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TOPOLOGICAL AND MEASURE PROPERTIES OF SOME SELF-SIMILAR SETS

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Abstract. Given a finite subset $\Sigma \subset \mathbb{R}$ and a positive real number $q < 1$ we study topological and measure-theoretic properties of the self-similar set $K(\Sigma; q) = \left\{ \sum_{n=0}^{\infty} a_n q^n : (a_n)_{n\in\omega} \in \Sigma^\omega \right\}$, which is the unique compact solution of the equation $K = \Sigma + qK$. The obtained results are applied to studying partial sumsets $E(x) = \left\{ \sum_{n=0}^{\infty} x_n \varepsilon_n : (\varepsilon_n)_{n\in\omega} \in \{0,1\}^\omega \right\}$ of multigeometric sequences $x = (x_n)_{n\in\omega}$. Such sets were investigated by Ferens, Morán, Jones and others. The aim of the paper is to unify and deepen existing piecemeal results.

1. Introduction

Suppose that $x = (x_n)_{n=1}^{\infty}$ belongs to $l_1 \setminus c_{00}$ which means that x is an absolutely summable sequence with infinitely many nonzero terms. Let

$$E(x) = \left\{ \sum_{n=1}^{\infty} \varepsilon_n x_n : (\varepsilon_n)_{n=1}^{\infty} \in \{0,1\}^{\mathbb{N}} \right\}$$

denotes the set of all subsums of the series $\sum_{n=1}^{\infty} x_n$, called *the achievement set* (or *a partial sumset*) of x. The investigation of topological properties of achievement

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sets was initiated almost one hundred years ago. In 1914 Soichi Kakeya [8] presented the following result:

Theorem 1.1 (Kakeya). *For any sequence* $x \in l_1 \setminus c_{00}$

(a) *$E(x)$ is a perfect compact set.*
(b) *If* $|x_n| > \sum_{i>n} |x_i|$ *for almost all* n*, then* $E(x)$ *is homeomorphic to the ternary Cantor set.*
(c) *If* $|x_n| \leq \sum_{i>n} |x_i|$ *for almost all* n*, then* $E(x)$ *is a finite union of closed intervals. In the case of non-increasing sequence* x*, the last inequality is also necessary for* $E(x)$ *to be a finite union of intervals.*

Moreover, Kakeya conjectured that $E(x)$ is either nowhere dense or a finite union of intervals. Probably, the first counterexample to this conjecture was given by Weinstein and Shapiro ([16]) and, independently, by Ferens ([5]). The simplest example was presented by Guthrie and Nymann [6]: for the sequence $c = ((5 + (-1)^n)/4^n)_{n=1}^{\infty}$, the set $T = E(c)$ contains an interval but is not a finite union of intervals. In the same paper they formulated the following theorem, finally proved in [12]:

Theorem 1.2. *For any sequence* $x \in l_1 \setminus c_{00}$*,* $E(x)$ *is one of the following sets*:

(a) *a finite union of closed intervals*;
(b) *homeomorphic to the Cantor set*;
(c) *homeomorphic to the set* T*.*

Note that the set $T = E(c)$ is homeomorphic to $C \cup \bigcup_{n=1}^{\infty} S_{2n-1}$, where S_n denotes the union of the 2^{n-1} open middle thirds which are removed from $[0, 1]$ at the n-th step in the construction of the Cantor ternary set C. Such sets are called Cantorvals (to emphasize their similarity to unions of intervals and to the Cantor set simultaneously). Formally, a *Cantorval* (more precisely, an $\mathcal{M}$-Cantorval, see [9]) is a non-empty compact subset S of the real line such that S is the closure of its interior, and both endpoints of any component with non-empty interior are accumulation points of one-point components of S. A non-empty subset C of the real line $\mathbb{R}$ will be called a *Cantor set* if it is compact, zero-dimensional, and has no isolated points.

Let us observe that Theorem 1.2 says that l_1 can be divided into 4 sets: c_{00} and the sets connected with cases (a), (b) and (c). Some algebraic and topological properties of these sets have been recently considered in [1].

We will describe sequences constructed by Weinstein and Shapiro, Ferens and Guthrie and Nymann using the notion of multigeometric sequence. We call