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STRONGLY DAMPED WAVE EQUATION AND ITS YOSIDA APPROXIMATIONS

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Abstract. In this work we study the continuity for the family of global attractors of the equations $u_{tt} - \Delta u - \Delta u_t - \varepsilon \Delta u_{tt} = f(u)$ at $\varepsilon = 0$ when Ω is a bounded smooth domain of $\mathbb{R}^n$, with $n \geq 3$, and the nonlinearity f satisfies a subcritical growth condition. Also, we obtain an uniform bound for the fractal dimension of these global attractors.

1. Introduction

We study the continuity of global attractors of the following semilinear evolution equation of second order in time

$$
(1.1) \qquad \begin{cases} u_{tt} - \Delta u - \Delta u_t - \varepsilon \Delta u_{tt} = f(u), & t > 0, \\ (u(0), u_t(0)) = (u_0, v_0), \\ u|_{\partial\Omega} = 0, \end{cases}
$$

and we give an uniform bound for the fractal dimension of these global attractors.

We know that, for $\varepsilon = 0$, this equation is the usual strongly damped wave equation, and its asymptotic dynamics – related to global atrtactors – has already been vastly explored; see for instance [6], [7], [9], [12], [15], [22], [23], [26]–[28].

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However, for each $\varepsilon > 0$ fixed, we have a special form of the improved Boussinesq equation (see [4], [19], [20], [25]) with damping $-\Delta u_t$, which, among other things, is used to describe ion-sound waves in plasma (see [20], [21]).

For each $\varepsilon > 0$ fixed, this equation has been studied in [8], in terms of existence and uniqueness of solutions, existence of global attractors and asymptotic bootstrapping; in this case, the linear part of the equation (after a change of variables) is a bounded operator. Here, since we want to study the continuity of attractors at $\varepsilon = 0$, we will use the properties of the limiting problem with $\varepsilon = 0$ (local and global well posedness, regularity and existence of global attractors) as reported in [6], [7].

Throughout this paper, we will assume that $f\colon \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function, respecting a growth condition with subcritical exponent; that is, there exist constants $c > 0$ and $\rho < (n+2)/(n-2)$ such that for all $s_1, s_2 \in \mathbb{R}$

$$|f(s_1) - f(s_2)| \leq c|s_1 - s_2|(1 + |s_1|^{\rho-1} + |s_2|^{\rho-1}), \tag{1.2}$$

and also, if λ_1 denotes the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions in Ω, we assume the following dissipation condition

$$\limsup_{|s|\to\infty} \frac{f(s)}{s} < \lambda_1. \tag{1.3}$$

To begin our study, we will write further A for $-\Delta$ with the Dirichlet boundary conditions. Our problem then takes the form

$$\begin{cases} u_{tt} + Au + Au_t + \varepsilon A u_{tt} = f(u), & t > 0, \\ (u(0), v(0)) = (u_0, v_0). \end{cases} \tag{1.4}$$

and it is well-known that $A\colon H^1_0(\Omega) \cap H^2(\Omega) \subset L^2(\Omega) \to L^2(\Omega)$ is a closed, densely define operator which has the following properties:

- (O1) A is self-adjoint with compact resolvent;
- (O2) A is an operator of positive type;
- (O3) $\sigma(A) = \sigma_p(A) = \{\lambda_n\}_{n\in\mathbb{N}}$, $\lambda_1 > 0$, $\lambda_i \leq \lambda_{i+1}$, for all $i \geq 1$ (repeated to take into account the multiplicity), $\lambda_n \xrightarrow{n\to\infty} \infty$ and if $v_n \in L^2(\Omega)$ are unitary eigenvectors associated with λ_n then $\{v_n\}_{n\in\mathbb{N}}$ constitutes an orthonormal basis for $L^2(\Omega)$.

Remark 1.1. We included in Appendix A the proof of the main results of functional analysis we will use, in order to make explicit the uniformity of the constants obtained for $\varepsilon \in [0,1]$.

The key point in our analysis is the observation that the differential equation in (1.4), for $\varepsilon > 0$, can be obtained from its limit, for $\varepsilon = 0$, with a suitable exchange of the unbounded operator A by its *Yosida approximation* Λ_ε (see