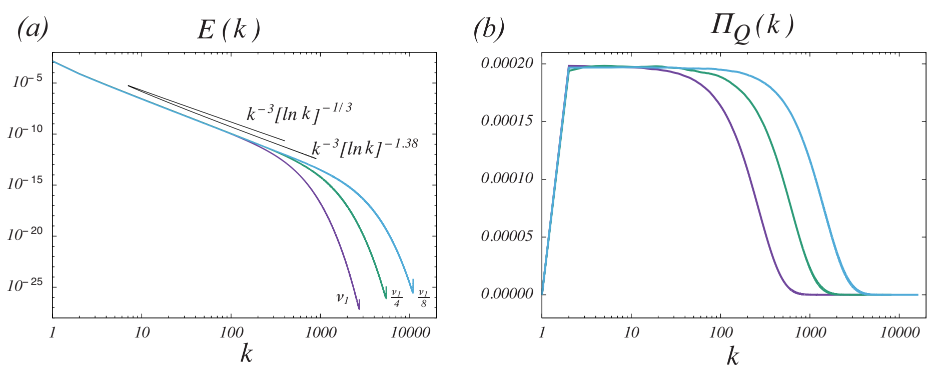

FIGURE 3. (a) Time-averaged energy spectra $E(k)$ of the solutions of the gCLMG equation (3.2) for the viscous coefficients, $\nu = \nu_1$, $\nu_1/4$, $\nu_1/8$, indicating that there appears an inertial range in the energy spectra. (b) Time-averaged enstrophy flux $\Pi_Q(k)$, which shows that the flux is almost constant independently on wavenumbers k in the inertial range. Hence, the enstrophy cascades from the region of low wavenumbers to that of high wavenumbers at a constant rate.

and imaginary parts are set to a Gaussian distribution. We also take the kinematic viscosity to be sufficiently small in the equation (3.2). We use the pseudo-spectral method and the fourth order Runge-Kutta method as the temporal integration to solve the equation numerically. The initial data is given by $\omega(x,0) = \sin x$. We then investigate the statistical properties of numerical solution after a long-time evolution when the solution becomes a statistically equilibrium state. The parameter a can be taken arbitrarily from $a \leq -1$, but we here use $a = -2$ where the L^2 norm of the solution, the model enstrophy, becomes the inviscid conserved quantity [61]. Figure 2 shows some snapshots of the numerical solutions for $\nu = \nu_1 := 2.5 \times 10^{-5}$, in which we observe sharp pulses moving randomly, which sometimes merge and emerge. We plot in Figure 3(a) the time averaged energy spectra,

$$E(k) = \langle E(k,t) \rangle = \left\langle \sum_{k \leq |k'| \leq k+1} \frac{1}{2} |\widehat{u}(k',t)|^2 \right\rangle,$$

for various viscous coefficients, where $\widehat{u}(k,t)$ denotes the Fourier coefficients of the model velocity field. We find the inertial range of the wavenumbers $10 < k < 1000$ with a self-similar constant slope. In order to check if the cascade of the model enstrophy occurs in this inertial range, introducing the enstrophy flux

$$\Pi_Q(k,t) = \sum_{\ell \leq k} \sum_{|k'|=l} \sum_{p+q=k'} \mathrm{Im}\left[\widehat{\omega}^*(k',t)(aq-p)\widehat{u}(p,t)\widehat{u}(q,t)\right],$$

we show the time averaged enstrophy flux $\Pi_Q(k) = \langle \Pi_Q(k,t) \rangle$ in Figure 3(b). In the inertial range $\Pi_Q(k)$ forms a plateau region, indicating the enstrophy transfers from low wavenumbers to high wavenumbers at a constant rate. Moreover, as the viscous coefficients decreases, the plateau region expands and the inertial range corresponding to the enstrophy cascade is certainly formed. The scaling law of the energy spectra in the inertial range deviates from $E(k) \sim k^{-3}$ obtained by the dimensional analysis as well as $E(k) \sim k^{-3} [\ln k]^{-1/3}$ derived from its logarithmic correction theory [53, 58] for 2D turbulence as shown in Figure 3(a).

In order to study the dependence of the emergence of the inertial range in the energy spectra on the large-scale forcing in (3.2), we switch the random forcing to a deterministic and stationary one, i.e., $f = C_0 \sin x$ for a constant $C_0 = -0.1$. We have found that the forced gCLMG equation has an asymptotically stable stationary solution, which has a localized vorticity pulse as shown in Fig. 4(a). We note that the steady solution looks similar to the randomly moving pulse solutions in Fig. 2. Furthermore, the energy spectrum of the steady solution is indistinguishable to that of the gGLMG turbulence in the inertial range as we see in Fig. 4(b). This indicates that the functional form of the turbulent $E(k)$ can be studied with the stationary solution, yielding $E(k) \sim k^{-3} [\ln k]^{-1.38}$, which is a better fit than the slope k^{-3} and $k^{-3}[\log k]^{-1/3}$ as in Fig. 3(a). Figure 4(b) also shows that the inertial range for the energy spectra of the steady solutions expands as $\nu \to 0$, indicating that the stationary solution seems to converge to a singular function (infinite vorticity) in the inviscid limit. The vorticity pulse solutions for various ν are numerically scaled as $\nu^{-0.2}\omega(\nu^{-0.6}(x - 2\pi))$ in the neighborhood of the pulse center as shown in Figure 4(c). Hence, the steady solution acquires a strong self-similarity at the center and its peak diverges as $\nu \to 0$. As a matter of fact, it is numerically confirmed that the steady solution in the zero viscous limit has a similar profile to a finite-time blowing-up solution of (3.2) with no viscosity $\nu = 0$ and no forcing $f = 0$. The

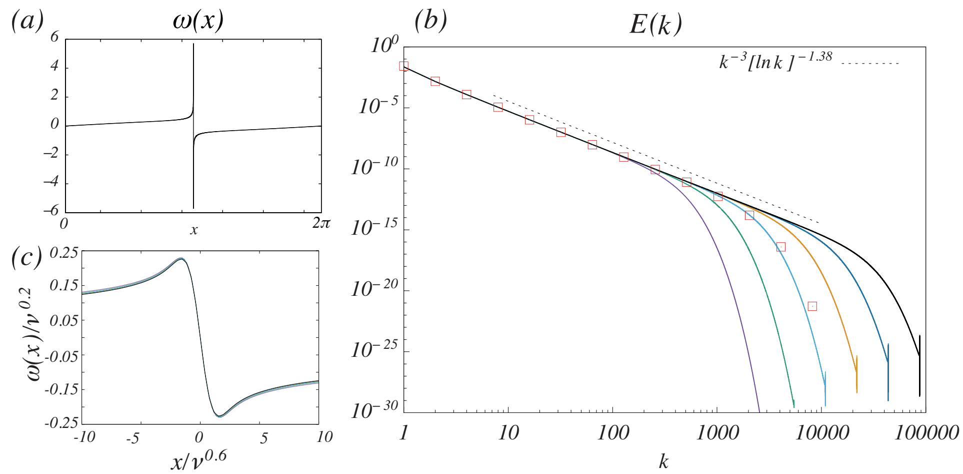

FIGURE 4. (a) A steady vorticity pulse $\omega(x)$, which is an asymptotically stable solution of (3.2) subject to the external force $f(x) = -0.1 \sin x$. (b) The energy spectra of the steady solutions to (3.2) with the deterministic forcing for various ν's. The squares represent data of the randomly forced case with $\nu_1/8$ shown in Fig. 3. (c) Scaled vorticity pulse of the stationary solution around the origin, $\nu^{-0.2}\omega(\nu^{-0.6}(x - 2\pi))$.

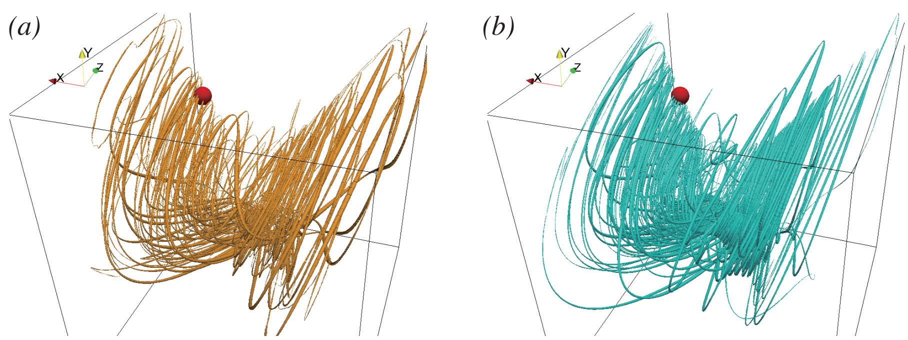

FIGURE 5. Orbits of the spectra $\mathrm{Im}[\omega(k,t)]$ of the randomly forced gCLMG turbulence projected on the three-dimensional phase space (k_1, k_2, k_3). The (red) sphere corresponds to the stationary solution with the same enstrophy and the viscosity. (a) $(k_1, k_2, k_3) = (4, 8, 16)$. (b) $(k_1, k_2, k_3) = (16, 32, 64)$.

observations above suggest some connections among asymptotically stable steady solutions of (3.2) with a sufficiently small viscosity and a deterministic forcing, blowing-up solutions of (3.2) with no viscosity and no forcing and statistical scaling law generated by the vorticity pulse obtained by (3.2) subject a large-scale random forcing.

Since the evolution of Fourier coefficients of the randomly forced gCLMG turbulence is regarded as an infinitely dimensional dynamical system of the spectra, we project the orbits of the spectra on a three-dimensional sub-space spanned by the chosen three wavenumbers in order to see the relation of the stationary solution to the gCLMG turbulence from the viewpoint of the dynamical system. Figure 5 shows the orbits of the spectra projected on the three-dimensional phase space of wavenumbers, (a) $(k_1, k_2, k_3) = (4, 8, 16)$ and (b) $(k_1, k_2, k_3) = (16, 32, 64)$ respectively. The red sphere at the upper left of the domain corresponds to the asymptotically stable stationary solution of the gCLMG equations with the deterministic forcing. We observe that the orbit is within a thin surface of "butterfly-like two leaves", and the stationary solution is located on one leaf of this attracting set. Significantly, the orbits in the two scale ranges are similar. With this self-similarity of the orbit, modeling of the gCLMG turbulence with a few degrees of freedom is conceivable.

What was found in the paper [61] for gCLMG turbulence with $a = -2$ is summarized as follows.

- **(1):** There appears the inertial range in the energy spectra corresponding to the cascade of the enstrophy that is the inviscid conserved quantity.
- **(2):** The time-averaged energy spectra in the inertial range for the randomly forced gCLMG turbulence is in good agreement with the energy spectra of the steady solution of the gCLMG equation with the deterministic forcing.
- **(3):** The asymptotically stable steady solution has a self-similar scaling and its peak blows up as $\nu \to 0$.
- **(4):** Orbits of the spectra of the the randomly forced gCMLG turbulence evolve within a characteristic manifold having a spatio-temporal self-similar structure in the infinite-dimensional phase space.

It is confirmed in [62] that these findings remain valid for $-4 < a \leq -1$, suggesting that the anomalous cascade of inviscid conserved quantity is understood through the singular steady solutions of gCLMG equations with the deterministic forcing.

Finally, in connection with Section 1, we formally apply the argument on dissipative weak solutions of the 3D Euler equations owing to Duchon and Robert [24] to the gCLMG equation with $a = -2$. Let us consider the inviscid gCLMG equation (3.2) without forcing, i.e., $\nu = f = 0$. For a compactly supported smooth function $\varphi(x) \in C_0^\infty(\mathbb{R})$, introducing a test function $\varphi^\varepsilon(x) = (1/\varepsilon)\varphi(x/\varepsilon)$, we regularize the model velocity v and the model vorticity ω as $v^\varepsilon = v * \varphi^\varepsilon$ and $\omega^\varepsilon = \omega * \varphi^\varepsilon$. Deriving the equations of v^ε and ω^ε from (3.2), we obtain the following equation for the enstrophy dissipation.

$$\frac{1}{2}(\omega\omega^\varepsilon)_t - (v\omega\omega^\varepsilon)_x + \frac{1}{2}\omega\left[-2(v\omega)^\varepsilon_x + (v\omega^\varepsilon)_x + v\omega^\varepsilon_x + (\omega v_x)^\varepsilon\right] = 0.$$

While the first and the second terms contribute to the conservation law of the enstrophy, the third one is regarded as a defect term $D_\varepsilon[v, \omega]$ yielding the anomalous enstrophy dissipation, which is defined as follows.

$$D_\varepsilon[v, \omega] = \frac{1}{2}\omega\left[-2(v\omega)^\varepsilon_x + (v\omega^\varepsilon)_x + v\omega^\varepsilon_x + (\omega v_x)^\varepsilon\right].$$

This term converges in the sense of distributions as $\varepsilon \to 0$.

$$D[v, \omega] = \lim_{\varepsilon \to 0} D_\varepsilon[v, \omega] = -(v\omega)_x\omega + \omega_x\omega v + \omega^2 v_x.$$

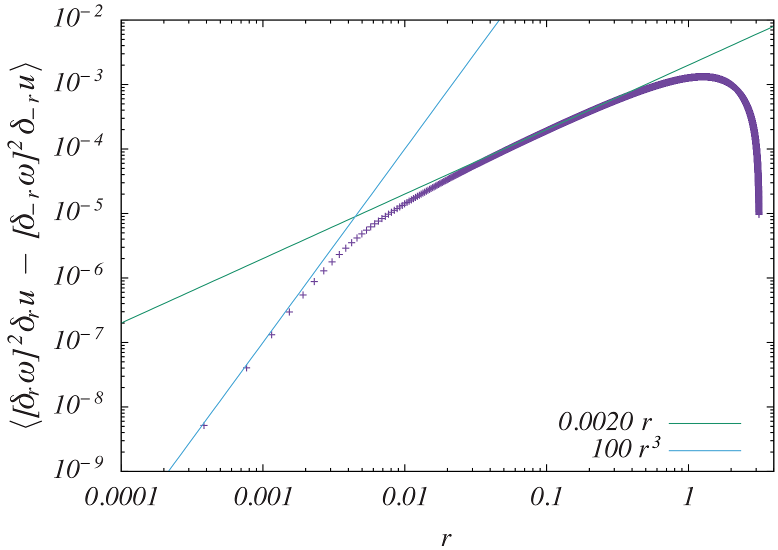

FIGURE 6. Plot of the spatially averaged statistical law of (3.4) for the gCLMG turbulence with $a = -2$, which is derived based on the argument by Duchon and Robert [24].

On the other hand, with the increments of the velocity $\delta_\xi v(x) = v(x+\xi) - v(x)$ and the vorticity $\delta_\xi \omega(x) = \omega(x+\xi) - \omega(x)$ at the two points x and $x+\xi$, let us introduce the following quantity.

$$E_\varepsilon[v,\omega](x,t) \equiv \int_{-\pi}^{\pi} (\delta_\xi \omega(x,t))^2 \delta_\xi v(x,t) \partial_\xi \varphi^\varepsilon(\xi) d\xi.$$

Then we have $E[v,\omega] := \lim_{\varepsilon\to 0} E_\varepsilon[v,\omega] = 2(v\omega)_x - 2\omega_x \omega v + 2\omega^2 v_x = 2D[v,w]$ in the sense of distributions, which gives rise to a formal relation between the ensemble averages of $E[v,\omega]$ and $D[v,\omega]$.

$$\mathbb{E}\left[\int_{-\pi}^{\pi} [(\delta_r\omega)^2\delta_r v - (\delta_{-r}\omega)^2\delta_{-r}v] d\xi\right] = 4r\mathbb{E}\left[D[v,w]\right]. \tag{3.4}$$

If the ensemble average of the defect term $\mathbb{E}[D[v,w]]$ is strictly positive for dissipative weak solutions of the gCLMG equation, the relation (3.4) is considered to be a statistical r^1-law. Figure 6 shows the relation (3.4) computed for the gCLMG turbulence, in which we see a clear agreement in the inertial range. However, the constant in the statistical law is not equivalent. Let us remark that the regularity condition in terms of Hölder continuity on dissipative weak solutions of the gCLMG equation with a positive defect term is uncertain, which will be another future direction for mathematical studies of this equation.

4. Epilogue : lessons of mathematical studies of turbulence

We summarize the mathematical studies of turbulent phenomena in terms of singular solutions of hydrodynamic equations presented in this article. In Section 1, we introduce the preceding mathematical studies of weak solutions with respect to Onsager's conjecture. The flows satisfying the non-vanishing energy dissipation rate in the inviscid limit, which is the assumption made by Kolmogorov in the theory of the isotropic turbulence, are characterized by weak solutions of the Euler equations and the Navier-Stokes equations with Hölder continuity of the exponent less than 1/3. Inspired by these works in regards to dissipative weak solutions, we introduce the two models to investigate the relation between of singular solutions of some hydrodynamic equations and turbulent flows. In Section 2, we show that an anomalous enstrophy dissipation of the incompressible and non-viscous flows in 2D turbulence is brought by the self-similar collapse of three point vortices, which is obtained as a limit of weak solutions of the dispersively regularized Euler equations for the initial data belonging to the space of Radon measures. In Section 3, we propose a one-dimensional hydrodynamic equation of turbulence, called the gCLMG equation. We demonstrate numerically that the scaling law of the time-averaged energy spectra corresponding to the cascade of enstrophy for the randomly forced gCLMG equation agrees with the spectra of the asymptotically stable steady solution of the gCLMG equation with a deterministic forcing. We also find that the steady solution diverges in the zero-viscous limit. This suggests that the anomalous dissipation of the inviscid conserved quantity in the turbulent phenomenon can be understood through the mathematical study of these steady singular solutions.

In each study, we have gained some interesting insights and future directions toward the understanding of turbulent phenomena. However, we here focus on nothing but one of characteristic properties of turbulent flows, and we then try to describe it in terms of singular solutions of hydrodynamic equations. That is to say, they are mathematical descriptions of turbulence from one limited aspect. Since our final goal is a complete theoretical understanding of turbulence as a whole, but it is beyond our reach. Nevertheless, we could have some lessons from these studies. We have observed that many self-similar structures appear in these studies: Self-similar collapse of three point vortices is a trigger of the anomalous enstrophy dissipation and we can describe the cascading phenomena in the inertial ranges via self-similar singular solutions of the gCLMG equation. This indicates that turbulent phenomenon seems to be described through beautiful scale-free singular solutions, where large scales and small scales are inseparable. This tells us that it is not plausible to understand the turbulent phenomena by using model equations describing a certain range of scales. Another aspect we found was the importance of an infinitely small "deviation" from complete self-similar structures. In Section 2, since the self-similar collapse is obtained as a limit of weak solutions of the dispersively regularized Euler equations, the magnitude of the enstrophy dissipation, say $-z_0$, is determined by weak solutions of the regularized equation, whose time evolutions is slightly away from the self-similar solution. In Section 3, while the scaling law of the energy spectra in the inviscid limit is well represented by self-similar singular inviscid steady solutions, we have found some intermittent behaviors in the pth moment of the vorticity difference in [61, 62], which is still an open problem of *intermittency*. After one interesting problem solves, another attractive problem emerges. Mathematical studies of turbulent flows never ends. Like this, the turbulence keeps our mathematical investigations of nonlinear hydrodynamic equations inspired.

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