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Free Definite Description Theory – Sequent Calculi and Cut Elimination

Abstract. We provide an application of a sequent calculus framework to the formalization of definite descriptions. It is a continuation of research undertaken in [20, 22]. In the present paper a so-called free description theory is examined in the context of different kinds of free logic, including systems applied in computer science and constructive mathematics for dealing with partial functions. It is shown that the same theory in different logics may be formalised by means of different rules and gives results of varying strength. For all presented calculi a constructive cut elimination is provided.

Keywords: sequent calculus; definite descriptions; free logic; definedness logic; partial terms; cut elimination

1. Introduction

In recent years a lot of research in proof theory has been concerned with the extension of proof-theoretic methods to formal theories. In particular, applications of the sequent calculus (SC) have shown a great usefulness in this field. Most of the work has been devoted to the formalization of particular theories, or classes of such theories axiomatizable by means of formulae of specific kinds. On the other hand, relatively little effort has been put into the adequate treatment of useful formal devices of more fundamental kinds and wide applicability, such as several types of operators.

Definite descriptions (DD) may serve as a significant example of such a neglected field in proof theory. An enormous number of books and papers have been devoted to providing an adequate solution of linguistic

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and philosophical problems connected with descriptions, but the number of formal systems and their studies is relatively modest. It is true that a number of natural deduction systems (ND) for DD have been provided, such as Kalish and Montague [26, 27], Słupecki and Borkowski [6], Stenlund [42, 43], Tennant [44, 45], Garson [15], Carlström [8], Francez and Więckowski [14], Kürbis [29, 30, 31]. But only a few of them (namely Tennant's and Kürbis' works) deal with DD by means of rules which allow for finer proof analysis and provide normalization proofs. Tennant's work [45] represents a particularly important contribution, where a general constructive neologicist theory of term-forming operators is discussed which extends specific theories of definite descriptions and set abstractors first developed in [44]. Tennant rightly noticed that the best framework for development of such theories is provided by free logic. In particular, he is using negative free logic, where all formulae with nondenoting terms are counted as false. This kind of free logic has many advantages, such as for work in constructive mathematics and computer science [see, e.g., 18].

There are also a few tableau calculi due to Bencivenga, Lambert and van Fraassen [5], Gumb [18], Bostock [7], Fitting and Mendelsohn [13] but all of them introduce DD by means of rather complex rules, and so are not really in the spirit of tableau systems methodology. Sequent calculi (SC) of some sort were proposed by Czermak [11], Gratzl [17]. This lack of interest is surprising because the application of formal machinery of modern proof theory, in particular of SC, may be advantageous for both sides. On the one hand, competing theories of DD may be shown in a new light. On the other hand, the behaviour of DD needs subtle syntactical analysis and this may enrich a toolkit of proof theory.

In [20, 22] we launched the program of formalizing different theories of DD in the setting of SC admitting cut elimination. The first paper was concerned with some modal theory due to Garson [15] and the second with a Fregean approach as developed by Kalish and Montague [26]. Recently, in [24], we also provided a cut-free SC for a hybrid version of the system of Fitting and Mendelsohn [13]. In this paper we continue such research and provide a proof-theoretic analysis of a theory of DD founded on the ground of free logic (FL), both in its negative (NFL) and positive (PFL) version. Moreover, all systems will be presented in classical and intiuitionistic versions. In section 2 we recall basic technical information concerning languages, logics and sequent calculi. In section 3 we briefly characterise the kind of theory of DD to be dealt with in this paper and

then (section 4) discusses difficulties connected with the formalization of DD in the setting of SC. Sections 5–8 contain a presentation of SC for several logics with DD. In particular, we start with SC for the Russellian theory of DD based on the definedness logic of Feferman [12] and Beeson [3]. Section 6 examines a similar system which gives much weaker theory of DD. A version of NFL equivalent to Tennant's ND system [44] and to Scott's logic of existence [40] is considered in section 7 and PFL in section 8. Section 9 focuses on the uniform proof of cut elimination theorem for all the presented systems. In section 10 we briefly consider prospects for further work and possible applications of our technical machinery.

2. Preliminaries

We will be concerned with logics formulated in a standard first-order predicate language with the following logical vocabulary:

- connectives: $\neg$, $\wedge$, $\vee$, $\rightarrow$;
- first-order quantifiers: $\forall$, $\exists$;
- predicates: E, $=$;
- iota-operator: ι.

An unary existence predicate E, although theoretically dispensable, is taken as primitive in all cases with one exception (see section 5). The category of terms covers variables and definite descriptions (briefly DD) built by means of iota-operator ι from formulae of the language. Since functions (including 0-ary ones, i.e. individual constants) may be always represented as descriptions, it is not necessary to consider them as additional terms. For convenience, variables are divided into bound $\mathrm{VAR} = \{x, y, z, \ldots\}$ and free (parameters) $\mathrm{PAR} = \{a, b, c, \ldots\}$. The definition of a term and formula is by simultaneous recursion on both categories.

In the metalanguage φ, ψ, χ denote any formulae and Γ, Δ, Π, Σ denote their multisets. In particular, we use a convention to the effect that $\Gamma_1, \ldots, \Gamma_i$ denote submultisets of Γ, whereas, e.g., $\Gamma_{1,2}$ denotes the multiset union of Γ_1 and Γ_2. It facilitates representation of schemata involving many-premiss rules. DD will be written as $\iota x\varphi$ where φ is a formula in the scope of a respective operator. Metavariables $t, t_1, \ldots$ denote arbitrary terms; moreover, we will use a metavariable d to denote any DD if its structure is not essential. In general, terms may fail to denote although for some of the logics under consideration (quasi-free

logics) all variables (bound and free) are assumed to denote. As for DD, denoting ones are called proper and nondenoting (or not unique) improper.

We define the complexity of an expression (term or formula) as the number of occurrences of symbols from the logical vocabulary. Note that as a consequence atomic formulae containing DD can be of different complexity and may be more complex than compound formulae.

$\varphi[t_1/t_2]$ is used for the operation of substitution of an arbitrary term t_2 for all occurrences of a variable/parameter t_1 in φ, and similarly $\Gamma[t_1/t_2]$ for a uniform substitution in all formulae in Γ. It is always assumed that the substitution thus represented is correct, i.e. if t_2 is a variable or contains variables, they remain free after substitution. To simplify matters, we will also freely be using the notation $\varphi(x)$, $\varphi(a)$, $\varphi(t)$. In particular, in proof schemata $\varphi(x)$ will be used to denote that φ (being a scope of some operator which binds x) contains at least one occurrence of free x, whereas $\varphi(t)$ will denote the result of substitution.

We provide an SC formalization of four kinds of different free logics, called here NQFL, PQFL, NFL and PFL (where N stands for negative, P for positive, Q for quasi), each in two versions — classical and intuitionistic. In negative free logics, in contrast to positive ones, atomic formulae with nondenoting terms are always evaluated as false or, equivalently, all predicates are strict, i.e. defined only on denoting terms. NQFL and PQFL are systems for quasi-free logics in the sense that only descriptions can fail to denote; variables are always denoting. If we can restrict instantiation in all quantifier rules to parameters it makes it possible to characterize them by means of standard (classical or intuitionistic) quantifier rules which justifies our use of the term 'quasi free'.

We briefly point out to which known logics these systems correspond. Negative quasi-free logic presented as the system NQFL is equivalent to the definedness logic (or logic of partial terms) of Beeson [3] and Feferman [12] which has been extensively studied and applied in computer science. Although it was rather developed in the context of constructive mathematics, Fefermann rightly noticed that it works without changes in the classical setting (in fact he was concerned only with classical semantics in [12]. PQFL is a positive variant of NQFL, i.e. not requiring that all predicates are strict. Its intuitionistic restricted version (no identity and descriptions) was studied proof-theoretically by Baaz and Iemhoff [2] and recently by Maffezioli and Orlandelli [35]. The remaining two systems NFL and PFL characterise absolutely free logics in the sense that

variables may also fail to denote. The logic of NFL has been, as the logic of existence [40], applied in computer sciences and in foundational studies [44, 45]. Again it works uniformly in classical and intuitionistic settings (Tennant [44] provides completeness proofs for both versions). PFL characterises the most popular version of free logic [see, e.g., 4, 33, 34] usually founded on classical logic and applied mainly in philosophical studies and as the basis of formalization of modal first-order logics [see, e.g., 15]. Note also that in contrast to quasi-free logics, absolutely free logics are also inclusive (or universally free) in the sense that they admit empty domains in their models. However, both SC may be easily modified to exclude such a possibility and we consider these noninclusive versions, too.

Recently, the problem of finding cut-free SCs for free logics was independently solved by Pavlović and Gratzl [38] (absolutely free logics) and by Indrzejczak [25] (also quasi-free logics) in languages without descriptions. However, all these logics (except PQFL) were investigated in a language with DD. Moreover, in all cases DD are characterised by the same axiom (L) due to [32]. It does not mean however that the obtained theory is the same. In negative logics (NQFL and NFL) the resulting theory is relatively strong and equivalent to a Russellian treatment of DD (in a sense to be explained in the next section), whereas in PFL it yields the weakest (or minimal) theory of free description theory called MFD. Particular systems examined in this article are modifications of some SCs introduced in [25] but extended with rules for DD. A crucial feature of these systems is that quantifier rules are restricted to parameters as instantiated terms. Such a restriction is possible thanks to the rule (EI) of introduction of existence formula to the antecedent of a sequent. In the system for NQFL which is here formulated in the language without an existence predicate some other rule is used which comprises the effect of (EI) (as well as the requirement of strictness for predicates). It was proved in [25] that SC with such rules and restricted quantifier rules are equivalent to systems with unrestricted rules. In the next section we explain why the elimination of unrestricted quantifier rules is vital for providing well-behaved sequent calculi for DD.

The propositional basis of all four systems is essentially Gentzen's LK for classical and LJ for intuitionistic version with some minor changes: (1) sequents of the form $\Gamma \Rightarrow \Delta$ are built not from sequences but from finite multisets to avoid inessential complications. (2) We also prefer to present all two-premiss rules in the multiplicative (or with independent contexts) version. (3) We display additionally rules for equivalence which

are not usually formulated but are crucial for our analysis of DD. The calculus LK consists of the following structural and logical rules:

$$(\mathrm{AX})\ \varphi \Rightarrow \varphi \qquad (\mathrm{Cut})\ \frac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}$$

$$(\mathrm{W}\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \mathrm{W})\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}$$

$$(\mathrm{C}\Rightarrow)\ \frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \mathrm{C})\ \frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}$$

$$(\neg\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\neg)\ \frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}$$

$$(\wedge\Rightarrow)\ \frac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\wedge)\ \frac{\Gamma \Rightarrow \Delta, \varphi \quad \Pi \Rightarrow \Sigma, \psi}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, \varphi \wedge \psi}$$

$$(\vee\Rightarrow)\ \frac{\varphi, \Gamma \Rightarrow \Delta \quad \psi, \Pi \Rightarrow \Sigma}{\varphi \vee \psi, \Gamma, \Pi \Rightarrow \Delta, \Sigma} \qquad (\Rightarrow\vee)\ \frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$$

$$(\rightarrow\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \varphi \quad \psi, \Pi \Rightarrow \Sigma}{\varphi \rightarrow \psi, \Gamma, \Pi \Rightarrow \Delta, \Sigma} \qquad (\Rightarrow\rightarrow)\ \frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \rightarrow \psi}$$

$$(\leftrightarrow\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \varphi, \psi \quad \varphi, \psi, \Pi \Rightarrow \Sigma}{\varphi \leftrightarrow \psi, \Gamma, \Pi \Rightarrow \Delta, \Sigma} \qquad (\Rightarrow\leftrightarrow)\ \frac{\varphi, \Gamma \Rightarrow \Delta, \psi \quad \psi, \Pi \Rightarrow \Sigma, \varphi}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, \varphi \leftrightarrow \psi}$$

In LJ all succedents are restricted to at most one formula so $(\Rightarrow\mathrm{C})$ is missing, Δ, Σ are either empty or singletons, and we have two rules for $(\Rightarrow\vee)$ and $(\leftrightarrow\Rightarrow)$:

$$(\Rightarrow\vee 1)\ \frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \varphi \vee \psi} \qquad (\Rightarrow\vee 2)\ \frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \vee \psi}$$

$$(\leftrightarrow\Rightarrow 1)\ \frac{\Gamma \Rightarrow \varphi \quad \psi, \Pi \Rightarrow \Sigma}{\varphi \leftrightarrow \psi, \Gamma, \Pi \Rightarrow \Sigma} \qquad (\leftrightarrow\Rightarrow 2)\ \frac{\Gamma \Rightarrow \psi \quad \varphi, \Pi \Rightarrow \Sigma}{\varphi \leftrightarrow \psi, \Gamma, \Pi \Rightarrow \Sigma}$$

We keep standard terminology with respect to proofs and rules; in particular, displayed formulae in the schemata of rules are active (principal in the conclusion and side-formulae in the premisses) and the remaining ones are parametric and together make a context. Proofs are defined in a standard way as finite trees with nodes labelled by sequents. The height of a proof $\mathcal{D}$ of $\Gamma \Rightarrow \Delta$ is defined as the number of nodes of the longest branch in $\mathcal{D}$. Moreover, $\vdash_k \Gamma \Rightarrow \Delta$ means that $\Gamma \Rightarrow \Delta$ has a proof of height k.

3. Definite Descriptions

We are concerned with a proof-theoretic characterization of DD so there is no space for wider presentation of different theories of DD and their philosophical or linguistic motivations. Nevertheless to make the paper self-contained some elementary information should be provided. We limit our considerations to only one approach with incidental remarks on other solutions; the reader may find a wider presentation in [4, 32, 33, 34]. The approach we are interested in is strongly connected with free logic and commonly called a free description theory. It is based on Lambert's axiom:

$$\forall x(\iota x\varphi(x) = x \leftrightarrow \forall y(\varphi(y) \leftrightarrow y = x)) \tag{L}$$

which is a universally quantified version of Hintikka axiom:

$$\iota x\varphi(x) = t \leftrightarrow \forall y(\varphi(y) \leftrightarrow y = t) \tag{H}$$

where instead of bound y an arbitrary term occurs. Hintikka axiom is too strong and with no restrictions leads to contradiction. For example, in the case when φ is a contradictory formula but we use $\iota x\varphi$ as t, and classical rule of universal quantifier elimination to it. Even its weaker version (L) when added to classical logic yields the same effect but on the ground of FL it is under control due to the weaker rule for universal quantifier elimination which allows for the instantiation of only those terms which are known to denote.

The most popular theories of DD were provided by Frege and Russell. We skip a discussion of Frege's account and direct the reader to Pelletier and Linsky [39] who attributed to Frege four different theories of definite descriptions; all very influential. One of them was formally developed by Kalish and Montague [26, 27] and formulated as cut-free SC by Indrzejczak [22]. The well-known Russellian approach is reductionist in the sense that it treats DD not as genuine terms but shows how they can be replaced in every context by complex first-order formulae. Its essential content may be expressed by means of the following axiom:

$$\psi(\iota x\varphi(x)) \;\leftrightarrow\; \exists y(\forall x(\varphi(x) \leftrightarrow x = y) \wedge \psi(y)) \tag{R}$$

Again if we let ψ be an arbitrary formula it leads to contradiction, so let it be an atomic formula. Even in such a restricted form (R) was often attacked as being too strong. The left-right implication means that if we state something about DD it implies that this description is

proper. According to the Strawson's well-known criticism if DD is used as an argument of a predicate its existence and uniqueness is presupposed rather than implied. Lambert's axiom is in general weaker than (R) and implies only the right-left implication of (R) which is commonly acceptable. But on the ground of NFL, (L) and (R) are equivalent so NFL with (L) is in fact quite a strong theory of DD and it is essentially Russellian. It is a consequence of the fact that in NFL all predicates are strict so the statement of an atomic formula implies that all terms occuring in it are denoting. On the other hand in PFL, (L) yields the weakest (minimal) theory of DD called MFD.

4. How to obtain SC Rules for Definite Descriptions?

Our aim is to provide cut-free SC with rules for DD which are provably equivalent to (L) and possibly close to standard rules applied in SC. Depending on the kind of underlying logic we will obtain different sets of rules realizing this task. In the context of standard Gentzen's SC it is difficult to characterize DD (or even identity) by means of rules that satisfy all conditions usually required from well-behaved rules of SC characterizing logical constants, like symmetry, separation, explicitness [see, e.g., 48]. Anyway we should try to obtain as much as possible; in particular, these additional rules should allow for proving the cut elimination theorem and satisfying the subformula property or some reasonable generalization of it. In the systems presented below cut elimination is the main adressed issue. In searching for satisfactory rules we encounter two main problems which must be overcome.

The first, and the main, problem is connected with the unrestricted instantiation of terms in quantifier rules. Let us consider the standard rules ($\forall\Rightarrow$) and ($\Rightarrow\exists$). If descriptions are allowed as instantiated terms we can infer from $\forall x Ax$ something like $A(\iota x(\exists y(Bxy \rightarrow \neg Cxy)))$. In the framework of SC it means that the subformula property is lost and, in particular for cut elimination proofs, that induction on the complexity of cut formula fails. The problem is similar to the one connected with so called Takeuti's conjecture concerning cut elimination for second-order logic. How to avoid trouble? In [20] cut elimination is proved in SC for a variant of first-order modal logic with DD where some solution of this problem was provided. The rules for quantifiers in that SC are restricted — we are allowed to instantiate them only with parameters.

It works due to the special rule which enables transmission between parameters and DD through the rules for identity. In consequence we can prove $\forall x\varphi \wedge \mathsf{E}t \to \varphi[x/t]$ for all terms t, not only for parameters. In that case the special rule reducing the instantiation was rather connected with the modal framework which is absent here. Fortunately it is still possible to apply a similar strategy. The main contribution of this work is the application of a single rule (EI) which enables such a restriction on the side of quantier rules without the lack of completeness. It works uniformly with all considered logics formulated in the language with an existence predicate. We also provide its special version for one of the logics in the language where an existence predicate is not needed (see section 5).

The second problem is connected with the shape of rules for DD and covers three different possible choices:

1. the choice of the principal formula;
2. the choice of side formulae;
3. one-sided rules versus symmetric rules.

As for the first issue we can choose a general form $\psi(\imath x\varphi)$ with ψ atomic [18] or more specialised ones, like $t = \imath x\varphi$ or $\mathsf{E}\imath x\varphi$. Following Tennant [44] we choose identity but the latter (i.e. $\mathsf{E}\imath x\varphi$) is also possible (as noted by Tennant as well). However there are serious practical reasons for the choice of identity as a specialised principal formula. Premisses of the rules will be obtained by decomposition of the right side of respective axioms, and in the case of (L) there is only one occurrence of $\forall$ to be dealt with. Since $\mathsf{E}\imath x\varphi$ is defined by Russell as $\exists x\forall y(\varphi[x/y] \leftrightarrow x = y)$ we would have to deal with two quantifiers. Moreover, by choosing identity we can obtain rules directly related to (L) which facilitates a syntactical proof of their equivalence to (L). The price is that such rules are not separate (in the sense of not exhibiting other logical constants), since identity is additionally involved in the schemata of rules for DD.

The choice of side formulae may be based not only on the right side of the respective axiom, as it was formulated above, but also on some other equivalent characterisations. In the case of (L) it may be either $\forall x(\varphi \leftrightarrow x = y)$ or $\varphi[x/y] \wedge \forall x(\varphi \to x = y)$. In most of the aforementioned systems such complex formulae are explicitly stated as premisses (or conclusions in tableaux) of respective rules. But in order to obtain a decent SC we need to make a decomposition of it into smaller parts devoid of other logical constants. Such a decomposition can lead to

different sets of premisses despite the fact that decomposed formulae are equivalent. Moreover, one more characterization of DD may be of interest, proposed by Goldblatt [16] in the context of first-order modal logics:

$$\imath x\varphi(x) = t \leftrightarrow \mathsf{E}t \wedge \forall x(\varphi(x) \leftrightarrow x = t) \tag{G}$$

Similarly as (H) it is stronger than (L) and leads to contradiction not only in the context of classical logic but even in PFL, if we instantiate t with $\imath x\varphi(x)$ and if φ is contradictory. However, Goldblatt uses NFL (moreover with some restriction on admissible t connected with modal distinctions between rigid and nonrigid terms) where it has no destructive consequences. It can be proved that in the context of NFL (and NQFL)(G) is equivalent to (L). Hence at least for both negative logics we can devise rules where premisses are obtained by decomposition of the right side of (G). However, for the sake of uniformity it is better to stay with (L). In the end, our choice will be $\forall x(\varphi \leftrightarrow x = y)$, since it is generally more convenient as it provides more informative premisses after its decomposition.

Concerning the last choice (i.e. one-sided versus symmetric rules), the standard solution applied in SC is to devise two (or more) symmetric rules which introduce respective constants to the antecedent and to the succedent of the conclusion. This is not always easy or even possible if we search for rules characterising either logical constants of some non-classical logics, or (non)logical constants of formal theories. Even the characterisation of identity in SC leads to problems with finding truly symmetric rules. For example one of the most popular set of rules for identity proposed by Negri and von Plato [37] and adopted in the second edition of Troelstra and Schwichtenberg [47], looks like that:

$$\text{(REF)}\ \frac{t = t, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \qquad \text{(REPL)}\ \frac{\varphi[x/t_2], t_1 = t_2, \varphi[x/t_1], \Gamma \Rightarrow \Delta}{t_1 = t_2, \varphi[x/t_1], \Gamma \Rightarrow \Delta}$$

where φ is restricted to atomic formulae to avoid troubles in the proof of cut elimination. Duplication of principal formulae in (REPL) is needed for the proof of admissibility of contraction. Since in the setting of LK contraction rules are primitive such contraction-absorbing rule may be simplified and it is enough to use:

$$\text{(REPL}')\ \frac{\varphi[x/t_2], \Gamma \Rightarrow \Delta}{t_1 = t_2, \varphi[x/t_1], \Gamma \Rightarrow \Delta}$$

This solution to the problem of identity formalization is an example of the general strategy applied by Negri and von Plato who successfully characterised several formal theories by means of rules introducing their specific constants always on the same side of a sequent. Let me call this solution the one-sided approach. It permits us to apply a general strategy for proving cut admissibility for several theories. In fact, for several auxiliary rules involving identities or existence formulae in the presented systems we have chosen the one-sided option; principal formulae are introduced always to the antecedent. However, for the free theory of DD based on (L) it is possible to devise a pair of symmetric rules introducing identities with DD to both sides of a sequent which, in our opinion, is a more natural solution for the characterization of logical constants in SC. Hence we have decided to use rules for DD that are symmetric but not separate, since descriptions are arguments of identities in principal formulae.

But the choice of symmetric rules for DD which involve identities leads to the situation where the one-sided approach cannot be consequently realised for the general identity rules if we want to save cut elimination. If we use a rule like (REPL′) (or similar) we encounter a serious problem which follows from the lack of separation of the rules for DD. If some other rules may introduce identities as principal formulae there is a possible clash which may destroy the possibility of cut elimination. Consider the following schematic cut application:

$$\dfrac{\dfrac{\Gamma_1 \Rightarrow \Delta_1 \ \ldots \ \Gamma_k \Rightarrow \Delta_k}{\Gamma \Rightarrow \Delta, d = t}\ (\mathrm{r1}) \qquad \dfrac{\Pi_1 \Rightarrow \Sigma_1 \ \ldots \ \Pi_n \Rightarrow \Sigma_n}{d = t, \Pi \Rightarrow \Sigma}\ (\mathrm{r2})}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\ (\mathrm{Cut})$$

where d is a definite description and both cut-formulae are principal formulae of (r1) and (r2). If one premiss is obtained via rule for DD (for example (r1)) and the other via other rule with identity as the principal formula (for example (REPL′), then in general we cannot make a reduction on the complexity of cut formula which is essential for constructive proofs of cut elimination.

If DD could be characterised by one-sided rules such a problem could be avoided, since identities would be introduced as principal formulae (of all respective rules) only to one side of a sequent. Such a formula in the second side of a sequent may be only parametric or introduced by W; in both cases there is no risk of a clash. Such a solution was applied to the Fregean theory of DD in [22] where all identities as principal formulae

are introduced only to succedents of sequents. But if the rules for DD are symmetric, as in the present case, we have a problem which must be avoided by changing the shape of other rules involving identities. In the particular case of (REPL′) there are seven interderivable equivalents (one being the axiomatic sequent). This follows from the general rule-generation theorem proved in [21]). Since we will refer to this theorem quite often we state it here without the proof:

THEOREM 1. *For a sequent* $\Gamma \Rightarrow \Delta$ *with* $\Gamma = \{\varphi_1, \ldots, \varphi_k\}$ *and* $\Delta = \{\psi_1, \ldots, \psi_n\}$, $k \geqslant 0$, $n \geqslant 0$, $k+n \geqslant 1$ *there are* $2^{k+n}-1$ *equivalent rules captured by the general schema*:

$$\frac{\Pi_1, \Rightarrow \Sigma_1, \varphi_1, \ \ldots, \ \Pi_i \Rightarrow \Sigma_i, \varphi_i \qquad \psi_1, \Pi_{i+1} \Rightarrow \Sigma_{i+1}, \ \ldots, \ \psi_j, \Pi_{i+j} \Rightarrow \Sigma_{i+j}}{\Gamma^{-i}, \Pi_1, \ldots, \Pi_i, \Pi_{i+1}, \ldots, \Pi_{i+j} \Rightarrow \Sigma_1, \ldots, \Sigma_i, \Sigma_{i+1}, \ldots, \Sigma_{i+j}\Delta^{-j}}$$

where $\Gamma^{-i} = \Gamma - \{\varphi_1, \ldots, \varphi_i\}$ *and* $\Delta^{-j} = \Delta - \{\psi_1, \ldots, \psi_j\}$ *for* $0 \leqslant i \leqslant k$, $0 \leqslant j \leqslant n$.

Informally, any equivalent rule can be obtained by taking a formula from the succedent (antecedent) of the sequent and providing a premiss with this formula in the antecedent (succedent). In particular in the case of (REPL′) it is one of the possible rules expressing the sequent $t_1 = t_2, \varphi[x/t_1] \Rightarrow \varphi[x/t_2]$; the remaining ones are:

$$(1)\frac{\Gamma \Rightarrow \Delta, \varphi[x/t_1]}{t_1 = t_2, \Gamma \Rightarrow \Delta, \varphi[x/t_2]} \qquad (2)\ \frac{\Gamma \Rightarrow \Delta, t_1 = t_2 \quad \Pi \Rightarrow \Sigma, \varphi[x/t_1]}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, \varphi[x/t_2]}$$

$$(3)\ \frac{\Gamma \Rightarrow \Delta, t_1 = t_2}{\varphi[x/t_1], \Gamma \Rightarrow \Delta, \varphi[x/t_2]} \qquad (4)\ \frac{\Gamma \Rightarrow \Delta, t_1 = t_2 \quad \varphi[x/t_2], \Pi \Rightarrow \Sigma}{\varphi[x/t_1], \Gamma, \Pi \Rightarrow \Delta, \Sigma}$$

$$(5)\ \frac{\Gamma \Rightarrow \Delta, \varphi[x/t_1] \quad \varphi[x/t_2], \Pi \Rightarrow \Sigma}{t_1 = t_2, \Gamma, \Pi \Rightarrow \Delta, \Sigma}$$

$$(=+)\ \frac{\Gamma \Rightarrow \Delta, t_1 = t_2 \quad \Pi \Rightarrow \Sigma, \varphi[x/t_1] \quad \varphi[x/t_2], \Lambda \Rightarrow \Theta}{\Gamma, \Pi, \Lambda \Rightarrow \Delta, \Sigma, \Theta}$$

where φ is atomic, t_1, t_2 are any terms.

Note that, with the exception of the last one, all of them have an explicit identity, or an atom φ which may be an identity, in the conclusion. Therefore, if we want to avoid a clash with one of the DD rules, schematically displayed above, we are left only with the last one. Hence (=+) will be our official rule.

5. The Russellian Theory of Definite Descriptions

We start with the system NQFL formulated in the standard language without an existence predicate E. It is adequately characterised as propositional LK (or LJ) with the following rules:

$$(\forall\Rightarrow)\ \frac{\varphi[x/b], \Gamma \Rightarrow \Delta}{\forall x\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\forall)^1\ \frac{\Gamma \Rightarrow \Delta, \varphi[x/a]}{\Gamma \Rightarrow \Delta, \forall x\varphi}$$

$$(\exists\Rightarrow)^1\ \frac{\varphi[x/a], \Gamma \Rightarrow \Delta}{\exists x\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\exists)\ \frac{\Gamma \Rightarrow \Delta, \varphi[x/b]}{\Gamma \Rightarrow \Delta, \exists x\varphi}$$

$$(\mathrm{STR})^2\ \frac{t_i = a, \Gamma \Rightarrow \Delta}{Rt_1 \dots t_n, \Gamma \Rightarrow \Delta} \qquad (\mathrm{STR}_=)^2\ \frac{t_i = a, \Gamma \Rightarrow \Delta}{t_1 = t_2, \Gamma \Rightarrow \Delta}$$

$$(=-')\ \frac{b = b, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

$$(=+')^3\ \frac{\Gamma_1 \Rightarrow \Delta_1, t_1 \approx t_2 \qquad \Gamma_2 \Rightarrow \Delta_2, \varphi[x/t_1] \qquad \varphi[x/t_2], \Gamma_3 \Rightarrow \Delta_3}{\Gamma \Rightarrow \Delta}$$

$$(\Rightarrow\iota)^1\ \frac{\varphi[x/a], \Gamma_1 \Rightarrow \Delta_1, a = c \qquad a = c, \Gamma_2 \Rightarrow \Delta_2, \varphi[x/a]}{\Gamma \Rightarrow \Delta, \iota x\varphi = c}$$

$$(\iota\Rightarrow)\ \frac{\Gamma_1 \Rightarrow \Delta_1, \varphi[x/b], b = c \qquad \varphi[x/b], b = c, \Gamma_2 \Rightarrow \Delta_2}{\iota x\varphi = c, \Gamma \Rightarrow \Delta}$$

1. where a is not in Γ, Δ and φ.
2. where a is not in Γ, Δ, $i \leqslant n$ and t_i is DD.
3. where φ is atomic, and $t_1 \approx t_2$ is either $t_1 = t_2$ or $t_2 = t_1$.

In the case of LJ-based version we must replace the last rule with two rules:

$$(\iota\Rightarrow 1)\ \frac{\Gamma_1 \Rightarrow \varphi[x/b] \qquad b = c, \Gamma_2 \Rightarrow \Delta}{\iota x\varphi = c, \Gamma \Rightarrow \Delta}$$

$$(\iota\Rightarrow 2)\ \frac{\Gamma_1 \Rightarrow b = c \qquad \varphi[x/b], \Gamma_2 \Rightarrow \Delta}{\iota x\varphi = c, \Gamma \Rightarrow \Delta}$$

The same procedure will apply to other systems in the intuitionistic version, so in general we omit later signalling of this necessary modification.

A few comments on some rules are in order. ($\mathrm{STR}_=$) (the name comes from 'strict') is in fact a special case of (STR) but it is better to display it as a rule on its own, since identities will play special role in our

considerations, as mentioned above. A side condition on t_i restricting it to descriptions is not necessary but is important for proof search to avoid the inessential infinite generation of identities with fresh parameters. In case of ($\mathrm{STR}_{=}$) this restriction is also important for the proof of cut elimination. This rule without respective restriction leads to the same kind of clash with ($\Rightarrow\imath$) as (REPL′). A way out of the trouble would be to apply a similar solution as with (=+) and introduce a rule:

$$(\mathrm{STR}'_{=})\ \frac{\Gamma_1 \Rightarrow \Delta_1, t_1 = t_2 \qquad t_i = a, \Gamma_2 \Rightarrow \Delta_2}{\Gamma \Rightarrow \Delta}$$

which does not need side condition concerning t_i. But in this case using a one-premiss rule with restriction does not lead to a problem with cut elimination so it seems to be a better option.

In both DD rules principal formulae are identities having only one d and an arbitrary parameter c as the other argument. In fact these rules are also correct in the version where an arbitrary second term (i.e. some other DD) is allowed. However, this restricted version is sufficient for completeness whereas the unrestricted one must be in some systems — as we will see in the next section — more complex.

The rule (=+′) is a strenghtened version of (=+) introduced in the last section. In fact this schema covers two rules: (=+) and its symmetric variant. This addition is needed for NQFL since otherwise we are unable to prove symmetry of identity for all four combination of terms. The problem is connected with restricted reflexivity rule (=−′) which is required to prove the symmetry. (=+) is sufficient to prove $a = b \to b = a$ or $a = d \to d = a$ for any parameters a, b and any definite description d. However to prove $d = a \to a = d$ we need a symmetric version:

$$\dfrac{\dfrac{d = a \Rightarrow d = a \qquad a = a \Rightarrow a = a \qquad a = d \Rightarrow a = d}{d = a, a = a \Rightarrow a = d}\ (=+')}{d = a \Rightarrow a = d}\ (=-')$$

whereas for proving $d = d' \to d' = d$ we need both versions; see the proof (I) on p. 520.

Note that the following is provable (for all systems we consider):

LEMMA 1. $\vdash t_1 = t_2, \varphi[x/t_1] \Rightarrow \varphi[x/t_2]$, *for any formula* φ.

PROOF. The proof is by structural induction on the shape of φ. In the basis φ is atomic and the claim follows directly from axioms by means of (=+). The inductive step is proved as in [37]. □

Essentially this system, but without rules for DD, was shown to be equivalent to the basic definedness logic of Feferman [12] (E^+ in [46]) in [25]. In particular, in spite of the restriction of ($\Rightarrow\exists$) and ($\forall\Rightarrow$) to parameters as instantiated terms it is complete due to rules (STR), ($STR_=$). Note that in the language of our system the existence (or definedness) predicate is missing in contrast to the formulation of Feferman. However it can be introduced by definition and its presence is not necessary in rules, since quantifier rules are restricted to parameters (always denoting) as instatiated terms. In a language without an existence predicate the present system is also equivalent to the ND system for Russellian theory provided by Kalish, Montague and Mar [27].

It remains to show that both rules for DD are interderivable with (L). The latter is provable in LJ in the way (II) on p. 520 (proofs for LK are obvious modifications).

The derivability of both rules in LJ (and similarly in LK) in the presence of (L) added as additional axiomatic sequents is straightforward. Notice first that:

(a) $\imath x\varphi = c, L \Rightarrow \forall x(\varphi \leftrightarrow x = c)$ and
(b) $\forall x(\varphi \leftrightarrow x = c), L \Rightarrow \imath x\varphi = c$ are provable.

PROOF. For (a):

$$(\leftrightarrow\Rightarrow 1)\ \frac{\imath x\varphi = c \Rightarrow \imath x\varphi = c \qquad \forall x(\varphi \leftrightarrow x = c) \Rightarrow \forall x(\varphi \leftrightarrow x = c)}{(\forall\Rightarrow)\ \dfrac{\imath x\varphi = c, \imath x\varphi = c \leftrightarrow \forall x(\varphi \leftrightarrow x = c) \Rightarrow \forall x(\varphi \leftrightarrow x = c)}{\imath x\varphi = c, \forall y(\imath x\varphi = y \leftrightarrow \forall x(\varphi \leftrightarrow x = y)) \Rightarrow \forall x(\varphi \leftrightarrow x = c)}}$$

By cut with (L), and cut with:

$$(\leftrightarrow\Rightarrow 1)\ \frac{\Gamma_1 \Rightarrow \Delta_1, \varphi[y/b] \qquad b = c, \Gamma_2 \Rightarrow \Delta_2}{(\forall\Rightarrow)\ \dfrac{\varphi[y/b] \leftrightarrow b = c, \Gamma \Rightarrow \Delta}{\forall x(\varphi \leftrightarrow x = c), \Gamma \Rightarrow \Delta}}$$

we obtain the conclusion of ($\imath\Rightarrow 1$). Obvious modifications of the last proof schema provide the derivability of ($\imath\Rightarrow 2$) or ($\imath\Rightarrow$) in LK.

The proof of (b) is similar:

$$(\leftrightarrow\Rightarrow 2)\ \frac{\forall x(\varphi \leftrightarrow x = c) \Rightarrow \forall x(\varphi \leftrightarrow x = c) \qquad \imath x\varphi = c \Rightarrow \imath x\varphi = c}{(\forall\Rightarrow)\ \dfrac{\forall x(\varphi \leftrightarrow x = c), \imath x\varphi = c \leftrightarrow \forall x(\varphi \leftrightarrow x = c) \Rightarrow \imath x\varphi = c}{\forall x(\varphi \leftrightarrow x = c), \forall y(\imath x\varphi = y \leftrightarrow \forall x(\varphi \leftrightarrow x = y)) \Rightarrow \imath x\varphi = c}}$$

The derivability of $\Gamma \Rightarrow \Delta, \forall x(\varphi \leftrightarrow x = c)$ from the premisses of ($\Rightarrow\imath$) is straightforward. Again by two cuts we derive the conclusions of respective rule. □

(I)

$$\dfrac{\dfrac{a=d\Rightarrow a=d \qquad d=d'\Rightarrow d=d' \qquad \dfrac{a=d'\Rightarrow a=d' \qquad a=d\Rightarrow a=d \qquad d'=d\Rightarrow d'=d}{a=d', a=d\Rightarrow d'=d}\,(=+)}{a=d, a=d, d=d'\Rightarrow d'=d}\,(=+')}{\dfrac{a=d, d=d'\Rightarrow d'=d}{\dfrac{d=d', d=d'\Rightarrow d'=d}{d=d'\Rightarrow d'=d}\,(\mathrm{C}\Rightarrow)}\,(\mathrm{Str}_{=})}\,(\mathrm{C}\Rightarrow)$$

(II)

$$\dfrac{\dfrac{\dfrac{(\imath\Rightarrow 1)\dfrac{\varphi[y/b]\Rightarrow\varphi[y/b] \qquad b=a\Rightarrow b=a}{\imath x\varphi=a, \varphi[y/b]\Rightarrow b=a} \qquad \dfrac{b=a\Rightarrow b=a \qquad \varphi[y/b]\Rightarrow\varphi[y/b]}{\imath x\varphi=a, b=a\Rightarrow\varphi[y/b]}\,(\imath\Rightarrow 2)}{(\Rightarrow\leftrightarrow)\dfrac{\imath x\varphi=a\Rightarrow\varphi[y/b]\leftrightarrow b=a}{\imath x\varphi=a\Rightarrow\forall y(\varphi\leftrightarrow y=a)}\,(\Rightarrow\forall)}}{\Rightarrow\imath x\varphi=a\leftrightarrow\forall y(\varphi\leftrightarrow y=a)}\,(\Rightarrow\leftrightarrow)\qquad \mathcal{D}}{\Rightarrow\forall x(\imath x\varphi=x\leftrightarrow\forall y(\varphi\leftrightarrow y=x))}\,(\Rightarrow\leftrightarrow)$$

where $\mathcal{D}$ is:

$$\dfrac{\dfrac{(\leftrightarrow\Rightarrow)\dfrac{b=a\Rightarrow b=a \qquad \varphi[y/b]\Rightarrow\varphi[y/b]}{\varphi[y/b]\leftrightarrow b=a, b=a\Rightarrow\varphi[y/b]} \qquad \dfrac{\varphi[y/b]\Rightarrow\varphi[y/b] \qquad b=a\Rightarrow b=a}{\varphi[y/b]\leftrightarrow b=a, \varphi[y/b]\Rightarrow b=a}}{(\forall\Rightarrow)\dfrac{\forall y(\varphi\leftrightarrow y=a), b=a\Rightarrow\varphi[y/b]}{(\Rightarrow\imath)\dfrac{}{\forall y(\varphi\leftrightarrow y=a)\Rightarrow\imath x\varphi=a}} \qquad \dfrac{\forall y(\varphi\leftrightarrow y=a), \varphi[y/b]\Rightarrow b=a}{}}$$

We stated that NQFL without rules for DD is an adequate SC for the basic definedness logic called LPT (the logic of partial terms) which was proved in [25]. Since (L) was also considered by Feferman [12] as a possible extension of LPT, as a result, NQFL with rules for DD provides an adequate SC characterization of this important logic with descriptions.

Two things are worth stressing. First, due to simple quantifier rules with no additional existence premisses which are usually present in the formulation of free logics, (L) may be characterised by means of simple two-premiss rules. Second, although (L) in general is treated as an axiom characterising the weakest (free) theory of DD, in the case of negative free logics it is in fact equivalent to (R). The present system is negative since the effect of (STR) is that all predicates are strict. It is quite easy to prove the equivalence of (L) and (R) and we omit it here.

6. Positive Quasi Free Logic with Descriptions

If we do not require that all predicates are strict but still treat all variables as denoting it has no impact on the quantifier rules. Therefore, the system PQFL contains the same rules for quantifiers. Anyway, it is easier to formulate a system in the language with a primitive existence (definedness) predicate E, since otherwise to express the fact that parameters have existential import we have to use the more complex formula $\exists x x = a$. It has also the same rules for DD and $(=+)$ $((=+')$ is not necessary). It differs from NQFL in the following way: instead of (STR), ($\mathrm{STR}_=$), $(=-')$ it has the following rules:

$$(\mathsf{EI})\ \frac{a = t, \Gamma \Rightarrow \Delta}{\mathsf{E}t, \Gamma \Rightarrow \Delta} \qquad (\mathsf{EE})\ \frac{\mathsf{E}b, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \qquad (=-)\ \frac{t = t, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

where a is not in $\Gamma, \Delta, \mathsf{E}t$ whereas b is any parameter and t any term.

Again some comments on the rules are in order. (EI) is a special rule which is a weaker version of (STR). It shows that, by definition, the predicate of existence is strict. It will be applied in all subsequent calculi. In general, thanks to this rule (which is admissible even in positive free logics) we can restrict the instantiation of terms to variables in all quantifier rules to parameters. Moreover, in PQFL we can get rid of additional existential premisses which are used in two quantifier rules provided by Baaz and Iemhoff [2]. Namely, in their system instead of $(\forall\Rightarrow)$, $(\Rightarrow\exists)$ we have:

$$(\forall \mathsf{E}\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \mathsf{E}t \quad \varphi[x/t], \Pi \Rightarrow \Sigma}{\forall x\varphi, \Gamma, \Pi \Rightarrow \Delta, \Sigma}$$

$$(\Rightarrow \exists \mathsf{E})\ \frac{\Gamma \Rightarrow \Delta, \mathsf{E}t \quad \Pi \Rightarrow \Sigma, \varphi[x/t]}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, \exists x\varphi}$$

which, by the application of the rule-generation theorem [21], can be replaced with the rules:

$$(\forall \mathsf{E}\Rightarrow)\ \frac{\varphi[x/t], \Gamma \Rightarrow \Delta}{\mathsf{E}t, \forall x\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \exists \mathsf{E})\ \frac{\Gamma \Rightarrow \Delta, \varphi[x/t]}{\mathsf{E}t, \Gamma \Rightarrow \Delta, \exists x\varphi}$$

These latter rules were in fact applied by Maffeziolli and Orlandelli [35] but in contraction-absorbing versions (i.e. with the repetition of active formulae from the conclusion in the premiss).

Since the proof of equivalence is simple, especially for the one-premiss variant of rules, we record it here. $(\forall \mathsf{E}\Rightarrow)$ (for $(\Rightarrow \exists \mathsf{E})$ the proof is similar) is derivable in our system:

$$\text{(Cut)}\ \frac{a = t, \varphi[x/a] \Rightarrow \varphi[x/t] \qquad \varphi[x/t], \Gamma \Rightarrow \Delta}{\text{(EI)}\ \dfrac{\text{(}\forall\Rightarrow\text{)}\ \dfrac{a = t, \varphi[x/a], \Gamma \Rightarrow \Delta}{a = t, \forall x\varphi, \Gamma \Rightarrow \Delta}}{\mathsf{E}t, \forall x\varphi, \Gamma \Rightarrow \Delta}}$$

where the leftmost leaf is derivable by Lemma 1.

For the other direction (EI) is derivable by $(\Rightarrow \exists \mathsf{E})$:

$$\text{(Cut)}\ \frac{\text{(}\Rightarrow\exists\mathsf{E}\text{)}\ \dfrac{\text{(}=-\text{)}\ \dfrac{t = t \Rightarrow t = t}{\Rightarrow t = t}}{\mathsf{E}t \Rightarrow \exists x\, x = t} \qquad \dfrac{a = t, \Gamma \Rightarrow \Delta}{\exists x\, x = t, \Gamma \Rightarrow \Delta}\ (\exists\Rightarrow)}{\mathsf{E}t, \Gamma \Rightarrow \Delta}$$

and $(\forall\Rightarrow)$ (as well as $(\Rightarrow\exists)$) is derivable by means of (EE):

$$\text{(EE)}\ \frac{\text{(}\forall\mathsf{E}\Rightarrow\text{)}\ \dfrac{\varphi[x/b], \Gamma \Rightarrow \Delta}{\mathsf{E}b, \forall x\varphi, \Gamma \Rightarrow \Delta}}{\forall x\varphi, \Gamma \Rightarrow \Delta}$$

(EE) is a rule corresponding to Beeson's axiom stating that all parameters denote. In NQFL this rule was dispensable, since it is derivable if we define $\mathsf{E}b$ as $b = b$ or $\exists x\, x = b$. Finally, since PQFL is not a negative logic, $(=-')$ is strengthened to express the unconditional reflexivity of identity. The side-effect of replacing $(=-')$ with $(=-)$ is that symmetry is proved in a uniform way for identities with arbitrary terms by means of $(=+)$ simply, and the strenghtening to $(=+')$ is not required.

It is worth stressing here the advantages from restricting the DD rules to principal formulae having always a parameter as the right argument of identity. If we use the unrestricted version with any t in this position, the rules from section 5 would be incorrect in PQFL. We should use instead something like this:

$$(\Rightarrow\iota)\ \frac{\varphi[x/a], \Gamma_1 \Rightarrow \Delta_1, a = t \qquad a = t, \Gamma_2 \Rightarrow \Delta_2, \varphi[x/a] \qquad \Gamma_3 \Rightarrow \Delta_3, \mathsf{E}t}{\Gamma \Rightarrow \Delta, \iota x\varphi = t}$$

$$(\iota\Rightarrow)\ \frac{\Gamma_1 \Rightarrow \Delta_1, \varphi[x/b], b = t \qquad \varphi[x/b], b = t, \Gamma_2 \Rightarrow \Delta_2 \qquad \Gamma_3 \Rightarrow \Delta_3, \mathsf{E}t}{\iota x\varphi = t, \Gamma \Rightarrow \Delta}$$

or this:

$$(\Rightarrow\iota)^1\ \frac{\varphi[x/a], \Gamma_1 \Rightarrow \Delta_1, a = t \qquad a = t, \Gamma_2 \Rightarrow \Delta_2, \varphi[x/a]}{\mathsf{E}t, \Gamma \Rightarrow \Delta, \iota x\varphi = t}$$

$$(\iota\Rightarrow)\ \frac{\Gamma_1 \Rightarrow \Delta_1, \varphi[x/b], b = t \qquad \varphi[x/b], b = t, \Gamma_2 \Rightarrow \Delta_2}{\mathsf{E}t, \iota x\varphi = t, \Gamma \Rightarrow \Delta}$$

with the same proviso concerning a (in case of LJ the last rule must be split into two differing only with respect to side formula in the middle premiss). The reason is that if t is a DD, then the explicit statement that it is proper is necessary. This is obtained by the addition of the rightmost premiss (or alternatively, of the existence formula to the conclusion) in both cases. But such rules are more complex and also the proof of their interderivability with (L) is more complicated. Restriction to parameters as second terms in principal formulae has the effect that this additional premiss (or additional active formula in the conclusion) is not needed, since all parameters denote. Therefore the same rules as in NQFL are sufficient for PQFL and the proof of adequacy is the same as in NQFL, since only the rules which are common to NQFL and PQFL are used for that aim.

The fact that both NQFL and PQFL share the same quantifier and DD rules may give the impression that we obtain the same theory of DD. But it is not true. The system NQFL considered in section 5 is Russellian in a twofold sense. By (STR) it comprises the Russellian denotation principle to the effect that atomic formulae with nondenoting terms are false and, as a result, the theory of DD although expressed by means of relatively weak principle (L) is in fact equivalent to (R). This may be seen as an advantage but not necessarily, as we observed in section 3. In this respect PQFL may be seen as offering a better theory of DD.

Despite the fact that (L) is expressed by means of the same rules, its theory of DD is essentially different. In particular, only one half of (R) is derivable since to obtain the left-right implication we must have all predicates strict. Thus we can claim that PQFL is the system of the minimal quasi-free DD.

7. Russellian Descriptions in the Logic of Existence

In the system NFL variables are not restricted in the range to existent objects but may be undefined like other terms, so in our nomenclature this logic is absolutely free. It has the effect that the presence of additional existence formulae is unavoidable in many cases, including the case of quantifier and DD rules. We must add the following rules to LK (or LJ):

$$(\forall \mathsf{E}\Rightarrow)\ \frac{\varphi[x/b], \Gamma \Rightarrow \Delta}{\mathsf{E}b, \forall x\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\forall \mathsf{E})^1\ \frac{\mathsf{E}a, \Gamma \Rightarrow \Delta, \varphi[x/a]}{\Gamma \Rightarrow \Delta, \forall x\varphi}$$

$$(\exists \mathsf{E}\Rightarrow)^1\ \frac{\mathsf{E}a, \varphi[x/a], \Gamma \Rightarrow \Delta}{\exists x\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\exists \mathsf{E})\ \frac{\Gamma \Rightarrow \Delta, \varphi[x/b]}{\mathsf{E}b, \Gamma \Rightarrow \Delta, \exists x\varphi}$$

$$(\mathrm{NEE})\ \frac{\mathsf{E}t_i, \Gamma \Rightarrow \Delta}{Rt_1 \ldots t_n, \Gamma \Rightarrow \Delta} \qquad (\mathrm{NEE}_=)^2\ \frac{\mathsf{E}t_i, \Gamma \Rightarrow \Delta}{t_1 = t_2, \Gamma \Rightarrow \Delta}$$

$$(\mathsf{EI})^1\ \frac{a = t, \Gamma \Rightarrow \Delta}{\mathsf{E}t, \Gamma \Rightarrow \Delta} \qquad (\mathsf{E}{=}{-})\ \frac{t = t, \Gamma \Rightarrow \Delta}{\mathsf{E}t, \Gamma \Rightarrow \Delta}$$

$$(\mathrm{STR}_=)^3\ \frac{t_i = a, \Gamma \Rightarrow \Delta}{t_1 = t_2, \Gamma \Rightarrow \Delta}$$

$$(=+')^4\ \frac{\Gamma_1 \Rightarrow \Delta_1, t_1 \approx t_2 \qquad \Gamma_2 \Rightarrow \Delta_2, \varphi[x/t_1] \qquad \varphi[x/t_2], \Gamma_3 \Rightarrow \Delta_3}{\Gamma \Rightarrow \Delta}$$

$$(\Rightarrow\imath)^1\ \frac{\mathsf{E}a, \varphi[x/a], \Gamma_1 \Rightarrow \Delta_1, a = c \qquad a = c, \Gamma_2 \Rightarrow \Delta_2, \varphi[x/a]}{\mathsf{E}c, \Gamma \Rightarrow \Delta, \imath x\varphi = c}$$

$$(\imath\Rightarrow)\ \frac{\Gamma_1 \Rightarrow \Delta_1, \varphi[x/b], b = c \qquad \varphi[x/b], b = c, \Gamma_2 \Rightarrow \Delta_2}{\mathsf{E}b, \imath x\varphi = c, \Gamma \Rightarrow \Delta}$$

1. where a is not in Γ, Δ and φ;
2. where t_i is a parameter;
3. where a is not in Γ, Δ, $i \leq n$ and t_i is DD;
4. where φ is atomic, and $t_1 \approx t_2$ is either $t_1 = t_2$ or $t_2 = t_1$.

Note that ($\imath\Rightarrow$) splits into two rules in LJ as in the preceding cases. (NEE) and (NEE$_=$) are counterparts of (STR) and (STR$_=$); again (NEE$_=$) is a special case of (NEE) but it is better to display it as a special rule for greater perspicuity. Both rules make all predicates strict so the logic is negative. This makes it necessary to weaken the rule for reflexivity of identity and hence we obtain (E=$-$).

One should notice that (NEE$_=$), in contrast to (NEE), is restricted to parameters. The weaker form of (NEE$_=$) is the reason that we keep (STR$_=$) with the same proviso as in NQFL. It is possible to use only (NEE$_=$) without any constraints on t_i (as in (NEE)). Such a choice, although more economical and uniform, leads again to a clash with ($\Rightarrow\imath$) in the proof of cut elimination. Also in this case we could avoid this problem by using the following rule:

$$(\mathrm{NEE}'_=)\ \frac{\Gamma_1 \Rightarrow \Delta_1, t_1 = t_2 \qquad \mathsf{E}t_i, \Gamma_2 \Rightarrow \Delta_2}{\Gamma \Rightarrow \Delta}$$

with no constraints on t_i. However, using two rules restricted to two different kinds of terms seems to be a better option which, as we shall see, creates no problems in the proof of cut elimination. One may easily notice that such a solution allows for the derivability of the unrestricted (NEE$_=$). In the case of identity with two DD as arguments we proceed in the following way:

$$(=+')\ \frac{d = a \Rightarrow d = a \qquad \mathsf{E}a \Rightarrow \mathsf{E}a \qquad \mathsf{E}d, \Gamma \Rightarrow \Delta}{\begin{array}{r}(\mathrm{NEE}_=)\ \dfrac{d = a, \mathsf{E}a, \Gamma \Rightarrow \Delta}{\begin{array}{r}(\mathrm{C}\Rightarrow)\ \dfrac{d = a, d = a, \Gamma \Rightarrow \Delta}{\begin{array}{r}(\mathrm{STR}_=)\ \dfrac{d = a, \Gamma \Rightarrow \Delta}{d = d', \Gamma \Rightarrow \Delta}\end{array}}\end{array}}\end{array}}$$

In the case where the identity has only one DD, the last application of (STR$_=$) is not needed. This proof shows also why, similarly as in NQFL, we need (=+$'$) instead of (=+).

Since all quantifier rules were changed the rules for DD are also changed. The reason is that premisses are obtained by decomposition of the right side of (L) by means of the rules of the respective logic. Note that the additional formula $\mathsf{E}a$ in the antecedent of the second premiss is needed. On the other hand, in the right premiss such an addition is not necessary, since such formula can be obtained in a bottom-up direction from $a = c$ by (NEE$_=$).

Again we must show that our characterization of (L) by rules is adequate. Tennant [44] introduced Gentzen-style natural deduction (ND) for negative free logic with rules for description and abstraction operators, later generalised in [45]. In the former case he notified his debts to Smiley [41] who provided axiomatic and semantic formulation of this theory. Scott's logic of existence with DD presented in axiomatic form in [40] is essentially the same logic which was formalised as the ND system by Tennant. There is a different characterization of identity but in the case where all predicates are strict it gives the same result.

Since Tennant has the set of ND rules for DD which are interderivable with (L) for the sake of variety we provide below the proof that our rules are interderivable with his rules instead of a direct proof with respect to (L). Tennant has one rule of introduction of DD and three rules of elimination. These rules when transformed into the sequent format may be directly expressed in SC in the following way:

$$(\Rightarrow\iota)\ \frac{\Gamma_1 \Rightarrow \Delta_1, \mathsf{E}c \quad \mathsf{E}a, \varphi[x/a], \Gamma_2 \Rightarrow \Delta_2, a = c \quad a = c, \Gamma_3 \Rightarrow \Delta_3, \varphi[x/a]}{\Gamma \Rightarrow \Delta, \iota x\varphi = c}$$

$$\frac{\Gamma_1 \Rightarrow \Delta_1, \mathsf{E}b \qquad \Gamma_2 \Rightarrow \Delta_2, \varphi[x/b] \qquad \Gamma_3 \Rightarrow \Delta_3, \iota x\varphi = c}{\Gamma \Rightarrow \Delta, b = c}$$

$$\frac{\Gamma_1 \Rightarrow \Delta_1, b = c \qquad \Gamma_2 \Rightarrow \Delta_2, \iota x\varphi = b}{\Gamma \Rightarrow \Delta, \varphi[x/b]}$$

$$\frac{\Gamma \Rightarrow \Delta, \iota x\varphi = c}{\Gamma \Rightarrow \Delta, \mathsf{E}c}$$

with the same proviso for a as the eigenvariable in the first rule.

His first rule is directly interderivable with our $(\Rightarrow\iota)$ by means of the rule-generation theorem. For the elimination of DD he uses three rules corresponding to the three premisses of the ι-introduction rule in order to obtain direct reduction schemata needed for normalization. The last one is just a special case of denotation principle needed for negative free logic. We can omit it since it is interderivable with our $(\mathrm{NEE}_{=})$.

To get a more standard SC we apply again the rule-generation theorem to the remaining two rules and obtain two left introduction rules for ι:

$$(\mathrm{T}\iota\Rightarrow 1)\ \frac{\Gamma_1 \Rightarrow \Delta_1, \mathsf{E}b \qquad \Gamma_2 \Rightarrow \Delta_2, \varphi[x/b] \qquad b = c, \Gamma_3 \Rightarrow \Delta_3}{\iota x\varphi = c, \Gamma \Rightarrow \Delta}$$

$$(\mathrm{T}\imath{\Rightarrow}2)\ \frac{\Gamma_1 \Rightarrow \Delta_1, b = c \qquad \varphi[x/b], \Gamma_2 \Rightarrow \Delta_2}{\imath x\varphi = c, \Gamma \Rightarrow \Delta}$$

In fact, these rules (with numerically restricted succedents) provide an intuitionistic version of NFL. Are they interderivable with our $(\imath{\Rightarrow})$? The present rule is derivable by means of both Tennant's rules. By the first and the second we get (III) and (IV) on p. 528, respectively, and we finish by combining them:

$$\dfrac{\dfrac{\imath x\varphi(x) = c, \Gamma \Rightarrow \Delta, \varphi(a) \qquad \mathsf{E}a, \imath x\varphi(x) = c, \varphi(a), \Gamma \Rightarrow \Delta}{\mathsf{E}a, \imath x\varphi(x) = c, \imath x\varphi(x) = c, \Gamma, \Gamma \Rightarrow \Delta, \Delta}\ (\mathrm{Cut})}{\mathsf{E}a, \imath x\varphi(x) = c, \Gamma \Rightarrow \Delta}\ (\mathrm{C})$$

Both Tennant's rules are derivable by the present rule. A proof of the first Tennant's rule looks like (V) on p. 528. For the second we can prove (VI) on p. 530. Hence both systems are equivalent in the setting of LK. Obvious modifications yield the same result for LJ.

In contrast to quasi free logics, NFL (and PFL considered in the next section) are universally free (or inclusive) in the sense that they are logics characterised by models admitting empty domains. We can strengthen NFL to obtain the system which excludes empty models. It is enough to add a rule which is a weaker version of (EE):

$$(\mathrm{E}\mathsf{E}')\ \frac{\mathsf{E}a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \qquad \text{where } a \text{ is not in } \Gamma, \Delta$$

With this rule one can immediately prove an axiom $\exists x \mathsf{E}x$. Let us call such a noninclusive version NFLN.

8. Free Description Theory in Positive Free Logic

PFL can be obtained from NFL simply by dropping rules (NEE), (NEE$_=$) and (STR$_=$), and changing the rules for identity and DD. Instead of (E=$-$) and (=$+'$) we have (=$-$) and (=$+$), exactly as in PQFL. The rules for quantifiers are just those of NFL. Since predicates are not strict the rules for DD from NFL are too weak and must be modified by the addition of suitable existence formulae:

$$(\imath{\Rightarrow})\ \frac{\mathsf{E}a, \varphi[x/a], \Gamma_1 \Rightarrow \Delta_1, a = c \qquad \mathsf{E}a, a = c, \Gamma_2 \Rightarrow \Delta_2, \varphi[x/a]}{\mathsf{E}c, \Gamma \Rightarrow \Delta, \imath x\varphi = c}$$

(III)

$$\dfrac{\dfrac{\dfrac{\mathsf{E}a \Rightarrow \mathsf{E}a \qquad \Gamma_1 \Rightarrow \Delta_1, a = c, \varphi(a) \qquad a = c, \varphi(a), \Gamma_2 \Rightarrow \Delta_2}{\mathsf{E}a, \imath x\varphi(x) = c, \varphi(a), \Gamma_{1,2} \Rightarrow \Delta_{2,3}, a = c}\,(\mathrm{T}\imath\Rightarrow 1) \qquad a = c, \varphi(a), \Gamma_2 \Rightarrow \Delta_2}{\mathsf{E}a, \imath x\varphi(x) = c, \varphi(a), \varphi(a), \Gamma_{1,2,2} \Rightarrow \Delta_{1,2,2}}\,(\mathrm{Cut})}{\mathsf{E}a, \imath x\varphi(x) = t, \varphi(a), \Gamma \Rightarrow \Delta}\,(\mathrm{C})$$

(IV)

$$\dfrac{\dfrac{\Gamma_1 \Rightarrow \Delta_1, \varphi(a), a = c \qquad \dfrac{\Gamma_1 \Rightarrow \Delta_1, \varphi(a), a = c \qquad \varphi(a), a = c, \Gamma_2 \Rightarrow \Delta_2}{\imath x\varphi(x) = c, a = c, \Gamma_{1,2} \Rightarrow \Delta_{1,2}, \varphi(a)}\,(\mathrm{T}\imath\Rightarrow 2)}{\imath x\varphi(x) = c, \Gamma_{1,1,2} \Rightarrow \Delta_{1,1,2}, \varphi(a), \varphi(a)}\,(\mathrm{Cut})}{\imath x\varphi(x) = c, \Gamma \Rightarrow \Delta, \varphi(a)}\,(\mathrm{C})$$

(V)

$$\dfrac{\Gamma_1 \Rightarrow \Delta_1, \mathsf{E}a \qquad \dfrac{\dfrac{\Gamma_2 \Rightarrow \Delta_2, \varphi(a)}{\Gamma_2 \Rightarrow \Delta_2, a = c, \varphi(a)}\,(\Rightarrow\mathrm{W}) \qquad \dfrac{a = c, \Gamma_3 \Rightarrow \Delta_3}{a = c, \varphi(a), \Gamma_3 \Rightarrow \Delta_3}\,(\mathrm{W}\Rightarrow)}{\mathsf{E}a, \imath x\varphi(x) = c, \Gamma_{2,3} \Rightarrow \Delta_{2,3}}\,(\imath\Rightarrow)}{\imath x\varphi(x) = c, \Gamma \Rightarrow \Delta}\,(\mathrm{Cut})$$

$$(\imath\Rightarrow)\ \frac{\Gamma_1 \Rightarrow \Delta_1, \varphi[x/b], b = c \qquad \varphi[x/b], b = c, \Gamma_2 \Rightarrow \Delta_2}{\mathsf{E}c, \mathsf{E}b, \imath x\varphi = c, \Gamma \Rightarrow \Delta}$$

where a is not in Γ, Δ, φ but b is any parameter. Again two variants of the second rule for LJ are needed.

The main difference with the rules from NFL is that in the present version of $(\Rightarrow\imath)$ we need $\mathsf{E}a$ also in the antecedent of the rightmost premiss and in $(\imath\Rightarrow)$ an additional occurrence of $\mathsf{E}c$ in the antecedent is necessary to ensure that c denotes. The reader can easily check why we cannot use the rules for DD from NFL. The reason is that in proofs of interderivability of the rules from the preceding section with Tennant's rules (equivalent to L) the rule ($\mathrm{NEE}_=$) was necessary.

Instead of the proof that these rules are interderivable with (L) on the basis of PFL we notice that this follows from the results in [20]. There a modal version of PFL was introduced where DD was characterised by means of the rules that we give in (VII) and (VIII) on p. 530.

These rules were shown to be equivalent to Garson's ND rules for DD from [15], and the latter are interderivable with (L). Since our present rules are nothing more than the simplification of the above rules obtained by the application of the rule-generation theorem we conclude that they adequately characterise the minimal free description theory MFD of Lambert. To obtain PFLN, a noninclusive variant adequate for nonempty models only, we must add ($\mathrm{E}\mathsf{E}'$) exactly as in the case of NFL.

9. Cut Elimination Theorem

We have shown the equivalence of the proposed SC with respective systems formulated as axiomatic or ND calculi but such proofs require numerous applications of cut. Below we present a constructive, uniform proof of the cut elimination theorem for all presented systems which is an extension of the proof in [25] for systems without DD. First note that the following auxiliary result holds for all considered systems:

LEMMA 2 (Substitution). *If* $\vdash_k \Gamma \Rightarrow \Delta$, *then* $\vdash_k \Gamma[a/b] \Rightarrow \Delta[a/b]$, *where* $\vdash_k \Gamma \Rightarrow \Delta$ *means that* $\Gamma \Rightarrow \Delta$ *has a proof of height* k.

The proof is standard by induction on the height of the proof of $\Gamma \Rightarrow \Delta$ [see, e.g., 37]. Note that we restrict the considerations to the substitution of parameters for parameters. The reason is that such a result is sufficient for our systems with all quantifiers restricted to the

(VI)

$$\dfrac{\dfrac{\dfrac{\Gamma_1 \Rightarrow \Delta_1, a = c}{\Gamma_1 \Rightarrow \Delta_1, a = c, \varphi(a)}\ (\Rightarrow\mathrm{W}) \qquad \dfrac{\varphi(a), \Gamma_2 \Rightarrow \Delta_2}{a = c, \varphi(a), \Gamma_2 \Rightarrow \Delta_2}\ (\mathrm{W}\Rightarrow)}{\iota x\varphi(x) = c, \mathsf{E}a, \Gamma \Rightarrow \Delta}\ (\iota\Rightarrow) \qquad \dfrac{\iota x\varphi(x) = c, \mathsf{E}a, \Gamma \Rightarrow \Delta}{\iota x\varphi(x) = c, a = c, \Gamma \Rightarrow \Delta}\ (\mathrm{NEE}_{=})}{\dfrac{\Gamma_1 \Rightarrow \Delta_1, a = c \qquad \iota x\varphi(x) = c, a = c, \Gamma \Rightarrow \Delta}{\iota x\varphi(x) = c, \Gamma_{1,1,2} \Rightarrow \Delta_{1,1,2}}\ (\mathrm{Cut})}$$

$$\dfrac{\iota x\varphi(x) = c, \Gamma_{1,1,2} \Rightarrow \Delta_{1,1,2}}{\iota x\varphi(x) = c, \Gamma \Rightarrow \Delta}\ (\mathrm{C})$$

(VII)

$$(\Rightarrow\iota)\ \dfrac{\Gamma_1 \Rightarrow \Delta_1, \mathsf{E}c \qquad \mathsf{E}a, \varphi[x/a], \Gamma_2 \Rightarrow \Delta_2, a = c \qquad \mathsf{E}a, a = c, \Gamma_3 \Rightarrow \Delta_3, \varphi[x/a]}{\Gamma \Rightarrow \Delta, \iota x\varphi = c}$$

(VIII)

$$(\iota\Rightarrow)\ \dfrac{\Gamma_1 \Rightarrow \Delta_1, \mathsf{E}c \quad \Gamma_2 \Rightarrow \Delta_2, \mathsf{E}b \qquad \Gamma_3 \Rightarrow \Delta_3, \varphi[x/b], b = c \qquad \varphi[x/b], b = c, \Gamma_4 \Rightarrow \Delta_4}{\iota x\varphi = c, \Gamma \Rightarrow \Delta}$$

instantiation of parameters. Moreover, the general version, with arbitrary t substituted for a does not hold for NQFL, PQFL and NFL, due to several rules which are correct only with the proviso that a parameter is involved (such as $(=-')$, (EE), $(\mathrm{NEE}_{=})$).

Moreover, we assume that all proofs are regular in the sense that every parameter which is fresh by the side condition on the respective rule must be fresh in the entire proof, not only on the branch where the application of this rule takes place. There is no loss of generality since every proof may be systematically transformed into a regular proof by the substitution lemma.

The strategy applied in [25] was originally introduced for hypersequent calculi by Metcalfe, Olivetti and Gabbay [36] and later extensively used in this framework [see, e.g., 10, 19, 23, 28]. However, it is also applicable to standard sequent calculi [see 20, 21]. It deals in an elegant way with the problems generated by contraction and proceeds by reducing either the height of one of the premisses (if a cut formula is not principal there) or the complexity of cut formula (if a cut formula is principal in both premisses).

Let us recall that the complexity of a formula is counted as the number of occurrences of logical constants including iota-operator, identity and E. Thus atomic formulae with DD are of complexity > 0 but it has no impact on the correctness of proof as we shall see. The complexity of a cut-formula φ is called its cut-degree and denoted as $d\varphi$. The proof-degree ($\mathrm{d}\mathcal{D}$) is the maximal cut-degree in the proof $\mathcal{D}$. The cut elimination theorem follows from two lemmata, where φ^k and Γ^k denote $k > 0$ occurrences of φ and Γ, respectively:

LEMMA 3 (Right reduction). *Let* $\mathcal{D}_1 \vdash \Gamma \Rightarrow \Delta, \varphi$ *and* $\mathcal{D}_2 \vdash \varphi^k, \Pi \Rightarrow \Sigma$ *with* $\mathrm{d}\mathcal{D}_1, \mathrm{d}\mathcal{D}_2 < d\varphi$, *and* φ *principal in* $\Gamma \Rightarrow \Delta, \varphi$. *Then we can construct a proof* $\mathcal{D}$ *such that* $\mathcal{D} \vdash \Gamma^k, \Pi \Rightarrow \Delta^k, \Sigma$ *and* $\mathrm{d}\mathcal{D} < d\varphi$.

LEMMA 4 (Left reduction). *Let* $\mathcal{D}_1 \vdash \Gamma \Rightarrow \Delta, \varphi^k$ *and* $\mathcal{D}_2 \vdash \varphi, \Pi \Rightarrow \Sigma$ *with* $\mathrm{d}\mathcal{D}_1, \mathrm{d}\mathcal{D}_2 < d\varphi$. *Then we can construct a proof* $\mathcal{D}$ *such that* $\mathcal{D} \vdash \Gamma, \Pi^k \Rightarrow \Delta, \Sigma^k$ *and* $\mathrm{d}\mathcal{D} < d\varphi$.

Both lemmata successively make a reduction first on the height of the right and then on the height of the left premiss of cut. The crucial point enabling suitable transformations is that all rules are substitutive and reductive. These notions were introduced by Ciabattoni [9] and applied for general form of cut elimination proof in hypersequent calculi

by Metcalfe, Olivetti and Gabbay [36] but can be also applied in the present setting. The former property is connected with the fact that multisets of formulae may be safely substituted for a cut formula which is parametric. It allows for induction on the height of a proof in cases when the cut formula is not principal in at least one premiss of cut. Rules with side conditions concerning fresh parameters are not fully substitutive as such but they are substitutive in regular proofs, so due to the substitution lemma this problem may be easily overcome.

The latter property, called coherency in [1], may be roughly defined as follows: a pair of introduction rules $(\Rightarrow\star)$, $(\star\Rightarrow)$ for a constant $\star$ is reductive if an application of cut on cut formulae introduced by these rules may be replaced by the series of cuts made on less complex formulae, in particular on their subformulae. Thus reductivity permits induction on cut-degree in the course of proving cut elimination. Note that the reductivity of DD rules is not sufficient for proving cut elimination in case of NQFL and NFL, since $\imath x\varphi = c$ may be also introduced by the application of $(\mathrm{NEE}_{=})$ or $(\mathrm{STR}_{=})$ in the right premiss of cut application, so these cases must be also examined.

THE PROOF OF LEMMA 3. By induction on the height of $\mathcal{D}_2$. The basis is trivial and induction step requires consideration of all cases of possible derivations of $\varphi^k, \Pi \Rightarrow \Sigma$ and the role of cut-formula in the transition. In the case where all occurrences of φ are parametric we simply apply the induction hypotheses to the premisses of $\varphi^k, \Pi \Rightarrow \Sigma$ and then apply to them the respective rule — here, the regularity of proofs prevents the violation of side conditions. If one of the occurrences of φ in the premiss(es) is a side formula of the last rule we must additionally apply weakening to restore the missing formula before the application of a rule. In the case where one occurrence of φ in $\varphi^k, \Pi \Rightarrow \Sigma$ is principal we make use of the fact that φ in the left premiss is principal too (note that for contraction and weakening it is trivial). We will show it only for the most complicated case of identity with DD in NFL. $\mathcal{D}_1$ ends with:

$$(\Rightarrow\imath)\ \frac{\mathsf{E}a, \varphi[x/a], \Gamma_1 \Rightarrow \Delta_1, a = c \qquad a = c, \Gamma_2 \Rightarrow \Delta_2, \varphi[x/a]}{\mathsf{E}c, \Gamma \Rightarrow \Delta, \imath x\varphi = c}$$

$\mathcal{D}_2$ ends with $\imath x\varphi = c^k, \Pi \Rightarrow \Sigma$ and we want to obtain $\mathsf{E}c, \Gamma^k, \Pi \Rightarrow \Delta^k, \Sigma$. There are three subcases. In the first, one occurrence of $\imath x\varphi = c$ is introduced by the application of $(\mathrm{NEE}_{=})$ and the premiss is $\mathsf{E}c, \imath x\varphi =$

$c^{k-1}, \Pi \Rightarrow \Sigma$. By the induction hypothesis we obtain $\mathsf{E}c, \Gamma^{k-1}, \Pi \Rightarrow \Delta^{k-1}, \Sigma$ which by weakening yields the result.

In the second subcase ($\mathrm{STR}_=$) is the last rule, the premiss is $\iota x\varphi = b, \iota x\varphi = c^{k-1}, \Pi \Rightarrow \Sigma$ where b is new. By the substitution lemma we obtain $\iota x\varphi = c, \iota x\varphi = c^{k-1}, \Pi \Rightarrow \Sigma$, so by the induction hypothesis (since the proof of this sequent is lower) we obtain the result.

The last rule is $(\iota\Rightarrow)$ so $\mathcal{D}_2$ ends with:

$$(\iota\Rightarrow)\ \frac{\iota x\varphi = c^i, \Pi_1 \Rightarrow \Sigma_1, \varphi[x/b], b = c \qquad \varphi[x/b], b = c, \iota x\varphi = c^j, \Pi_2 \Rightarrow \Sigma_2}{\iota x\varphi = c^k, \mathsf{E}b, \Pi' \Rightarrow \Sigma}$$

where $i + j = k$ and $\mathsf{E}b$ is either in Π_1 or in Π_2. By the substitution lemma applied to premisses of $(\Rightarrow\iota)$ we obtain:

(a) $\mathsf{E}b, \varphi[x/b], \Gamma_1 \Rightarrow \Delta_1, b = c$ and
(b) $b = c, \Gamma_2 \Rightarrow \Delta_2, \varphi[x/b]$.

By the induction hypothesis applied to premisses of $(\iota\Rightarrow)$ we obtain:

(c) $\Gamma^i, \Pi_1 \Rightarrow \Sigma_1, \Delta^i, \varphi[x/b], b = c$ and
(d) $\varphi[x/b], b = c, \Gamma^j, \Pi_2 \Rightarrow \Sigma_2, \Delta^j$.

By cut on (a) and (c) we obtain:

(e) $\mathsf{E}b, \Gamma^i, \Gamma_1, \Pi_1 \Rightarrow \Sigma_1, \Delta^i, \Delta_1, b = c, b = c$.

By cut on (b) and (d) we obtain:

(f) $b = c, b = c, \Gamma^j, \Gamma_2, \Pi_2 \Rightarrow \Sigma_2, \Delta^j, \Delta_2$.

(e) and (f) after contraction and cut on $b = c$, some contractions and (W⇒) to add $\mathsf{E}c$ yield the desired sequent. Since all these new cuts have lower degree than $d(\iota x\varphi = c)$, the new proof has lower degree and we are done. □

THE PROOF OF LEMMA 4. The proof is similar to the proof of Lemma 3 but by induction on the height of $\mathcal{D}_1$. Here the cut formula in the right premiss does not need to be principal. Therefore, when cut-formula is principal in the left premiss we apply first the induction hypothesis and next the rule in question to side-formulae. The new proof of the left premiss satisfies the assumption of the right reduction lemma, so we can safely apply it and, possibly after some applications of structural rules, obtain the result. We omit the details. □

THEOREM 2. *Cut is eliminable from proofs in* NQFL, PQFL, NFL, PFL, NFLN, PFLN, *where the last two systems are noninclusive variants of* NFL *and* PFL.

PROOF. The proof follows from the left reduction lemma by double induction: primary on $\mathrm{d}\mathcal{D}$ and subsidiary on the number of maximal cuts (in the basis and in the inductive step of the primary induction). We always take the topmost maximal cut and apply the left reduction lemma to it. By successive repetition of this procedure we diminish either the degree of a proof or the number of maximal cuts in it until we obtain a proof with $\mathrm{d} = 0$. □

10. Comments

We conclude this paper with the remark that this kind of analysis can be extended to other kinds of operators in accordance with the strategy described by Tennant [45]. In particular, already in [44] he formulated analogous ND rules for the abstraction operator in the basic set theory in NFL. Their SC counterparts may be devised in an analogous way as we did with DD:

$$(\Rightarrow\iota)\ \frac{\mathsf{E}a, \varphi[x/a], \Gamma_1 \Rightarrow \Delta_1, a \in c \qquad a \in c, \Gamma_2 \Rightarrow \Delta_2, \varphi[x/a]}{\mathsf{E}c, \Gamma \Rightarrow \Delta, \{x : \varphi\} = c}$$

$$(\iota\Rightarrow)\ \frac{\Gamma_1 \Rightarrow \Delta_1, \varphi[x/b], b \in c \qquad \varphi[x/b], b \in c, \Gamma_2 \Rightarrow \Delta_2}{\mathsf{E}b, \{x : \varphi\} = c, \Gamma \Rightarrow \Delta}$$

where a is not in Γ, Δ, φ but b is any parameter. Both rules are also reductive hence an NFL-based version of Tennant's basic set theory can be also proved to be cut-free.

Another task requiring further study is the construction of decent SC for stronger theories of DD which were studied by Lambert [33] and others on the basis of MFD. For some of them it is not hard to provide rules [see, e.g., 20] but certainly more involved research should be carried out. Just for illustration we show how we can formalise FD1 which is an extension of MFD by means of the addition of the cancellation law: $t = \iota x(x = t)$ to MFD. One can add the following rule in PFL:

$$(\mathrm{Can})\ \frac{t = \iota x(x = t), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

It is easy to check that its addition cannot destroy cut elimination proof for PFL. Moreover, we can dispense with (=−), since it is derivable by means of (Can) and (=+).

The last task worth further research is to find solutions which are better-behaved from the standpoint of actual proof-search. We have shown that the application of the rule-generation theorem [25] may be helpful in finding rules with a smaller branching factor. The main problem in the present systems is with (=+) which generates three branches in proofs and violates the subformula property. The example of two other rules involving identity as the principal formula — ($\mathrm{STR}_{=}$) and ($\mathrm{NEE}_{=}$) — shows that by suitably placed restrictions on active terms one can obtain better characterization. In particular, in NFL by division of the labour between these two rules restricted to different kind of terms, we obtain a more satisfactory solution. We conjecture that by introducing more specialised versions of (=+), dealing with different kinds of identities, we can gain stricter control over the process of cut elimination and avoid the clash with DD-rules in a similar way. This is a non-trivial task which requires further research.

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