Logic and Logical Philosophy (2022)
DOI: 10.12775/LLP.2022.003
Published online: January 30, 2022

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Axiomatization of BLRI Determined by Limited Positive Relational Properties

Abstract. In the paper a generalised method for obtaining an adequate axiomatic system for any relating logic expressed in the language with Boolean connectives and relating implication (**BLRI**), determined by the limited positive relational properties is studied. The method of defining axiomatic systems for logics of a given type is called an algorithm since the analysis allows for any logic determined by the limited positive relational properties to define the adequate axiomatic system automatically, step-by-step. We prove in the paper that the algorithm really works and we show how it can be applied to **BLRI**.

Keywords: algorithm α; Boolean logics with relating implication; BLRI; relating logic; relating semantics

1. Introduction

In this study, we present a generalised method for obtaining a consistent and complete axiomatic system for any relating logic expressed in a language with Boolean connectives and relating implication defined by the so-called limited positive relational properties.

Conditions of this kind take the form of a general conditional sentence with an antecedent in the form of relational expression conjunctions, i.e., expressions built with binary predicate and variables traversing formulas, and a consequent in the form of a relational expression. Multiple examples of such properties can be found in [Epstein, 1990; Jarmużek and Klonowski, 2021, submitted-a; Jarmużek and Malinowski, 2019a], where it has been shown how relating semantics, with the appropriate conditions for the considered type, can allow for analysing implication

Special Issue: Relating Logic and Relating Semantics. II. Guest Editors: T. Jarmużek & F. Paoli

that takes into account content relations of the expressions, causal implication, and connexive implication.

We call the method of obtaining axiomatic systems, for logics of a given type, the α *algorithm*, since our analysis allows for any logic of a given type to determine step-by-step the adequate axiomatic system. The proof of completeness of axiomatic systems obtained by applying the α algorithm that we will present constitutes a modification of Henkin-style completeness proofs for zero-order logic. Such proofs, for various types of related logic, were presented in [Epstein, 1979, 1990; Klonowski, 2019, 2021a].[1] All of those cases, however, made use of the fact of the expressivity of the relating relation in the language of the analysed logic. Our proof does not use the expressivity of the relating relation. By means of an appropriate transformation, we will show how to transform the relational conditions that determine a given logic into axioms. In addition to axioms, in some cases, we must additionally consider a rule that allows us to transform axioms in a way that corresponds to deducing relational conditions from the given initial conditions.

The paper consists of an introduction, five sections, and a conclusion. In Section 1, we introduce the language of the analysed logics, the necessary notations, and the notion of Boolean logic with its corresponding implication. For the latter, we also define the type of relational condition of interest. In Section 2, we define an axiomatic system and use examples to describe and demonstrate the α algorithm's operation. In Sections 3 and 4, we will deal respectively with the proof of consistency and completeness of the axiomatic systems obtained using α.

2. Boolean logics with relating implication

In this paper, we will focus on a certain family of Boolean logics with relating implications. In general, by Boolean logics with relating implication (**BLRI**), we mean relating logics with classical negation, conjunction, and disjunction in which there is only one relating connective — the relating implication. That is, an implication whose interpretation necessitates taking into account hypothetical relationships between the antecedent and the consequent.

The *language* of Boolean logic with relating implication is a zero-order language consisting of sentence variables and the following con-

[1] Constructive proofs were examined in [Paoli, 1996; Klonowski, 2018].

nectives: negation $\neg$, conjunction $\wedge$, disjunction $\vee$ and implication $\rightarrow$, as well as parentheses:), (. Let us denote the set of sentence variables as Var. Let us define the set of formulas in a standard way and denote them as For. The variables A, B, C and D will constitute the set of formulas For. The variables F, G, H, and I will traverse formulas.

In turn, X, Y and Z will traverse the power set of the set For, i.e. $\mathcal{P}(\mathsf{For})$. We will drop parentheses in formulas according to the standard convention of conjunctive relational strength and define the following abbreviations: $A \supset B := \neg A \vee B$ and $A \equiv B := (\neg A \vee B) \wedge (\neg B \vee A)$.

In our discussion, we will often use the iterated conjunction $A_1 \wedge \ldots \wedge A_n$, which we will also write as follows: $\bigwedge_{i=1}^{n} A_i$. On the other hand, the conjunction obtained from $\bigwedge_{i=1}^{n} A_i$ by excluding the parts $A_{j_1}, \ldots, A_{j_m}$ will be written as follows: $\bigwedge_{i \neq j_1, \ldots, j_m, i=1}^{n} A_i$.

Let us assume that in the case where $n = 1$ a formula $\bigwedge_{i=1}^{n-1} A_i * B$, where $* \in \{\supset, \wedge\}$, is synonymous to B, i.e. $\bigwedge_{i=1}^{n-1} A_i * B = B$.

A *model* of the analysed language is an ordered pair $\langle v, R\rangle$, such that, $v\colon \mathsf{Var} \longrightarrow \{1, 0\}$ is a classical valuation. While $R \subseteq \mathsf{For} \times \mathsf{For}$ is a binary relation, called *relational relation*.[2] In this paper, we will use the notation $R(A, B)$ and $\sim R(A, B)$, to imply, that A is in the relation R to B and A is not in the relation R to B, respectively. The relational symbol R can be used to state various relations between, what we refer to as, sentences.

A formula $A \in \mathsf{For}$ is *true in model* $\mathfrak{M} = \langle v, R\rangle$ (in symb.: $\mathfrak{M} \models A$; $\mathfrak{M} \not\models A$, if false) iff for every $B, C \in \mathsf{For}$:

$$
\begin{array}{ll}
v(A) = 1, & \text{if } A \in \mathsf{Var} \\
\mathfrak{M} \not\models B, & \text{if } A = \neg B \\
\mathfrak{M} \models B \text{ and } \mathfrak{M} \models C, & \text{if } A = B \wedge C \\
\mathfrak{M} \models B \text{ or } \mathfrak{M} \models C, & \text{if } A = B \vee C \\
[\mathfrak{M} \not\models B \text{ or } \mathfrak{M} \models C] \text{ and } R(B, C), & \text{if } A = B \rightarrow C.
\end{array}
$$

Let $X \subseteq \mathsf{For}$. We will write $\mathfrak{M} \models X$ instead of, for every $A \in X$, $\mathfrak{M} \models A$.[3]

We adopt standard definitions of semantic consequence relations and valid formulas. Let $X \cup \{A\} \subseteq \mathsf{For}$ and $\mathbf{M}$ be a set of models. Then:

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[2] For a discussion on the relation R see [Estrada-González et al., 2021].

[3] We will say that a formulaic scheme is false when there is some formula in the form, of that scheme, that is false.

- A is a *semantic consequence of* (*entailed by*) X *within the class* $\mathbf{M}$ (in symb.: $X \models_{\mathbf{M}} A$) iff for every $\mathfrak{M} \in \mathbf{M}$, if $\mathfrak{M} \models X$ then $\mathfrak{M} \models A$.
- A is a *valid formula within the class* $\mathbf{M}$ (in symb.: $\models_{\mathbf{M}} A$) iff $\emptyset \models_X A$.[4]

By *logic*, we will mean an ordered pair consisting of a set of formulas and a semantic consequence relation. Since we will focus on one set of formulas, the For set, we can identify logic with the semantic consequence relation. *A Boolean logic with a relating implication* is any logic $\models \subseteq \mathcal{P}(\mathsf{For}) \times \mathsf{For}$.

In this paper, we will focus on Boolean logics with relating implications, determined by sets of models satisfying some kind of relational conditions.[5] The conditions we will focus on will be the limited positive relational properties, i.e., the relational properties of the form of a general sentence built with a large quantifier ($\forall$), a metalinguistic implication ($\Rightarrow$), a metalinguistic conjunction (and), a metalinguistic disjunction (or), and atomic expressions built with the relational symbol R, as well as variables traversing formulas from the set For and conjunctions of the subject language. However, because there is no metalinguistic negation in them ($\sim$, it is not the case), we call the analysed properties positive. However, we only consider positive properties that are contained in the resulting a conjunction of atomic expressions or an atomic expression.[6]

Of course, we approach the general quantifier and the indicated conjunctions classically. We denote the set of expressions built with the relational symbol R and the variables traversing formulas from the set For as Var^+. We will denote the set of formulas defined in the standard way, including expressions from the set Var^+ and the conjunctions of conjunction (and), as well as the disjunction (or) as For^+. The variables φ, ψ and χ will traverse formulas from the set For^+, as well as formulas built from formulas from the set For^+ using implication and the big quantifier. In the case where $A_1, \ldots, A_n$ are the only variables in a φ formula, we will write $\varphi(A_1, \ldots, A_n)$. As with formulas from the For set,

[4] We will conclude that a formula schema, in particular an axiom schema, is a semantic consequence of (derived from) some set (resp. valid) in some system when any formula of the form, of that schema, is a semantic consequence of (derived from) a given set (resp. valid) in that system.

[5] In other words, we analyse only those logics that we can define by models whose relations satisfy any fixed conditions of a certain type.

[6] We use various standard symbols for metalinguistic functors, sometimes using symbols, sometimes using natural language expressions. In the case of relational conditions, we will use the symbols indicated.

we will omit outer brackets for formulas from the For^+ set and formulas created using formulas from For^+. We will also use iteration. We adopt similar conventions for meta-linguistic conjunction iterations as for object language conjunction iterations. In addition to φ_1 and ... and φ_n, we will also use the following notation $\mathrm{AND}_{i=1}^n \varphi_i$.

We can now introduce the type definition of the relational properties of interest. We shall state that φ is *limited positive relational property* (LPR) iff there exist $\psi, \chi \in \mathsf{For}^+$ such that $\chi = \chi_1$ and ... and $\chi_n \in \mathsf{Var}^+$, where $\chi_1, \dots, \chi_n \in \mathsf{Var}^+$, $\varphi := \forall_{A_1,\dots,A_m,B_1,\dots,B_l}(\psi(A_1, \dots, A_m) \Rightarrow \chi(B_1, \dots, B_l))$, for $n, m, l \geqslant 1$; or, there exists $\psi \in \mathsf{For}^+$, such that $\psi = \psi_1$ and ... and ψ_n, where $\psi_1, \dots, \psi_n \in \mathsf{Var}^+$ and $\varphi := \forall_{A_1,\dots,A_m} \psi(A_1, \dots, A_m)$, for $n, m \geqslant 1$.

Furthermore, when writing down some LPR, we will omit the large quantifier. We have the following examples of expressions that are sLPR:

$$R(A, A) \tag{LPR1}$$

$$R(\neg A, A) \tag{LPR2}$$

$$R(A, \neg A) \tag{LPR3}$$

$$R(A, B) \text{ and } R(B, C) \Rightarrow R(A, C) \tag{LPR4}$$

$$R(A, B) \Rightarrow R(\neg A, \neg B) \tag{LPR5}$$

$$R(A, B) \text{ or } R(A, C) \Rightarrow R(A, B \vee C) \tag{LPR6}$$

$$R(A \wedge B, C \wedge D) \text{ or } R(A \wedge B, C \vee D) \Rightarrow R(A, C) \text{ and } R(B, D) \tag{LPR7}$$

Clearly, any LPR is equivalent to some LPR whose antecedent and consequent have normal form. We conclude that φ has *positive conjunctive normal form* (PDN) iff $\varphi \in \mathsf{Var}^+$ or $\varphi = \psi_1$ and ... and ψ_n ($\varphi = \psi_1$ or ... or ψ_n), where $n \geqslant 1$ for all $i \leqslant n$, $\psi_i = R(F_1, F_2)$ or ... or $R(F_{2m+1}, F_{2m})$ ($\psi_i = R(F_1, F_2)$ and ... and $R(F_{2m+1}, F_{2m})$), for $m \geqslant 1$.

According to classical logic, we know that any expression built from conjunction, disjunction, and some atoms is classically equivalent to an expression of normal form. Thus, the following fact can be derived:

COROLLARY 2.1. 1. *For any φ that is LPR, if $\varphi \in \mathsf{For}^+$, then such ψ exists that has PDN and such χ exists that has PCN such that for any $R \subseteq \mathsf{For} \times \mathsf{For}$, R fulfils φ iff R fulfils ψ, as well as R fulfils φ iff R fulfils χ.*

2. *For any φ that is LPR, if $\varphi \notin \mathsf{For}^+$, then exists such ψ that has PDN and exists such χ that has PCN such that for any $R \subseteq \mathsf{For} \times \mathsf{For}$, R fulfils φ iff R fulfils $\psi \Rightarrow \chi$.*

Using Corollary 2.1, it is easy to see that any LPR can be simplified to one or more sLPR whose antecedents are conjunctions and whose consequents are an element from the set Var^+. We conclude that φ is a *simplified restricted positive relational property* (sLPR) iff there exist $\psi_1, \ldots, \psi_n, \chi \in \mathsf{Var}^+$, where $n \geqslant 1$, such that $\varphi := \psi_1$ and $\ldots$ and $\psi_n \Rightarrow \chi$ or $\varphi \in \mathsf{Var}^+$. The first five expressions of the LPR examples given above, i.e. (LPR1)–(LPR5), are also sLPR.

Based on Corollary 2.1 and classical logic we obtain:

Corollary 2.2. *For any φ being LPR, there exist multiple $\psi_1, \ldots, \psi_n$, for $n \geqslant 1$, being sLPR such that for any $R \subseteq \mathsf{For} \times \mathsf{For}$, R fulfils φ iff R fulfils $\psi_1, \ldots, \psi_n$.*

To illustrate Corollary 2.2, let us consider the examples (LPR6)–(LPR7). Clearly, the condition (LPR6) has the following corresponding LPR conditions:

$$R(A, B) \Rightarrow R(A, B \vee C) \tag{LPR6.1}$$

$$R(A, C) \Rightarrow R(A, B \vee C) \tag{LPR6.2}$$

and condition (LPR7) has the following corresponding sLPR conditions:

$$R(A \wedge B, C \wedge D) \Rightarrow R(A, C) \tag{LPR7.1}$$

$$R(A \wedge B, C \wedge D) \Rightarrow R(B, D) \tag{LPR7.2}$$

$$R(A \wedge B, C \vee D) \Rightarrow R(A, C) \tag{LPR7.3}$$

$$R(A \wedge B, C \vee D) \Rightarrow R(B, D). \tag{LPR7.4}$$

3. Axiomatic systems

At a later stage, we will refer to the notion of derivability. We shall now define the notion of an axiomatic system of Boolean logic with relating implication. For this purpose, we will use the set of classical laws expressed in the Boolean language, i.e., the laws of Boolean logic. Such a set will be denoted as BL.

By *axiomatic system* (*Boolean logic with relating implication*) we shall mean the set of formulas $X \subseteq \mathsf{For}$ satisfying the following conditions:

- $\mathsf{BL} \subseteq X$,
- X contains any formula with the form of an Implication Elimination Rule.

$$(A \to B) \supset (A \supset B) \tag{E $\to$}$$

- X is closed to the rule of material detachment, i.e. the following rule:

$$\frac{\begin{array}{c} A \\ A \supset B \end{array}}{B} \tag{MD}$$

i.e., for any $A, B \in \mathsf{For}$, if $A \supset B, A \in X$, to $B \in X$.[7]

The scheme (E $\to$) allows us to eliminate the relating implication or weaken the relating implication to an abbreviation classically equivalent to the material implication. Note that since axiomatic system contains formulas of the form (E $\to$) and is closed to (MD), it is also closed to Modus Ponens rule:

$$\frac{\begin{array}{c} A \\ A \to B \end{array}}{B} \tag{MP}$$

Let X be an axiomatic system and $Y \cup \{A\} \subseteq \mathsf{For}$. Then:

- A is *thesis based on the system* X iff $A \in X$
- A is *syntactic consequence* (*derivable from*) Y *based on system* X (in symb.: $Y \vdash_X A$) iff exists $n \in \mathbb{N}$ such that $B_1, \ldots, B_n \in Y$ and $\bigwedge_{i=1}^{n} B_i \supset A \in X$.[8]

We can state that $A \in X$ iff $\emptyset \vdash_X A$ iff $X \vdash_X A$.

Let us denote the smallest axiomatic system as $\mathbf{W}_{\to}$. In turn, let us denote the smallest axiomatic system containing all formulas of the form of any of the schemes of formulas $(x_1), \ldots, (x_n)$ as $\mathbf{W}_{\to} \oplus \{(x_1), \ldots, (x_n)\}$.

[7] If any set X will contain all formulas with the form of some scheme F, we shall refer to it as $F \in X$ or $(x) \in X$, if (x) will denote F.

[8] We shall say that a formula scheme, in particular an axiom scheme, is a theorem (corresponding syntactic consequence of (is derived from) some set) in some system when any formula of the form of that scheme is a theorem (corresponding syntactic consequence of (is derived from) a given set) in that system. We shall then use the same notation as for formulas.

4. Algorithm α — moving from relational conditions to axioms and inference rules

In this section, we will define a method for transforming arbitrary sLPR into schemes of formulas that will serve as schemes of axioms of logics defined by given relational conditions. In doing so, we will focus on properties that are sLPR.

Let us start by transforming the expressions that can occur in the antecedent of conditions that are sLPR. Let

$$\alpha_a(\varphi) := \begin{cases} F \to G & \text{if } \varphi = R(F,G) \\ \bigwedge_{i=1}^{n}(\alpha_a(F_{2i-1})) \to \alpha_a(F_{2i})) & \text{if} \varphi = \mathrm{And}_{i=1}^{n} R(F_{2i-1}, F_{2i}) \end{cases}$$

Let us now look at the expression transformations that can occur as a result of sLPR. Let α_c be a function that transforms expressions from the set Var^+ in the following way:

$$\alpha_c(R(F,G)) = (F \to G) \vee (F \wedge \neg G).$$

Using α_1 and α_2 we define a function that transforms any sLPR into schemes of formulas (axiom schemes):

$$\alpha(\varphi) = \begin{cases} \alpha_c(\varphi) & \text{if } \varphi = R(F,G) \\ \alpha_a(\psi) \supset \alpha_c(\chi) & \text{if } \varphi = \psi \Rightarrow \chi \text{ where } \psi = \mathrm{AND}_{i=1}^{n} R(F_{2i-1}, F_{2i}) \\ & \text{and } \chi = R(G,H). \end{cases}$$

In addition, we define a function β that transforms schemas expressing sLPR on an object language basis to sLPR:

$$\beta(F) = \begin{cases} R(G,H) & \text{if } F = (G \to H) \vee (G \wedge \neg H) \\ \mathrm{AND}_{i=1}^{n} R(G_{2-1}, G_{2i}) \Rightarrow R(H,I) & \text{if } F = \bigwedge_{i=1}^{n}(G_{2i-1} \to G_{2i}) \supset \\ & (H \to I) \vee (H \wedge \neg I) \end{cases}$$

Based on Corollary 2.2, for any LPR φ with, α we can transform φ into certain types of formulae. Let us consider some examples, transforming the conditions (LPR1)–(LPR7). Condition (LPR1) can be transformed into the following schema:

$$(A \to A) \vee (A \wedge \neg A) \tag{A1}$$

Referring to classical logic, (A1) can be reduced to the following form:

$$A \to A$$

i.e., (A1) $\equiv (A \to A) \in \mathsf{BL}$.

Conditions (LPR2) and (Rα) (LPR3) can be transformed into the following schemas:

$$(\neg A \to A) \vee (\neg A \wedge \neg A) \tag{A2}$$
$$(A \to \neg A) \vee (A \wedge \neg\neg A) \tag{A3}$$

Once again, referring to classical logic, we can reduce (A2) and (A3) to the following:

$$(\neg A \to A) \vee \neg A$$
$$(A \to \neg A) \vee A,$$

i.e. (A2) $\equiv (\neg A \to A) \vee \neg A \in \mathsf{BL}$ and (A3) $\equiv (A \to \neg A) \vee A \in \mathsf{BL}$.

The condition (LPR4) can be transformed in the following way:

$$(A \to B) \wedge (B \to C) \supset (A \to C) \vee (A \wedge \neg C. \tag{A4}$$

In this case, we can also make a reduction, this time to the following scheme:

$$(A \to B) \wedge (B \to C) \supset (A \to C).$$

However, in addition to classical logic, we need to apply (E$\to$) and (MD), i.e. (A4) $\equiv (A \to B) \wedge (B \to C) \supset (A \to C) \in \mathbf{W}_{\to}$.

Condition (LPR5) is transformed in the following way.

$$(A \to B) \supset (\neg A \to \neg B) \vee (\neg A \wedge \neg\neg B). \tag{A5}$$

In this case, we can only make the following minor reduction:

$$(A \to B) \supset (\neg A \to \neg B) \vee (\neg A \wedge B),$$

resulting in (A5) $\equiv (A \to B) \to (\neg A \to \neg B) \vee (\neg A \wedge B) \in \mathsf{BL}$.

Condition (LPR6) can be transformed indirectly with (LPR6.1) and (LPR6.2):

$$(A \to B) \supset ((A \to B \vee C) \vee (A \wedge \neg(B \vee C)) \tag{A6.1}$$
$$(A \to C) \supset ((A \to B \vee C) \vee (A \wedge \neg(B \vee C)). \tag{A6.2}$$

Once again referring to the classical logic (E$\to$) and (MD) (A6.1) and (A6.2) can be modified in the following way:

$$(A \to B) \supset (A \to B \vee C)$$
$$(A \to C) \supset (A \to B \vee C)$$

i.e., (A6.1) $\equiv (A \to B) \supset (A \to B \vee C) \in \mathbf{W}_{\to}$ and (A6.2) $\equiv (A \to C) \supset (A \to B \vee C) \in \mathbf{W}_{\to}$.

Similarly, in the case of condition (LPR7), we can transform conditions (LPR7.1)–(LPR7.4):

$$(A \wedge B \to C \wedge D) \supset (A \to C) \vee (A \wedge \neg C) \tag{A7.1}$$

$$(A \wedge B \to C \wedge D) \supset (B \to D) \vee (B \wedge \neg D) \tag{A7.2}$$

$$(A \wedge B \to C \vee D) \supset (A \to C) \vee (A \wedge \neg C) \tag{A7.3}$$

$$(A \wedge B \to C \vee D) \supset (B \to D) \vee (B \wedge \neg D). \tag{A7.4}$$

In this case, we are unable to make any reductions similar to those described above.

Let us note that, from the point of view of the α transformation, the (LPR1), (LPR4) and (LPR6) cases are similar in some respects. Namely, in the given cases, we have been able to reduce the axiom schemes obtained with α to schemes in which only relating implications exist, either in the scheme itself or in the antecedent and consequent of the scheme, and in which there is no alternative with a single member of the form of a relating implication. In this way, we obtained schemes describing the well-known laws of various implications: reflexivity, transitivity, and the introduction of alternatives.

However, it should be stressed that the axiom schemes obtained by α do not always provide a complete axiomatization of a given logic. A problem may arise when we start to consider logics determined by several relational conditions from which a new condition (or conditions) can be deduced which, when transformed by α, allows us to obtain a valid schema (or schema) which we will not derive using the axioms obtained with α. For example, note that if a relation satisfies (LPR2)–(LPR4), it also satisfies (LPR1). However, from the set of formulas in the form of schemes (A2)–(A4) we will not derive (A1) on the basis of $\mathbf{W}_{\to}$. We can easily show that (A1) is independent of (A2)–(A4). For this purpose, it suffices to consider the classical matrices for $\neg, \wedge, \vee$, and the following matrix for $\to$:

$\to$	1	0
1	1	0
0	1	0

Under the given interpretation, all elements of the set BL, as well as (E $\to$), are true, and so is (MD). Moreover, (A2)–(A4) will also be true. In turn, schema (A1) is false, e.g., $p \to p$ is false if p is assigned 0.

Similarly, if a relation satisfies the condition (LPR5), it also satisfies the following condition:

$$R(A, B) \Rightarrow R(\neg\neg A, \neg\neg B).$$

The condition given allows the validity of the following scheme to be demonstrated:

$$(A \to B) \supset (\neg\neg A \to \neg\neg B). \tag{A5.1}$$

However, from (A5) we will not be able to derive the schema (A5.1) on the ground of $\mathbf{W}_{\to}$. The scheme (A5.1) is indeed independent of (A5). Let us consider a relating model $\langle v, R\rangle$ such that for any $A \in \mathsf{Var}$, $v(A) = 1$ and for any $A, B \in \mathsf{For}$, $R(A, B)$ iff for any $C \in \mathsf{For}$, $A = C \wedge \neg C$ and $B = C \vee \neg C$. Such a model satisfies the following condition:

$$R(A, B) \Rightarrow R(\neg A, \neg B) \text{ or } (\langle v, R\rangle \not\models A \text{ and } \langle v, R\rangle \models B).$$

In the given model, all elements of the set BL , (E$\to$) and (MD) are true. The scheme (A5) is true as well. In turn, the scheme (A5.1) is false. Namely, since $\sim R(\neg\neg(p \wedge \neg p), \neg\neg(p \vee \neg p))$, then the formula $p \wedge \neg p \to p \vee \neg p) \supset (\neg\neg(p \wedge \neg p) \to \neg\neg(p \vee \neg p))$ is false.

As a result, we introduce the following rules that will allow us to use the obtained axiom schemes to prove the formulas that we will obtain using α from the conditions that determine the logic in question. For any $n, m \in \mathbb{N}$, for any formulas $B_1, \ldots, B_4$, for any $m_0 \leqslant m$ such that $C_{2m_0-1} = B_1$ and $C_{2m_0} = B_2$:

$$\frac{\begin{array}{c}\bigwedge_{i=1}^{n-1}(A_{2i-1} \to A_{2i}) \supset ((B_1 \to B_2) \vee (B_1 \wedge \neg B_2)) \\ \bigwedge_{i=1}^{m}(C_{2i-1} \to C_{2i}) \supset ((B_3 \to B_4) \vee (B_3 \wedge \neg B_4))\end{array}}{\bigwedge_{i=1}^{n-1}(A_{2i-1} \to A_{2i}) \wedge \bigwedge_{i \neq m_0, i=1}^{m}(C_{2i-1} \to C_{2i}) \supset ((B_3 \to B_4) \vee (B_3 \wedge \neg B_4))} \tag{Rα}$$

The applicability of the rule (Rα) must be limited accordingly. The following inference based on (Rα) shows that our rule need not always lead from thesis to thesis:

$$\frac{\begin{array}{c}(p \to q) \supset (\neg p \vee p \to p \wedge \neg p) \vee ((\neg p \vee p) \wedge \neg(p \wedge \neg p)) \\ (\neg p \vee p \to p \wedge \neg p) \supset (q \to p) \vee (q \wedge \neg p)\end{array}}{(p \to q) \supset (q \to p) \vee (q \wedge \neg p)}$$

However, the analysed rule will not allow falsity if we apply it to substitutions of axioms and/or formulas obtained by means of it. Let us demonstrate this property in the next section by showing the consistency of the axiomatic system obtained by the α algorithm.

Due to the indicated restriction, we need to introduce a specific notion of the set closure on (Rα). Let X be an axiomatic system whose only axiom schemes obtained with α are $(x_1), \ldots, (x_n)$. *Closure on* (Rα) *in relation to* X (denoted by $\mathrm{R}\alpha(X)$) is the smallest set $Y \subseteq \mathsf{For}$ such as $(x_1), \ldots, (x_n) \in Y$ and Y are closed on (Rα). Let us note that for any axiomatic system X containing at least one axiom scheme obtained with, α there exists a closure on (Rα) in relation to X. We conclude that X is *axiomatically closed on* (Rα) iff closure on (Rα) in relation to X is contained within X, i.e. $\mathrm{R}\alpha(X) \subseteq X$. Let as denote the smallest axiomatic system containing the axiom schemes $(x_1), \ldots, (x_n)$ and axiomatically closed on (Rα) as $\mathbf{W}_{\rightarrow} \oplus \{(x_1), \ldots, (x_n); (\mathrm{R}\alpha)\}$.

Let us note that the rule (Rα) allows us not only to derive formulas from other formulas, but also formula schemes from other formula schemes. Moreover, we can see that an axiomatically closed set contains only formula schemes, i.e., any formula belonging to this set falls under a scheme that also belongs to this set. In order to prove this property, let us consider the following metarule, which allows us to move from schemas to schemas. For any $n, m \in \mathbb{N}$, for any schemas $G_1, \ldots, G_4$, for any $m_0 \leqslant m$ such that $H_{2m_0-1} = G_1$ and $H_{2m_0} = G_2$:

$$\frac{\begin{array}{c}\bigwedge_{i=1}^{n-1}(F_{2i-1} \rightarrow F_{2i}) \supset ((G_1 \rightarrow G_2) \vee (G_1 \wedge \neg G_2)) \\ \bigwedge_{i=1}^{m}(H_{2i-1} \rightarrow H_{2i}) \supset ((G_3 \rightarrow G_4) \vee (G_3 \wedge \neg G_4))\end{array}}{\bigwedge_{i=1}^{n-1}(F_{2i-1} \rightarrow F_{2i}) \wedge \bigwedge_{i \neq m_0, i=1}^{m}(H_{2i-1} \rightarrow H_{2i}) \supset ((G_3 \rightarrow G_4) \vee (G_3 \wedge \neg G_4))} \quad (\mathrm{R}\alpha^{+})$$

Let X be an axiomatic whose only axiom schemes obtained with α are $(x_1), \ldots, (x_n)$. *Closure on* (Rα^+) *in relation to* X (denoted as $\mathrm{R}\alpha^+(X)$) is the smallest set $Y \subseteq \mathsf{For}$ such that $(x_1), \ldots, (x_n) \in Y$ and Y is closed on (Rα^+). Therefore, $\mathrm{R}\alpha^+(X)$ is a set consisting of schemes itself, i.e., any formula belonging to $\mathrm{R}\alpha^+(X)$ falls under some scheme which is either some scheme of axioms or is a scheme obtained by means of (Rα^+).

FACT 4.1. *Let X be an axiomatic system, then $R\alpha(X) = R\alpha^+(X)$.*

PROOF. "⇒" We will prove that $\mathrm{R}\alpha^+(X)$ is closed on ($\mathrm{R}\alpha$). Let us take any $n, m \in \mathbb{N}$ and any formulas $A_1, \ldots, A_{2n-2}, B_1, \ldots, B_4, C_1, \ldots, C_{2m}$. Suppose there exists $m_0 \leqslant m$ such that $C_{2m_0-1} = B_1$ and $C_{2m_0} = B_2$ and the formulas of the following form belong to $\mathrm{R}\alpha^+(X)$:

$$\textstyle\bigwedge_{i=1}^{n-1}(A_{2i-1} \to A_{2i}) \supset ((B_1 \to B_2) \vee (B_1 \wedge \neg B_2)) \tag{1}$$

$$\textstyle\bigwedge_{i=1}^{m}(C_{2i-1} \to C_{2i}) \supset ((B_3 \to B_4) \vee (B_3 \wedge \neg B_4)) \tag{2}$$

We will show that the following formula belongs to $\mathrm{R}\alpha^+(X)$:

$$(\textstyle\bigwedge_{i=1}^{n-1}(A_{2i-1} \to A_{2i}) \wedge \bigwedge_{i\neq m_0, i=1}^{m}(C_{2i-1} \to C_{2i})) \supset ((B_3 \to B_4) \vee (B_3 \wedge \neg B_4)) \tag{3}$$

By the definition of $\mathrm{R}\alpha^+(X)$ there are some schemes with the following forms:

$$\textstyle\bigwedge_{i=1}^{n-1}(F_{2i-1} \to F_{2i}) \supset ((G_1 \to G_2) \vee (G_1 \wedge \neg G_2)) \tag{1'}$$

$$\textstyle\bigwedge_{i=1}^{m}(H_{2i-1} \to H_{2i}) \supset ((G_3 \to G_4) \vee (G_3 \wedge \neg G_4)) \tag{2'}$$

belonging to $\mathrm{R}\alpha^+(X)$, under which (1) and (2) fall respectively, i.e., (1), (2) can be obtained from (1′) and (2′) by replacing the schemes with the corresponding formulas. Since for $m_0 \leqslant m$ it is such that $C_{2m_0-1} = B_1$ and $C_{2m_0} = B_2$, then $H_{2m_0-1} = G_1$ and $H_{2m_0} = G_2$. Therefore, by using ($\mathrm{R}\alpha^+$) do (1′), (2′), the following schema belongs to $\mathrm{R}\alpha(X)$:

$$(\textstyle\bigwedge_{i=1}^{n-1}(F_{2i-1} \to F_{2i}) \wedge \bigwedge_{i\neq m_0, i=1}^{m}(H_{2i-1} \to H_{2i})) \supset ((G_3 \to G_4) \vee (G_3 \wedge \neg G_4)) \tag{3'}$$

(3) falls under (3′). Therefore, (3) belongs to $\mathrm{R}\alpha^+(X)$. We shall prove that $\mathrm{R}\alpha(X)$ is closed on ($\mathrm{R}\alpha^+$).

Let us take any $n, m \in \mathbb{N}$ and any $F_1, \ldots, F_{2n-2}, G_1, \ldots, G_4, H_1, \ldots, H_{2m}$. Suppose there exists $m_0 \leqslant m$ such that $H_{2m_0-1} = G_1$ and $H_{2m_0} = G_2$ and the following schemes belong to $\mathrm{R}\alpha^+(X)$:

$$\textstyle\bigwedge_{i=1}^{n-1}(F_{2i-1} \to F_{2i}) \supset ((G_1 \to G_2) \vee (G_1 \wedge \neg G_2)) \tag{$1_\star$}$$

$$\textstyle\bigwedge_{i=1}^{m}(H_{2i-1} \to H_{2i}) \supset ((G_3 \to G_4) \vee (G_3 \wedge \neg G_4)) \tag{$2_\star$}$$

Let us also assume that a scheme of the following form does not belong to $\mathrm{R}\alpha(X)$:

$$(\bigwedge_{i=1}^{n-1}(F_{2i-1} \to F_{2i}) \wedge \bigwedge_{i\neq m_0,i=1}^{m}(H_{2i-1} \to H_{2i})) \supset ((G_3 \to G_4) \vee (G_3 \wedge \neg G_4)) \quad (3_\star)$$

Thus, there are formulas such that substituting for the substitutions in (Rα) ($3_\star$) allows one to obtain the following formula:

$$(\bigwedge_{i=1}^{n-1}(A_{2i-1} \to A_{2i}) \wedge \bigwedge_{i\neq m_0,i=1}^{m}(C_{2i-1} \to C_{2i})) \supset ((B_3 \to B_4) \vee (B_3 \wedge \neg B_4)) \quad (3'_\star)$$

Thus, the formula ($3'_\star$) does not belong to R$\alpha(X)$. However, since the formulas ($1_\star$) and ($1_\star$) belong to R$\alpha(X)$, then the following formulas belong to R$\alpha(X)$:

$$\bigwedge_{i=1}^{n-1}(A_{2i-1} \to A_{2i}) \supset ((B_1 \to B_2) \vee (B_1 \wedge \neg B_2)) \quad (1_{\star\star})$$

$$\bigwedge_{i=1}^{m}(C_{2i-1} \to C_{2i}) \supset ((B_3 \to B_4) \vee (B_3 \wedge \neg B_4)), \quad (2_{\star\star})$$

where ($1_{\star\star}$), ($2_{\star\star}$) fall respectively under ($1_\star$), ($2_\star$). Applying (Rα) to ($1_{\star\star}$), ($2_{\star\star}$) we see that the formula ($3'_\star$) also belongs to R$\alpha(X)$. ⊣

Let us consider some examples of rules that are special cases of the rule (Rα). For example, we have the following rules:

$$\frac{(\neg A \to A) \vee (\neg A \wedge \neg A) \qquad (\neg A \to A) \wedge (A \to \neg A) \supset (A \to A) \vee (A \wedge \neg A)}{(A \to \neg A) \supset (A \to A) \vee (A \wedge \neg A)} \quad \text{(R1)}$$

$$\frac{(A \to \neg A) \vee (A \wedge \neg\neg A) \qquad (A \to \neg A) \supset (A \to A) \vee (A \wedge \neg A)}{(A \to A) \vee (A \wedge \neg A)} \quad \text{(R2)}$$

It is easy to see that using (R1), (A2) and (A4) we derive the following scheme:

$$(A \to \neg A) \supset (A \to A) \vee (A \wedge \neg A).$$

Using additionally (R2) and (A3) we can derive (A1). Thus, we can conclude that (A1) $\notin \mathbf{W}_{\to} \oplus \{$(A2), (A3), (A4)$\}$, but (A1) $\in \mathbf{W}_{\to} \oplus \{$(A2), (A3), (A4); (R$\alpha$)$\}$.

Let us further consider the following rule:

$$\frac{(A \to B) \supset (\neg A \to \neg B) \vee (\neg A \wedge \neg\neg B) \qquad (\neg A \to \neg B) \supset (\neg\neg A \to \neg\neg B) \vee (\neg\neg A \wedge \neg\neg\neg B)}{(A \to B) \supset (\neg\neg A \to \neg\neg B) \vee (\neg\neg A \wedge \neg\neg\neg B)} \quad \text{(R3)}$$

Using (R3) and twice (A5) we shall derive the following scheme:

$$(A \to B) \supset (\neg\neg A \to \neg\neg B) \vee (\neg\neg A \wedge \neg\neg\neg B),$$

which using classical logic can be reduced to (A5.1), i.e. (A5.1) $\equiv (A \to B) \supset (\neg\neg A \to \neg\neg B) \vee (\neg\neg A \wedge \neg\neg\neg B) \in \mathsf{BL}$. Therefore, we can conclude that (A5.1) $\notin \mathbf{W}_{\to} \oplus \{$(A5)$\}$ but (A5.1) $\in \mathbf{W}_{\to} \oplus \{$(A5); (R$\alpha$)$\}$.

5. Soundness theorem

In order to show the soundness, we show that the properties of the relation that can obtain by applying the (Rα) rule and the transformation are satisfied by any relation satisfying any fixed sLRP.

LEMMA 5.1. *Let the sLPR $\varphi_1, \ldots, \varphi_n$ and the axiomatic system $X = \mathbf{W}_{\to} \oplus \{\alpha(\varphi_1), \ldots, \alpha(\varphi_n);$ (Rα)$\}$. Then, for any relation R satisfying the conditions $\varphi_1, \ldots, \varphi_n$ and for any formula scheme F, if $F \in R\alpha(X)$, then R fulfils $\beta(F)$.*

PROOF. Let us make all the following assumptions. Let F be an arbitrary formulaic scheme and $F \in \mathrm{R}\alpha(X)$. By virtue of Fact, 4.1 $F \in \mathrm{R}\alpha^+(X)$. Moreover, by virtue of the definition, $\mathrm{R}\alpha^+(X)$, there exists a finite binary tree T such that:

- the root is F
- from each non-leaf node n_1 on which a scheme F_1 is located there are exactly two edges leading to nodes n_2, n_3, on which there are schemes F_2, F_3 such, that on the basis of (Rα^+) if $F_2, F_3 \in \mathrm{R}\alpha^+(X)$, then $F_1 \in \mathrm{R}\alpha^+(X)$
- leaves are any of schemes $\alpha(\varphi_1)$, ..., $\alpha(\varphi_n)$.

We inductively define the notion of the n-th application level (Rα^+) in the tree T:

- for $n = 1$, it is the application (Rα^+) for the tree's leaves T
- for any $n > 1$, n-th level (Rα^+) would be applied (Rα^+) for expressions obtained after $n - 1$ level of (Rα^+) application.

We show inductively that for any $n \in \mathbb{N}$, if the scheme G has been obtained at n-th level of application (Rα^+), then R fulfils $\beta(G)$.

Output step. Firstly, assume that the scheme G has the following form:

$$\bigwedge_{i=1}^{n-1}(F_i \to F_{i+1}) \wedge \bigwedge_{i\neq m_0, i=1}^{m}(H_i \to H_{i+1}) \supset (G_3 \to G_4) \vee (G_3 \wedge \neg G_4)) \quad (\dagger)$$

for any $n, m \in \mathbb{N}$ such that for any $m_0 \leqslant m$ $H_{2m_0-1} = G_1$ and $H_{2m_0} = G_2$ and has been obtained by applying the rule (Rα^+) for the following schemas:

$$\bigwedge_{i=1}^{n-1}(F_{2i-1} \to F_{2i}) \supset ((G_1 \to G_2) \vee (G_1 \wedge \neg G_2)) \quad (\star)$$

$$\bigwedge_{i=1}^{m}(H_{2i-1} \to H_{2i}) \supset ((G_3 \to G_4) \vee (G_3 \wedge \neg G_4)). \quad (\star\star)$$

The scheme ($\dagger$) by applying β allows obtaining the following sLPR:

$$\mathrm{AND}_{i=1}^{n-1}R(F_{2i-1}, F_{2i}) \text{ and } \mathrm{AND}_{i\neq m_0, i=1}^{m}R(H_{2i-1} \to H_{2i}) \Rightarrow R(G_3, G_4). \quad (\dagger')$$

Schemas ($\star$) and ($\star\star$) are two schemas out of $\alpha(\varphi_1), \ldots, \alpha(\varphi_n)$.

If R fulfils all the conditions $\varphi_1, \ldots, \varphi_n$, then it fulfils the following conditions in particular:

$$\mathrm{AND}_{i=1}^{n-1}R(F_{2i-1}, F_{2i}) \Rightarrow R(G_1, G_2) \quad (\star')$$

$$\mathrm{AND}_{i=1}^{m}R(H_{2i-1}, H_{2i}) \Rightarrow R(G_3, G_4) \quad (\star\star')$$

Therefore, if $H_{2m_0-1} = G_1$, $H_{2m_0} = G_2$ and R fulfils conditions ($\star'$), ($\star\star'$), then R fulfils ($\dagger'$).

Induction assumption. Let us assume, that for any $n \in \mathbb{N}$, if the scheme G is obtained on a n-th application level then R fulfils ($\dagger'$).

Induction step. Second, let us consider any scheme obtained on the $n+1$ application level of the (Rα^+) rule. We apply the induction assumption, and reason as in the initial step. $\dashv$

We can now proceed to show that any axiom system obtained using the axiom α is consistent. We carry out the proof in the standard way, by showing that the axiom schemes and inference rules preserve the validity of the models of the logics in question.

Theorem 5.2. *Let $\models$ be any Boolean logic with a relational implication determined by sLPR $\varphi_1, \ldots, \varphi_n$ and let $X = \{\alpha(\varphi_1), \ldots, \alpha(\varphi_n); (\mathrm{R}\alpha)\}$ be an axiomatic system. Then, for any $Y \cup \{A\} \subseteq \mathsf{For}$, if $Y \vdash_X A$, then $Y \models A$.*

Proof. All classical laws of Boolean logic are valid, and the rule (MD) preserves validity in any Boolean logic, with relational implication as well. It is also easy to see that in any Boolean logic with relational implication, (E →) is also valid.

We will show that $\models \{\varphi_1, \ldots, \varphi_n\}$ and $\models \mathrm{R}\alpha(X)$. Let $i \leqslant n$. We consider two cases.

Let us assume that the property φ_i has the following form

$$R(F, G)$$

for F, G schemes. Then, $\alpha(\varphi_i)$ has the following form:

$$(F \to G) \vee (F \wedge \neg G).$$

Let us assume, that F consists of $n \geqslant 1$ variables and G z $m \geqslant 1$ variables. Let us also take any formulas $A_1, \ldots, A_n$ and $B_1, \ldots, B_m$. Let F' and G' be formulas obtained respectively from F and G through substitution of A_i i-nth variable in F and B_i i-nth variable in G.

Let us take any model $\langle v, R\rangle$ of logic $\models$.

If $\langle v, R\rangle \models F' \wedge \neg G'$, then $\langle v, R\rangle \models (F' \to G') \vee (F' \wedge \neg G')$. If $\langle v, R\rangle \not\models F' \wedge \neg G'$, then $\langle v, R\rangle \models F' \to G'$, if R fulfils φ_i and $R(F', G')$.

Let us assume that the property φ_i has the following form

$$\mathrm{And}_{i=1}^{m} R(F_{2i-1}, F_{2i}) \Rightarrow R(G, H),$$

for some schemes $F_1, \ldots, F_{2m}$ ($m \in \mathbb{N}$) and some schemes G, H. Then $\alpha(\varphi_i)$ has the following form:

$$\bigwedge_{i=1}^{m}(F_{2i-1} \to F_{2i}) \supset (G \to H) \vee (G \wedge \neg H).$$

Assume that F_i consists of $n_i \geqslant 1$ variables, G of $o \geqslant 1$ variables, H of $u \geqslant 1$ variables. Let us take any formulas $A_{1_1}, \ldots, A_{n_1}, \ldots, A_{1_m}, \ldots, A_{n_m}, B_1, \ldots, B_o$ and $C_1, \ldots, C_u$. Let F_i', G' and H' be formulas obtained respectively from F_i, G and H through substituting j-nth variable in F_i formula A_{j_i}, j-nth variable in G formula B_j and j-nth variable in H formula C_j.

Let us take any model $\langle v, R\rangle$ of logic $\models$. Assume that $\langle v, R\rangle \models \bigwedge_{i=1}^{n}(F'_{2i-1} \to F'_{2i})$.

If $\langle v, R\rangle \models G' \wedge \neg G'$, then $\langle v, R\rangle \models (F' \to G') \vee (F' \wedge \neg G')$. If $\langle v, R\rangle \not\models F' \wedge \neg G'$, to $\langle v, R\rangle \models F' \to G'$, if R fulfils φ_i and $\mathrm{AND}_{i=1}^{n} R(F'_{2i-1}, F'_{2i})$.

Let us take any scheme, say, $F \in \mathrm{R}\alpha(X)$ and any model $\langle v, R\rangle$ logic $\models$. On the basis of Lemma 5.1 R fulfils $\beta(F)$. Thus, by reasoning similarly to the axiom schemes, $\alpha(\varphi_1), \ldots, \alpha(\varphi_n)$, any formula of the form F is true in $\langle v, R\rangle$. $\dashv$

6. Completeness theorem

For completeness analysis, we introduce the standard notions of non-contradictory and maximally non-contradictory sets. Let X be an axiomatic system and $Y \subseteq \mathsf{For}$. Then:

- Y is X-*consistent* iff $Y \not\vdash_X p \wedge \neg p$,
- Y is X-*inconsistent* iff Y is not X-consistent.

The following fact is the standard one:

FACT 6.1. *Let X be an axiomatic system and $Y \cup \{A\} \subseteq \mathsf{For}$. Then, $Y \cup \{\neg A\}$ is X-consistent iff $Y \not\vdash_X A$.*

The notion of maximal X-consistent set is defined in the standard way. Let X be an axiomatic system and $Y \subseteq \mathsf{For}$. Y is *maximal X-consistent* iff the following conditions are satisfied:

- Y is X-consistent
- for every $Z \subseteq \mathsf{For}$, if $Y \subset Z$, then Z is X-inconsistent.

A set of all maximal X-consistent sets is denoted by Max_X. Maximally non-contradictory sets are obviously theories, i.e., they are closed to their own logical consequences:

FACT 6.2. *Let X be an axiomatic system, $Y \in \mathsf{Max}_X$ and $A \in \mathsf{For}$. Then, $A \in Y$ iff $Y \vdash_X A$.*

By virtue of Fact 6.2 and the axiomatic system definition, we can show that the maximal X-consistent sets are saturated with respect to connectives $\neg$, $\wedge$, $\vee$.

FACT 6.3. *Let X be an axiomatic system and $Y \in \mathsf{Max}_X$. Then, for every $A, B \in \mathsf{For}$:*

1. *$\neg A \in Y$ iff $A \notin Y$*
2. *$A \wedge B \in Y$ iff $A \in Y$ and $B \in Y$*
3. *$A \vee B \in Y$ iff $A \in Y$ or $B \in Y$.*

The next theorem is Lindenbaum's lemma, which we will use in our proof of completeness:

Fact 6.4. *Let X be an axiomatic system and $Y \subseteq \mathsf{For}$. Then, if Y is X-consistent, then there is $Z \subseteq \mathsf{For}$ such that $Y \subseteq Z$ and $Z \in \mathsf{Max}_X$.*

Let us now proceed to define the canonical model. Let X be an axiomatic system and $Y \in \mathsf{Max}(X)$. We define the valuation of the sentential variables as v_Y:

$$v_Y(A) = \begin{cases} 1, & \text{if } A \in Y \\ 0, & \text{if } A \notin Y \end{cases}$$

We define an array of relating relations $(R_n)_{n \in \mathbb{N}}$:

- $R_1(A, B)$ iff $A \to B \in Y$
- $R_{n+1}(A, B)$ iff at least one of the following holds:
  (1) $\langle A, B\rangle \in \bigcup_{i \leqslant n} R_i$
  (2) there is $m \in \mathbb{N}$ such that $C_1, \ldots, C_{2m} \in \mathsf{For}$, $\langle v_Y, \bigcup_{i \leqslant n} R_i\rangle \models \bigwedge_{i=1}^{m}(C_{2i-1} \to C_{2i})$ and $\bigwedge_{i=1}^{m}(C_{2i-1} \to C_{2i}) \supset ((A \to B) \vee (A \wedge \neg B)) \in \mathrm{R}\alpha(X)$
  (3) $(A \to B) \vee (A \wedge \neg B) \in \mathrm{R}\alpha(X)$.

Let us denote the sum of the defined sequence relations in the following way: $\breve{R}_Y := \bigcup_{n \in \mathbb{N}} R_n$.

The canonical model determined with respect to Y (in short: Y*-model*, in symb.: $\mathfrak{M}_Y$) is the model $\langle v_Y, \breve{R}_Y\rangle$. Assume that for any $n \in \mathbb{N}$, n*-the canonical model determined with respect to* Y (in short: n*-model*, in symb.: $\mathfrak{M}_Y^n$) is the model $\langle v_Y, R_n\rangle$. Let us notice that if for any $n \in \mathbb{N}$, $R_n \subseteq R_{n+1}$, to $\mathfrak{M}_Y^n = \langle v_Y, \bigcup_{m \leqslant n} R_m\rangle$.

Notice that $\mathrm{R}\alpha(\mathbf{W}_{\to}) = \emptyset$ and for any $Y \in \mathsf{Max}(\mathbf{W}_{\to})$, $(R_n)_{n \in \mathbb{N}} = R_1$, therefore $\mathfrak{M}_Y = \mathfrak{M}_Y^1$.

Note also that the 1-model has the property that any formula belonging to the maximally non-contradictory set with respect to which a given canonical model is determined is true in the 1-model.

Fact 6.5 (Klonowski, 2021a). *Let X be an axiomatic system and $Y \in \mathsf{Max}(X)$. Then, for any $A \in \mathsf{For}$, $\mathfrak{M}_Y^1 \models A$ iff $A \in Y$.*

We will now show that the same formulas are true in each canonical model as in the 1-model.

LEMMA 6.6. *Let X be an axiomatic system and $Y \in \mathsf{Max}(X)$. Then, for any $n \in \mathbb{N}$, for any $A \in \mathsf{For}$, $\mathfrak{M}^1_Y \models A$ iff $\mathfrak{M}^n_Y \models A$.*

PROOF. Consider an axiomatic system X such that $\mathrm{R}\alpha(X) \neq \emptyset$.In the case, where $X = \mathbf{W}_{\rightarrow}$, the property is satisfied in an obvious way.

We will carry out a double inductive proof. We will start the initial step of the first induction with $n = 2$.

Initial step. Let $n = 2$.

1.1. *Initial step.* Let $A \in \mathsf{For}$ and the complexity of A is equal to 1, therefore $A \in \mathsf{Var}$. From definition $\mathfrak{M}^1_Y$ and $\mathfrak{M}^2_Y$, $\mathfrak{M}^1_Y \models A$ iff $\mathfrak{M}^2_Y \models A$.

1.2. *Induction assumptions.* Let $m \in \mathbb{N}$. Suppose that for any $A \in \mathsf{For}$, if complexity A is not bigger than m, then $\mathfrak{M}^1_Y \models A$ iff $\mathfrak{M}^n_Y \models A$.

1.3. *Induction step.* Let $A \in \mathsf{For}$ and complexity A equals $m + 1$. If $A = \neg B$ or $A = B * C$, where $* \in \{\wedge, \vee\}$, then on the basis of the induction assumption 1.2 $\mathfrak{M}^1_Y \models A$ iff $\mathfrak{M}^2_Y \models A$.

Let us consider the case where $A = B \rightarrow C$.

"$\Rightarrow$" Suppose that $\mathfrak{M}^1_Y \models A \rightarrow B$. Then, by virtue of the definition of the formula true in the model, ($\mathfrak{M}^1_Y \not\models B$ or $\mathfrak{M}^1_Y \models C$) and $R_1(B, C)$. By virtue of inductive assumption 1.2, 1.2 $\mathfrak{M}^2_Y \not\models B$ or $\mathfrak{M}^2_Y \models C$.By virtue of the relational relations definition, $(R_n)_{n \in \mathbb{N}}$ we obtain that $R_2(B, C)$. Thus, by virtue of the true model formula definition, we obtain that $\mathfrak{M}^2_Y \models B \rightarrow C$.

"$\Leftarrow$" Suppose that $\mathfrak{M}^2_Y \models B \rightarrow C$. Then, by virtue of the true model formula definition ($\mathfrak{M}^2_Y \not\models B$ or $\mathfrak{M}^2_Y \models C$) and $R_2(B, C)$. On the basis of the inductive assumption, 1.2 $\mathfrak{M}^1_Y \not\models B$ or $\mathfrak{M}^1_Y \models C$. Let's not explicitly assume that $\sim R_1(B, C)$. If $R_2(B, C)$ and $\sim R_1(B, C)$, then by virtue of the relational relations sequence definition $(R_n)_{n \in \mathbb{N}}$ we have the following possibilities:

(a) there are $k, l \in \mathbb{N}$ such that $D_1, \ldots, D_{2k} \in \mathsf{For}$,
$\mathfrak{M}^1_Y \models \bigwedge_{i=1}^{k}(D_{2i-1} \rightarrow D_{2i})$ and
$\bigwedge_{i=1}^{k}(D_{2i-1} \rightarrow D_{2i}) \supset ((B \rightarrow C) \vee (B \wedge \neg C)) \in \mathrm{R}\alpha(X)$

(b) $((B \rightarrow C) \vee (B \wedge \neg C)) \in \mathrm{R}\alpha(X)$.

In case (a) since $\bigwedge_{i=1}^{k}(D_{2i-1} \rightarrow D_{2i}) \supset ((B \rightarrow C) \vee (B \wedge \neg C)) \in \mathrm{R}\alpha(X)$, then $\bigwedge_{i=1}^{k}(D_{2i-1} \rightarrow D_{2i}) \supset ((B \rightarrow C) \vee (B \wedge \neg C)) \in Y$. Therefore, on the basis of Fact 6.5, $\mathfrak{M}^1_Y \models \bigwedge_{i=1}^{k}(D_{2i-1} \rightarrow D_{2i}) \supset ((B \rightarrow C) \vee (B \wedge \neg C))$. Therefore, if $\mathfrak{M}^1_Y \models \bigwedge_{i=1}^{k}(D_{2i-1} \rightarrow D_{2i})$, then $\mathfrak{M}^1_Y \models (B \rightarrow C) \vee (B \wedge \neg C)$. But, on the virtue of the inductive assumption 1.2 $\mathfrak{M}^1_Y \not\models B \wedge \neg C$. Therefore $\mathfrak{M}^1_Y \models B \rightarrow C$. On the basis of the true formula definition in model $R_1(B, C)$. In case (b) we reason similarly as in case (a).

Inductive assumption. Let $n \in \mathbb{N}$. Suppose that for any arbitrary $A \in \mathsf{For}$, $\mathfrak{M}^1_Y \models A$ iff $\mathfrak{M}^n_Y \models A$. *Inductive step.* Let us consider model $\mathfrak{M}^{n+1}_Y$. We can inductively show, that for any $A \in \mathsf{For}$, $\mathfrak{M}^n_Y \models A$ iff $\mathfrak{M}^{n+1}_Y \models A$.

2.1. *Initial step.* Reasoning as in 1.1.

2.2. *Inductive assumptions.* Let $m \in \mathbb{N}$. Consider that for any $A \in \mathsf{For}$, if complexity A is not bigger than m, then $\mathfrak{M}^n_Y \models A$ iff $\mathfrak{M}^{n+1}_Y \models A$.

2.3. *Inductive step.* Let $A \in \mathsf{For}$ and complexity A be equal to $m+1$. As in 1.3, if $A = \neg B$ or $A = B * C$, where $* \in \{\wedge, \vee\}$, then by the virtue of inductive assumption 2.2 $\mathfrak{M}^n_Y \models A$ iff $\mathfrak{M}^{n+1}_Y \models A$.

Let us consider a case where $A = B \to C$.

"$\Rightarrow$" As in 1.3 „$\Rightarrow$". By virtue of the relational relations sequence definition, $(R_n)_{n\in\mathbb{N}}$, we obtain that if $R_n(B, C)$, then $R_{n+1}(B, C)$.

"$\Leftarrow$" Reasoning as in 1.3 "$\Leftarrow$". Let us consider that $\mathfrak{M}^{n+1}_Y \models B \to C$. Then by the virtue of true formula definition in model, ($\mathfrak{M}^{n+1}_Y \not\models B$ or $\mathfrak{M}^{n+1}_Y \models C$) and $R_{n+1}(B, C)$. By virtue of inductive assumption $\mathfrak{M}^n_Y \not\models B$ or $\mathfrak{M}^n_Y \models C$. Let us (inexplicitly) assume that $\sim R_n(B, C)$. If $R_{n+1}(B, C)$ and $\sim R_n(B, C)$, then by virtue of the relational relations sequence definition $(R_n)_{n\in\mathbb{N}}$ we have possibilities (a) and (b) from 1.3.

In case (a) $\bigwedge_{i=1}^{k}(D_{2i-1} \to D_{2i}) \supset ((B \to C) \vee (B \wedge \neg C)) \in \mathrm{R}\alpha(X)$, to $\bigwedge_{i=1}^{k}(D_{2i-1} \to D_{2i}) \supset ((B \to C) \vee (B \wedge \neg C)) \in Y$. Therefore, by virtue of Fact 6.5, $\mathfrak{M}^1_Y \models \bigwedge_{i=1}^{k}(D_{2i-1} \to D_{2i}) \supset ((B \to C) \vee (B \wedge \neg C))$. Thus, if $\mathfrak{M}^1_Y \models \bigwedge_{i=1}^{k}(D_{2i-1} \to D_{2i})$, then $\mathfrak{M}^1_Y \models (B \to C) \vee (B \wedge \neg C)$. Thus, by virtue of the main inductive assumption, $\mathfrak{M}^n_Y \models (B \to C) \vee (B \wedge \neg C)$. But by virtue of the inductive assumption 2.2 $\mathfrak{M}^n_Y \not\models B \wedge \neg C$. Therefore, $\mathfrak{M}^n_Y \models B \to C$. This, by virtue of the true formula definition in model, $R_n(B, C)$.

In case (b), we reason similarly to in case (a).

Since for any $A \in \mathsf{For}$, $\mathfrak{M}^n_Y \models A$ iff $\mathfrak{M}^{n+1}_Y \models A$, then on basis of inductive assumption for any $A \in \mathsf{For}$, $\mathfrak{M}^1_Y \models A$ iff $\mathfrak{M}^{n+1}_Y \models A$. ⊣

By virtue of Lemma 6.6 we can prove the following:

LEMMA 6.7. *Let X be an axiomatic system and $Y \in \mathsf{Max}(X)$. Then for any $A \in \mathsf{For}$, $\mathfrak{M}^1_Y \models A$ iff $\mathfrak{M}_Y \models A$.*

PROOF. *Initial step.* Let $A \in \mathsf{For}$ and complexity A equals 1, therefore $A \in \mathsf{Var}$. By the virtue of $\mathfrak{M}^1_Y$ and $\mathfrak{M}_Y$, $\mathfrak{M}^1_Y \models A$ iff $\mathfrak{M}_Y \models A$.

Inductive assumptions. Let $m \in \mathbb{N}$. Consider that for any $A \in \mathsf{For}$, if complexity A is no larger than m, then $\mathfrak{M}^1_Y \models A$ iff $\mathfrak{M}_Y \models A$.

Inductive step. Let $A \in \mathsf{For}$ and complexity A equals $m+1$. If $A = \neg B$ or $A = B * C$, where $* \in \{\wedge, \vee\}$, then by virtue of inductive assumption $\mathfrak{M}^1_Y \models A$ iff $\mathfrak{M}_Y \models A$.

Let us consider a case, where $A = B \to C$.

"$\Rightarrow$" Assume that $\mathfrak{M}^1_Y \models A \to B$. Then by virtue of true formula definition in model, ($\mathfrak{M}^1_Y \not\models B$ or $\mathfrak{M}^1_Y \models C$) and $R_1(B, C)$. By virtue of inductive assumption $\mathfrak{M}_Y \not\models B$ or $\mathfrak{M}_Y \models C$. By virtue of relational relation sequence definition, $(R_n)_{n\in\mathbb{N}}$ we obtain that $\breve{R}_Y(B, C)$. Therefore, by virtue of model true formula definition, $\mathfrak{M}_Y \models B \to C$.

"$\Leftarrow$" Assuming that $\mathfrak{M}_Y \models B \to C$. Then, by virtue of model true formula definition, ($\mathfrak{M}_Y \not\models B$ or $\mathfrak{M}_Y \models C$) and $\breve{R}_Y(B, C)$. By virtue of inductive assumption, $\mathfrak{M}^1_Y \not\models B$ or $\mathfrak{M}^1_Y \models C$. By virtue of model true formula definition $\mathfrak{M}^1_Y \models B \supset C$. If $\breve{R}_Y(B, C)$, then exists $n \in \mathbb{N}$ such that $R_n(B, C)$. By virtue of Lemma 6.6, $\mathfrak{M}^n_Y \models B \supset C$. Therefore, by virtue of model true formula definition, $\mathfrak{M}^n_Y \models B \to C$. Once again, referring to Lemma 6.6 $\mathfrak{M}^1_Y \models B \to C$. ⊣

We now show that the canonical model determined with respect to the maximally non-contradictory set determined with respect to a given axiomatic system satisfies the relevant conditions that determine the given logic, for which we have determined by the α algorithm the given axiomatic system.

LEMMA 6.8. *Let $X = \mathbf{W}_{\to} \oplus \{\alpha(\varphi_1), \ldots, \alpha(\varphi_n); (\mathrm{R}\alpha)\}$ be the axiomatic system with given sLPR $\varphi_1, \ldots, \varphi_n$. Then, for any $Y \in \mathsf{Max}(X)$, $\breve{R}_Y$ fulfils conditions $\varphi_1, \ldots, \varphi_n$.*

PROOF. Let φ be any property out of $\varphi_1, \ldots, \varphi_n$. Assume that φ has the following form $R(F, G)$ for some schemas F, G. Then $\alpha(\varphi)$ has the following form:

$$(F \to G) \vee (F \wedge \neg G).$$

Let us assume that F consists of $n \geqslant 1$ variables and G out of $m \geqslant 1$ variables. Taking any formulas $A_1, \ldots, A_n$ and $B_1, \ldots, B_m$. Let F' and G' be formulas obtained respectively from F and G by replacing i-nth variable in F by the formula A_i and i-nth variable in G by B_i.

If $X = \{\alpha(\varphi); (\mathrm{R}\alpha)\}$, then $(F' \to G') \vee (F' \wedge \neg G') \in \mathrm{R}\alpha(X)$. Thus, by the virtue of sequence definition $(R_n)_{n\in\mathbb{N}}$ for any $j > 1$, $R_j(F', G')$. Thus $\breve{R}_Y(F', G')$.

Assume that φ has the following form:

$$\mathrm{And}_{i=1}^{n} R(F_{2i-1}, F_i) \Rightarrow R(G, H)$$

for some schemas $F_1, \ldots, F_{2n}$ ($n \in \mathbb{N}$) and some schemas G and H. Then $\alpha(\varphi)$ has the following form:

$$\bigwedge_{i=1}^{n}(F_{2i-1} \rightarrow F_{2i}) \supset ((G \rightarrow H) \vee (G \wedge \neg H)). \tag{1_*}$$

Let us assume that F_i consists of $m_i \geqslant 1$ variables, G out of $j \geqslant 1$ variables and H out of $k \geqslant 1$ variables. Taking any formulas $A_{1_1}, \ldots, A_{m_1}, \ldots, A_{1_n}, \ldots, A_{m_n}, B_1, \ldots, B_j$ and $C_1, \ldots, C_k$. Let F_i', G' and H' be formulas obtained respectively from F_i, G and H through substitution of the l-nth variable in F_i with the formula A_l, l-nth variable in G with the formula B_l and l-nth variable in H with the formula C_l.

Assume that $\langle F_1', F_2'\rangle, \ldots, \langle F_{2n-1}', F_{2n}'\rangle \in \breve{R}_Y$. We can consider the following possibilities:

(a) for any $i \leqslant n$, $\langle F_{2i-1}', F_{2i}'\rangle \in R_1$
(b) for any $i \leqslant n$, $\langle F_{2i-1}', F_{2i}'\rangle \notin R_1$
(c) exists $i \leqslant n$ such that $\langle F_{2i-1}', F_{2i}'\rangle \in R_1$ and exists $i \leqslant n$ such that $\langle F_{2i-1}', F_{2i}'\rangle \notin R_1$.

Let us consider case (a). Let $i \leqslant n$. If $\langle F_{2i-1}', F_{2i}'\rangle \in R_1$, then by virtue of relational relations sequence definition $F_{2i-1}' \rightarrow F_{2i}' \in Y$. Therefore, on the basis of Fact 6.3, $\bigwedge_{i=1}^{n}(F_{2i-1}' \rightarrow F_{2i}') \in Y$. Thus, if ($1_*$) belongs to Y, then by virtue of Fact 6.3 $(G' \rightarrow H') \vee (G' \wedge \neg H') \in Y$. Assume that $G' \wedge \neg H' \notin Y$. Then by the virtue of Fact 6.3, $G' \rightarrow H' \in Y$.Then by virtue of the relational relations sequence definition, $R_1(G', H')$. Thus $\breve{R}_Y(G', H')$. Assume that $G' \wedge \neg H' \in Y$. Formula (1_*) belongs to $\mathrm{R}\alpha(X)$. If $\bigwedge_{i=1}^{n}(F_{2i-1}' \rightarrow F_{2i}') \in Y$, then by virtue of Fact 6.5 $\mathfrak{M}_Y^1 \models \bigwedge_{i=1}^{n}(F_{2i-1}' \rightarrow F_{2i}')$. Thus, on the basis of Lemma 6.6, $\mathfrak{M}_Y^2 \models \bigwedge_{i=1}^{n}(F_{2i-1}' \rightarrow F_{2i}')$. Thus, by virtue of the relational relations sequence definition, $R_Y^2(G', H')$. Thus $\breve{R}_Y(G', H')$.

Let us consider case (b). For any $i \leqslant n$, exists the smallest l_i such that $\langle F_{2i-1}', F_{2i}'\rangle \in R_{l_i}$ and following possibilities occur:

(b1) there is $s_i \in \mathbb{N}$ such that $C_{1_i}, \ldots, C_{2s_i} \in \mathsf{For}$, $\langle v_Y, \bigcup_{t<l_i} R_t\rangle \models \bigwedge_{t=1_i}^{s_i}(C_{2t-1} \rightarrow C_{2t})$ and $\bigwedge_{t=1_i}^{s_i}(C_{2t-1} \rightarrow C_{2t}) \supset ((F_{2i-1}' \rightarrow F_{2i}') \vee (F_{2i-1}' \wedge \neg F_{2i}')) \in \mathrm{R}\alpha(X)$
(b2) $(F_{2i-1}' \rightarrow F_{2i}') \vee (F_{2i-1}' \wedge \neg F_{2i}') \in \mathrm{R}\alpha(X)$.

Assume that for $i_1, \dots, i_u \leqslant n$, case (b1) occurs, while for $i_{u+1}, \dots, i_{u+w} \leqslant n$ occurs (b2). Thus:

$$\bigwedge_{t=1_{i_1}}^{s_{i_1}} (C_{2t-1} \to C_{2t}) \supset ((F'_{2i_1-1} \to F'_{2i_1}) \vee (F'_{2i_1-1} \wedge \neg F'_{2i_1})) \tag{i_1}$$

$$\vdots$$

$$\bigwedge_{t=1_{i_u}}^{s_{i_u}} (C_{2t-1} \to C_{2t}) \supset ((F'_{2i_{u+w}-1} \to F'_{2i_{u+w}}) \vee (F'_{2i_{u+w}-1} \wedge \neg F'_{2i_{u+w}})) \tag{i_u}$$

belong to $\mathrm{R}\alpha(X)$. Let us consider the following formulas:

$$\bigwedge_{t=1_{i_1}}^{s_{i_1}} (C_{2t-1} \to C_{2t}) \wedge \bigwedge_{t=1, t\neq i_1}^{n} (F'_{2t-1} \to F'_{2t}) \supset ((G' \to H') \vee (G' \wedge \neg H')) \tag{i_1^+}$$

$$\vdots$$

$$\bigwedge_{t=1_{i_1}}^{s_{i_1}} (C_{2t-1} \to C_{2t}) \wedge \dots \wedge \bigwedge_{t=1_{i_{u-1}}}^{s_{i_{u-1}}} (C_{2t-1} \to C_{2t}) \wedge \bigwedge_{t=1,\, t\neq i_1,\dots,i_{u-1}}^{n} (F'_{2t-1} \to F'_{2t}) \supset ((G' \to H') \vee (G' \wedge \neg H')) \tag{i_{u-1}^+}$$

Using ($\mathrm{R}\alpha$) do (1_*) and (i_1) we obtain (i_1^+), thus (i_1^+) belongs to $\mathrm{R}\alpha(X)$. Using ($\mathrm{R}\alpha$) for (i_2) and (i_1^+) we obtain (i_2^+), therefore (i_2^+) belongs to $\mathrm{R}\alpha(X)$. Let Consider that (i_{u-z}^+) belongs to $\mathrm{R}\alpha(X)$. Using ($\mathrm{R}\alpha$) for ($i_{u-(z-1)}$) and (i_{u-z}^+) we obtain ($i_{u-(z-1)}^+$). Therefore, ($i_{u-(z-1)}^+$) belongs to $\mathrm{R}\alpha(X)$. Thus, (i_{u-1}^+) belongs to $\mathrm{R}\alpha(X)$. Using ($\mathrm{R}\alpha$) for (i_{u-1}^+) and (i_u) and we obtain the following formula:

$$\bigwedge\nolimits_{t=1_{i_1}}^{s_{i_1}} (C_{2t-1} \to C_{2t}) \wedge \dots \wedge \bigwedge\nolimits_{t=1_{i_u}}^{s_{i_u}} (C_{2t-1} \to C_{2t}) \wedge \bigwedge\nolimits_{t=i_{u+1}}^{i_{u+w}} (F'_{2t-1} \to F'_{2t}) \supset ((G' \to H') \vee (G' \wedge \neg H')). \tag{$*$}$$

Therefore ($*$) belongs to $\mathrm{R}\alpha(X)$. The $\mathrm{R}\alpha(X)$ consists also of the following formulae:

$$(F'_{2i_{u+1}-1} \to F'_{2i_{u+1}}) \vee (F'_{2i_{u+1}-1} \wedge \neg F'_{2i_{u+1}}) \tag{i_{u+1}}$$

$$\vdots$$

$$(F'_{2i_{u+w}-1} \to F'_{2i_{u+w}}) \vee (F'_{2i_{u+w}-1} \wedge \neg F'_{2i_{u+w}}). \tag{i_{u+w}}$$

Let us consider the following formulas:

$$\bigwedge_{t=1_{i_1}}^{s_{i_1}} (C_{2t-1} \to C_{2t}) \wedge \cdots \wedge \bigwedge_{t=1_{i_u}}^{s_{i_u}} (C_{2t-1} \to C_{2t}) \wedge \bigwedge_{t \neq i_{u+1}, t=i_{u+2}}^{i_{u+w}} (F'_{2t-1} \to F'_{2t}) \supset ((G' \to H') \vee (G' \wedge \neg H')) \quad (i^+_{u+1})$$

$\vdots$

$$\bigwedge_{t=1_{i_1}}^{s_{i_1}} (C_{2t-1} \to C_{2t}) \wedge \cdots \wedge \bigwedge_{t=1_{i_u}}^{s_{i_u}} (C_{2t-1} \to C_{2t}) \wedge (F'_{2i_{u+w}} \to F'_{2i_{u+w}}) \supset ((G' \to H') \vee (G' \wedge \neg H')) \quad (i^+_{u+(w-1)})$$

Using (Rα) for ($*$) and (i_{u+1}) we obtain (i^+_{u+1}), therefore (i^+_{u+1}) belongs to R$\alpha(X)$.

Using (Rα) for (i_{u+2}) and (i^+_{u+1}) we obtain (i^+_{u+2}), therefore (i^+_{u+2}) belongs to R$\alpha(X)$. Let Assume that ($i^+_{(u+(w-z)}$) belongs to R$\alpha(X)$. Using (Rα) for ($i_{u+(w-(z-1))}$) and ($i^+_{u+(w-z)}$) we obtain ($i^+_{u+(w-(z-1))}$). Therefore, ($i^+_{u+(w-(z-1))}$) belongs to R$\alpha(X)$. Thus, ($i^+_{u+(w-1)}$) belongs to R$\alpha(X)$. Applying (Rα) for (i_{u+w}) and ($i^+_{u+(w-1)}$) we obtain the following formula:

$$\bigwedge_{t=1_{i_1}}^{s_{i_1}} (C_{2t-1} \to C_{2t}) \wedge \ldots \wedge \bigwedge_{t=1_{i_u}}^{s_{i_u}} (C_{2t-1} \to C_{2t}) \supset ((G' \to H') \vee (G' \wedge \neg H')) \quad (**)$$

Therefore ($**$) belongs to R$\alpha(X)$. Let us take the largest $i_1, \ldots, i_u$ and denote it as z. By the virtue of Lemma 6.6 $\langle v_Y R_{l_z} \rangle \models \bigwedge_{t=1_{i_1}}^{s_{i_1}} (C_{2t-1} \to C_{2t}) \wedge \ldots \wedge \bigwedge_{t=1_{i_u}}^{s_{i_u}} (C_{2t-1} \to C_{2t})$. Moreover, the formula ($**$) belongs to R$\alpha(X)$. Thus, by virtue of the relational relation sequences definition, $\breve{R}_Y(G', H')$.

The cases when for all $i \leqslant n$ (b1) occurs and the case when for all $i \leqslant n$ occurs (b2) are a simple modifications of the possibility under our consideration. In the first case, we will obtain the formula ($*$) without conjunction. Then, $\bigwedge_{t=i_{u+w}}^{i_u} (F'_{2t-1} \to F'_{2t})$, in the second case, the formula ($**$) without conjunction $\bigwedge_{t=1_{i_1}}^{s_{i_1}} (C_{2t-1} \to C_{2t}) \wedge \ldots \wedge \bigwedge_{t=1_{i_u}}^{s_{i_u}} (C_{2t-1} \to C_{2t})$, simply $(G' \to H') \vee (G' \wedge \neg H')$.

Let us consider case (c). Let $i_1, \ldots, i_{u+w}$ be all indices for which are satisfied $\langle F'_{2i-1}, F'_{2i} \rangle \notin R_1$ and $i_{u+w+1}, \ldots, i_{u+w+z}$ for which fulfilled are $\langle F'_{2i-1}, F'_{2i} \rangle \in R_1$. As in (b) assume that for $i_1, \ldots, i_u \leqslant n$, case (b1) occurs, in turn for $i_{u+1}, \ldots, i_{u+w} \leqslant n$ case (b2) occurs. We reason analogously to case (b) and obtain that the following formula belongs to $\mathrm{R}\alpha(X)$:

$$\bigwedge_{t=1_{i_1}}^{s_{i_1}} (C_{2t-1} \to C_{2t}) \wedge \cdots \wedge \bigwedge_{t=1_{i_u}}^{s_{i_u}} (C_{2t-1} \to C_{2t}) \wedge \bigwedge_{t=i_{u+w+1}}^{i_{u+w+z}} (F'_{2t-1} \to F'_{2t}) \supset ((G' \to H') \vee (G' \wedge \neg H')). \quad (***)$$

Let us take the largest $i_1, \ldots, i_u$ and denote it as z. By the virtue of Lemma 6.6, $\langle v_Y R_{l_z} \rangle \models \bigwedge_{t=1_{i_1}}^{s_{i_1}} (C_{2t-1} \to C_{2t}) \wedge \cdots \wedge \bigwedge_{t=1_{i_u}}^{s_{i_u}} (C_{2t-1} \to C_{2t})$.

Moreover, if $\bigwedge_{s=u_{u+w+1}}^{i_{u+w+z}} (F'_{2i-1} \to F'_{2i}) \in Y$, by virtue of Fact 6.5, $\mathfrak{M}^1_Y \models \bigwedge_{s=u_{u+w+1}}^{i_{u+w+z}} (F'_{2i-1} \to F'_{2i})$. Therefore, by virtue of Lemma 6.6, $\mathfrak{M}^{l_z}_Y \models \bigwedge_{s=u_{u+w+1}}^{i_{u+w+z}} (F'_{2i-1} \to F'_{2i})$. Thus $\mathfrak{M}^{l_z}_Y \models \bigwedge_{t=1_{i_1}}^{s_{i_1}} (C_{2t-1} \to C_{2t}) \wedge \cdots \wedge \bigwedge_{t=1_{i_u}}^{s_{i_u}} (C_{2t-1} \to C_{2t}) \wedge \bigwedge_{s=u_{u+w+1}}^{i_{u+w+z}} (F'_{2i-1} \to F'_{2i})$. Formula ($***$) belongs to $\mathrm{R}\alpha(X)$. Therefore, by relating relations sequence definition $\breve{R}_Y(G', H')$. ⊣

Using Lemmas 6.7 and 6.3 we can easily prove the completeness of any axiomatic system obtained by the α algorithm.

THEOREM 6.9. *Let $\models$ be any Boolean logic with relating implication set by sLPR $\varphi_1, \ldots, \varphi_n$ and let $X = \{\alpha(\varphi_1), \ldots, \alpha(\varphi_n); (\mathrm{R}\alpha)\}$ be the axiomatic system. Then for any $Y \cup \{A\} \subseteq \mathsf{For}$, if $Y \models A$, then $Y \vdash_X A$.*

PROOF. Assume that $Y \nvdash_X A$. On the basis of Fact 6.1 $Y \cup \{\neg A\}$ is X-non-contradictory. Therefore by the virtue of Fact 6.4 there exists $Z \in \mathsf{Max}(X)$ such that $Y \cup \{\neg A\} \subseteq Z$. By the virtue of Fact 6.5 $\mathfrak{M}^1_Y \models Y \cup \{\neg A\}$. By the virtue of Lemma 6.7, $\mathfrak{M}_Y \models Y \cup \{\neg A\}$. And once again, referring to Lemma 6.8, $\breve{R}_Y$ fulfils $\varphi_1, \ldots, \varphi_n$. Therefore $Y \not\models A$. ⊣

7. Summary

In our paper we studied a generalised method for obtaining an adequate axiomatic system for any relating logic expressed in the language with

Boolean connectives and relating implication, determined by the limited positive relational properties.

The method of obtaining axiomatic systems for logics of a given type is called an algorithm, since the analysis allows for any logic of a given type (determined by the limited positive relational properties) to define the axiomatic system adequate for it. We call this algorithm α.

The proof of completeness of axiomatic systems obtained by applying the α algorithm that we presented is a modification of Henkin-style completeness proofs for propositional logics. The proof in the paper does not use expressivity of the relating relation, since in many cases of limited relational properties the relation R is not expressible.

Our proposal is a partial answer to the problem formulated during the 1st Workshop on Relating Logic[9], called problem α: axiomatization of logical systems defined by relating semantics (by given classes of valuations/relations).

We call the answer *partial* because it concerns only the relating implication and the properties that are LPR. To have more, we also need to consider other relating connectives and non-limted relational properties, including negative properties. We take up this challenge in [Jarmużek and Klonowski, submitted-b].

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