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Measures in Euclidean Point-Free Geometry (an exploratory paper)

Abstract. We face with the question of a suitable measure theory in Euclidean point-free geometry and we sketch out some possible solutions. The proposed measures, which are positive and invariant with respect to movements, are based on the notion of infinitesimal masses, i.e. masses whose associated supports form a sequence of finer and finer partitions.

Keywords: measures; point-free structures; region-based theories of space

1. Introduction

In [3] we explored the possibility of defining measures in a mereological space (this question is also addressed in [1, 2, 11]). Namely, we proposed "rough" approximate measures in Dempster-Shafer style based on a mass whose support is a partition of the universe. Unfortunately, these measures are inadequate since they do not ensure the measurability of a sufficient number of regions. Mostly, they do not allow endless approximation processes, a fundamental tool in measure theory. Furthermore, the invariance of a measure with respect to movements (a crucial aspect of geometric measurements) was not taken into account.

In this paper we propose a more powerful tool based on the notion of infinitesimal masses, i.e. masses whose associated supports form a sequence of finer and finer partitions. We test this tool on point-free foundation of Euclidean geometry as formulated in [6, 7]. Moreover, since in [7] the notion of a movement is given, we can define a class of measures invariant with respect to movements. To simplify our presentation, we

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always refer to the Euclidean plane but what is said easily extends to every dimension.

The paper is organized as follows: In Section 2 we recall the main definitions and results given in [3] where approximate measures in a mereological space are discussed. In Section 3 a new notion of approximate measure based on infinitesimal masses is proposed. Some remarks on these measures are drawn in Section 4. In order to test these measures in the framework of point-free geometry, in Section 5 we define a prototypical point-free geometrical structure extending the one proposed in [6] according to [7] (see [3, 8], too). In Section 6 we give a long list of notions for point-based geometry. Via these notions it is not difficult to define the usual point-based Euclidean geometry in Hilbert style. In Section 7 we introduce approximate measures in the prototypical structure via square-tessellation. In Section 8 we discuss a possible axiomatization of point-free geometry inspired by the prototypical structure. Namely, thanks to Axiom 1, we can define a class of point-free structures in which it is possible introduce all the definitions early proposed for the prototypical structure. In Section 9 we define measures which are based on square-tessellations. Section 10 is dedicated to some open questions.

2. Approximate measures in a mereological space in Dempster-Shafer style

Given an ordered set, we denote by $\mathcal{O}$ the overlapping relation and by $\mid$ its negation. We write $x\mathcal{O}y$ if there is $z \neq 0$ such that $z \leq x$ and $z \leq y$, and we denote by $\mathcal{O}(x)$ the set of regions overlapping x.

We call an *extended mereological space* a bounded separative ordered set $\boldsymbol{R} = (Re, \leq, 0, 1)$ where $\leq$ is an order relation such that

1. 0 is the minimum,
2. 1 is the maximum,
3. $\forall z\,(z\mathcal{O}x \Rightarrow z\mathcal{O}y) \Rightarrow x \leq y$.

We can rewrite this last condition as follows,

$$x \nleq y \Rightarrow \exists z(z \neq 0 \,\wedge\, z \leq x \,\wedge\, z \mid y).$$

We call *regions* the elements of *Re*, *empty region* the minimum 0, *universe* the maximum 1. In what follows we use the set-theoretical language in denoting the notions of this structure. So, we call *inclusion* the relation $\leq$ and if $x \leq y$ we say that x *is contained in* y. We say that two regions x, y are *disjoint* if $x|y$ and sometimes we use the words *union* and *intersection* instead of join and meet. We say that a set Π of nonempty regions is a *partition* of a region x provided the regions of Π are pairwise disjoint, their join exists and it is equal to x.

The definition of an approximate measure proposed in [3] is based on the notion of a mass.

DEFINITION 2.1. Let $\boldsymbol{R}$ be a distributive extended mereological space. A map $m\colon Re \to [0, +\infty)$ with $m(0) = 0$ is called *mass*. We call *focal* a region $z \in Re$ such that $m(z) \neq 0$ and we denote by $Foc(m)$ the class of all focal regions of m. Let $\tau\colon Re \to Re$ be a one-to-one map, we say that m is *invariant with respect to* τ whenever $m(x) = m(\tau(x))$, for every $x \in Re$. We say that m is invariant with respect to a group of transformation if it is invariant with respect to all the maps of the group.

DEFINITION 2.2. The *lower* and *upper approximation measures* associated with a mass m are the functions $int\colon Re \to [0, \infty]$ and $ext\colon Re \to [0, \infty]$ such that, for every region x,

$$int(x) = \sum_{z \leq x} m(z) \quad ; \quad ext(x) = \sum_{z \mathcal{O} x} m(z).$$

Notice that by adding the condition $\sum_{z \in Re} m(z) = 1$ to these definitions we obtain the definitions of *belief* and *plausibility* functions in Dempster-Shafer theory.

DEFINITION 2.3. Denote by $I([0, \infty])$ the class of all closed intervals contained in $[0, \infty]$. Then the *interval approximate measure* is the function $\mu\colon Re \to I([0, \infty])$ defined by setting $\mu(x) = [int(x), ext(x)]$ for every region x. We say that x is *measurable* whenever $int(x) = ext(x)$ and in this case we put $\mu(x) = int(x) = ext(x)$.

If a mass m is invariant with respect to a group, then the associated measure is invariant, too. In [3] we proved the following proposition whenever $\boldsymbol{R}$ is a Boolean algebra.

PROPOSITION 2.1. *The functions int and ext are superadditive and subadditive, respectively.*

Namely, for every pair of disjoint regions x, y,

$$int(x) + int(y) \leq int(x \vee y) \quad ; \quad ext(x) + ext(y) \geq ext(x \vee y).$$

Equivalently,

$$\mu(x \vee y) \subseteq \mu(x) \oplus \mu(y).$$[1]

Moreover, there are also extended mereological spaces admitting a mass such that ext is not subadditive.

Proof. To prove that *int* is superadditive we observe that, taking into account that there is a focal element z such that $z \leq x$ and $z \leq y$,

$$int(x) + int(y) = \sum_{z \leq x} m(z) + \sum_{z \leq y} m(z) \leq \sum_{z \leq x \vee y} m(z) = int(x \vee y).$$

To prove that *ext* is subadditive, we claim that

$$\mathcal{O}(x \vee y) = \mathcal{O}(x) \cup (\mathcal{O}(y) \setminus \mathcal{O}(x)).$$

Indeed, trivially $\mathcal{O}(x) \cup (\mathcal{O}(y) \setminus \mathcal{O}(x)) \subseteq \mathcal{O}(x \vee y)$. Assume that $z \in \mathcal{O}(x \vee y)$, hence there exists a $z \neq 0$ such that $z \leq z$ and $z \leq x \vee y$. The inequality $z \leq x \vee y$ entails $z = z \wedge (x \vee y) = (z \wedge x) \vee (z \wedge y)$ and therefore, since $z \neq 0$, either $(z \wedge x) \neq 0$ or $(z \wedge y) \neq 0$. Then either $z \in \mathcal{O}(x)$ or $z \in \mathcal{O}(y)$.

Thanks to the just-proved equality, we have

$$\begin{aligned} ext(x \vee y) &= \sum_{z \in \mathcal{O}(x)} m(z) + \sum_{z \in (\mathcal{O}(y) \setminus \mathcal{O}(x))} m(z), \\ &\leq ext(x) + \sum_{z \in \mathcal{O}(y)} m(z) = ext(x) + ext(y). \end{aligned}$$

To prove the last part of the proposition, consider the "diamond lattice", a non-distributive lattice with five elements which we denote by S, A, B, C, V where S is the maximum, V is the minimum and $A \wedge B = V$, $A \wedge C = V$, $C \wedge B = V$, $A \vee B = S$, $A \vee C = S$, $C \vee B = S$. Since $C \wedge (A \vee B) = C \wedge S = C$ while $(C \wedge A) \vee (C \wedge B) = \emptyset \vee \emptyset = \emptyset$, the lattice is not distributive. It is immediately verifiable that that it is an extended mereological space.

Define a mass m by $m(A) = m(B) = m(C) = 1$, $m(S) = \lambda$ and $m(V) = 0$. Then

$$\begin{gathered} ext(A \vee B) = ext(S) = \lambda + 1 + 1 + 1 = \lambda + 3, \\ ext(A) = ext(B) = 1 + \lambda \end{gathered}$$

[1] Given two sets X and Y of real numbers we put $X \oplus Y = \{x+y : x \in X, y \in Y\}$, in the case of two intervals $[a, b]$ and $[a', b']$, we have $[a, b] + [a', b'] = [a + a', b + b']$.

and therefore $ext(A \vee B) \leq ext(A) + ext(B) \leftrightarrow \lambda + 3 \leq 2\lambda + 2 \leftrightarrow \lambda \geq 1$. That holds true for any disjoint pairs of elements, and it shows that, depending on the choice of the mass in the diamond we obtain subadditivity by putting $\lambda \leq 1$, superadditivity by putting $\lambda \geq 1$ and additivity by putting $\lambda = 1$. [2] □

Proposition 2.2. *The function μ is additive on the set of measurable regions, i.e. if x and y are two disjoint measurable regions, then $x \vee y$ is measurable and*

$$\mu(x \vee y) = \mu(x) + \mu(y).$$

Proof. The proof is an immediate consequence of Proposition 2.1. □

Definition 2.4. We call *normal* a mass m such that $Foc(m)$ is a partition of the universe.

The following proposition suggests the importance of normal masses.

Proposition 2.3. *Let m be a mass in a complete mereological space $(Re, \leq)$. Then m is normal if and only if every focal region x is measurable and $\mu(x) = m(x)$.*

Proof. Let $f \in Foc(m)$ be measurable, then all the focal elements overlapping f are contained in f. On the other hand, if $\mu(f) = m(f)$, then there is no focal region f' strictly contained in f. This means that f is the unique focal element overlapping f. Hence the elements of $Foc(m)$ are pairwise disjoint. Consider now the join fo of the elements of $Foc(m)$, and assume that $fo \neq 1$, i.e. $1 \not\leq fo$. By the definition of a mereological space there exists $z \neq 0$ such that $z | fo$. By the density hypothesis, there exists a focal element z' contained in z and therefore disjoint from fo, a contradiction.

Vice versa, if $Foc(m)$ is a partition of the universe, then obviously every $x \in Foc(m)$ is measurable and its measure is $m(x)$. □

Unfortunately, even in the case of normal masses the situation remains unsatisfactory. That occurs, for example, whenever the focal regions are too big with respect to the regions we have to measure. In this paper we will explore an alternative path where a fundamental role is played by a succession of normal masses with increasingly finer corresponding partitions.

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[2] Notice that while $C \in \mathcal{O}(A \vee B)$, $C \notin \mathcal{O}(A)$ and $C \notin \mathcal{O}(B)$.

3. Measures based on infinitesimal masses

Recall that the *refinement relation* $\preceq$ in the class of partitions of a nonempty set is defined by putting $\Pi' \preceq \Pi$ whenever every element of Π is the join of elements of Π'. Trivially, $\preceq$ is an order relation.

Definition 3.1. Given two normal masses m, m' we say that m' is a *refinement* of m, and we write $m' \preceq m$, if $Foc(m') \preceq Foc(m)$ and, for every region $z \in Foc(m)$,

$$\sum_{x \leq z,\, x \in Foc(m')} m'(x) = m(z).$$

Informally, $m' \preceq m$ means that m' is obtained by a fragmentation of m preserving the additive property.

Definition 3.2. Let $m = (m_n)_{n \in \mathbb{N}}$ be a sequence of normal masses and put $Foc(m) = \cup_{n \in \mathbb{N}} Foc(m_n)$. We say that m is an *infinitesimal mass* provided that

(i) $m_n \preceq m_{n-1}$, for every $n \in \mathbb{N}$;
(ii) there is no partition Π such that $\Pi \preceq Foc(m_n)$, for every $n \in \mathbb{N}$;
(iii) $Foc(m)$ is *dense in Re*, i.e. every region contains an element of $Foc(m)$.

Given an infinitesimal mass m and $n \in \mathbb{N}$, we denote by int_n, ext_n, μ_n the functions associated with m_n as in Definitions 2.2 and 2.3. Obviously, given a region x, $(int_n(x))_{n \in \mathbb{N}}$ is order-preserving, $(ext_n(x))_{n \in \mathbb{N}}$ is order-reversing and, for every $n \in \mathbb{N}$, $int_n(x) \leq ext_n(x)$. As a consequence, $(\mu_n(x))_{n \in \mathbb{N}}$ is a nested sequence of intervals.

Definition 3.3. The *lower approximation* and *upper approximation measures* associated with the infinitesimal mass m are the functions *int* and *ext* defined by setting

$$int(x) = \lim_{n \to \infty} int_n(x) \qquad \text{and} \qquad ext(x) = \lim_{n \to \infty} ext_n(x),$$

respectively. The *approximate measure* μ is defined by

$$\mu(x) = [int(x), ext(x)] = \cap_{n \in \mathbb{N}} \mu_n(x).$$

We say that x is *measurable* whenever $int(x) = ext(x)$ and in this case we put $\mu(x) = int(x) = ext(x)$.

It is evident that if x is a finite join of elements of $Foc(m)$, then x is measurable and there exists $k \in \mathbb{N}$ such that $\mu(x) = \mu_k(x)$.

PROPOSITION 3.1. *Assume that Re is distributive and let int and ext be the functions associated with an infinitesimal mass, then* $int(x) \leq ext(x)$. *Moreover, int is superadditive and ext is subadditive, i.e., for every disjoint regions x and y, we have*

$$int(x \vee y) \geq int(x) + int(y) \qquad \text{and} \qquad ext(x \vee y) \leq ext(x) + ext(y).$$

Equivalently,

$$\mu(x \vee y) \subseteq \mu(x) \oplus \mu(y).$$

PROOF. Given two disjoint regions x, y, since, $int_n(x \vee y) \geq int_n(x) + int_n(y)$, for every $n \in \mathbb{N}$, we have

$$\begin{aligned} int(x \vee y) &= \lim_{n\to\infty} int_n(x + y) \geq \lim_{n\to\infty}(int_n(x) + int_n(y)) \\ &= int(x) + int(y). \end{aligned}$$

Hence *int* is superadditive. Since, for every $n \in \mathbb{N}$, we have $ext_n(x \vee y) \leq ext_n(x) + ext_n(y)$, we get

$$\begin{aligned} ext(x \vee y) &= \lim_{n\to\infty} ext_n(x \vee y) \leq \lim_{n\to\infty}(ext_n(x) + ext_n(y)) \\ &= \lim_{n\to\infty} ext_n(x) + \lim_{n\to\infty} ext_n(y) = ext(x) + ext(y). \end{aligned}$$

Hence *ext* is subadditive. □

PROPOSITION 3.2. *The function μ is additive over the set of measurable regions, i.e. if x and y are two disjoint measurable regions and $x \vee y$ exists, then $x \vee y$ is a measurable region and*

$$\mu(x \vee y) = \mu(x) + \mu(y).$$

PROOF. The proof is an immediate consequence of Proposition 3.1. □

4. A few remarks on measuring by infinitesimal masses

The notion of a positive measure is the subject of a large and interesting literature.[3] The following proposition shows that the measures associated with an infinitesimal mass are positive.

PROPOSITION 4.1. *Let m be an infinitesimal mass. Then $int(x) > 0$ for every region x and therefore, the measure μ associated with m is*

[3] A measure μ is *positive* if $\mu(x) \neq 0$ for every measurable region $x \neq 0$.

positive. Moreover, $int(x) < int(y)$ *whenever* $x < y$*, whence* $\mu(x) < \mu(y)$ *whenever* $x < y$*.*[4]

Proof. Since $Foc(m)$ is dense in Re, for every nonempty region x there is $n \in \mathbb{N}$ and $f \in Foc(m_n)$ such that $f \leq x$. Therefore $\mu(x) = int(x) \geq m_n(f) > 0$.

Assume now that $x < y$. Since $(Re, \leq, 0, 1)$ is a mereological space and $y \nleq x$, there exists a nonempty region z, disjoint from x, such that $z \leq y$. Given $f \in Foc(m)$ such that $f \leq z$, we have $int(x) + m_n(f) \leq int(y)$, therefore $int(x) < int(y)$. □

We conclude the section by saying that, in mereology much remains to be done concerning measures based on infinitesimal masses. However, we do not want to continue to work on these options, but in the remaining part of the paper we prefer to explore this approach in the framework of point-free geometry.

5. Testing the proposed measures in a prototypical point-free geometrical structure

To test the idea of an approximate measure via an infinitesimal mass, we refer to the approach to point-free geometry proposed in [6] and [7] which, in turn, is related to Hilbert's point-based axiomatization of geometry. In this approach a basic role is played by a prototypical point-free structure in which the regions are represented by the regular closed subsets of $\mathbb{R}^2$ (as usual in the literature on point-free geometry).

Definition 5.1. Given a topological space, denote by c and i the closure and the interior operators, respectively. Then the *regularization operator* r is defined by setting $r(X) = c(i(X))$ for every subset X. A *regular closed subset* is a fixed point of r.

One proves that $r(r(X)) = r(X)$, hence $r(X)$ is a regular closed subset, for every subset X of the topological space.

Theorem 5.1. *Let* RC *be the class of the regular closed subsets of* $\mathbb{R}^2$*. Then the structure* $(RC, \subseteq)$ *is an atomless complete Boolean algebra*

[4] This means that, in accordance with Euclid's statement (Euclid, Elements, Book I, Common Notion), "The whole is greater than the part."

(equivalently a complete mereological space). The Boolean operations are defined by putting, for every $C \subseteq RC$ and $X \in RC$,

$$\wedge C = r(\cap C), \quad \vee C = r(\cup C), \quad \neg X = r(-X).$$

We call *regions* the elements of RC, *empty region* the empty set, *universe* the maximum $\mathbb{R}^2$ of RC. In what follows we prefer the ring style notation, writing $X+Y, X \cdot Y$, X^{-1} instead of $X \vee Y$, $X \wedge Y$, $\neg X$. It is evident that all the figures usually considered in Euclidean geometry (triangles, rectangles, circles ...) can be represented as elements of RC, while points and lines do not.

Since we consider two different Boolean algebras of sets, namely $(RC, \subseteq)$ and $(\mathcal{P}(\mathbb{R}^2), \subseteq)$, we will use the prefix RC to name the operations in $(RC, \subseteq)$. We call *RC-intersection*, *RC-union* and *RC-complement* the meet, the join and the complement in $(RC, \subseteq)$, respectively. In accordance, we say that two regions X and Y are *RC-disjoint* whenever $X \cdot Y = \emptyset$, i.e. the interior of $X \cap Y$ is empty. It is evident the meaning of expressions like *RC-partition*, *RC-overlap* and so on.

A further step is to enrich the Boolean structure $(RC, \subseteq)$ with the class *Conv* of the convex elements of RC according to [6, 7, 8].[5] The regions of this class are also called *ovals* according to Whitehead's terminology.

Finally, to define the basic notion of congruence, we introduce a group acting on the set RC of regions corresponding to the group $(Is, \circ, {}^{-1}, i)$ of the plane isometries [see 7]. Indeed, for every map f in Is, we define the map $f^*\colon \mathcal{P}(\mathbb{R}^2) \to \mathcal{P}(\mathbb{R}^2)$ by setting $f^*(X) = \{f(P) : P \in X\}$, for every $X \in RC$.[6] Consider now $(Is^*, \circ, {}^{-1}, i)$, where $Is^* = \{f^* : f \in Is\}$, $\circ$ is the composition and i the identity map. One proves that this structure is a group which is isomorphic to the group of isometries.

DEFINITION 5.2. We say that $\boldsymbol{Is^*} = (Is^*, \circ, {}^{-1}, i)$ is the *group of the movements of a region.*

We are now ready to give our main definition.

DEFINITION 5.3. We call a *prototypical point-free plane*, in brief *PPF*, the structure $(RC, \subseteq, Conv, Is^*)$.

[5] A different choice has been made by A. Śniatycki in [14] where a half-plane is a primitive notion.

[6] We prefer the notation $f^*(X)$ instead of the common notation $f(X)$ which is misleading since f is different from f^*.

This point-free structure is relevant since through it we can define a point-based model of the Euclidean plane. We will show this by referring to the primitive notions of Hilbert's system of axioms for the Euclidean plane.[7]

6. The definitional power of the prototypical point-free plane: lines, points, incidence

In [6, 7] it is shown that in the prototypical point-free plane it is possible to define a model of Hilbert's system of axioms for plane geometry. To do this we have to give a list of definitions corresponding to the ones assumed as primitives in Hilbert's book. We start with "straight lines", "points" and the "incidence" relation.

Definition 6.1. An *RC-half-plane* is an element H in *Conv* such that H^{-1} is in *Conv*. Given a region X, the *RC-boundary* of X is the set $\{X, X^{-1}\}$. We call an *RC-straight line*, in short *RC-line*, the boundary $l = \{H, H^{-1}\}$ of an RC-half-plane H and we say that H and H^{-1} are the *RC-sides* of l.

We denote by Hp the class of RC-half-planes and by RCL the class of RC-lines.

Definition 6.2. Two RC-lines l_1, l_2 are called *parallel* whenever they have two disjoint RC-sides, otherwise they are called *incident* or *intersecting*.

The following items are obviously equivalent:

- l_1 is parallel with l_2;
- l_1 and l_2 have two sides whose RC-union is the universe;
- l_1 and l_2 have two comparable sides.

We can consider the RC-partition of the universe generated by the sides of n lines. In the following proposition we consider the cases $n = 1$ and $n = 2$.

Proposition 6.1. *Let* $l_H = \{H, \neg H\}$ *and* $l_K = \{K, \neg K\}$ *be RC-lines. Then the associated partition of the universe is the class* Π *of*

[7] However, one can proceed in a similar way for all the main proposals of the Euclidean geometry axiomatization.

the nonempty regions in $\{H \cdot K, H \cdot (\neg K), (\neg H \cdot K), (\neg H) \cdot (\neg K)\}$. *More precisely,*

– *If* $l_H = l_K$, *then* Π *has two elements.*
– *If* $l_H \neq l_K$ *and they are parallel, then* Π *has three elements.*
– *If* $l_H \neq l_K$ *and they are incident, then* Π *has four elements.*

PROOF. The first part of the proposition is straightforward.

If $l_H = l_K = l$, then Π consists of the sides of l. If $l_H \neq l_K$ and l_H, l_K are parallel (for example, if H and K are RC-disjoint), then Π consists of the half-planes H and K and the region $(-H) \cdot (-K)$. Finally, if $l_H \neq l_K$ and l_H, l_K are incident, then the partition has four elements. □

Defining a point is our next step and for this we give the definition of a representative of a point.

DEFINITION 6.3. Let us call a *representative of a point*, briefly an *r-point*, a pair $P = \{r, t\}$ of intersecting RC-lines. We denote by RP the set of representatives of a point.

Then an r-point is defined as the intersection of two lines, in a sense. The notion of a point is obtained by introducing a suitable equivalence relation in RP.

DEFINITION 6.4. Given an r-point P, we say that P is an *interior* point of a convex region X whenever X overlaps all four angles determined by P; P is an *interior* point of a region Y whenever it is an interior point of an oval contained in Y.

DEFINITION 6.5. We say that two r-points P and Q are *separable* if there are two disjoint ovals X_1 and X_2 such that P is interior to X_1 and Q is interior to X_2. We say that P and Q are $r-$ *equivalent* and we write $P \equiv_r Q$ it they are not separable. An *RC-point* is an equivalence class $[P]$ modulo $\equiv_r$.[8] We denote by Po the set of RC-points.

DEFINITION 6.6. An RC-point is an *interior* point of a region X provided that all its representative elements are internal to X.[9]

[8] As already mentioned, Whitehead and most point-free geometry scholars proposed a different definition of a point based on the notion of an abstraction class. In this paper we prefer the definition of a point as proposed by A. Śniatycki in [14] and adopted in [6, 7].

[9] As a trivial consequence of the definition of $\equiv_r$, if an r-point $P = \{l_1, l_2\}$ is an interior point of a region X, then all the r-points equivalent to P are interior in X.

Definition 6.7. A point $[P]$ *lies* on the boundary of a region X if it is not an interior point of X and it is not an interior point of X^{-1}. In particular, $[P]$ *lies* on a line $l = \{H, H^{-1}\}$ if it is neither an interior point of H nor an interior point of H^{-1}.

Let's continue with the definition of the remaining primitive notions of Hilbert geometry.

Definition 6.8. The *betweeness relation* is the ternary relation *bet* defined in *Po* by setting $bet([A], [B], [C])$ under the condition that when $[A]$ and $[C]$ are interior points of a convex subset X, they imply that $[B]$ is an interior point of X.

Finally, we must define congruence between angles and segments.

Definition 6.9. Given two intersecting *RC*-lines, we call an *angle* every region of the associated partition.

Definition 6.10. Let $[P]$, $[Q]$ be two points, then the set $\{[P], [Q]\}$ is called an *RC-segment with endpoints* $[P]$ and $[Q]$. We denote by PQ this segment and we say that it is *internally contained in an oval* X if both $[P]$ and $[Q]$ are interior points of X. We say that PQ is *internally contained* in a region X if it is internally contained in an oval contained in X.

Definition 6.11. The *congruence relation* $\equiv_c$ between regions (in particular between angles) is defined by setting $X \equiv_c X'$ whenever there is $\tau^* \in Is^*$ such that $\tau^*(X) = X'$.

It then only remains to define congruence between segments.

Definition 6.12. The congruence relation $\equiv_c$ between segments is defined by setting $PQ \equiv_c P'Q'$ if there is $\tau^* \in Is^*$ such that, for every oval X, PQ is internally contained in X if and only if $P'Q'$ is internally contained in $\tau^*(X)$.[10]

Definition 6.13. We call *prototypical point-based Euclidean plane associated with PPF* the structure $PPB = (Po, RCL, \epsilon, \equiv_c, bet)$.

[10] Apparently, we could define congruence between segments by putting $PQ \equiv_c P'Q'$ whenever there is $\tau \in Is$ such that $P'Q' = \tau(P)\tau(Q)$. However, we are interested in definitions using only primitive notions of the prototypical structure. Hence we may use Is^* but not Is.

PPF can be viewed as an interpretation of the language $\mathcal{L}_H$ used by Hilbert in his theory. In [7] the authors prove the following theorem.

THEOREM 6.1. *The prototypical point-based structure PPB associated with PPF is isomorphic to the analytic model of Euclidean plane.*[11] *Consequently, this structure is a model of Hilbert's system of axioms.*

PROOF. Consider the analytic model of the Euclidean plane. Denote by L the set of lines and define the map $i\colon \mathbb{R}^2 \cup L \to Po \cup RCL$ that associates

(i) every straight line l with the RC-line $\{H, H^{-1}\}$, where H is a closed half-plane;
(ii) every point $P = (\underline{x}, \underline{y})$ in $\mathbb{R}^2$ with the RC-point $[\{l_1, l_2\}]$ defined by the lines $l_1 := \{(x, y) \in \mathbb{R}^2 : x = \underline{x}\}$ and $l_2 := \{(x, y) \in \mathbb{R}^2 : y = \underline{y}\}$.

Obviously, i is an isomorphism. Since isomorphic models satisfy the same propositions, *PPB* is a model of Hilbert's system of axioms for plane geometry. □

This theorem shows that there exists a point-free geometrical structure able to define the usual point-based Euclidean geometry.

7. Measures in the prototypical point-free plane via masses associated with a square-tessellation

In this section we test on the prototypical structure our definition of a measure based on the notion of an infinitesimal mass. We will consider infinitesimal masses associated with partitions whose elements are squares defined as follow.

DEFINITION 7.1. We say that two incident lines are *perpendicular* if the associated four angles are pairwise congruent. In this case we say that these angles are *right angles*.

DEFINITION 7.2. Given two parallel RC-lines, let H and K be the related disjoint sides. Then we say that $(-H)\dot{(}-K)$ is a *stripe*. Two stripes are called *perpendicular* provided that the lines defining a stripe are perpendicular to the lines defining the other stripe.

[11] We refer to the model of Euclidean geometry in which a point is an element of $\mathbb{R}^2$, a straight line is the set of points satisfying a linear equation in two variables and so on.

Definition 7.3. A *square* is a region obtained by intersecting two perpendicular congruent stripes. A *square-tessellation* is a partition of the plane whose elements are pairwise congruent squares.

Definition 7.4. Given a square-tessellation Π, a sequence $(\Pi_n)_{n\in\mathbb{N}}$ of square-tessellations is defined by induction as follows

– $\Pi_1 = \Pi$;
– Π_n is the square-tessellation obtained by dividing each square of Π_{n-1} into four congruent squares.

In turn, $(\Pi_n)_{n\in\mathbb{N}}$ is associated with the sequence $m = (m_n)_{n\in\mathbb{N}}$ defined by putting

– $m_1(X) = 1$ if X is a square in Π_1 and $m_1(X) = 0$ otherwise;
– $m_n(X) = m_{n-1}(X')/4$ if X' is a square in Π_{n-1} containing X and $m_n(X) = 0$ otherwise.

The proof of the following theorem is straightforward.

Theorem 7.1. *The just defined sequence $m = (m_n)_{n\in\mathbb{N}}$ is an infinitesimal mass invariant with respect to the group Is^* of transformation. As a consequence, the associated measure μ is invariant with respect to this group.*

Remark 7.1. It is evident that μ coincides with the restriction of Jordan's measure μ^* to the class *RC* of regular closed subsets. Nevertheless, this does not imply that the behaviour of μ coincides with μ^*. This happens since these measures are defined in different Boolean algebras in spite of the fact that the domain of these algebras is the same. Indeed, two disjoint regular closed subsets may be not disjoint in the power set of $\mathbb{R}^2$.

8. The axiomatic point of view

We will utilize the prototypical point-free plane to individuate a system of axioms for point-free geometry. The purpose is to capture structures in which it is possible to repeat what has been done starting from *PPF*, in particular the definition of a measure based on the infinitesimal masses.

To do this, we denote by $\mathcal{L}_{PF}$ a language containing names for the primitive notions of *PPF*. Then an interpretation of this language is a structure of type $(Re, \leq, Ov, Is^*)$ in which *Re* is a nonempty set, $\leq$ is a binary relation on *Re*, *Ov* a subset of *Re* and Is^* a set of functions from

Re into *Re*. Obviously, according to [6, 7], we fix our axioms among the properties of *PPF* and we start from the following basic axiom.

Axiom 1. $(Re, \leq)$ is an atomless complete Boolean algebra, *Ov* is an algebraic closure system in this algebra and *Is* is a subgroup of the group of automorphisms of $(Re, \leq, Ov)$.[12]

Definition 8.1. We call *basic point-free structure* any model of Axiom 1. We call *regions* the elements of *Re*, *ovals* the elements in *Ov*, *movements* the functions in Is^* and *inclusion* the relation $\leq$.

In a basic point-free structure we can define a very large list of notions, geometrical in nature. In particular, we can define all the primitive notions of Hilbert theory. Here is a list of definitions coinciding with the early proposed list given in the prototypical point-free structure. Unlike the case of prototypical structure, we use lowercase letters x, y, z, ... to denote variables for regions.

- A *half-plane* is an oval h whose complement h^{-1} is still an oval.
- The pair $\{x, x^{-1}\}$ is the *boundary* of a region x.
- A *straight line*, in short a *line*, is the boundary $l = \{h, h^{-1}\}$ of a half-plane h. Moreover, h and h^{-1} are named the *sides* of l.
- Two lines l_1, l_2 are called *parallel* if a side of l_1 is disjoint from a side of l_2, otherwise they are called *incident*.
- Let $l_1 \neq l_2$ be two lines. If they are incident, we call *angle* a region obtained intersecting a side of l_1 with a side of l_2. If they are parallel, we call *strip* the complement of the join of their disjoint sides.
- An *r-point* is a set $P = \{l_1, l_2\}$ of two incident lines.
- An r-point $P = \{l_1, l_2\}$ is an *interior point of an oval* x if x overlaps all the four angles defined by the sides of l_1 and l_2. P is *an interior point of a region* x if it is an interior point of an oval contained in x. We denote by $\in_{in}$ the so defined relation between points and regions.
- In the class of r-points denote by $\equiv_p$ the equivalence relation defined by setting $P \equiv_p Q$ provided that, for every region x, $P \in_{in} x$ if and only if $Q \in_{in} x$.
- We call *point* an equivalence class $[P]$ of an r-point P modulo $\equiv_p$.

[12] A subset $\mathcal{C}$ of a Boolean algebra B is a *closure system* whenever it is closed under finite and infinite joins and $1 \in \mathcal{C}$. A closure system is *algebraic* whenever it is closed under infinite joins of totally ordered subsets. Finally, a bijective function $\tau : Re \to Re$ is an *automorphism* of $(Re, \leq, Ov)$ whenever it is an automorphism of the Boolean algebra $(Re, \leq)$ such that $x \in Ov$ if and only if $\tau(x) \in Ov$.

- Let $[P]$ be a point. We call $[P]$ an *interior point* of a region x, in brief $[P] \in_{in} x$, if there is $P' \in [P]$ such that $P' \in_{in} x$;
- Given two points $[P], [Q]$, $[P] \neq [Q]$, we say that $\{[P], [Q]\}$ is a *segment* and we denote by PQ this segment. We say that PQ is *internally contained* in a region x, in short $PQ \in_{in}$, if both $[P]$ and $[Q]$ are interior points of x.
- An r-point $P = \{l_1, l_2\}$ *lies on the boundary of a region* x if it is neither an interior point of x nor an interior point of x^{-1}. In particular P *lies on a line* $l = \{h, h^{-1}\}$ if it is not an interior point of h and it is not an interior point of h^{-1}. We denote by $\in_{on}$ the relation so defined.
- A point $[P]$ *lies* on the boundary of a region x, in brief $[P] \in_{on} x$, if there is $P' \in [P]$ such that $P' \in_{on} x$.[13]
- The *betweeness relation* is the ternary relation *bet* defined in *Po* by setting $bet([A], [B], [C])$ under the condition that when $[A]$ and $[C]$ are interior points of a convex subset X, they imply that $[B]$ is an interior point of X.
- We say that the regions x and x' are *congruent*, and we write $x \equiv x'$, if there is a movement τ such that $\tau(x) = x'$.
- We say that the segment PQ is *congruent* to the segment $P'Q'$ if there is a movement τ such that, for every oval x, $PQ \in_{in} x$ if and only if $P'Q' \in_{in} \tau(x)$.

Once we have seen that all primitive notions of Hilbert's approach are definable, we can use a simple trick to extend Axiom 1 into a satisfactory theory. This trick, which is named the *"cannibalization of a theory"* in [5], was adopted by Tarski in his famous paper [15]. In Tarski's paper one cannibalizes Pieri's system of axioms for Euclidean geometry. In our case the trick applies to Hilbert's system of axioms for plane geometry.

Indeed, denote by $\mathcal{L}$ a language containing the language $\mathcal{L}_{PF}$ for the basic point-free structures and Hilbert language $\mathcal{L}_H$. Then in $\mathcal{L}$ we can define a theory HT with formulas expressing Axiom 1, formulas expressing all the definitions of the primitive notions of Hilbert theory and mainly all the axioms of Hilbert theory. HT is consistent since the

[13] Observe that the relation $\equiv_p$ is compatible with both $\in_{in}$ and $\in_{on}$, i.e.

$P \in_{in} x$ and $Q \equiv_p P \Rightarrow Q \in_{in} x$,

$P \in_{on} x$ and $Q \equiv_p P \Rightarrow Q \in_{on} x$.

As a consequence, a point $[P]$ is an interior point of a region x (lies on the boundary of a region x) if and only if all the r-points in $[P]$ are interior points of x (lie on the boundary of x).

union of *PPF* and *PPB* is an interpretation of $\mathcal{L}$ which is a model of HT. Moreover, all models of HT admit a reduct which satisfies Hilbert's system of axioms. This means that the axiom system HT is an point-free approach to Euclidean geometry which is adequate from a logical point of view.

As a matter of fact, as highlighted by Tarski in [15], the cannibalization method is far from being satisfactory, since it seems preferable to individuate a system of axioms expressing intrinsic properties of regions.[14] In this paper we will consider only two axioms enabling us to extend measures defined in *PPF* to the whole class of the basic point-free structures.

9. Measuring by square-tessellation

We now extend some definitions, early given in *PPF*, to the basic point-free structure.

- Two incident lines are *perpendicular* whenever the associated angles are pairwise congruent. In this case these angles are named *right angles*.
- Two stripes are *perpendicular* whenever the lines defining the first strip are perpendicular to the lines defining the second strip.
- A *square* is the meet of two perpendicular and congruent stripes.
- A *square-tessellation* is a partition of the universe whose elements are pairwise congruent squares.

At this point two existence axioms are necessary.

Axiom 2. For every square s, there is a square-tessellation of the universe whose elements are congruent to s.

Axiom 3. For every square s, there exists a partition of s formed by four pairwise congruent squares.

In what follows we fix a square u, called *unit square*, and a square-tessellation Π, whose elements are congruent to u, called *unitary tessellation*. Axioms 2 and 3 allow us to give the following definition.

Definition 9.1. Given a unitary tessellation Π we obtain a sequence $(\Pi_n)_{n\in\mathbb{N}}$ of tessellations by setting

[14] An attempt in this direction was been made in [7].

- $\Pi_1 = \Pi$
- Π_k is obtained by substituting each square s in Π_{k-1} with its partition formed by four pairwise congruent squares.

In turn, we can define an infinitesimal mass.

Definition 9.2. Given a unitary tessellation Π, we define the sequence $m = (m_n)_{n\in\mathbb{N}}$ by putting $m_k(x) = 4^{1-k}$ whenever x is an element of the partition Π_k and $m_k(x) = 0$ otherwise.

The proof of the following proposition is straightforward.

Proposition 9.1. *Let $m = (m_n)_{n\in\mathbb{N}}$ be as in Definition 9.2. Then m is an infinitesimal mass which is invariant with respect to the group of movements. Consequently, the associated measure is invariant with respect this group.*

10. Conclusions and open questions

As the title claims, this paper, as well as our paper [3], is exploratory in nature. Mainly, it aims to indicate the existence of a problem:

the need of a satisfactory measure theory for mereological structures and, mainly, in point-free geometry.

This means that our proposals are just one way to draw attention to this question and to suggest some ideas. We have no claims to consider our proposals to be complete and even less the only ones possible. In this paper we have sketched measures in point-free geometry, which are based on an infinitesimal mass derived by a square tessellation. However, nothing prevents us from considering a different tessellation; for example, a tessellation based on an equilateral triangle taken as unitary. In this regard it is interesting to observe that, with reference to *PPF*, both the formula $Area = l^2$ to calculate the area of a square with side l (by applying square-tessellation) and the formula $Area = l^2$ to calculate the area of an equilateral triangle with the same side (by applying triangle-tessellation) are correct. Indeed, fix an equilateral triangle t, then there exists a tessellation formed by triangles which are congruent to t. Moreover, since every equilateral triangle is a union of four pairwise congruent equilateral triangles, we can define an infinitesimal mass exactly as in the case of squares. It is immediate to see that, with respect to this mass, an equilateral triangle with side $l = 2$ is a union of four pairwise congruent equilateral triangles, therefore its measure is 2^2. Likewise, if $l = 3$, then its measure is 3^2, if $l = 1/3$, then its measure is $(1/3)^2$ and so on.

A further question that could be addressed is the extension of this idea of measurement for a point-free approach to non-Euclidean geometry. In this case the notion of a tessellation might be again the basis of a definition of measures.

Finally, one could investigate the possibility of substituting measures understood as an assignment of a real number to a region, with "measures" understood as a comparison between extensions of regions, based on the relation of equidecomposability.[15]

Recall that, roughly speaking, two figures A and B are called *equidecomposable* whenever B can be obtained from A by cutting A into a finite number of pieces and then recomposing them by suitable movements. More precisely:

Definition 10.1. Two regions x, x' are called *equidecomposable* if there are a partition $\{x_1, \ldots, x_n\}$ of x and a partition $\{x'_1, \ldots, x'_n\}$ of x' such that x'_i is congruent to x_i, for $i \in \{1, \ldots, n\}$.

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