Spencer Johnston

PER SE MODALITY AND NATURAL IMPLICATION — AN ACCOUNT OF CONNEXIVE LOGIC IN ROBERT KILWARDBY

Abstract. We present a formal reconstruction of the theories of the medieval logician Robert Kilwardby, focusing on his account of accidental and natural inferences and the underlying modal logic that gives rise to it. We show how Kilwardby's use of an essentialist modality underpins his connexive account of implication.

Keywords: Robert Kilwardby; connexive logic; modal logic; medieval logic

1. Introduction

When philosophers in the Middle Ages took up the study of Aristotle's writings on logic, they encountered some ideas that were known to them from other sources and other ideas that were novel.

When it came to the study of hypotheticals by medieval authors in the 13th century, the newfound awareness of the *Prior Analytics* provided some new material to further motivate their study of principles such as 'Aristotle's Thesis' ("No proposition entails its own negation"). Remember that for medieval authors a hypothetical is an expression that is used to join terms or propositions and include, e.g., 'and', 'or', 'if. . . then' etc. Principles such as Aristotle's Thesis are, in modern logic, deeply connected with connexive implication [see, e.g., 13, 14]. It should be recalled that connexive logics are a family of contra-classical logics. Following [9, p. 1], a logic is said to be *contra-classical* (in the deep sense, which requires the logic under discussion to not be inconsistent [see 9, pp. 2–3]) if not everything provable in the logic is provable in classical

logic. Contra-classical logics, then, are an unusual family of non-classical logics, since nearly all of the non-classical logics that are discussed (e.g., relevance and para-consistent logics, intuitionistic logic, and many multi-valued logics) are *not* contra-classical. They reject particular tautologies and/or theorems that classical logic claims are valid. However, they do *not* include any validities that are not valid in classical logic. They are proper sub-logics of classical logic. This is not the case for connexive logics. First, they accept as valid some theorems that are invalid in classical logic. Second, for such logics to be non-trivial they also need to reject some theorems of classical logic. The easiest way to see this is to recall that classical logic is Post-complete, i.e. that there are no logical systems that can consistently extend classical logic. Because of this, if we were to add any of the distinctive theorems of connexive logic (i.e. any of the theorems of connexive logic that are not theorems of classical logic) to classical logic, the logic would become trivial.

What are some of the theorems that are taken to characterise connexive systems? The hallmark of connexive logic systems is:

> The definition of connexive implication is transmitted to us by Sextus Empiricus:
>> And those who introduce the notion of connexion say that a conditional is sound when the contradictory of its consequent is incompatible with its antecedents.
>
> It is characteristic of this variety of implication that no proposition connexively implies or is implied by its own negation, since it is never incompatible with its own double negation, nor is its own negation incompatible with itself.   [13, p. 415]

Using $\rightarrow$ as the symbol for implication, this approach to logic is often seen to give rise to the following theses:

Aristotle's Thesis: $\neg(\neg A \rightarrow A)$
Boethius' Thesis: $(A \rightarrow B) \rightarrow \neg(A \rightarrow \neg B)$
Abelard's Thesis: $\neg((A \rightarrow \neg B) \wedge (A \rightarrow B))$

Medieval authors were familiar with Boethius' remarks about hypotheticals from his writings on logic, e.g., *De Topicis Differentiis* and the remarks about implication that he made which led to a connexive principle being named in his honour. However, with the resurfacing of the *Prior Analytics* in the Latin West [6, p. 1][1], medieval authors now found

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[1] Eberson writes: "The translation of the Prior Analytics that the scholastics used was the one Anicius Manlius Boethius had produced in the early sixth century.

connexive principles in the writings of Aristotle as well. As we shall see in this paper, what this gave rise to was an attempt to understand these logical principles in light of the theory of logic that was present in the 13th century. For the 13th century English Dominican Robert Kilwardby, who wrote an important commentary on Aristotle's *Prior Analytics*, the way to understand connexive implication hinges in part on drawing a distinction between two kinds of implication. One of these notions of implication is grounded in an analysis of necessity where the consequent needs to be contained (in a way that will need to be further explained) in the antecedent. For Kilwardby, the kind of necessity that preserves this notion of containment is called 'per se' necessity and the class of implication is called either natural implication or essential implication. These implications gives rise to a connexive logic.

Kilwardby contends that the correct reading of Aristotle requires that necessity in the modal syllogism be understood only as *per se* necessity. Kilwardby argues that the truth-conditions for this interpretation yield a consistent and intelligible interpretation of the modal syllogism in the *Prior Analytics*.

Kilwardby then goes on to generalise this and distinguish between two notions of logical consequence. According to Kilwardby, the former consequence relationship is what we would now call a connexive consequence relationship. According to Kilwardby natural consequences do not validate the principle of explosion and do validate Aristotle's Thesis (as well as other connexive principles).

Up until this point, formal treatments of Kilwardby's logical works have focused on his theory of the syllogism [see 15, 17, 18]. For example, Paul Thom provides a syntactic treatment of Kilwardby's logic, but does not formalise the distinction between *natural* and *accidental* consequences [see 17]. In this paper we develop a semantic reconstruction of Kilwardby's modal logic, drawing connections between Kilwardby's logical theory and recent work done on the logic of essence. This paper will start by situating Kilwardby's distinction between *per se* and *per accidens* within its historical setting, and then will draw out the relevant properties that such modal operations have. After this we will discuss how these notions are generalised to the distinction between *natural* and *accidental* consequences. With this in place, we will provide a formal re-

Having been totally forgotten, it seems, for centuries, it resurfaced in the second quarter of the twelfth century." [6, p. 1]

construction of Kilwardby's logic. This will be done by drawing inspiration from the recent literature on the logic of essence and from Marko Malink's treatment of Aristotle's modal logic. We will then show that Kilwardby's theory of *natural* consequences gives rise to a connexive logic.

2. Kilwardby's modal logic

2.1. Historical context

The English Dominican Robert Kilwardby is an interesting figure in the history of logic. The exact date and location of Robert Kilwardby's birth are unknown, however, given the dates that we do know about him, it seems unlikely that he was born much earlier than 1200.[2] Likewise, the location of Kilwardby's birth is unclear. Kilwardby is clearly English, however none of our historical sources further elaborate on where he is from. The name 'Kilwardby' is a reference to his place of birth, though it appears with many variations in spelling among the various sources. According to Sommer-Seckendorff, by the 15th century, Kilwardby named at least two English villages, one in Leicestershire and the other in Yorkshire, and it seems likely that Kilwardby was from one of these places [16, p. 1].

The early dates of Kilwardby's life are also unclear and rest on some conjecture. We know that he completed his Master of Arts degree at the University of Paris and later completed his theology degree at Oxford, however, an exact chronology of this time is not well-known or clear. For example, one theory holds that Kilwardby completed his arts degree in the early 1220's, wrote the majority of his works on Aristotle and later took the Dominican habit some time between 1240 and 1245. It is during this time that Kilwardby also completed his theological studies at Oxford and stayed on to lecture there. With Kilwardby's appointment as a provincial prior in 1261, his career as an academic came to an end, while his life as an ecclesiastical figure began.[3]

Kilwardby went on to be elected Archbishop of Canterbury in 1272. His time as Archbishop was marked by two noteworthy events, the coro-

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[2] One source gives his date of birth as 1204, but does not provide a reference [see 16, p. 47, fn. 7].

[3] For further exposition of this position and references to other theories see [16] and the references therein.

nation of Edward I on August 19$^{\text{th}}$ 1276 and Kilwardby's issuing of a number of prohibitions to teach in 1277. A number of important questions in the history of science revolve around Kilwardby's role in issuing these prohibitions to teach and how they related to the condemnations issued by Bishop Stephen Tempier in Paris earlier that year. While the reasons why Kilwardby issued these prohibitions to teach do not have any bearing on this work, their content does include some purported logical principles.[4] Kilwardby was promoted to Cardinal in 1279 and died later that year.

Kilwardby's most important logical work is his *Commentary on the Prior Analytics.* Written around 1240 while Kilwardby was at the University of Paris, the commentary is the first complete medieval commentary on the *Prior Analytics* that has survived [see 18, p. 2; and 6, p. 101]. A two volume critical edition of the commentary and English translation have recently been published [10].

The reception of the *Prior Analytics* in the Latin West is an interesting story and a brief summary of it helps illuminate the historical importance of Kilwardby's commentary. The *Prior Analytics* is one of six Aristotelian texts that made up the *Organon*.[5] These texts were often 'introduced' by Porphyry's *Isagoge* [2, p. 1] After the collapse of the Roman Empire, only some of these texts were transmitted and closely studied in the Latin West. The *Prior Analytics* was one of the texts that was not transmitted directly. Before the 10$^{\text{th}}$ century, those working in the Latin West had to make do with second-hand accounts. It was not until c.980 that a copy of Boethius' *De Syllogismo Categorico* was recovered [2, p. 2] and it was not until the writings of Abelard that we again find explicit references to the *Prior Analytics* [6, p. 2].

However, during the 11$^{\text{th}}$ century the influence of the *Prior Analytics* was slight. It was not until the 12$^{\text{th}}$ century that the *Prior Analytics* began to exert some influence on medieval thinkers. For example, Peter Abelard seems to have made some study of the text and his pupil Otto Freisingen is said to have brought a copy of the text to Germany [6, p. 97]. The reception of the *Prior Analytics* seems to have been slowed by a number of factors. For one, it seems that many 12$^{\text{th}}$ century scholars

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[4] See [19] for further discussions on the relationship between the prohibitions to teach issued in 1277 and logic.

[5] The other five texts were the *Categories*, *On Interpretation*, the *Posterior Analytics*, the *Topics*, and the *Sophistical Refutations*.

believed the views of the book were better articulated in other works. For example, John of Salisbury remarks that:

> Although we need its doctrine, we do not need the book itself that much. For whatever is contained there is presented in an easier and more reliable manner elsewhere, though nowhere in a truer or more forceful manner.[6]   [6, p. 97]

Very few commentaries on the *Prior Analytics* have survived from the 12th century and those that have, only in fragments [6, p. 99]. Thus it is Kilwardby's *Commentary on the Prior Analytics* that, so far as we are aware, is the first completely preserved commentary on this work [6, p. 101]. As such, it offers us insights into the role of this text and its function in medieval academic practice.

Kilwardby's treatment of Aristotle's work, as we shall see, is sophisticated, interesting, and exerted considerable influence on the logicians and philosophers that followed Kilwardby. The work itself is massive, coming to almost 300,000 words. In this work, Kilwardby provides commentary on and offers an exposition of Aristotle's logic.

2.2. Modal logic

In his treatment of Aristotle's modal logic, Kilwardby attempts to explain and elucidate how Aristotle conceived of modality. Kilwardby saw in Aristotle's ontology the interpretive key for understanding what Aristotle understood by necessity. According to Kilwardby:

> For, a *per se* necessity-proposition requires the subject to be *per se* some of the predicate itself. But when it is said "All who are literate are of necessity men", the subject is not *per se* some of the predicate itself; but it is granted that it is necessary, because the literate are not separate from what is some of man. But this is a *per accidens* necessity.[7]   [18, p. 20]

When speaking about *per se* expressions, it is important to be clear about what these terms signify. Kilwardby's view of signification is a

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[6] According to Ebbesen the work to which John refers is probably Boethius' *On Categorical Syllogisms* [6, p. 97].

[7] "Propositio enim per se de necessario exigit subiectum per se esse aliquid ipsius predicati. Cum autem dicitur 'omne grammaticum de necessitate est homo' ipsum subiectum non est aliquid per se ipsum predicate. Sed quia grammaticum non separator ab eo quodest aliquid ipsius hominis, ideo conceditur esse necessaria. Sed que sic est de necessario per accidens est de necessario." [10, p. 130]

fairly standard medieval realist reading of signification [see 3, pp. 70–72]. On this view, written terms immediately signify mental terms. Mental terms, in turn, immediately and *naturally* signify things in the world. Hence, for Kilwardby, there are two ways of understanding the signification of terms. On the one hand we can think of terms as the names that are given to various things.[8] This corresponds to the mediate sense of signification. On this account we can think of terms as composed of utterances, which in turn can be combined with other utterances to form propositions. For example, on this reading, the term 'man' can be used to signify the various men in the world (if there are any). On the other hand we can think of the signification of terms as expressing relationships between various kinds of concepts in a mental language. In this case we are interested in the immediate signification of the words. On this view, terms signify thoughts or concepts. Both of these are important for signification, because it means that either a term can signify a class of objects, or it can refer to the concept the term expresses. Both notions of signification will be important for our discussions about logical consequence and for how we formalise Kilwardby's modal logic. Because of this view of signification, the notion of being *per se* or *per accidens* can be applied for both the mediate sense (picking out objects) and the immediate sense (picking out terms in a mental language). As Thom points out [see 18, pp. 20–21], there are two important features of *per se* necessary propositions that should be observed. First, when Kilwardby speaks of the subject being *per se* some of the predicate itself, the idea is that one should only focus on the term in isolation and not in relationship to other terms that it may be combined with [18, p. 21]

The second thing Thom observes is that 'being a *per se* term' transfers, in the sense that if the subject of a categorical proposition is *per se*, then so is the predicate. This leads Thom to suggest that:

> A true syllogistic necessity-proposition has to state a truth of Aristotelian ontology, as "All men are animals" does but "All animals are men" does not. Thus, assuming the correctness of Aristotelian ontology, affirmative per se necessity-propositions have to express *de dicto* necessities with *per se* terms. [18, p. 21]

It is this ontologically-driven interpretation of *per se* modalities that we will follow in this paper.

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[8] i.e. 'est sermo de termino ut est praedicabilis'

The distinction between *per se* and *per accidens* modalities was not original to Kilwardby. It is based on Aristotle's distinction between *per se* and *per accidens* predication in the *Posterior Analytics* 73$^\text{A}$34–73$^\text{B}$9. Here Aristotle writes that:

> One thing belongs to another in itself [translated into Latin as 'per se'] both if it belongs to it in what it is e.g. line to triangle and point to line (for their substance depends on these and they belong in the account which says what they are) and also if the things it belongs to themselves belong in the account which make clear what it is.   [1, p. 118]

At the heart of this distinction between *per se* and *per accidens* is the idea that a term is (truly) *per se* another term if the first term is part of the definition of the second term. This notion of definitional containment is something that Kilwardby accepts, and, as we shall see, is an important part of formalising Kilwardby's theory. In particular, Kilwardby argues [10, p. 129] that Aristotle's understanding of necessity in the modal syllogism should be restricted to *per se* necessary propositions. It should also be observed that "it is a characteristic of per se terms that, as a matter of necessity, whatever falls under them does so of necessity..." [18, pp. 21 and 27].

As for the relationship between terms and things in such propositions, Kilwardby says that:

> Another doubt. It seems that there are counter-examples against the conversion of the universal affirmative necessity-proposition in propositions like 'Every grammarian of necessity is human''...because while these are taken to be true their converses are false. To this it can be said that the conversion is blocked because of the different ways of taking the terms in subject-position and in predicate position. For when it is said 'Every grammarian of necessity is human' the term 'grammarian' stands for what it refers to; for if it where to stand for the quality co-signified by the name the proposition would be false. However, when it is stated conversely 'Some human of necessity is a grammarian', the term 'grammarian' is taken for the quality alone. And these different ways of taking terms like grammarian in subject- and predicate-position obstruct the conversion.   [10, p. 129]

This passage is interesting for a number of reasons. Here we see a standard example of a true *per accidens* necessary proposition, 'Every grammarian is human', and also of a standard worry about the conversion of such propositions. However, the main thing to observe for

our purposes here is that, when a term occurs in the subject position, it supposits for the thing, while when a term occurs in the predicate position it is taken for the quality.

From Kilwardby's remarks we can gleam the following 'conditions': for categorical propositions [see 10, pp. 473 and 513–515]:

Proposition	Gloss
Every A is necessarily B	A is per se included in B
No A is necessarily B	A is per se excluded from B
Some A is necessarily B	Some part of A is per se included in B
Not every A is necessarily B	Some part of A is per se excluded from B

There are a number of important things to observe about Kilwardby's theory. First, in this case, "is necessarily" is the same thing as"of necessity is" and "is of necessity". These are all intended to capture the case where the necessity is modifying the copula, as opposed to occurring as one of the terms. Second, Kilwardby treats negative propositions of necessity and assertorics differently. According to Kilwardby:

> In a negative assertoric proposition the predicate is actually denied of those things that are actually under the subject and not of those for which its contingent to be under it... But it is otherwise with a negative necessity proposition. For on account of the fact that the predicate is necessarily denied of the subject, the predicate in that proposition is also actually denied, under a modality of necessity, of all things that are under the subject, and of all things for which its contingent to be under the subject. [10, pp. 514–515]

The key thing to observe here is that in negative necessary propositions, what is denied of the subject must be such that it excludes those things that contingently would fall under the subject, not just those things which in fact do fall under the subject.

Kilwardby also lists a number of principles that he gives as rules for understanding when a syllogism is valid. According to Kilwardby, the following rules given necessary conditions for validities in the apodictic and assertoric syllogisms, where the apodictic necessities are restricted to *per se* necessities. This is important to note, as we will only focus on *per se* necessary propositions here [see 18, pp. 176–177]

P1. In every syllogism, one premise must be universal.
P2. In every syllogism, one premise must be affirmative.
P3. In first figure syllogisms, the major must be universal.
P4. In first figure syllogisms, the minor must be affirmative.

P5. In second figure syllogisms, the major must be universal.

P6. In second figure syllogisms, one of the premises must be negative.

P7. In third figure syllogisms, the minor must be affirmative.

P8. In first figure assertoric/necessity syllogisms, the necessity-proposition must be major.

P9. In second figure assertoric/necessity syllogisms, one premise must be a universal negative necessity proposition.

P10. In affirmative third figure assertoric/necessity syllogisms, the necessity premise must be a universal affirmative.

P11. In negative third figure assertoric/necessity syllogisms, the necessity premise must be a universal negative.

However, Kilwardby is aware that the reading he offers for modal propositions does not validate all of the principles that he discussies in his commentary. In particular, there is a problem with principle Thom gives as P9. The following passage from Thom nicely summarises the issue:[9]

> Aristotle says that Baroco XLL can be shown invalid by the same terms that were used earlier to refute Camestres LXL [. . .]. This counter-example is the one in Aristotle discussion of LX2 moods where the assertoric premise is not as-of-now. Moreover, the example is not convincing because if there is a white thing that can not be an animal, then presumably it could not be a man either, and so if the premises were true the conclusion would be true too [. . .]. So he [Kilwardby] proposes a Major that is true merely as-of-now. . . ; and he proposes a minor that states the *per se* necessity that some animal cannot be a man [. . .]. So the premises of his example fit his semantics whereas the premises of Aristotle's example did not. But what of the conclusion? Certainly the conclusion, that some animal cannot be white, would be regarded as true by Aristotle, since he takes it that no raven is possibly white. Whether Kilwardby would agree is not clear.   [18, pp. 168–169]

2.3. Natural and accidental inferences

Kilwardby was well aware that there were logical principles of inference that were not syllogistic in nature. While much of his discussion of

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[9] In this passage Thom uses the following notation. He uses the standard medieval menomic names for the various syllogistic inferences. He uses X to denote an assertoric proposition and L to denote a necessary proposition. In other passages he uses M to denote a possible proposition and Q to denote a contingent one.

the *Prior Analytics* is focused on syllogisms, Kilwardby does discuss a number of non-syllogistic inferences. In particular, Kilwardby spends a fair bit of space discussing hypothetical consequences in a few places. Kilwardby's remarks about the inferences that we now would think of as connexive are prompted as a discussion of the arguments that Aristotle makes in Book 2 57^{a}36–57^{b}17. In particular, the passage where Aristotle says:

> I mean, for example, that it is impossible that B should necessarily be great if A is white and that B should necessarily be great if A is not white. For whenever if this, A, is white it is necessary that that, B, should be great, and if B is great that C should not be white, then it is necessary if A is white that C should not be white. And whenever it is necessary, if one of the two things is, that the other should be, it is necessary if the latter is not, that the former should not be. If then B is not great A cannot be white. But if A is not white, it is necessary that B should be great, it necessarily results that if B is not great, B itself is great. But this is impossible. For if B is not great, A will necessarily not be white. If then if this is not white B must be great, it results that if B is not great [. . .].   [1, p. 91]

Kilwardby summarises part of the argument in this passage as follows:

> Second he [Aristotle] proves it as follows. If from A's being white it follows of necessity that B is great, then from the denial of the consequent, if B is not great, A is not white. But ex hypothesi it follows 'if A is not white, B is great'. So from first to last, it follows 'If B is not great, B is great'. But this is impossible. So it is impossible for something to follow of necessity from the one thing's being so, and not being so.[10]   [10, p. 1139]

After treating Aristotle's remarks about a similar argument with three terms, Kilwardby raises the following doubts:

> There is a doubt about the major proposition in his argument. For it seems that one and the same thing does follow from the same thing's being so and not being so, because if you are seated, then God exists, and if you are not seated then God exists, because the necessary follows

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[10] "Secundo ostendit eam sic. Si ad A esse album sequitur ex necessitate B esse magnum, ergo a destructione consequentis, si B non est magnum, A non est album. Sed sequitur ex ypotesi 'Si A non est album, B est magnum'. Ergo, a primo ad ultimum, sequitur 'Si B non est magnum, B est magnum'. Hoc autem impossibile. Ergo impossibile est ad idem esse et non esse ex necessitate aliquid sequi." [10, p. 1138, l. 49–55]

from anything. Further, if you are sitting then one of the following is true: that you are sitting, that you are not sitting. And if you are not sitting one of them is true. Further, a disjunctive follows from either of its parts, and in a natural inference. Hence it follows 'If you are sitting then you are sitting or not sitting' and 'If you are not sitting, then you are sitting or not sitting' and thus one and the same thing follows in a natural inference.[11] [10, p. 1141]

Kilwardby raises these objections to the position that, he argues, Aristotle endorses here. Earlier in the text, Kilwardby quotes Aristotle as endorsing the following: "He [Aristotle] states it. And this is BUT ⟨ IT'S IMPOSSIBLE THAT ONE AND THE SAME THING IS NECESSARILY SO WHEN THE SAME THING IS [...]. He expounds it, saying that when it follows of necessity If 'A is white, B is great' it does not follow of necessity 'If A is not white B is great' "[12] [10, p. 1139]. It should be observed that this is a 'contraposed' form of Boethius' Thesis, i.e. it is of the form 'if A then B, then it is not the case that if not A then B'. Our focus will be on the first and the third argument, as they are the most important ones for what follows. As Kilwardby's responses make clear, he is intending to endorse the principle of reasoning and defend it against three initial objections. As is clear from the formulation, what Kilwardby is going to argue for is the principle sometimes referred to as Abelard's Thesis ("It is impossible for the same thing to be of necessity both when a certain thing is and when that same thing is not" [10, p. 1139]), as well as for the validity of Aristotle's Thesis ("No proposition entails its own negation"), as he sees both occurring in the text of Aristotle.

The first doubt is terse, but is clearly a counter-example to Boethius' Thesis ("If A implies B, then A does not imply not B"). The first ar-

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[11] "dubitatur de maiore propositione sue rationis. Videtur enim ad idem esse et non esse sequi idem, quia si tu sedes deus est et non sedes deus est, quia necessarium sequitur ex quolibet. Adhuc si tu sedes alterum istorum est uerum, te sedere, te non sedere. Et si tu non sedes, alterum eorum est uerum. Adhuc disiunctiua sequitur ad utramque sui partem, et hoc naturali consequentia. Quare sequitur 'Si tu sedes, tu sedes uel tu non sedes' et 'Si tu non sedes, tu sedes uel non sedes'. Et ita naturali consequentia sequitur idem ad idem esse et non esse, et ita ex necessitate." [10, p. 1140, l. 68–77]

[12] "Ponit. Et hoc est IDEM AUTEM CUM SIT ⟨ ET NON SIT IMPOSSIBILE EX NECESSITATE IDEM ESSE ⟩ Exponit, dicens quod cum ex necessitate sequatur 'Si A est album, B est magnum', non sequitur ex necessitate 'Si A non est album, B est magnum' ".

gument purports to create a counter-example to the thesis by observing that a necessary truth is entailed by any proposition. Both:

1. If you are sitting, then God exists.
2. If you are not sitting, then God exists.

follow because 'God exists' is necessarily true, and a necessary truth is entailed by any proposition.

The third argument hinges on the role of disjunctions, and is also important because it notes the introduction of the term *natural*. This will be important for what follows. Boethius' Thesis appears to be false, since a disjunction follows from either of its disjuncts. But in that case, from P we can infer P or not P, and from not P we can infer P or not P.

Kilwardby begins his defence of Aristotle's Thesis by making the following distinction:

> To the first objection it should be said that inferences are of two types, viz. essential or natural (as when the consequent is naturally understood in the antecedent) and incidental [accidens] inferences. Now it is inferences of the latter type according to which we say that the necessary follows from anything; and Aristotle's remarks are not to be understood as being about these.[13]   [10, p. 1141]

Kilwardby's response is to distinguish two kinds of inferences. On the one hand, we have inferences that are essential or natural (*essentialis uel naturalis*) and on the other, we have inferences that are accidental (*accidens*, translated "incidental"). Natural inferences are those where the consequent is said to be understood in the antecedent, while accidental ones are not. Likewise, accidental inferences allow for a necessary truth to be entailed by any principle, while, presumably, natural inferences do not, since if they did, then it is not clear how this would be a response to the first objection.

In the case of the second and third objections, Kilwardby's responses are closely linked. He writes

> To the second objection it should be said that the same thing can follow in two ways, viz. either by virtue of the same thing in it (and

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[13] "Ad primum dicendum quod duplex est consequentia, scilicet essentialis uel naturalis (sicut quando consequens naturaliter intelligitur in antecedente), et consequentia accidentalis. Talis autem est consequentia secundum quam dicimus necessarium sequi ad quodlibet; et de tali non est intelligendus sermo Aristotelis." [10, p. 1140, l. 79–84]

in this way one and the same thing cannot follow from the same thing affirmed and denied, and this is what Aristotle means), or by virtue of different things in it (and in this way one and the same thing can follow from the same thing affirmed and denied, and this is not what Aristotle means; but the objection proceed on this basis)

The third is also solved by this means, because a disjunctive follows from both of its parts by virtue of different things in it, not by virtue of the same thing. So Aristotle understands that something does not follow of necessity from the same thing's being so and not being so, in a natural inference, and by virtue of the same thing.[14]

[10, pp. 1142–1143]

In the third argument we also see an important feature about natural consequences. Kilwardby's claim that "when a proposition follows in natural consequences from each of two contradictory antecedents, this is in virtue of two different things" is best understood in the following way: Say we start a deduction by assuming A and B. For ease of the example, assume we have one rule for or-introduction, namely from 'ϕ' infer 'ϕ or ψ' and another rule that allows us to infer 'ψ or ϕ' from 'ϕ or ψ'. Then we can infer 'A or B' from 'A' and we can infer 'B or A' from 'B', from which it follows that 'A or B'. Kilwardby's point here is that this inference is naturally valid, but the grounds or the basis for inferring the disjunction is different in each case. In one case it is based on the content of 'A' and in the other case it is based on the content of 'B'. This again is further evidence for thinking of natural consequences as preserving the content of the antecedent from which a particular consequent is inferred. The idea here is that natural consequences are sensitive to which propositions are used to ground or justify the inference that follows.

This concludes this historical discussion of Kilwardby's work. In the next section we will draw together two different logical theories that will be necessary to formalise Kilwardby's theory. As we have already seen, the distinction between *per se* and *per accidens* necessities is driven by the presence or lack of an essential connection between subject and predicate. As we shall see, one way of making sense of natural and accidental

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[14] "Ad secundum dicendum quod idem potest sequi dupliciter, scilicet aut gratia eiusdem in ipso (et sic non potest sequi idem ad idem affirmatiuu, et negatiuum, et hoc intendit Aristoteles; sic sutem processit oppositio).

Per hoc etiam soluitur tertium, quia disiunctiua sequitur ad utramque sui partem non gratia eiusdem in ea, sed gratia diuersorum. Intelligit ergo Aristoteles quod ad idem esse et non esse non sequitur ex necessitate aliquid naturali consequentia et gratia eiusdem." [10, p. 1142, l. 91–96]

inferences is to view them as a generalised form of the *per se*/*per accidens* distinction. The key idea here is that, in a natural consequence, the content of the consequent is contained in the antecedent, i.e. there is a (generalised) *per se* relationship between the antecedent and the consequent.

3. Semantics for Kilwardby

At the heart of our approach to Kilwardby's logic will be an augmentation of the usual possible worlds semantics with additional lattice-theoretic machinery that then imposes constraints on how objects are assigned to predicates. Our inspiration for this approach is due to recent treatments of Aristotle's modal logic [see, e.g., 11, 12] on the one hand, and recent literature on the logics of essence [7, 8] on the other. The order relations are only pre-orders (i.e. the order relationship is reflexive and transitive, but need not be anti-symmetric) on the set of terms. In what follows we only need our essential predications to be pre-orders on the set of terms, downwardly closed under accidental predication. Unlike Malink, as we shall see below, we will add an additional relationship that captures the idea of an essential incompatibility between terms. There is insufficient space to properly explore the relationship between our reconstruction and some of the ideas that Malink's work attribute to Aristotle. However, we will briefly sketch Malink's work in what follows:

The aim of Malink in [11] is:

> to provide a single formal model that exactly captures Aristotle's claims on (in)validity and inconclusiveness in the whole modal syllogistic. This model is intended to be not without a certain explanatory value for our understanding of why Aristotle's modal syllogistic looks the way it does; but in the following we shall focus on the logical reconstruction and sketch the explanatory background only in a cursory manner.
>
> [11, p. 96]

Our main focus here will be to discuss the logical reconstruction offered by Malink.

Malink starts by introducing the following three primitive relations Υ, E and $\tilde{E}$, where Υ stands for accidental predication, E for substantial essential predication, i.e. predication in the category of substance and $\tilde{E}$ for non-substantial essential predication, i.e. predication in one of the other Aristotelian categories. The discussion of how Malink uses these terms is interesting and the reader should see [11, pp. 97–102]. The

symbol $a\,\hat{E}\,b$ is a shorthand used by Malink for the disjunction $a\,E\,b$ or $a\,\tilde{E}\,b$. Malink's system is governed by the following five axioms:

ax 1. $\forall a\ a\,\boldsymbol{\Upsilon}\,a$
ax 2. $\forall a,b,c((a\,\boldsymbol{\Upsilon}\,b \wedge b\,\boldsymbol{\Upsilon}\,c) \to a\,\boldsymbol{\Upsilon}\,c)$
ax 3. $\forall a,b(a\,\hat{E}\,b \to a\,\boldsymbol{\Upsilon}\,b)$
ax 4. $\forall a,b,c((a\,E\,b \wedge b\,\boldsymbol{\Upsilon}\,c) \to a\,E\,c)$
ax 5. $\forall a,b,c((a\,\tilde{E}\,b \wedge b\,\boldsymbol{\Upsilon}\,c) \to a\,\tilde{E}\,c)$

Slightly less formally, $\boldsymbol{\Upsilon}$ is a reflexive and transitive relation; E and $\tilde{E}$ are transitive subrelations of $\boldsymbol{\Upsilon}$ that are downwardly closed under $\boldsymbol{\Upsilon}$. This also gives us an easy way of thinking about the class of models that this gives rise to. Let A be a non-empty set (of terms), and $\boldsymbol{\Upsilon}$, E and $\tilde{E}$ be subsets of A^2. Then, $\mathcal{A} = \langle A, \boldsymbol{\Upsilon} \rangle$ is a pre-order on A with E and $\tilde{E}$ as designated down-sets of $\mathcal{A}$.[15]

Malink is able to show that axioms 1–5 are sufficient to prove all of the validities that Aristotle claims are valid in the *Prior Analytics* and that the class of models satisfying 1–5 are sufficient to prove counterexamples for all of the syllogisms that Aristotle says are invalid.

What is important for our purposes is the idea that at the heart of Aristotle's modal logic are the various relationships that obtain between terms. Of particular interest in the case of the modal syllogistic are the relationships of being essentially predicated and of being accidentally predicated. It is these two notions that form the basis of how we should understand the predication relationships.

Second, the modal logic that Malink proposes for Aristotle rests on a very close connection between the truth of necessary propositions and the predicate being essentially predicated of the subject if the proposition is affirmative and the predicate being essentially incompatible with the subject, if the proposition is negative. Malink writes:

> We have seen that the universal affirmative necessity $\mathbb{N}^a ab$ is not obtained from the corresponding assertoric proposition $\mathbb{X}^a ab$ by adding modal sentential operators, but by replacing the $\boldsymbol{\Upsilon}$-copula of accidental predication by the $\hat{E}$-copula of essential predication.   [11, p. 106]

We may suspect Kilwardby would wholeheartedly agree with this. The basic idea of thinking of *per se* necessities in terms of the essential predication of one term for another, or the incompatibility of one term for another, is very much within the spirit of Kilwardby's project. What

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[15] Recall that, given an pre-order $\leq$ on S the down-set of y is $\{x \in S : x \leq y\}$.

we will draw on is the underlying idea that the terms of the modal logic are, in some sense 'ordered' by relationships of accidental containment and essential containment.

3.1. Semantic reconstructions

Our semantic reconstruction of Kilwardby's remarks will proceed as follows: First, we will introduce a general language $\mathfrak{L}_K$ for discussing these propositions as they are used in syllogistic propositions. Second, we will introduce an order relationship on terms, $\mathfrak{T}$, to give the structural relationships between inferences involving the various terms. This will then be augmented with machinery to allow us to discussion ontological structure. In the case of immediate signification, this will be handled by $\mathbb{T}$. Semantics for the various propositions will then be given and critically assessed. In the final section of the reconstruction, we will expand the analysis to allow for treatment of the connectives 'and', 'or', 'not', 'if . . . , then' etc.

First, let us define the language that we will be working in for this section. We define $\mathfrak{L}_K$ in the following way.

A language for Kilwardby models. Let $\mathfrak{L}_K = \langle \text{TERM}, \mathfrak{a}, \mathfrak{e}, \mathfrak{i}, \mathfrak{o}, \text{ps}, \text{pa} \rangle$ where:

- TERM is a countable set of terms;
- '$\mathfrak{a}$', '$\mathfrak{e}$', '$\mathfrak{i}$' and '$\mathfrak{o}$' are operators used to form the usual categorical propositions 'Every A is B', 'No A is B', 'Some A is B', and 'Not every A is B', as the formation sequence will make clear;
- 'ps' and 'pa' are modal operations corresponding to *per se* necessity and *per accidens* necessity.

The set of well-formed formulae $\text{WFF}_{\mathfrak{L}_K}$ is the smallest set closed under the following conditions for all $A, B \in \text{TERM}$ and $\circ \in \{\mathfrak{a}, \mathfrak{e}, \mathfrak{i}, \mathfrak{o}\}$:

- $A \circ B \in \text{WFF}_{\mathfrak{L}_K}$;
- $A \overset{\text{ps}}{\circ} B \in \text{WFF}_{\mathfrak{L}_K}$;
- $A \overset{\text{pa}}{\circ} B \in \text{WFF}_{\mathfrak{L}_K}$.

At the heart of our semantics are the following two ideas. The first is the usual idea from modal logic that, at various points, objects may fall under different predicates. This will be used to capture the idea of accidental necessity and possibility. The second idea is that the relationships between terms determine the essential and accidental relationships

that hold between terms. The idea here is that we interpret 'Every A is
per se B' as true if the definition of A is part of the definition of B. To
that end we will introduce three relationships. The first is one of generic
predication, which we will denote $\leq$. Given terms A, B, $A \leq B$ should
be read as 'A is predicated of B', or 'A is accidentally predicated of B',
where 'accidentally' is understood in an inclusive sense, i.e. it does not ex-
cluded essential predication.[16] The second operation, $\trianglelefteq$, is a relationship
of essential or definitional containment [again, see 11, p. 98]. As such,
$A \trianglelefteq B$ should be read as saying 'A is part of the essence of B'. Finally,
$|$ is the relationship of being *per se* repugnant or excluded by definition.

With these glosses in place, we will start by introducing the algebraic
machinery we will be using. Most of the ideas from lattice theory that
we will employ in this paper are standard, and can be found in [4].

Let T be an arbitrary set and $\leq$, $\trianglelefteq$ and $|$ be binary relations on T.
We require that $\leq$ is a pre-order (i.e. it is reflexive and transitive), $\trianglelefteq$ is
transitive and $|$ is irreflexive and symmetric. We further require that:

(1) for all $x, y \in T$, if $x \trianglelefteq y$, then $x \leq y$;
(2) for all $x, y, z \in T$, if $x \leq y$ and $y \trianglelefteq z$, then $x \trianglelefteq z$;
(3) for all $x, y \in T$, if $x \leq y$, then not $x \mid y$;
(4) for all $x, y, z \in T$, if $x \leq y$ and $y \mid z$, then $x \mid z$.

In what follows we shall refer to these as order properties (1)–(4). The
basic idea behind these structures is that we use the relations $\leq$, $\trianglelefteq$, and
$|$ to represent the principles of accidental predication, essential predi-
cation, and (definitional) incompatibility, respectively. The set T corre-
sponds to the set of terms that we are evaluating. Property (1) says that
if y is essentially predicated of x, then y is accidentally predicated of x.
Property (2) says that if y is accidentally predicated of x and z is essen-
tially predicated of y, then z is also essentially predicated of x. Property
(3) tells us that if y is accidentally predicated of x, then y cannot be
definitionally incompatible with x. By properties (1) and (3), this also
follows for essential predication. Finally, property (4) tells us that if y is
accidentally predicated of x and z is definitionally incompatible with y,
then z is also definitionally incompatible with x.

In this context, each of our conditions makes good sense. Every term
is contained in itself accidentally, however a term does not need to be de-
fined in terms of itself (and so the relation does not need to be reflexive),

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[16] $\leq$ is in the same spirit as γ that Malink uses [see 11, p. 97] for the introduction
of the term and his remarks about the inclusive sense of accidental predication.

and similarly for essential containment. In the case of incompatibility, we require that no term is incompatible with itself (in essence, a consistency requirement) and that incompatibility is symmetric (i.e. given terms A, B, if A is incompatible with B, then B is also incompatible with A). With these conditions in place, we can give truth-conditions for the various assertoric and modal propositions that Kilwardby treats as follows.

Semantics for assertoric and *per se* immediate signification. Let $\mathfrak{T} = \langle T, \leq, \trianglelefteq, | \rangle$ and $\mathbb{T} = \langle \mathfrak{T}, c \rangle$, where c is a function from TERM into T (for any term A, instead of '$c(A)$' we will write 'A^c'). Then we can give truth-conditions for the immediate sense of signification as follows:

$\mathbb{T} \vDash A\mathsf{a}B$ iff $A^c \leq B^c$,

$\mathbb{T} \vDash A\mathsf{e}B$ iff $\neg \exists t \in T(t \leq A^c \,\&\, t \leq B^c)$,

$\mathbb{T} \vDash A\mathsf{i}B$ iff $\exists t \in T(t \leq A^c \,\&\, t \leq B^c)$,

$\mathbb{T} \vDash A\mathsf{o}B$ iff $A^c \nleq B^c$,

$\mathbb{T} \vDash A\overset{\text{ps}}{\mathsf{a}}B$ iff $A^c \trianglelefteq B^c$,

$\mathbb{T} \vDash A\overset{\text{ps}}{\mathsf{e}}B$ iff $A^c \mid B^c$,

$\mathbb{T} \vDash A\overset{\text{ps}}{\mathsf{i}}B$ iff $\exists t \in T(t \trianglelefteq A^c \,\&\, t \trianglelefteq B^c)$,

$\mathbb{T} \vDash A\overset{\text{ps}}{\mathsf{o}}B$ iff $\exists t \in T\big(t \leq A^c$ and either $t \mid B^c$ or

$\qquad\qquad \exists t' \in T(B^c \leq t \,\&\, A^c \leq t') \,\&\, \forall t'' \in T(A^c \leq t'' \Rightarrow t'' \mid t)\big)$.

What this account does not provide for us is a natural way to separate *per se* necessity from *per accidens* necessity. For that, we will make use of the normal possible worlds semantics. We will also give semantics for the mediate sense of signification, i.e. the case where the word signifies the objects that fall under a given term. To do that we will make use of constant domain modal logic:

Semantics for assertoric mediate signification. Let $\mathcal{T} = \langle D, W, R, v \rangle$ where:

- D and W are non-empty sets (informally, D is a domain and W is a set of worlds),
- R is a reflexive binary relation on W,
- $v \colon W \times \text{TERM} \to \wp(D)$, where $\wp(D)$ is the power set of D.

Informally, we can think of $A\mathsf{a}B$ as saying that the interpretation of A, i.e. A^c, is accidentally predicated of the interpretation of B, i.e. of B^c.

In the case of AeB informally, the idea is that there is no element of T that is predicated of both the interpretations of A and B. Similar points apply for AoB and AiB. The modal cases are also similar, substituting essential for accidental in the case of $\overset{\text{ps}}{\text{a}}$ and incompatible for $\overset{\text{ps}}{\text{e}}$. We can then give semantic definitions for mediate signification as follows:[17]

$$\mathcal{T}, w \vDash AaB \text{ iff } v(w, A) \neq \emptyset \ \& \ v(w, A) \subseteq v(w, B),$$
$$\mathcal{T}, w \vDash AeB \text{ iff } v(w, A) \cap v(w, B) = \emptyset,$$
$$\mathcal{T}, w \vDash AiB \text{ iff } v(w, A) \cap v(w, B) \neq \emptyset,$$
$$\mathcal{T}, w \vDash AoB \text{ iff } v(w, A) = \emptyset \text{ or } v(w, A) \nsubseteq v(w, B),$$
$$\mathcal{T}, w \vDash A\overset{\text{pa}}{\text{a}}B \text{ iff } \forall x \in R[w] \ \mathcal{T}, x \vDash AaB,$$
$$\mathcal{T}, w \vDash A\overset{\text{pa}}{\text{e}}B \text{ iff } \forall x \in R[w] \ \mathcal{T}, x \vDash AeB,$$
$$\mathcal{T}, w \vDash A\overset{\text{pa}}{\text{i}}B \text{ iff } \forall x \in R[w] \ \mathcal{T}, x \vDash AiB,$$
$$\mathcal{T}, w \vDash A\overset{\text{pa}}{\text{o}}B \text{ iff } \forall x \in R[w] \ \mathcal{T}, x \vDash AoB,$$

where $R[w] := \{x \in W : wRx\}$.

Here the informal reading of these proposition is the usual medieval one, e.g., AaB is true if there is at least one thing that is A, and everything that is A is B, and similarly for the other categorical propositions. The modal propositions are ones where necessity has wide scope, e.g., it is necessary that every A is B, and likewise for the other cases. Now, in order to bring *per se* and *per accidens* modalities together into a common system, we impose for all terms A and B the following further conditions:

1. $A^c \leq B^c$ if and only if for some $w \in W$: $v(w, A) \subseteq v(w, B)$;
2. if $A^c \trianglelefteq B^c$ then for all $w \in W$: $v(w, A) \subseteq v(w, B)$;
3. if $A^c \mid B^c$ then for all $w \in W$: $v(w, A) \cap v(w, B) = \emptyset$.

In this section we bring together the role and function of how terms are being handled in the immediate model and combine them with a more standard semantic interpretation of the relationship between terms and objects by adding three new conditions. The idea is that this is a way

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[17] There is an important set of questions around the role that existential import plays with propositions in the mediate and immediate sense [see 18, p. 24, for some discussion of this in addition to Kilwardby's discussion]. A brief summary would be to say: "Kilwardby's doctrine of affirmative necessity-propositions doesn't commit him to saying that such propositions imply that their subjects apply to something that exists in nature" [18, p. 24]. As such, the semantics given below do *not* require *per se* or *per accidens* necessities to have existential import.

of bringing together the remarks about per se, which depends on the relationships between terms, and per accidens.

Kilwardby models for immediate signification. Let $\mathfrak{K}_{\mathrm{I}} = \{D, W, T, R, \leq,$ $\trianglelefteq, |, c, v\}$, where:

- D and W are non-empty sets (D is our domain; W is a set of worlds),
- T is a non-empty set of subsets of D, i.e. $\emptyset \neq T \subseteq \wp(D)$,
- R is a binary relation on W,
- $\leq$, $\trianglelefteq$ and $|$ are binary relations on T which satisfy the conditions previously given for them,
- $c \colon \mathrm{TERM} \to T$,
- $v \colon W \times \mathrm{TERM} \to \wp(D)$.

Then we have the following truth-conditions for the various formulae:

$\mathfrak{K}_{\mathrm{I}}, w \vDash A\mathsf{a}B$ iff $A^c \leq B^c$,

$\mathfrak{K}_{\mathrm{I}}, w \vDash A\mathsf{e}B$ iff $\neg\exists t \in T(t \leq A^c \ \& \ t \leq B^c)$,

$\mathfrak{K}_{\mathrm{I}}, w \vDash A\mathsf{i}B$ iff $\exists t \in T(t \leq A^c \ \& \ t \leq B^c)$,

$\mathfrak{K}_{\mathrm{I}}, w \vDash A\mathsf{o}B$ iff $A^c \nleq B^c$,

$\mathfrak{K}_{\mathrm{I}}, w \vDash A\overset{\mathrm{ps}}{\mathsf{a}}B$ iff $A^c \trianglelefteq B^c$,

$\mathfrak{K}_{\mathrm{I}}, w \vDash A\overset{\mathrm{ps}}{\mathsf{e}}B$ iff $A^c \mid B^c$,

$\mathfrak{K}_{\mathrm{I}}, w \vDash A\overset{\mathrm{ps}}{\mathsf{i}}B$ iff $\exists t \in T((t \leq A^c \ \& \ t \trianglelefteq B^c) \text{ or } (t \trianglelefteq A^c \ \& \ t \leq B^c))$,

$\mathfrak{K}_{\mathrm{I}}, w \vDash A\overset{\mathrm{ps}}{\mathsf{o}}B$ iff $\exists t \in T(t \mid A^c \ \& \ t \mid B^c)$,

$\mathfrak{K}_{\mathrm{I}}, w \vDash A\overset{\mathrm{pa}}{\mathsf{a}}B$ iff $\forall x \in R[w] \ v(x, A) \subseteq v(x, B)$,

$\mathfrak{K}_{\mathrm{I}}, w \vDash A\overset{\mathrm{pa}}{\mathsf{e}}B$ iff $\forall x \in R[w] \ v(x, A) \cap v(x, B) = \emptyset$,

$\mathfrak{K}_{\mathrm{I}}, w \vDash A\overset{\mathrm{pa}}{\mathsf{i}}B$ iff $\forall x \in R[w] \ v(x, A) \cap v(x, B) \neq \emptyset$,

$\mathfrak{K}_{\mathrm{I}}, w \vDash A\overset{\mathrm{pa}}{\mathsf{o}}B$ iff $\forall x \in R[w] \ v(x, A) \nsubseteq v(x, B)$.

With this in place, we can define logical consequence for these models as follows. Let Γ be a set of formulae and ϕ be a formula. We say that ϕ is a *logical consequence* of Γ and write $\Gamma \vDash \phi$ iff for any model $\mathfrak{K}_{\mathrm{I}} = \{D, W, T, R, \leq, \trianglelefteq, |, c, v\}$ and any $w \in W$: if $\mathfrak{K}_{\mathrm{I}}, w \vDash \gamma$ for any $\gamma \in \Gamma$, then $\mathfrak{K}_{\mathrm{I}}, w \vDash \phi$.

3.2. Adequacy of our model

As we have shown in our discussion of Kilwardby's modal logic (above), Kilwardby gives rules that govern when a syllogism is valid [see 18, pp. 176–177]. According to Kilwardby, these rules given necessary conditions for validities in the apodictic and assertoric syllogisms, where the apodictic necessities are restricted to *per se* necessities. This is important to note, as we will only focus on *per se* necessary propositions here. As we have already discussed, Kilwardby is aware that his model does not entirely track these conclusions. When we limit our attention to *per se* necessary propositions, as Kilwardby does, we can prove that most of these hold. In particular, properties P1–P7 are standard features of assertoric syllogisms and can be shown to follow from our semantics. We will focus here on the modal claims.

Before we move on to the proof of this, we will often appeal to the following lemma, whose proof is routine.

LEMMA 1. *Let $\circ$ range over the categorical operations 'a', 'e', 'i','o' and let $*$ range over 'ps', 'pa' and blank. Then:*

1. $A\overset{\text{ps}}{\circ}B \vDash A\overset{\text{pa}}{\circ}B$
2. $A\overset{\text{pa}}{\circ}B \vDash A\circ B$
3. $A\overset{\text{ps}}{a}B \vDash A\overset{*}{i}B$
4. $A\overset{\text{ps}}{e}B \vDash A\overset{*}{o}B$
5. $A\overset{\text{pa}}{a}B \vDash A\overset{\text{pa}}{i}B$
6. $A\overset{\text{pa}}{e}B \vDash A\overset{\text{pa}}{o}B$
7. $A\overset{\text{pa}}{a}B \vDash AiB$
8. $A\overset{\text{pa}}{e}B \vDash AoB$
9. $AaB \vDash AiB$
10. $AeB \vDash AoB$
11. $A\overset{\text{ps}}{e}B \vDash A\overset{*}{e}B$
12. $A\overset{*}{i}B \vDash A\overset{*}{i}B$

A triple $S = \langle M, m, C \rangle$ consists of a pair $\langle M, m \rangle$ of premises from which a conclusion C can be drawn. We require that M, m satisfy the following conditions [this definition is based on the one found in 20]:

1. M, m, and C are categorical propositions,
2. M, m, and C have exactly three terms,
3. the predicate of C occurs in M,
4. the subject of C occurs in m,
5. M and m share a common term that does not occur in C.

1st figure	XXX	LLL	LXL	XLL
Barbara	**25b37**	**29b36**	**30a17**	*30a23*
Celarent	**25b40**	**29b36**	**30a17**	*30a23*
Darii	**26a23**	**29b36**	**30a17**	*30b2*
Ferio	**26a25**	**29b36**	**30b1**	*30b5*
2nd Figure				
Cesare	**27a5**	**29b36**	**30b9**	*30b18*
Camestres	**27a9**	**29b36**	*30b20*	**30b14**
Festino	**27a32**	**29b36**	**31a5**	
Baroco	**27a36**	**30a6**	*30a10*	*31a15*
3rd Figure				
Darapti	**28a17**	**29b36**	**31a24**	**31a31**
Felapton	**28a26**	**29b36**	**31a33**	*31a37*
Disamis	**28b7**	**29b36**	*31b31*	**31b12**
Datisi	**28b11**	**29b36**	**31b19**	*31b20*
Bocardo	**28b15**	**30a7**	*32a4*	*31b40*
Ferison	**28b31**	**29b36**	**31b33**	*32a1*

Figure 1. Apodictic syllogisms in Aristotle

It should be observed that the definition of S given here is entirely syntactic. In what follows, a triple will count as a syllogism even if one or both of the premises in the syllogism are false.

We say that a triple $S = \langle M, m, C \rangle$ is *valid* iff C is a logical consequence of $\{M, m\}$, i.e. $M, m \vDash C$. We will denote this by $\vDash S$. Every valid triple is called a *syllogism*. With these definitions in place it is straight-forward to verify that each of P1–P8, P11, and P12 hold for modal syllogisms. One simply needs to check all of the relevant cases.

This completes our treatment of the modal syllogisms that Kilwardby outlines in his treatment of Aristotle's modal logic. There is an interesting implication of our analysis that we should mention at this point. In [11, p. 117] Malink provides a standard table that tracks all of the LXL, LLL, XLL, and XXX syllogisms that Aristotle claims are valid as well as references to where those validities are affirmed in the *Prior Analytics*. We reproduce a modified version of the table in Figure 1, omitting the numbers in square brackets (which are references to theorems in Malink's paper), stating the syllogisms using their medieval mnemonic names, and using L where Malink uses N. Valid syllogisms are in bold font while invalid syllogisms are in italics. If the square is blank, Aristotle does not say if the syllogism is valid or not.

As the reader can verify from what we have shown above, our reconstruction of Kilwardby correctly tracks the validities in Aristotle's modal logic, with the exception of those relating to P9. In particular, XLL Baroco does not follow. This disconnect tracks a similar set of issues that Kilwardby identifies in his interpretation of Aristotle.[18]

4. Connexive implication

As we have already observed, Kilwardby provides a characterisation of natural inferences that makes use of logical connectives other than the categorical operations 'ɑ', 'e', 'i' and 'o'. In particular, Kilwardby discusses implication and disjunction. The main results of Kilwardby's theory can be summarised as follows:

1. The natural implication relationship validates Aristotle's Thesis and Abelard's Thesis.
2. Disjunction introduction is a natural inference.
3. The natural implication relationship does not validate *ex impossibile quodlibet*.

Given what we have seen in our model, we can motivate this sort of implication as a generalisation of the notion of meaning containment. Just as *per se* necessary propositions capture the idea that the meaning of one term is contained in the other, the more generalised idea is that the meaning of the consequent of a hypothetical should be contained in the antecedent. In a similar spirit, because the account of logic is derived from syllogistic theory, the failure of *ex falso* in the case of the syllogism should be preserved when it is extended to natural consequences. We have also seen textual support in Kilwardby endorsing this for natural inferences.

Inferences like disjunction introduction (and also, as we shall see, conjunction elimination) preserve this notion of meaning containment, since in each case the antecedent 'contains in it' the meaning relationship necessary to ground the truth of the consequent. In the framework we are working with here, this idea becomes the notion that the definition of any term does not contain the negation of that term, or, taken from

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[18] I would like to thank one of the reviewers of this paper for pointing out that P9 does not follow on the semantics given.

the object side, for no term, if an object satisfies the definition of that term, does it then fail to satisfy the definition of that term.[19]

The primary challenge is in formulating the notion of negation that we want to work with. What we want to do is use the relation | to define a notion of negation that captures the idea of a proposition being *per se* impossible.[20] This is not a notion that Kilwardby treats in a general setting, but is only discussed (as far as I am aware) in the context of categorical propositions. We can think of it as saying that a term or idea is *per se* impossible if there is something incompatible about the way that the idea expressed by the proposition has been defined.

Quasi-first-order language. Let $\mathcal{L}_{\text{qfo}} = \{\text{TERM}, \mathfrak{a}, \mathfrak{e}, \mathfrak{i}, \mathfrak{o}, \neg, \Box, \boxplus, \wedge, \vee, \rightarrow\}$. The definition of well-formed formulae of $\mathcal{L}_{\text{qfo}}$ is a natural generalisation of what we have already seen, where we now use '$\boxplus$' to mean per se necessary and '$\Box$' to mean necessary. The set $\text{WFF}_{\mathfrak{L}_{\text{qfo}}}$ of well-formed formulae is the smallest set that satisfies the following conditions:

- if $A, B \in \text{TERM}$ and $\circ \in \{\mathfrak{a}, \mathfrak{e}, \mathfrak{i}, \mathfrak{o}\}$, then $A \circ B$, $A \overset{\text{ps}}{\circ} B$ and $A \overset{\text{pa}}{\circ} B$ belong to $\text{WFF}_{\mathfrak{L}_{\text{qfo}}}$,
- if $\phi, \psi \in \text{WFF}_{\mathfrak{L}_{\text{qfo}}}$, then $\neg\phi$, $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$, $\Box\phi$ and $\boxplus\phi$ belong to $\text{WFF}_{\mathfrak{L}_{\text{qfo}}}$.

The formulas of the form $A \circ B$, $\Box A \circ B$ and $\boxplus A \circ B$ will be called *atomic*.

$\mathcal{L}_{\text{qfo}}$ is a quasi-first-order language, because all its atomic formulae are categorical propositions. Most importantly, it does not allow for singular terms. The choice to limit our attention to languages with categorical terms as the basic elements is primarily driven by the ease with which one can generalise the *Ref* relationship in what follows.

Kilwardby models for $\mathcal{L}_{\text{qfo}}$. Our Kilwardby models for the language $\mathcal{L}_{\text{qfo}}$ will be similar to the previous models we considered. The only thing that will need to change is how we assign truth-conditions to newly added operations.

Let $\mathfrak{L} = \{D, W, T, R, \leq, \trianglelefteq, |, c, v\}$ be the same as $\mathfrak{K}_{\text{I}}$ from p. 469. With the exception of *per se* necessity, truth-conditions for the logical operations are defined as usual for $\circ \in \{\mathfrak{a}, \mathfrak{e}, \mathfrak{i}, \mathfrak{o}\}$:

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[19] It should be noted that one very natural class of terms that *do* seem to violate this condition are those with liar-like properties, e.g., the membership condition of the Russell set.

[20] Informally, we can think of *per se* impossibility as a kind of definition incompatibility.

$\mathfrak{L}, w \vDash A \circ B$ iff as in our treatment of the modal syllogism,

$\mathfrak{L}, w \vDash A \overset{\text{pa}}{\circ} B$ iff as in our treatment of the modal syllogism,

$\mathfrak{L}, w \vDash A \overset{\text{ps}}{\circ} B$ iff as in our treatment of the modal syllogism,

$\mathfrak{L}, w \vDash \neg \phi$ iff $\mathfrak{L}, w \nvDash \phi$,

$\mathfrak{L}, w \vDash \phi \wedge \psi$ iff $\mathfrak{L}, w \vDash \phi$ & $\mathfrak{L}, w \vDash \psi$,

$\mathfrak{L}, w \vDash \phi \vee \psi$ iff $\mathfrak{L}, w \vDash \phi$ or $\mathfrak{L}, w \vDash \psi$,

$\mathfrak{L}, w \vDash \phi \to \psi$ iff $\mathfrak{L}, w \nvDash \phi$ or $\mathfrak{L}, w \vDash \psi$,

$\mathfrak{L}, w \vDash \Box \phi$ iff $\forall x \in R[w]$ $\mathfrak{L}, x \vDash \phi$.

In order to define how the *per se* operations work, we will make use of the following. Let $Ref \colon \text{WFF}_{\mathfrak{L}_{\text{qfo}}} \to \wp(T)$ governed by the following conditions for all $A, B \in \text{TERM}$, $\circ \in \{\mathfrak{a}, \mathfrak{e}, \mathfrak{i}, \mathfrak{o}\}$ and $\phi, \psi \in \text{WFF}_{\mathfrak{L}_{\text{qfo}}}$:

- $Ref(A \overset{*}{\circ} B) = \{A^c, B^c\}$,
- $Ref(\phi \vee \psi) = Ref(\phi) \cup Ref(\psi)$,
- $Ref(\phi \wedge \psi) = Ref(\phi) \cap Ref(\psi)$,
- $Ref(\phi \to \psi) = \{t \in Ref(\phi) : \exists u \in Ref(\psi)\ t \trianglelefteq u\}$,
- $Ref(\neg \phi) = \{t \in T : \forall u \in Ref(\phi)\ t \mid u\}$,
- $Ref(\boxplus \phi) = Ref(\phi) = Ref(\Box \phi)$.

Informally, we can think of *Ref* as the operation that collects the interpretations of various terms that occur in formulae.

Finally, we generalise $\trianglelefteq$ and $\mid$ to subsets of T as follows: Let $\mathbb{A}, \mathbb{B}$ be sets of elements of T and m, n be elements of T then:

- $\mathbb{A} \trianglelefteq \mathbb{B}$ if and only if $\forall t \in \mathbb{A} \exists u \in \mathbb{B}$: $t \trianglelefteq u$.
- $\mathbb{A} \mid \mathbb{B}$ if and only if $\forall t \in \mathbb{A} \forall u \in \mathbb{B}$: $t \mid u$.
- $m \trianglelefteq \mathbb{B}$ if and only if $\exists u \in \mathbb{B}$: $m \trianglelefteq u$.
- $m \mid \mathbb{B}$ if and only if $\forall u \in \mathbb{B} : m \mid u$.
- $\mathbb{A} \trianglelefteq n$ if and only if $\forall t \in \mathbb{A}$: $t \trianglelefteq n$.
- $\mathbb{A} \mid n$ if and only if $\forall t \in \mathbb{A}$: $t \mid n$.

The idea here is that of ϕ is contained in ψ just in case the content of ϕ is part of the content of ψ. Likewise, ϕ and ψ are incompatible (we might say strongly incompatible) if none of their content is consistent.

The idea in each case will then be to use the operation $\boxplus$ to change the truth-conditions under which the main connective of the formula is evaluated. In the cases with categorical formulae, it transforms them into *per se* formulae if they are not, and leaves them unchanged if they are already *per se*. In the case of the new operations, we use the *Ref*

clauses to give the relevant truth-conditions. The truth-conditions for $\overset{\text{ps}}{\circ}, \overset{\text{pa}}{\circ}$ are as before.

$$\mathfrak{L}, w \vDash \boxplus A \overset{\text{ps}}{\circ} B \text{ iff } \mathfrak{L}, w \vDash A \overset{\text{ps}}{\circ} B,$$

$$\mathfrak{L}, w \vDash \boxplus A \overset{\text{pa}}{\circ} B \text{ iff } \mathfrak{L}, w \vDash A \overset{\text{ps}}{\circ} B,$$

$$\mathfrak{L}, w \vDash \boxplus A \circ B \text{ iff } \mathfrak{L}, w \vDash A \overset{\text{ps}}{\circ} B,$$

$$\mathfrak{L}, w \vDash \boxplus \neg \phi \text{ iff } Ref(\phi) = \emptyset,$$

$$\mathfrak{L}, w \vDash \boxplus(\phi \to \psi) \text{ iff } Ref(\phi) \trianglelefteq Ref(\psi),$$

$$\mathfrak{L}, w \vDash \boxplus(\phi \wedge \psi) \text{ iff } Ref(\phi) \cap Ref(\psi) \neq \emptyset,$$

$$\mathfrak{L}, w \vDash \boxplus(\phi \vee \psi) \text{ iff } Ref(\phi) \cup Ref(\psi) \neq \emptyset.$$

The only clause that should require some comment is our definition of $\boxplus\neg\phi$. Here, the idea is that $\boxplus\neg$ tells us that the formula that follows it is '*per se* false'. The key idea to observe is that $Ref(\phi)$ will be empty when their is a term A which occurs in ϕ such that $A^c \in Ref(\phi)$ and $A^c \in Ref(\neg\phi)$, i.e. the term A^c occurs in both ϕ and its negation. This leads to a *per se impossibility*, because $A^c | A^c$ is always false.

   With these in place, we will now show that using $\boxplus$ and $\to$ we can define a notion of *natural* implication. As before, we start with the standard account of logical consequence. We say that:

$\Gamma \vDash \phi$ iff for any model $\mathfrak{L}$ and any $w \in W$, if $\mathfrak{L}, w \vDash \Gamma$ then $\mathfrak{L}, w \vDash \phi$.

Moreover, we say that ϕ is a *natural implication of* Γ (denoted by $\vDash_N \Gamma \to \phi$) iff for any model $\mathfrak{L}$ and $w \in W$ we have $\mathfrak{L}, w \vDash \bigwedge_{\gamma \in \Gamma} \boxplus\gamma \to \boxplus\phi$. In other words, one obtains a natural consequence relation by taking each element of Γ, placing $\boxplus$ in front of each element, taking the conjunction of all the elements in Γ. We can express this by generalising the notion of Ref to sets of formulae by saying that $Ref(\Gamma) = \{Ref(\gamma) : \gamma \in \Gamma\}$ and seeing if $\Gamma \trianglelefteq \phi$, i.e. $\forall t \in Ref(\Gamma) \exists u \in Ref(\psi): t \trianglelefteq u$.

   As we will show, for this notion of implication, we obtain the main connexive features of Kilwardby's systems.

   There is another option, which we highlight here but will not develop in further detail. The notion of a natural consequence relation, $\vDash_N$, can also be defined in a spirit similar to the notion of implication. It should be noted, however, that if this is done in the natural way, i.e.:

$$\Gamma \vDash_N \phi \text{ if and only if } \boxplus\Gamma \vDash \boxplus\phi,$$

where $\boxplus\Gamma := \{\boxplus\gamma : \gamma \in \Gamma\}$. It will not capture some of the inferences

that Kilwardby desires to hold.[21] The following definition seems closer in spirit to the idea of natural consequence:

$$\Gamma \vDash_N \phi \text{ if and only if } \Gamma \trianglelefteq \phi.$$

The following lemma will be helpful:

LEMMA (Incompatibility Lemma). *For no $t \in T$ do we have $t \in Ref(\phi)$ and $t \in Ref(\neg\phi)$.*

PROOF. Assume for a contradiction, so that for some t, $t \in Ref(\phi)$ and $t \in Ref(\neg\phi)$. By the definition of $Ref(\neg\phi)$ it follows that $t \mid Ref(\phi)$. But $t \in Ref(\phi)$ and $t \mid Ref(\phi)$ entail $t \mid t$, by the requirement of strong incompatibility. But this is impossible as $\mid$ is irreflexive. □

It should be remarked that $\vDash_N$ is an unusual kind of consequence relationship and that we lack the property: if $\Gamma \cup \{\phi\} \vDash_N \psi$ then $\Gamma \vDash_N \phi \rightarrow \psi$. To see this, first observe that $\phi \wedge \neg\phi \vDash_N \psi$. This follows since $Ref(\phi) \cap Ref(\neg\phi) = \emptyset$, by the incompatibility lemma. However, it does not follow that $\vDash_N (\phi \wedge \neg\phi) \rightarrow \psi$. To see this, take any model where $Ref(\psi)$ is non-empty. Since $Ref(\phi) \cap Ref(\neg\phi) = \emptyset$, it follows that $(\phi \wedge \neg\phi) \ntrianglelefteq \psi$. The way that the various operations are defined is not as one would expect in classical logic.

For example, $\phi \wedge \psi \vDash \phi$ holds because $Ref(\phi \wedge \psi) \subseteq Ref(\phi)$. From this it follows that if $Ref(\phi \wedge \psi) \subseteq Ref(\phi)$ and $Ref(\phi \wedge \psi) \neq \emptyset$ then $Ref(\phi) \neq \emptyset$.

The following three theses capture the main connexive features of Kilwardby's notion of implication:

Aristotle's Thesis. We claim that $\vDash_N \neg(\neg\phi \rightarrow \phi)$: this holds, if and only if $\vDash \boxplus\neg(\neg\phi \rightarrow \phi)$. We will show that $\neg\exists A^c \in Ref(\neg\phi \rightarrow \phi)$ such that A occurs in ϕ. We will show that $Ref(\neg\phi \rightarrow \phi) = \emptyset$. To see this, observe that $Ref(\neg\phi \rightarrow \phi) = \{A^c \in Ref(\neg\phi) : \exists B^c \in Ref(\phi), \text{ and } A^c \trianglelefteq B^c\}$. Now, assume for a contradiction that $\{A^c \in Ref(\neg\phi) : \exists B^c \in Ref(\phi), \text{ and } A^c \trianglelefteq B^c\}$ is non-empty and call the witness of this C^c. Then $C^c \in Ref(\neg\phi)$, and there is some B^c, such that $B^c \in Ref(\phi)$ and $C^c \trianglelefteq B^c$. It then follows by the definition of Ref that $C^c \mid Ref(\phi)$ and so $C^c \mid B^c$. However, since $C^c \trianglelefteq B^c$, it follows that $C^c \leq B^c$ and so not $C^c \mid B^c$ which is a contradiction. Hence $Ref(\neg\phi \rightarrow \phi) = \emptyset$ and so $\neg\exists A^c \in Ref(\neg\phi \rightarrow \phi)$.

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[21] This was helpfully suggested by one of the reviewers.

Abelard's Thesis. We claim that $\vDash_N \neg((\phi \rightarrow \neg\psi) \wedge (\phi \rightarrow \psi))$. It suffices to show that $Ref(\phi \rightarrow \neg\psi) \cap Ref(\phi \rightarrow \psi) = \emptyset$. Assume for a contradiction that there is an A^c (such that A occurs in ϕ) such that $A^c \in Ref(\phi \rightarrow \neg\psi) \cap Ref(\phi \rightarrow \psi)$, i.e. the following both hold:

1. $A^c \in Ref(\phi)$ and there is a B^c such that $B^c \in Ref(\psi)$ and $A^c \trianglelefteq B^c$,
2. $A^c \in Ref(\phi)$ and there is a B^c such that $B^c \in Ref(\neg\psi)$ and $A^c \trianglelefteq B^c$.

Let G^c be a witness to 1 and E^c be a witness to 2. Then, by 1, we have $A^c \trianglelefteq G^c$ and $G^c \in Ref(\psi)$; and, by 2, we have $A^c \trianglelefteq E^c$ and $E^c \in Ref(\neg\psi)$. By the definition of $Ref(\neg\psi)$ it follows that $E^c \mid Ref(\psi)$ and so $E^c \mid G^c$, by the definition of $\mid$. Since we have $A^c \trianglelefteq E^c$, we also have $B^c \leq E^c$, by order property (1); and so $B^c \mid G^c$, by order property (1). However, $A^c \trianglelefteq G^c$, and so $A^c \leq G^c$ by order property (4); and so $A^c \mid G^c$ does not hold, by order property (3). This is a contradiction. Hence $Ref(\phi \rightarrow \neg\psi) \cap Ref(\phi \rightarrow \psi) = \emptyset$.

Boethius' Thesis. We claim that $\vDash_N ((\phi \rightarrow \psi) \rightarrow \neg(\phi \rightarrow \neg\psi))$. This holds if and only if for all A^c, if $A^c \in Ref(\phi \rightarrow \psi)$ then there is a B^c such that $B^c \in Ref(\neg(\phi \rightarrow \neg\psi))$ and $A^c \trianglelefteq B^c$.

Take an arbitrary term, C and assume that $C \in Ref(\phi \rightarrow \psi)$. Then $C^c \in Ref(\phi)$ and there is a G^c such that $G^c \in Ref(\psi)$ and $C^c \trianglelefteq G^c$. Now, we claim that $G^c \in Ref(\neg(\phi \rightarrow \neg\psi))$. It suffices to show that $G^c \mid Ref(\phi \rightarrow \neg\psi)$. So, take an arbitrary E^c such that $E^c \in Ref(\phi \rightarrow \neg\psi)$. It then follows that there is a B^c such that $E^c \trianglelefteq B^c$ and $B^c \mid Ref(\psi)$. Call this F^c. Then $E^c \trianglelefteq F^c$ and $F^c \mid Ref(\psi)$. From the second conjunct it follows by the definition of $\mid$ that $F^c \mid G^c$. Hence by order properties (1) and (4), it follows that $G^c \mid E^c$. As E^c was arbitrary, this holds for all $E^c \in Ref(\phi \rightarrow \neg\psi)$. Hence, by the definition of $\mid$, we have $G^c \mid Ref(\phi \rightarrow \neg\psi)$ as claimed.

It is also easy to see that disjunction introduction is a natural relationship. To see this, observe that $Ref(\phi) \trianglelefteq Ref(\phi \vee \psi)$, since for any $A^c \in Ref(\phi)$ it follows that $A^c \in Ref(\phi) \cup Ref(\psi)$ (for any ψ), and hence $A^c \in Ref(\phi \vee \psi)$.

5. Conclusion

Our aim in this paper has been to discuss how Robert Kilwardby understood connexive logic and to argue that Kilwardby's understanding of implication is grounded, in part, on his understanding of necessity. In

particular, the two kinds of necessity that Kilwardby develops serve as a basis for reconstructing his distinction between natural and accidental implications. As we have shown in our formal reconstruction, one can obtain a connexive account of implication that is also in keeping with Kilwardby's remarks on *per se* necessity. It should be observed, in concluding, that Kilwardby was not the only medieval philosopher in this period to endorse connexive principles (e.g., both Peter Abelard and Peter of Spain endorsed connexive theses [5, p. 231–239]). It is not clear how far an essentialist interpretation of modality may have been guiding other connexive logicians. However, for Kilwardby the notion of *per se* necessity and the notion of natural entailment are deeply connected both with each other and with an underlying ontological interpretation of necessity, which is closely connected with ideas in both the *Topics*, and the *Posterior Analytics*. This provides an interesting perspective on connexive logic that both helps us understand the history of connexive logic, and, I hope, may suggest interesting and different ways to approach connexive logic in a modern context.

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SPENCER JOHNSTON
University of Cambridge
Faculty of Philosophy
scj43@cam.ac.uk
spencer.johnston.8603@gmail.com