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Disjoint Logics

Abstract. We will present all the *mixed* and *impure* disjoint three-valued logics based on the Strong Kleene schema. Some, but not all of them, are (inferentially) empty logics, while one of them is trivial. We will compare them regarding their relative strength. We will also provide a recipe for building philosophical interpretations for each of these logics, and show why the kind of permeability that characterises them is not such a bad feature. Finally, we will present a three-side sequent system for most of these logics.

Keywords: disjoint logic; three-valued logics; substructural logics; epistemic commitments; permeability

1. Introduction

A logic is usually defined as a language plus a consequence relation. And what has came to be known as "the Tarskian paradigm" provides a clear answer to the question of what a consequence relation is. Nevertheless, the Tarskian paradigm has been challenged in different ways. Here, we will concentrate on one particular reform of it, that generates what we call *disjoint logics*. But to fully understand what it takes for a logic to be disjoint, it will be convenient to start by defining what "the Tarskian paradigm" is.

For the sake of simplicity, we will focus on propositional logics. For $\mathcal{L}$ a propositional language and Var a countably infinite set of propositional variables, we denote by $\mathrm{FOR}(\mathcal{L})$ the absolutely free algebra of formulae of $\mathcal{L}$, whose universe is $\mathrm{FOR}(\mathcal{L})$. In what follows, the propositional language will be fixed to be the set $\{\neg, \wedge, \vee\}$.

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A *Tarskian consequence relation over a propositional language* $\mathcal{L}$ is a relation $\vDash \subseteq \wp(\mathrm{FOR}(\mathcal{L})) \times \mathrm{FOR}(\mathcal{L})$ obeying the following conditions for all $A \in \mathrm{FOR}(\mathcal{L})$ and for all $\Gamma, \Delta \subseteq \mathrm{FOR}(\mathcal{L})$:[1]

1. $\Gamma \vDash \Delta$ if there is an $A \in \Delta$ such that $A \in \Gamma$ (reflexivity)
2. If $\Gamma \vDash \Delta$ and $\Gamma \subseteq \Gamma'$, $\Delta \subseteq \Delta'$, then $\Gamma' \vDash \Delta'$ (monotonicity)
3. If $\Gamma \vDash \Delta, A$ and $\Gamma', A \vDash \Delta'$, then $\Gamma, \Gamma' \vDash \Delta, \Delta'$ (cut)

Additionally, a (Tarskian) consequence relation $\vDash$ is substitution-invariant whenever if $\Gamma \vDash \Delta$, and σ is a substitution on $\mathrm{FOR}(\mathcal{L})$, then $\{\sigma(A) : A \in \Gamma\} \vDash \{\sigma(B) : B \in \Delta\}$. A *Tarskian logic over a propositional language* $\mathcal{L}$ is an ordered pair $(\mathrm{FOR}(\mathcal{L}), \vDash)$, where $\vDash$ is a substitution-invariant Tarskian consequence.

Many scholars have argued that the Tarskian conception of logic (i.e., that a logic is just a *Tarskian logic*) is actually quite narrow. For example, Avron [1] and Gabbay [15] believe that the condition of monotonicity should be relaxed; whereas Malinowski [24] and Frankowski [13] argue for a liberalization of it that allows logics to drop reflexivity and/or cut. The former are known as non-reflexive logics, while the latter are referred to as non-transitive proposals.[2] But there are many other options in the menu. There are also logics that give up Weakening, or even Exchange.[3]

In this paper we will be presenting a variety of, for all we know, new logics. Most of them will also qualify as "substructural".[4] We will call them *disjoint logics*. In a nutshell, what is peculiar of disjoint logics is that the standard for premises does not share any truth-value with the standard for conclusions.[5] At this point, an obvious question arises: why

[1] Though a Tarskian consequence relation is usually understood as single-conclusioned, as we will be working in a multi-conclusion framework, we prefer this more general definition — which actually should be more accurately called "Scottian" than "Tarskian", due to the multi-conclusion schema.

[2] Non-transitive approaches to logical consequence were discussed, previously, in many works — to which the authors refer in their papers. Some of these are due to Strawson (as referred in [7, 8, 9, 13, 16, 17, 27, 29, 32, 33, 34, 35]. Non-reflexive logics are discussed, for example, in [7, 14, 25, 30].

[3] Non-monotonic logics are presented in many places. For example, in [4, 18, 21, 23]. Theories without Exchange are more unusual. An example of them can be found in [3, 19].

[4] Below, we will show why most of these logics are non-reflexive, and why some of them are non-transitive.

[5] As we will explain below, a standard is just a set of truth-values. In Section 3 we will explain in detail how these logics work.

would anyone care about a logic with a *disjoint* consequence relation? Here is a quick answer to such a question.[6]

It is common — and reasonable — to think that, in a valid inference — or at least in sound ones — premises provide reasons to accept the conclusions. Or, to put it more clearly, that to *accept* the premises provide reasons for *accepting* the conclusions. Nevertheless, this way to understand things does not represent the only kind of epistemic-support scenario that is faced in everyday — or in philosophical — life.

For example, sometimes the *rejection* of certain sentences provide good reasons to *accept* others. Or the fact that one *do not accept nor reject* some set of sentences provide reasons to either *accept* or *reject* some conclusion. These situations cannot be straightforwardly represented by valid or sound inferences in logics that aim to correspond to truth-preservation facts, in one way or another. Nevertheless, they can be represented by valid inferences in other logics. And some of them can be captured by valid inferences in disjoint logics.[7]

The structure of this article is as follows. Section 2 is devoted to a characterization of mixed, impure and disjoint logics, while a detailed presentation of the so-called *strong-Kleene disjoint logics* is given in Section 3. We show that almost every single one of them makes reflexivity invalid, and, if certain conditions are met, they also invalidate cut. This justifies calling them *substructural.* In Section 4 we provide a philosophical interpretation for these logics, while in Section 5 we evaluate some potential objections against this project. In Section 6 we present a sound and complete proof-theory for many of the strong-Kleene disjoint logics that have been previously introduced. Finally, in Section 7, we present some concluding remarks.

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[6] The details of how disjoint logics can be applied to this task will be defer to Section 4, after we have have presented the logics we will be exploring.

[7] Though not all of these constraints are apt to be recovered by disjoint logics. For example, sometimes accepting some set of sentences obliged one to either accept or reject some other set of sentences. This kind of situations would be suitably captured by a logic that, for reasons that will become obvious later, might be called $\mathbf{s}$–$\bar{\mathbf{n}}$. But this is not a disjoint logic.

An anonymous referee has pointed to us that what would be interesting is to explore a logic that preserves value $\frac{1}{2}$. Intuitively, this logic tells you that if you doubt some premises, you also have to doubt the conclusion, and this provides a criterion of rationality for the attitude of doubt. We think that this is a very interesting logic indeed. Nevertheless, this is not a disjoint logic, but a *pure* one. Therefore, it falls outside the scope of this paper.

2. Mixed, impure and disjoint consequence relations

In [6] Chemlá et al. present the notion of *mixed consequence relation*. Below, a *standard* is a set of truth values. A consequence relation $\models$ (for a propositional language $\mathcal{L}$) is *mixed* if and only if there are standards $\mathcal{D}^+$, $\mathcal{D}^-$ such that for all sets Γ, Δ we have:[8]

$$\Gamma \models \Delta \text{ iff for any valuation } v \text{, if for any } \gamma \in \Gamma \text{ we have } v(\gamma) \in \mathcal{D}^+, \\ \text{then there is a } \delta \in \Delta \text{ such that } v(\delta) \in \mathcal{D}^-.$$

Another way to understand the *standards* for premises and conclusion is as specifying which are the values each formula in a *sound* argument or inference can adopt. If $\mathcal{D}^+ \neq \mathcal{D}^-$, then the mixed consequence relation is *impure*. Finally, if $\mathcal{D}^+$ and $\mathcal{D}^-$ does not have any elements in common, then we will say that the mixed consequence relation is *disjoint* (this last distinction is ours).

For matters of simplicity, we will work with a propositional language with the usual truth-functional connectives with standard three-valued strong-Kleene interpretation.

	$\neg$
1	0
$\frac{1}{2}$	$\frac{1}{2}$
1	0

$\wedge$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
0	0	0	0

$\vee$	1	$\frac{1}{2}$	0
1	1	1	1
$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
0	1	$\frac{1}{2}$	0

Moreover, the functions $\to$ and $\leftrightarrow$ are definable via the usual definitions.

Two of the most well known mixed logics are **ST** and **TS**. In the first we have $\mathcal{D}^+_{\mathbf{ST}} = \{1\}$ and $\mathcal{D}^-_{\mathbf{ST}} = \{1, \frac{1}{2}\}$, while in the second case we have $\mathcal{D}^+_{\mathbf{TS}} = \{1, \frac{1}{2}\}$ and $\mathcal{D}^-_{\mathbf{TS}} = \{1\}$. In both of these cases, the **S** stands for *strict*, while the **T** stands for *tolerant*. The logic **ST** is non-transitive, **TS** is non-reflexive — and, in fact, empty, at least if the language contains neither $\top$ nor $\bot$ as a 1 and 0 constants, respectively.[9] But despite

[8] As an anonymous referee has reminded us, this way of understanding a consequence relation can be (and has been) paraphrased as "if each member of Γ is designated, then some member of Δ is anti-designated". Notice that it is not assumed that the set of anti-designated values is the complement of the designated ones.

[9] **ST** and **TS** are labelled as *substructural* because at least one *structural* feature of a Tarskian consequence relation is given up by them. As we mentioned, **ST** abandons cut, while **TS** drops reflexivity. The logic **ST** can be portrayed as a *p-logic*, as devised by Frankowski in [13] as a means to characterize logical systems where valid derivations are such that the degree of strength of the conclusions can be smaller than that of the premises. For an extensive presentation of **ST**, see also [7, 8, 27].

being both *mixed* and *impure*, they are, in fact, *non-disjoint*, because the value 1 belongs to $\mathcal{D}^+_{\mathbf{ST}}$ and $\mathcal{D}^-_{\mathbf{ST}}$, as well as to $\mathcal{D}^+_{\mathbf{TS}}$ and $\mathcal{D}^-_{\mathbf{TS}}$. What we will explore here are logics that do not share such a feature with **ST** and **TS**.

There are twenty seven different non-empty three-valued logics based on the strong-Kleene scheme. In the sequel, we will characterize and provide some interesting results involving them.

3. Disjoint logics

3.1. The non-reflexivity (and non-transitivity) of disjoint logics

Before characterizing each of these logics, let us mention two general facts about disjoint logics (that does not even depend on the number of truth-values in the valuations or models used to characterize them): reflexivity is invalid in every disjoint logic meeting one important constraint that will be make explicit below, and cut is invalid in a significant subclass of them.

Fact 3.1. (i) *Reflexivity is invalid in any disjoint logic in which* $\mathcal{D}^+ \neq \varnothing \neq \mathcal{D}^-$*.*[10]

Another interesting generalization of Tarskian consequence relations is the notion of q-consequence relation, due to [24]. **TS** is the 3-valued q-matrix logics associated to the 3-element Kleene algebra. **TS** is discussed by, e.g., Cobreros et al. in [7], and Chemlá et al. in [6] in the context of the more general discussion of what represents a "proper" consequence relation between formulae. Moreover, it was also discussed by Malinowski in [25] as a tool to model empirical inference with the aid of the 3-valued Kleene algebra, and more recently was stressed by French in [14], in connection with the paradoxes of self-reference.

[10] One could question the claim that the fail of reflexivity does not depend on the number of truth-values in the valuations or models used to characterize them. In particular, take a three-valued semantics with truth, falsity, and both truth and falsity as distinct values, a consequence relation such that if all the premise are both true and false, then at least one of the conclusions is true, but if we recast the semantics relationally, then reflexivity holds, since if all premise are related to both true and false, then at least one is related to true. Thus, whether reflexivity fails depends after all on the models used to characterize them.

Nevertheless, if we recast the semantics in this way, but we keep our definition of a disjoint logic as a mixed logic with no truth-values belonging to both $\mathcal{D}^+$ and $\mathcal{D}^-$, then this is not a disjoint logic after all, because the truth-value *true* belongs to both sets. Of course, it is possible to claim that *being true and false* is a truth-value different from both *true* and *false*. But the best way to make sense of this claim is

(ii) *Cut is invalid in any at least three-valued disjoint logic in which* $\mathcal{D}^+ \neq \varnothing \neq \mathcal{D}^-$ *and there is a value* i *such that* $i \notin \mathcal{D}^+ \cup \mathcal{D}^-$.

PROOF. For any disjoint logic we have $\mathcal{D}^+ \cap \mathcal{D}^- = \varnothing$.

(i) Consider an instance of reflexivity with p. We use a valuation such that $v(p) \in \mathcal{D}^+$ but $v(p) \notin \mathcal{D}^-$.

(ii) Consider an instance of cut: $p \vDash q$, $q \vDash r \gg p \vDash r$. Let v be such that $v(p) \in \mathcal{D}^+$, $v(p) \notin \mathcal{D}^-$, $v(q) \in \mathcal{D}^-$, $v(q) \notin \mathcal{D}^+$, $v(r) \notin \mathcal{D}^+ \cup \mathcal{D}^-$. □

Thus, most disjoint logics we will be talking about are non-reflexive, and the first six of the list that we will present are non-transitive.

In what follows, we will focus on the inferential consequence relations of some disjoint logics, and leave aside for the moment every reference to their metainferential properties — besides those we have already mention, namely that each of them are non-reflexive, and many of them are non-transitive.[11]

3.2. Strong-Kleene disjoint logics

As we have already mentioned, we will be exploring three-valued disjoint logics. There are twenty of them that are based on the strong-Kleene scheme. Thus, we can call them *strong-Kleene disjoint logics*.

In order to improve readability and keep track of which logic is being discussed, we will rename the standards we will be taking about — and therefore the logics they determine. We will keep the substructural terminology, and use $\mathbf{s}$ and $\mathbf{t}$ for $\{1\}$ and $\{1, \frac{1}{2}\}$, respectively. We will use $\mathbf{n}$ for $\{\frac{1}{2}\}$ and the sign $\boldsymbol{\emptyset}$ for the empty set $\varnothing$. Finally, we will use an operation $\overline{\boldsymbol{x}}$, that provides the complement of $\boldsymbol{x}$ from $\{\mathbf{s}, \mathbf{t}, \mathbf{n}, \boldsymbol{\emptyset}\}$. Here is a list of the new vocabulary introduced: $\mathbf{s} := \{1\}$, $\overline{\mathbf{s}} := \{\frac{1}{2}, 0\}$, $\mathbf{n} := \{\frac{1}{2}\}$, $\overline{\mathbf{n}} := \{1, 0\}$, $\mathbf{t} := \{1, \frac{1}{2}\}$, $\overline{\mathbf{t}} := \{0\}$, $\boldsymbol{\emptyset} := \varnothing$, and $\overline{\boldsymbol{\emptyset}} := \{1, \frac{1}{2}, 0\}$. Using these new abbreviations, we will now present a list of all strong-Kleene disjoint logics. While the first sign stands for the $\mathcal{D}^+$ of a given logic, the second represents its $\mathcal{D}^-$:

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by characterizing the semantics in a non-relational way. But if you do that, then reflexivity turns out to be invalid after all. I would like to thank the anonymous referee for a discussion on this matter.

[11] This does not mean that we think that metainferences, or metainferential properties, are not important, but developing a complete picture of the metainferential behaviour of these logics deserves a whole independent piece of work.

(1) $\mathbf{sn}$	(7) $\mathbf{s\overline{s}}$	(13) $\overline{\boldsymbol{\emptyset}}\boldsymbol{\emptyset}$	(19) $\boldsymbol{\emptyset}\mathbf{n}$	(25) $\mathbf{n}\boldsymbol{\emptyset}$
(2) $\mathbf{ns}$	(8) $\mathbf{\overline{s}s}$	(14) $\boldsymbol{\emptyset}\overline{\boldsymbol{\emptyset}}$	(20) $\boldsymbol{\emptyset}\mathbf{\overline{t}}$	(26) $\mathbf{\overline{t}}\boldsymbol{\emptyset}$
(3) $\mathbf{\overline{t}n}$	(9) $\mathbf{n\overline{n}}$	(15) $\boldsymbol{\emptyset}\mathbf{t}$	(21) $\mathbf{t}\boldsymbol{\emptyset}$	(27) $\boldsymbol{\emptyset}\boldsymbol{\emptyset}$
(4) $\mathbf{n\overline{t}}$	(10) $\mathbf{\overline{n}n}$	(16) $\boldsymbol{\emptyset}\mathbf{\overline{n}}$	(22) $\mathbf{\overline{n}}\boldsymbol{\emptyset}$	
(5) $\mathbf{s\overline{t}}$	(11) $\mathbf{\overline{t}t}$	(17) $\boldsymbol{\emptyset}\mathbf{\overline{s}}$	(23) $\mathbf{\overline{s}}\boldsymbol{\emptyset}$	
(6) $\mathbf{\overline{t}s}$	(12) $\mathbf{t\overline{t}}$	(18) $\boldsymbol{\emptyset}\mathbf{s}$	(24) $\mathbf{s}\boldsymbol{\emptyset}$	

We will not present or comment on all these logics, but just on a few of them that either are representative of some subset of the rest of all strong-Kleene disjoint logics, or are somehow philosophically intriguing. The first logic we will present is **sn**. An inference $\Gamma \vDash \Delta$ *is valid in* **sn** if and only if for any valuation v, if $v(\gamma) = 1$ for any $\gamma \in \Gamma$, there is a $\delta \in \Delta$ such that $v(\delta) = \frac{1}{2}$.

The logic **sn** is not empty (at the inferential level) — i.e., it has at least one valid inference. For instance, any inference with a classical contradiction as a premise will be valid — e.g., $p \wedge \neg p \vDash q$. In fact, the same happens in any disjoint three-valued logic with **s** as the standard for premises.

When given a philosophical interpretation of these logics, we will concentrate both on **sn** and **ns**. Thus, it will be convenient to explicitly introduce this logic. An inference $\Gamma \vDash \Delta$ *is valid in* **ns** if and only if for any valuation v, if $v(\gamma) = \frac{1}{2}$ for any $\gamma \in \Gamma$, then $v(\delta) = 1$ for some $\delta \in \Delta$.

If $\top$ or $\bot$ belong to the language, then **ns** is not an empty logic either. In particular, any inference with $\bot$ as a premise will be valid. But in their absence, the valuation v that gives value $\frac{1}{2}$ to any propositional letter will invalidate any inference (and sentence).

How serious is that? At least, **ns**'s situation seems not worst than, for example, **TS**'s, the well-known logic in the field of substructural solutions to semantic paradoxes. **TS** is an empty logic, but also a well established one.

The logic $\mathbf{\overline{t}t}$ is representative of another group of logics. For it any truth-value either belongs to $\mathcal{D}^+$ or $\mathcal{D}^-$. An inference $\Gamma \vDash \Delta$ is valid in $\mathbf{\overline{t}t}$ if and only if for any valuation v, if $v(\gamma) = 0$ for any $\gamma \in \Gamma$, there is a $\delta \in \Delta$ such that $v(\delta) \in \mathbf{t}$.

The logic $\mathbf{\overline{t}t}$ is not empty. In fact, every classically valid sentence — and therefore, also every inference with a classical tautology as one of its conclusions — is valid in it. Moreover, every inference with a classical tautology as one of its premises is also valid. In fact, if no $\frac{1}{2}$-operator λ

is available, a sentence — or an inference with an empty set of premises — is valid in $\overline{\mathbf{t}}\mathbf{t}$ if and only if it is classically valid.

The logic $\overline{\boldsymbol{\emptyset}}\boldsymbol{\emptyset}$ is another curious logic. An inference $\mathit{\Gamma} \vDash \mathit{\Delta}$ is valid in $\overline{\boldsymbol{\emptyset}}\boldsymbol{\emptyset}$ if and only if for any valuation v, if $v(\gamma) \in \{1, \frac{1}{2}, 0\}$ for any $\gamma \in \mathit{\Gamma}$, there is a $\delta \in \mathit{\Delta}$ such that $v(\delta) \in \varnothing$.

Obviously, $\overline{\boldsymbol{\emptyset}}\boldsymbol{\emptyset}$ is an empty logic — and it remains so no matter what constants are added to the language. In fact, to say that it is empty does not sufficiently describe how this logic works. It has the important feature that that any valuation is a counterexample to any inference. In this sense, it is unique, and unusually strong.

The logic $\boldsymbol{\emptyset}\overline{\mathbf{s}}$ is one of the strongest strong-Kleene logics. In fact, it is only weaker than $\boldsymbol{\emptyset}\overline{\boldsymbol{\emptyset}}$. An inference $\mathit{\Gamma} \vDash \mathit{\Delta}$ is valid in $\boldsymbol{\emptyset}\overline{\mathbf{s}}$ if and only if for any valuation v if $v(\gamma) \in \boldsymbol{\emptyset}$ for any $\gamma \in \mathit{\Gamma}$, there is a $\delta \in \mathit{\Delta}$ such that $v(\delta) \in \overline{\mathbf{s}}$.

Every classical contradiction will be valid in $\boldsymbol{\emptyset}\overline{\mathbf{s}}$ — and every classical tautology will be invalid in it. Moreover, every inference without an empty set of premises will be valid in it.

Our next logic is the last that we will present, and it might look even stranger than the rest. It is a very special case, indeed. We are talking about $\boldsymbol{\emptyset}\boldsymbol{\emptyset}$. An inference $\mathit{\Gamma} \vDash \mathit{\Delta}$ is valid in $\boldsymbol{\emptyset}\boldsymbol{\emptyset}$ if and only if for any valuation v, if $v(\gamma) \in \varnothing$ for any $\gamma \in \mathit{\Gamma}$, there is a $\delta \in \mathit{\Delta}$ such that $v(\delta) \in \varnothing$.

It might be thought that, as the standard for premises and conclusions is the same, this is not a disjoint logic. And this is rightly so, at least if we understand disjoint logics as logics such that the standard for premises is different from the standard for conclusions. Nevertheless, given an alternative definition of disjoint logics, $\boldsymbol{\emptyset}\boldsymbol{\emptyset}$ surely qualifies as such. According to this criterion, a logic is a disjoint if and only if the intersection of the standards for premises and conclusions is empty. And this is a feature that $\boldsymbol{\emptyset}\boldsymbol{\emptyset}$ exhibits. $\mathcal{D}^+$ and $\mathcal{D}^-$ do not have any elements in common.

Whether or not is convenient to qualify $\boldsymbol{\emptyset}\boldsymbol{\emptyset}$ as a disjoint logic seems a terminological question. If this last way of understanding disjoint logics seems preferable, then it might be interesting to see how this logic behaves. For the reasons already mentioned, $\boldsymbol{\emptyset}\boldsymbol{\emptyset}$ is as trivial with respect to inferences with a non-empty set of premises as any other logic with the empty set as a standard for premises. Moreover, as no sentence will satisfy the standard for conclusions, it has no tautologies, though every inference without an empty set of premises will be valid in it.

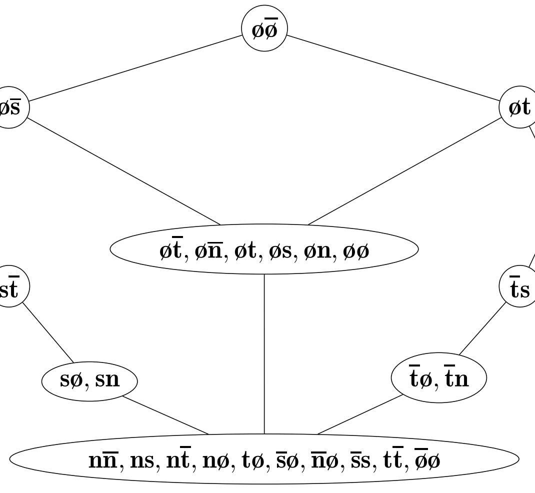

Figure 1. A Hasse diagram of the lattice of strong-Kleene disjoint logics

The logics that we have presented — and the ones that we have only mentioned — are all the three-valued disjoint logics based on a Strong Kleene scheme. What we need to do now is to specify what the relations and relative strength of these logics are. The next subsection is devoted to these tasks.

3.3. An order for strong-Kleene disjoint logics

It seems natural to ask how all these disjoint logics relate to each other. One important way to approach this task is to order them by their relative inferential strength, i.e., by the *inclusion* relation between the different sets of the valid inferences of these logics. To make it easy to visualize this order, we will present a Hasse diagram of the lattice of these disjoint logics (see Figure 1) — ordered, as we have said, by inferential strength.

As the diagram shows, **øø̄** is the strongest strong-Kleene disjoint logic, because it validates every non-empty inference — i.e, every inference with at least one premise or at least one conclusion.[12]

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[12] Nevertheless, it is non-trivial, because the empty inference — i.e., the inference with an empty set of premises and an empty set of conclusions — is not valid. In fact, the empty inference is not valid in any mixed logic, pure or impure. An inference $\Gamma \vDash_{\mathcal{D}^+,\mathcal{D}^-} \Delta$ is valid in a mixed logic $\mathcal{D}^+$, $\mathcal{D}^-$ if and only if for any valuation v, if

The logics $\mathbf{\emptyset\bar{s}}$ and $\mathbf{\emptyset t}$ are the two remaining strongest. The two of them validate every inference with a non-empty set of premises — because no valuation can meet such a standard when applied to formulas (in this case, premises of an inference). Nevertheless, not every empty-premise inference is valid in them. While $\mathbf{\emptyset\bar{s}}$ validates every empty-premise inference just in case the conclusions are a set of classically unsatisfiable sentences, $\mathbf{\emptyset t}$ makes valid every empty-premise inference just in case the conclusions form a set of sentences satisfiable by every classical valuation. So, in particular, every inference with a classical contradiction will be valid in $\mathbf{\emptyset\bar{s}}$, and every inference with a classical tautology will be valid in $\mathbf{\emptyset t}$.

The logics $\mathbf{s\bar{s}}$ and $\mathbf{\bar{t}t}$ are two of the stronger sublogics of $\mathbf{\emptyset\bar{s}}$ and $\mathbf{\emptyset t}$, respectively. $\mathbf{s\bar{s}}$ validates every inference with a set of classically unsatisfiable premises, or a set of conclusions unsatisfiable in classical logic. Thus, every inference with a classical contradiction as a premise, or as a conclusion, will be valid in the logic. In a parallel way, $\mathbf{\bar{t}t}$ validates every inference with a set of premises such that no valuation can be a classical counterexample to every formula in the set. Also, $\mathbf{\bar{t}t}$ makes valid every inference with a set of conclusions such that no valuation can be a counterexample of every formula according to classical logic (i.e., such that no Boolean valuation is a counterexample to every conclusion). So, every inference with a classical tautology as a premise, or as a conclusion, will be valid in the logic. Therefore, these two logics are incomparable.[13]

The logics $\mathbf{\emptyset\bar{t}}$, $\mathbf{\emptyset\bar{n}}$, $\mathbf{\emptyset t}$, $\mathbf{\emptyset s}$, $\mathbf{\emptyset n}$ and $\mathbf{\emptyset\emptyset}$ are equivalent, sublogics of both $\mathbf{\emptyset\bar{s}}$ and $\mathbf{\emptyset t}$, and incomparable with $\mathbf{s\bar{s}}$ and $\mathbf{\bar{t}t}$. As they have $\mathbf{\emptyset}$ as the standard for premises, they validate every non-empty premise inference. But no empty premise inference will be valid in them. The valuation v that gives

for any $\gamma \in \Gamma$ we have $v(\gamma) \in \mathcal{D}^+$, then there is a $\delta \in \Delta$ such that $v(\delta) \in \mathcal{D}^-$. As the Γ of the empty inference contains no formula, any valuation v will satisfy the antecedent of the previous conditional. But, as Δ is also empty, it is not possible for v to satisfy the consequent. Thus, if a logic validates the empty inference, then it is not a mixed logic. Notice that not every logic that validates the empty inference is trivial. Da Ré [10] presents two (single-premise) logics of this kind — and both of them can be expanded non-trivially with a transparent truth predicate.

[13] This does not mean that these conditions exhaust the valid inferences in each logic. For example, the following schemes involving some form of double negation are valid in these two logics: $A \vDash \neg\neg A$, $\neg\neg A \vDash A$. And none of them necessarily involve neither unsatisfiable sets of formulas (in the premises or in the conclusions) nor sets of formulas (in the premises or in the conclusions) such that no valuation can be a counterexample to every formula in the set.

the value $\frac{1}{2}$ is a counterexample to all those inferences in $\mathbf{\varnothing\overline{t}}$, $\mathbf{\varnothing\overline{n}}$, $\mathbf{\varnothing s}$ and $\mathbf{\varnothing\varnothing}$, while any Boolean valuation will be a counterexample to them in $\mathbf{\varnothing n}$.

The logic $\mathbf{s\overline{t}}$ is strictly weaker than $\mathbf{s\overline{s}}$, but stronger than the rest of its sublogics. Every counterexample in $\mathbf{s\overline{t}}$ is also a counterexample in $\mathbf{s\overline{s}}$, but not the other way around. In fact, every inference with a set of classically unsatisfiable conclusions, and without a set of premises that are classically unsatisfiable, will be valid in $\mathbf{s\overline{s}}$, but not in $\mathbf{s\overline{t}}$. Once again, the valuation that gives the value $\frac{1}{2}$ to every formula will be a counterexample to this inferences. $\vDash p \wedge \neg p$ is a relevant example.

Similarly, the logic $\mathbf{\overline{t}s}$ is strictly weaker than $\mathbf{\overline{t}t}$, but stronger than the rest of its sublogics. Every counterexample to an inference in $\mathbf{\overline{t}s}$ is also a counterexample in $\mathbf{\overline{t}t}$, but not the other way around. Every inference with a set of conclusions satisfiable by every classical valuation or without a set of premises classically unsatisfiable, will be valid in $\mathbf{\overline{t}t}$, but not in $\mathbf{\overline{t}s}$. The valuation that gives the value $\frac{1}{2}$ to every formula witness this fact. $\vDash p \vee \neg p$ is one example of this kind.

The logics $\mathbf{\overline{t}\varnothing}$ and $\mathbf{\overline{t}n}$ are the two logics strictly weaker than $\mathbf{\overline{t}t}$, but stronger than the rest of its sublogics. The three logics validate inferences with a set of premises that is satisfiable by every valuation in classical logic. Thus, it is satisfiable by every strong-Kleene valuation, and not just by every Boolean one. Nevertheless, neither $\mathbf{\overline{t}\varnothing}$ nor $\mathbf{\overline{t}n}$ validates no other inference, while, as we have seen, $\mathbf{\overline{t}t}$ does.

The logics $\mathbf{s\varnothing}$ and $\mathbf{sn}$ have regarding $\mathbf{s\overline{t}}$ a similar relation than the ones $\mathbf{\overline{t}\varnothing}$ and $\mathbf{\overline{t}n}$ have with respect to $\mathbf{\overline{t}t}$, as it is easy to see.

Finally, the logics $\mathbf{n\overline{n}}$, $\mathbf{ns}$, $\mathbf{n\overline{t}}$, $\mathbf{n\varnothing}$, $\mathbf{t\varnothing}$, $\mathbf{\overline{s}\varnothing}$, $\mathbf{\overline{n}\varnothing}$, $\mathbf{\overline{s}s}$, $\mathbf{t\overline{t}}$ and $\mathbf{\overline{\varnothing}\varnothing}$ are empty; and so they are sublogics of every other strong-Kleene disjoint logic. The valuation that gives the value $\frac{1}{2}$ to every formula will be a counterexample to every inference in each one of these logics but $\mathbf{\overline{n}\varnothing}$. Every Boolean valuation will play a similar role in this logic.

A central task now is to provide a philosophical interpretation for these logics. That is the goal of the next section.

4. An epistemic reading of three-valued logics

In this section, we will present some of the philosophical interpretations available for these consequence relations.

It is standard to think that each value of a three-valued logic can be suitable related to a specific epistemic or metaphysical state. Specifically, the value 1 can be related to being true, but also to being certain — i.e., to the epistemic state of certainty that a specific agent might be in with respect to a specific proposition — or to the epistemic attitude of acceptance of that proposition. $\frac{1}{2}$, in turn, can be related to the metaphysical state of being neither true nor false, or the dialetheic situation of being both true and false. But it can also be understood as a state of uncertainty, of withholding belief, or one in which the agent has as many reasons to accept the proposition as she has to reject it. Finally, the value 0 is usually related to the metaphysical situation of being false, but also to the epistemic state of rejection — or the probably different one of being certain of the falsity of the proposition.[14] With these different ways to understand these traditional three values, it is possible to easily build philosophical interpretations for all of these logics. In particular, if we attach these values to the epistemic attitudes usually associated with them — i.e., as accepting, neither accepting nor rejecting, or as rejecting a certain (set of) sentence(s) — we will more or less automatically develop epistemic interpretations for all of these logics.

Take **sn**, for instance. This logic can be read as qualifying as valid inferences of *neither accepted nor rejected* premises from *accepted* sentences — i.e., inferences such that if every premise is accepted, then at least one conclusion is neither accepted nor rejected.

Another example: the logic **ns** can be understand as qualifying as valid inferences of *accepted* sentences from *neither accepted nor rejected* premises — i.e., inferences such that if every premise is neither accepted nor rejected, then at least one conclusion is accepted.[15]

[14] These attitutes may or may not be identical to the one of accepting the negation of the proposition rejected. As this discussion is not central to the point we want to state here, we will leave it aside for the moment.

[15] An anonymous referee has pointed out that this is not a very interesting logic, because it only tells us that if we doubt a premise, we have to accept that we doubt it. She points out that, if one has a "doubt" operator in ones language, then some intuitively valid instances of it include "Jane is at the dentist; therefore, I doubt Jane is at the dentist". Nevertheless, if we look at what the theory *invalidates*, then the theory might be more informative. For example, it tells us that if we doubt that p, we should not be certain about whether p holds. We may accept that the normative relations between what we withhold and what we accept may not be that interesting. Nevertheless, if the logic works well — and, maybe, is supplied with interesting operators, like "doubt", then the things that will hold in it will be exactly what are supposed to. In any case, even if this is not an interesting logic — because the normative relations between what we withhold and what we accept are not be that interesting — this does not mean that the rest of these logics are not interesting either.

Can these logics, interpreted in this way, be applied to a specific case — to the epistemic situation of some particular person? We believe they can. And it is not that hard to understand what needs to be done in these cases — from a technical point of view, at least.

Take, for example, $\mathbf{n\bar{t}}$. This logic validates the inferences from a set of sentences that are neither accepted nor rejected, to set of sentences that contained at least one rejected sentence. Thus, in order to obtain which sentences must be rejected by a particular agent in some specific situation, given the set of sentences she neither accepts nor rejects, the models/valuations must be restricted to the ones that give the value $\frac{1}{2}$ to the sentences that are in fact neither accepted nor rejected. The conclusions of the valid inferences based on sets of sentences that received the value $\frac{1}{2}$ in all the remain models, will be the ones that must be rejected — i.e., those whose rejection the subject is committed to, given the set of sentences she neither accepts nor rejects.

Thus, we have philosophical interpretations for all of these logics or at least for most of them. In particular, we are unsure whether this kind of interpretation can be extended to mixed logics involving $\{1, \frac{1}{2}, 0\}$ or $\boldsymbol{\varnothing}$ as a standard for premises or conclusions. In the cases where $\boldsymbol{\varnothing}$ plays the part of the standard for premises, we get a trivial logic that validates every inference with a non-empty set of premises. Moreover, it is not clear what the empty set might represent, as an attitude or a mix of attitudes. There seems to be a similar problem with logics that have $\overline{\boldsymbol{\varnothing}}$ as a standard for conclusions. In some of these cases we get a trivial logic. What kind of commitment might a trivial logic represent?

Nevertheless, in the case of logics that have the set $\overline{\boldsymbol{\varnothing}}$ as a standard for premises, there seems to be a clear philosophical reading of the kind we have already introduced. These logics validate just the inferences such that there is at least one conclusion that can be accepted/rejected/neither accepted or rejected come what may, or a combination of them, depending on the case. Thus, if our focus is on sentences rather than on inferences, these logics seem to be useful.

So far, not only have we provided a recipe to build philosophical interpretations for most of these logics, but we have also offered a good case for claiming that it is not the case that *everything goes* — i.e., that every possible logic will have a substantial philosophical interpretation. Remember that there seems to be no useful reading of logics that has the set containing every single value as a standard for conclusions. This might be a good case for arguing against taking these systems as expressing *genuine* — or *real*, or at least *interesting* — logics.

It is worth noticing that this epistemic reading of the truth-values, in terms of acceptance, rejection, and withholding judgement, is closely related to well-known philosophical interpretations of mixed-consequence relations — specifically, to either non-reflexive or non-transitive ones. Though the sets $\mathcal{D}^+$ and $\mathcal{D}^+$ are not attached any particular philosophical interpretation, in Malinowski's discussion of q-matrices [as can be seen in 25], they are often taken to represent, respectively, the set of accepted and rejected elements. Valid inferences, as Malinowski puts it in [24], should be read like this: if no statement of the conclusion is accepted, then some of the premises should be rejected. Whereas, in Frankowski's discussion of p-matrices they are usually taken to represent, respectively, the set of values that represents the degree of strength of the premises and the set of values representing the degree of strength of the conclusion [as can be seen in 13]. As he understand it, valid inference of a logic with a p-consequence relation should be read like this: if all premises are accepted, then some statements of the conclusion are not rejected. In a line inspired by Malinowski, French in [14] reads valid inferences (or sequents) as the ones such that if we do not *reject* every premise, we should *accept* some conclusion. Shramko and Wansing [30], while discussing q-matrices, identifies the truth-values belonging to $\mathcal{D}^+$ as representatives of a generalized notion of truth, and the truth-values belonging to $\mathcal{D}^-$ as representatives of a generalized notion of falsity. It is worth keeping in mind that none of these readings of the consequence relations correspond to a relation that may characterize a disjoint logic — at least if the things accepted are included in the non-refuted and the non-rejected ones.

Each one of these readings explains validity in terms of two attitudes: acceptance and rejection. We do the same, but expand the set of relevant attitudes used to apply or make sense of these consequence relations, with one more attitude: withholding belief. Moreover, the authors mentioned in the previous paragraph explain validity in terms of two kinds of attitudes that can be taken with respect to premises and conclusions, such that one of them implies the other one. For example, *acceptance* is a kind of *non-rejection*. Nevertheless, the sets of attitudes that are mentioned in the application of our disjoint logics, are such that none of them is included in the other one.

These reading of the consequence relations of the disjoint logics we have presented can be strengthened if some epistemic bridge principles in the normativity of logic literature are adopted. Though we prefer to remain neutral with respect to this important discussion, we will present some examples of how these interaction could work. As MacFarlane mentions in his classic paper [22], a bridge principle can have a wide or narrow scope. They might be presented, respectively, under these two general forms:

(NS) If $\Gamma \vDash \Delta$, then if you have the epistemic attitude X with respect to Γ, then you must have the epistemic attitude X with respect to Δ

(WS) If $\Gamma \vDash \Delta$, then either we have the epistemic attitude X with respect to Γ, or we must have the epistemic attitude Y with respect to Δ.

In (WS), X and Y are mutually exclusive attitudes, like rejection and acceptance.

If we adopt a disjoint logic, both kind of bridge principles should be presented under a different form. (NS) should be recast like this:

(NS′) If $\Gamma \vDash \Delta$, then if you have the epistemic attitude X with respect to Γ, then you must have the epistemic attitude X′ with respect to Δ

And the attitudes X and X′ might not be identical.

Similar changes should be made if we choose a wide scope bridge-principle. Now X and Y in (WS) need not be mutually exclusive attitudes.

As an example, if we adopt the logic **sn**, these two principles might look like this:

(NS-**sn**) If $\Gamma \vDash \Delta$, then if you have *accepted* Γ, then you must *either accept or reject* Δ

(WS-**sn**) If $\Gamma \vDash \Delta$, then either you *withhold judgement or reject* Γ, or you *neither accept nor reject* Δ [for more about bridge principles see 12, 20, 28, 31].

Before ending this section, we will like to refer to one further issue. Though some of the strong-Kleene disjoint logics are empty, this does not mean that they are all the same. For example, we might still discriminate between them if suitable new vocabulary is added, or if the valuations are restricted in such a way that, for example, the particular rejections of an agent are represented. And by this we mean that, for example, if we add a $\frac{1}{2}$ constant to the vocabulary, it will not be the case that two former empty logics will keep on validating the same inferences. Consider, for example, logics $\mathbf{n\bar{t}}$ and **ns**. If a nullary constant $\bot$ representing the value 0 is added to the language, then $\vDash \bot$, for example, will be valid in $\mathbf{n\bar{t}}$, but not in **ns**. But if a nullary constant $\top$ representing the value 1 is added, then $\vDash \top$ will be valid in **ns**, but not in $\mathbf{n\bar{t}}$. Thus, this two empty logics do not react in the same way when new constants are added to the language, and therefore cannot be taken as the same logic.

Finally, it is worth pointing out an important feature of strong-Kleene logics. A transparent truth predicate — plus a suitable mechanism to achieve self-reference — can be safely added to all of these logics — at least if there is no problematic logical constant around, such as a consistency/classicallity operator.[16] Though we will not say more about this issue here, this happens because they are defined through Strong Kleene valuations (or models, or some other way to interpret predicated and names, as the addition of a truth predicate force us to do).

So far, then, we have provide a way to achieve philosophical inter-

[16] As an anonymous referee has pointed out, this is not immediate. First, because these are propositional, and quantified versions of these logics haven't been discussed. Second, because If PA is based on a disjoint logic, the diagonal lemma will likely fail. And lastly, as modus ponens fails for many of these logics, it would not be easy to understand transparency in terms neither of the arrow nor of the turnstile. Nevertheless, it would not be problematic to present first order versions of these logics. Changes should be made, because we will need to expand the semantics with models. But notice that all of these logics are defined in terms of the strong-Kleene schema. The quantifiers, then, might be interpreted in a pretty straightforward way, as infinite conjunctions and infinite disjunctions. Probably PA is not the safest option as a mechanism to achieve self-reference, but this is not the option available. Ripley [26] and French [14], for example, use a metalinguistic function from names to sentences. Moreover, Barrio et al. [2] adapts this way to achieve self-reference in a first-order theory to build a truth-theory in a language without quantifiers but with propositional constants, based on a (paraconsistent) logic of formal inconsistency with a strong — and, thus, suitable to express revenge sentences — negation. (For more about strong negations, see, for example, [11].) So it is not necessary to be in a first-order context to express traditional paradoxical sentences like the Liar. Moreover, worries about a suitable conditional vanishes once we choose this way to achieve self-reference over PA's Diagonal Lemmas. In addition, it is worth noticing that three-valued theories can represent the Liar sentence through a $\frac{1}{2}$ constant. This has the additional virtue of seeing if the Liar trivializes the theory. But it is easy to see that every non-trivial strong-Kleene disjoint logic can be expanded with a $\frac{1}{2}$ constant without becoming trivial.

pretations for most of the three-valued disjoint logics that we have presented. Nevertheless, there is one major criticism that can be found in the literature to this kind of approach to logic. The next section is devoted to it.

5. Permeability (and why it is not such a big problem)

The following objection against disjoint logics can be traced back to Chemlá et al. [5], and can be summarized like this: all disjoint logics are permeable. But no *permeable* theory can be a true logic. Therefore, disjoint logics are not *real* or *fully-fledged* logics. To fully understand it, we need first to introduce some definitions.

A consequence relation is *permeable* if and only if it is *left-to-right* or *right-to-left* permeable, in the following sense:

left-to-right permeability: for all $\Gamma, \Delta, \Sigma : \Gamma, \Sigma \vDash \Delta \Rightarrow \Gamma \vDash \Sigma, \Delta$
right-to-left permeability: for all $\Gamma, \Delta, \Sigma : \Gamma \vDash \Delta, \Sigma \Rightarrow \Gamma, \Sigma \vDash \Delta$

As they proved, a truth-relation is polarized if and only if it is non-permeable. Disjoint logics are non-polarized truth-relations. Therefore, they are permeable.

A mixed truth-relation $\vDash_{\mathcal{D}^+,\mathcal{D}^-}$ is *polarized* if and only if it is T-polarized and F-polarized, in the following sense:

T-polarization: $\mathcal{D}^+ \cap \mathcal{D}^- \neq \varnothing$
F-polarization: $V \setminus (\mathcal{D}^+ \cup \mathcal{D}^-) \neq \varnothing$

More specifically, they proved the following:

Theorem 5.1 (5, Theorem 2.28). *A mixed semantics is sound and complete with respect to a non-permeable logic if and only if it is polarized.*

So now we can see exactly what is happening here: disjoint logics are permeable because they are not T-polarized (by definition).[17]

We do not consider this objection too compelling. In order to understand why, it is useful to check the main reasons why permeability should, initially, be avoided. Both of them are captured, respectively, by the following passage:

> If a logic is not permeable, then its consequence relation is neither universal nor trivial (when negated, either constraint implies non-triviality

[17] And some of them are even not F-polarized.

> as per the antecedent and non-universality as per the denied consequent). [5, p. 4]

Before presenting an answer, it will be useful to keep in mind that when they talk about *Universality*, they mean that *every* inference is valid, and when they refer to *Triviality*, they mean that *no* inference is valid. This is why, as they say, the negation of any of the two permeability constraints — i.e., of any of the two conditionals — implies that there is at least one valid inference — thus denying triviality — and at least one invalid inference — thus denying universality.

If the only reason to reject permeable logics is to avoid both triviality and universality, then the scope of the attack is narrower that what it might initially be supposed to be. Though it is true that there are some trivial disjoint logics, just as there are universal logics of this kind, many of the disjoint logics that we have presented are truly informative, i.e., neither trivial nor universal. Thus, if this is the only reason to avoid permeability, these logics show us that there is nothing to worry about. But, of course, the authors think that there are some other reasons to reject permeability. One of them is displayed in the following remark:

> We may describe a permeable consequence relation as one that would confuse the role of premises with the role of conclusions, or the reverse. [5, p. 4]

We think that this objection is kind of exaggerated. Take, for example, **sn**. Though this logic is not left-to-right permeable, it is right-to-left permeable. Does this means that **sn** confuses premises with conclusions? How serious a situation is this?

We think that being right-to-left permeable though not left-to-right permeable, does not mean that we are dealing with a logic that confuses the role of premises with the role of conclusions. If that were the case, then it would be the case, for example, $p \vDash q \wedge \neg q$ if and only if $q \wedge \neg q \vDash p$. But while the first one is invalid, the second one is valid. Thus, it is not true that *permeability* equals *confusion between premises and conclusions.*

This does not mean that being, for example, right-to-left permeable is not an undesirable feature. But in the particular case of **sn**, this happens because this logic has no tautologies whatsoever, and in fact every valid inference involves a set of classically unsatisfiable set of premises. But the real question here is whether this is enough to disqualify some system as a *real* logic, or something like that. This might well be the case if the system confuses premises with conclusions, but, as we have show, this should not be equal with being permeable. Until some other reasons were provided, we do not think the dismissal of these logics is justified.

Moreover, the only reason why a logic like $\mathbf{n\overline{n}}$ is permeable, is that no inference is valid in it. Thus, the antecedents of both permeability conditions will be false, and, thus, both conditionals will come out true. But it is a little bit misleading to say that an empty inferential logic, like $\mathbf{n\overline{n}}$, messes up the role of premises and conclusions, just because no inference is valid according to it. If that is the case, then another fairly well established substructural logic, like **TS** — which is another empty logic, if the language of the logic does not contain neither $\top$ nor $\bot$ — must also be disregarded as a true logic. Nevertheless, French [14] has defend it as a suitable solution to semantic paradoxes (and provide a compelling bilateralist reading of it). Regarding permeability, both **TS** and $\mathbf{n\overline{n}}$ stand and fall together.

As Chemlá et al. [6] define the notion, a logic can be permeable just because it is right to left permeable, though not left-to-right permeable — and neither universal nor trivial. **sn** is a logic like this. Suppose that we accept the claim of Chemla et al. that permeable logics confuse the role of premises with the role of conclusions. How bad is this? Is it always equally bad? Does **sn** confuse it in the same way as an empty logic like $\mathbf{n\overline{n}}$ does? There seems to be, at least, a matter of degree, but also a matter of different types involved — a left-to-right, but not right-to-left permeable logic seems very different from a a right-to-left, but not left-to-right permeable logic — something that the criteria given in [5] do not capture.

A third passage might give further reasons to reject permeable logics:

> On an inferentialist perspective, however, it is often desirable to single out two special truth values, True and False, matching propositions with specific inferential roles. The value False is attached to the principle that from a contradictory proposition anything follows, but also to the principle that if an argument is not valid, then the addition of a contradiction to the conclusions won't make it valid. The roles for the True are dual: a tautology should follow from any premise whatsoever, but if an argument is not valid, adding it among the premises will not make it valid either. Using 1 and 0 to represent the True and the False, those principles correspond to:
>
> (T1) $\forall\gamma,\delta : \gamma \models \delta, 1$ (T2) $\forall\gamma,\delta : \gamma, 1 \models \delta$ implies $\gamma \models \delta$
> (F1) $\forall\gamma,\delta : \gamma, 0 \models \delta$ (F2) $\forall\gamma,\delta : \gamma \models \delta, 0$ implies $\gamma \models \delta$.

> However, none of the semantic properties we have introduced so far for truth-relations guarantees that three will be truth values playing the roles of the True and the False.[18] [5, pp. 9–10]

The first thing worth saying about this is that truth and falsity are primarily semantic notions that are not initially defined by their inferential roles. At least, the burden of the proof falls on those who want to reduce them to proof-theoretic rules. If anything truth and falsity seem to be more connected with attitudes such as acceptance and rejection[19] than with specific rules (without interpretation in a consequence relation).

But suppose that it is in fact the case that Truth and Falsity are indeed accurately represented by principles T1, T2, F1 and F2. Thus, if a logic can be expanded with constants that satisfy these principles, they can represent both Truth and Falsity. Well, it is actually the case that many disjoint logics may be expanded with constants like 1, 0, despite being permeable and having non-polarized consequence relations. Nevertheless, it is not easy to associate those constants with Truth and False, at least not in those logics. Thus, either it is not true that Truth and Falsity are indeed accurately represented by principles T1, T2, F1 and F2, or it is not true that permeable and non-polarized consequence relations cannot represent them.

For example, in **sn**, a constant 1 — representing truth, or acceptance — can be added to the language satisfying T1 and T2, while a constant $\frac{1}{2}$ — representing neither truth nor falsity or neither acceptance nor rejection — can be added to the language. And that constant will satisfy F1 and F2.

In a similar vein, in $\mathbf{n\bar{t}}$, a constant $\frac{1}{2}$ — representing neither truth nor false, or or neither acceptance nor rejection — can be added to the language. That constant will satisfy both T1 and T2, while a constant 0 — representing false, or rejection — can be added to the language. This new constant will satisfy F1 and F2.

It is worth noticing that, though the language for these disjoint logics may include a 1-constant $\top$ and a 0-constant $\bot$, there is no guarantee

[18] To clarify, what the authors are claiming is that when one adds constants for the top and bottom elements of the lattice, a permeable logic will not satisfy one of those four principles, and thus will not adequately capture truth or falsity.

[19] Or with the following possible norms of the practice of assertion: accepting (the truth) and rejecting (the false).

that T1 and T2 will hold for the former, and F1 and F2 will hold for the latter. In fact, it is the $\frac{1}{2}$-constant — and not $\top$ — that satisfies T1 and T2 in $\mathbf{n\bar{t}}$, while in the **sn**'s case that constant — and not $\perp$ — is the one that satisfies principles F1 and F2.

Let's sum up. We have shown that being left-to-right (or right-to-left) permeable is not enough to confuse the role of premises and conclusions, and as many disjoint logics can be expanded with constants satisfying, respectively, principles T1, T2, F1 and F2, Chemlá and Egré seem to lack a forceful reason to reject every permeable logic just for being permeable. Thus, we not only have provide some positive reasons for accepting these logics, but also reject the available criticisms they are subject to, given that they are permeable logics.

6. A proof system for disjoint logics

Though this project is about how some new logics — i.e., the strong-Kleene disjoint logics — can be characterized through semantic means, it is possible to design proof systems for them. In what follows we will present a calculus that can be used for the strong-Kleene disjoint logics that we have introduced. Our target proof theory will be the three-sided disjunctive sequent system **DL** (for "disjoint logics", of course).

We now specify how disjunctive sequents behave.[20]

A disjunctive sequent $\Gamma \mid \Sigma \mid \Delta$ is satisfied by a valuation v if and only if $v(\gamma) = 0$ for some $\gamma \in \Gamma$, or $v(\sigma) = \frac{1}{2}$ for some $\sigma \in \Sigma$, or $v(\delta) = 1$ for some $\delta \in \Delta$. A sequent is valid if and only if it is satisfied by every valuation. A valuation is a counterexample to a sequent if the valuation does not satisfy the sequent.

As we have said, **DL** can be used as a proof system for our disjoint logics. To exemplify how this works, we will take a closer look to the case of $\mathbf{s\bar{s}}$. $\mathbf{s\bar{s}}$'s consequence relation is such that an inference from Γ to Δ is valid if and only if there is no valuation such that every formula in Γ receives the value 1 and every formula in Δ also receives the value 1. Thus, there is a strong relation between **DL**'s valid sequents and $\mathbf{s\bar{s}}$'s valid inferences:

$$\Gamma \models_{\mathbf{s\bar{s}}} \Delta \text{ if and only if } \Gamma, \Delta \mid \Gamma, \Delta \mid \varnothing \text{ is valid.}$$

[20] This system strongly resembles — and is obviously based on — the one Ripley present in [26] for the truth theories based on **ST**, **TS**, **LP** and **K3**.

This fact follows from the definition of **s$\bar{\text{s}}$**'s validity and the definition of validity of a three-side sequent.

The proof system we are about to present includes, as usual, some axioms and rules. A sequent is provable if and only if it follows from the axioms by some number (possibly zero) of applications of the rules. As we are working with sets, the effects of the structural rules of Exchange and Contraction are built in, and Weakening is built into the axioms. Still, to make things easy, we will include a Structural Rule of Weakening. We will have three versions of a three-sided cut rule, and also a Derived Cut rule (that can be inferred from the three basic rules of cut) that will be a key part of the Completeness Proof. Id is the only axiom-schema of **DL**. Weak, cut1, cut2, cut3 and Derived Cut are structural rules. The rest of them are **DL**'s operational rules.

$$\text{Id}\ \frac{}{A, \Gamma \mid A, \Sigma \mid A, \Delta} \qquad \text{Cut 1}\ \frac{\Gamma, A \mid \Sigma \mid \Delta \qquad \Gamma \mid \Sigma, A \mid \Delta}{\Gamma \mid \Sigma \mid \Delta}$$

$$\text{Weak}\ \frac{\Gamma \mid \Sigma \mid \Delta}{\Gamma, \Gamma\prime \mid \Sigma, \Sigma' \mid \Delta, \Delta'} \qquad \text{Cut 2}\ \frac{\Gamma \mid \Sigma \mid \Delta, A \qquad \Gamma \mid \Sigma, A \mid \Delta}{\Gamma \mid \Sigma \mid \Delta}$$

$$\text{Cut 3}\ \frac{\Gamma, A \mid \Sigma \mid \Delta \qquad \Gamma \mid \Sigma \mid \Delta, A}{\Gamma \mid \Sigma \mid \Delta}$$

$$\text{Derived Cut}\ \frac{\Gamma, A \mid \Sigma, A \mid \Delta \qquad \Gamma \mid \Sigma, A \mid \Delta, A \qquad \Gamma, A \mid \Sigma \mid \Delta, A}{\Gamma \mid \Sigma \mid \Delta}$$

$$\text{L}\neg\ \frac{\Gamma \mid \Sigma \mid \Delta, A}{\Gamma, \neg A \mid \Sigma \mid \Delta} \qquad \text{L}\wedge\ \frac{\Gamma, A, B \mid \Sigma \mid \Delta}{\Gamma, A \wedge B \mid \Sigma \mid \Delta}$$

$$\text{M}\neg\ \frac{\Gamma \mid \Sigma, A \mid \Delta}{\Gamma \mid \Sigma, \neg A \mid \Delta} \qquad \text{R}\wedge\ \frac{\Gamma \mid \Sigma \mid \Delta, A \qquad \Gamma \mid \Sigma \mid \Delta, B}{\Gamma \mid \Sigma \mid \Delta, A \wedge B}$$

$$\text{R}\neg\ \frac{\Gamma, A \mid \Sigma \mid \Delta}{\Gamma \mid \Sigma \mid \Delta, \neg A}$$

$$\text{M}\wedge\ \frac{\Gamma \mid \Sigma, A \mid \Delta, A \qquad \Gamma \mid \Sigma, B \mid \Delta, B \qquad \Gamma \mid \Sigma, A, B \mid \Delta}{\Gamma \mid \Sigma\, A \wedge B \mid \Delta}$$

As $\vee$ can be defined in terms of the former, we will not specify rules for it.

The following are some important properties of **DL**:

Theorem 6.1 (Soundness). *If a sequent $\Gamma \mid \Sigma \mid \Delta$ is provable in* **DL***, then it is valid.*

Proof. The axioms are valid, and validity is preserved by the rules, as can be checked without too much trouble. □

But the system **DL** is also complete. In Appendix we prove:

Theorem 6.2 (Completeness). *If a sequent $\Gamma \mid \Sigma \mid \Delta$ is valid, then it is provable in* **DL**.

7. Conclusion

We have presented all the different *mixed* and *impure* disjoint three-valued metainferential consequence relations based on the strong-Kleene schema that exist. Most of them are non-reflexive, while some of them are non-transitive, thus qualifying as substructural. Some, but not all of them, are (inferentially) empty logics, others are trivial, while even others are neither. We have compared them regarding their validities, and also provided a recipe to build philosophical interpretations for every single one of these logics, and shown why the kind of permeability that characterized them is not such a bad feature. Finally, we have given a disjunctive sequent-system for one of these strong-Kleene disjoint logics.

Appendix: The completeness proof

We will use the method of reduction trees,[21] that allows us to build for any given sequent, either a proof of that sequent, or a counterexample to it. The method also provides of a way of building the eventual counterexample. We will introduce the notions of subsequent and sequent union, that will be used in the proof:

A sequent $S = \Gamma \mid \Sigma \mid \Delta$ is a *subsequent* of a sequent $S' = \Gamma' \mid \Sigma' \mid \Delta'$ (written $S \sqsubseteq S'$) if and only if $\Gamma \subseteq \Gamma'$, $\Sigma \subseteq \Sigma'$ and $\Delta \subseteq \Delta'$.

A sequent $S = \Gamma \mid \Sigma \mid \Delta$ is the *sequent union* of a set of sequents $\{\Gamma_i \mid \Sigma_i \mid \Delta_i\}_{i\in I}$ (written $S = \bigsqcup\{\Gamma_i \mid \Sigma_i \mid \Delta_i\}_{i\in I}$) iff $\Gamma = \bigcup_{i\in I} \Gamma_i$, $\Sigma = \bigcup_{i\in I} \Sigma_i$ and $\Delta = \bigcup_{i\in I} \Delta_i$.

[21] For similar proofs, see [26].

The construction starts from a root sequent $S_0 = \mathit{\Gamma}_0 \mid \mathit{\Sigma}_0 \mid \mathit{\Delta}_0$, and then builds a tree in stages, applying at each stage all the operational rules that can be applied, plus Derived Cut "in reverse", i.e. from the conclusion sequent to the premise(s) sequent(s). For the proof, we use an enumeration of the formulae. We will reduce, at each stage, all the formulae in the sequent, starting from the one with the lowest number, then continuing with the formula with the second lowest number, and moving on in this way until the formula with the highest number in the sequent is reduced. In the case where a formula appears in more than one side of the sequent, we will start by reducing the formula that appears on the left side and then proceed to the middle and the right side, respectively. The final step, at each stage n of the reduction process, will be an application of the Derived Cut rule to the n-formula in the enumeration. If we apply a multi-premise rule, we will generate more branches that will need to be reduced. If we apply a single-premise rule, we just extend the branch with one more leaf. We will only add formulae at each stage, without erasing any of them. As a result of the process just described, every branch will be ordered by the subsequent relation. Any branch that has an axiom as it topmost sequent will be closed. A branch that is not closed is considered open. This procedure is repeated until every branch is closed, or until there is an infinite open branch. If every branch is closed, then the resulting tree itself is a proof of the root sequent. If there is an infinite open branch Y, we can use it to build a counterexample to the root sequent. Thus, stage 0 will just be the root sequent S_0. If it is an axiom, the branch is closed. For any stage $n + 1$, one of two following things might happen:

1. For all branches in the tree after stage n, if the tip is an axiom, the branch is closed.

2. For open branches: For each formula A in a sequent position in each open branch, if A already occurred in that sequent position in that branch (i.e. A has not been generated during stage $n + 1$), and A has not already been reduced during stage $n + 1$, then reduce A as is shown below. There are three possible positions in which a formula can appear in a sequent: either on (i) the left side, or on (ii) the middle, or on (iii) the right side. We need to consider all these possible cases.

- If A is a propositional letter, then do nothing.
- If A is a negation $\neg B$, then: if A is in the left/ middle/ right position, extend the branch by copying its current tip and adding B to the right/ middle/ left position.

- If A is a conjunction $B \wedge C$, then: (i) if A is in the left position, extend the branch by copying its current tip and adding both B and C to the left position. (ii) If A is in the middle position, split the branch in three: extend the first by copying the current tip and adding B to both the middle and right positions; extend the second by copying the current tip and adding C to the middle and right positions; and extend the third by copying the current tip and adding both B and C to the middle position. (iii) If A is in the right position, split the branch in two: extend the first by copying the current tip and adding B to the right position; and extend the second by copying the current tip and adding C to the right position.

We will also apply the Derived Cut rule at each step. Consider the nth formula in the enumeration of formulae and call it A. Now extend each branch using the Derived Cut rule. For each open branch, if its tip is $\Gamma \mid \Sigma \mid \Delta$, split it in three and extend the new branches with the sequents $(\Gamma, A \mid \Sigma, A \mid \Delta)$, $(\Gamma, A \mid \Sigma \mid \Delta, A)$, and $(\Gamma \mid \Sigma, A \mid \Delta, A)$, respectively.

Now we need to repeat this procedure until every branch is closed, or, if that does not happen, until there is an infinite open branch. If the first scenario is the actual one, then the tree itself is a proof of the root sequent, because each step will be the result of an application of a structural or operational rule to the previous steps. If the second scenario is the actual one, we can use the infinite open branch to build a counterexample.

If in fact there is an infinite open branch Y, then the Derived Cut rule will have been used infinitely many times. Thus, every formula will appear at some point in the branch for the first time, and will remain in every step afterwards. Now, we first collect all sequents of the infinite open branch Y into one single sequent $S_\omega = \Gamma_\omega \mid \Sigma_\omega \mid \Delta_\omega = \bigsqcup\{S : S \text{ is a sequent of } Y\}$. Notice that, as Derived Cut has been applied infinitely many times in the construction of the branch, every formula will occur in exactly two places in S_ω.[22] Thus, there will be a valuation such that no formula in the sequent gets the value associated with the place where it occurs (i.e. 0 if the formula occurs in the left, $\frac{1}{2}$ if it occurs in the middle, 1 if it occurs in the right). Hence, for each formula A in the

[22] It cannot occur in the three places, because then there will be some finite stage n where the formula appears for the first time in the branch in the three sides. But then that sequent will be an axiom, and therefore the branch will be closed.

sequent, v will give to A a value different from the ones corresponding to the sides where A appears in the sequent. But that includes all the formulae in the initial and finite sequent S_0. That valuation, then, will also be a counterexample to S_0. Therefore that valuation will be a counterexample to the sequent being considered.

Thus, for atomic formulae A (propositional letters), $v(A) = 0$ or $\frac{1}{2}$ or 1, respectively, if and only if A does not appear in Γ_ω or Σ_ω or Δ_ω, respectively. Let us now take A to be any formula whatsoever.

The rules for reducing formulae can be used to show by induction that, if none of the components of complex formulae receive the value associated with any place in which they appear in S_ω, neither will the compound. We will not see, due to limitations of space, how this method works in detail. We will just consider the case of conjunctions $B \wedge C$ as an example.

We need to consider three possible situations: (i) either the conjunction appears in both the left and the right sides, or (ii) it appears both in the left and in the middle sides, or (iii) it appears on the middle and the right sides. We will just check what happens with case (i), and leave (ii) and (iii) to the reader. In this case, eventually, $B \wedge C$ will be reduced from a sequent like $\Gamma, B \wedge C \mid \Sigma \mid \Delta, B \wedge C$. The reduction of the conjunction on the left side will demand to copy the current tip, and also the addition of B and C on the left. But, as $B \wedge C$ appears also in the right side, this demands to split the branch in two, and to extend the first by copying the current tip and adding B to the right position, and also to extend the second by copying the current tip and adding C also to the right position. Thus, the two new sequents will be:

$$\dfrac{\Gamma, B \wedge C, B, C \mid \Sigma \mid \Delta, B \wedge C, B}{\vdots} \qquad \dfrac{\Gamma, B \wedge C, B, C \mid \Sigma \mid \Delta, B \wedge C, C}{\vdots}$$

Thus, these are two new branches. The complexity of B and C is less than the complexity of $B \wedge C$, hence the inductive hypothesis can be applied to them. Therefore, B will get the value $\frac{1}{2}$ on the valuation corresponding to the first branch — because the only place where it does not appear is the middle one —, while C will get the value $\frac{1}{2}$ in the valuation corresponding to the second branch — because, once again, the only place where it does not appear is the middle one. Therefore, it does not matter whether C, in the first case, or B, in the second case, gets the value 1 or $\frac{1}{2}$. In each of these cases, $B \wedge C$ will get the value $\frac{1}{2}$. Thus, none of these formulae in the branch receives the value associated with the sides of the sequent in which they appear.

By completing the induction along these lines, we can show that we can construct a valuation such that no formula receives the value associated with any place where it appears in S_ω. But, as we know, that includes all the formulae in the initial sequent S_0. That valuation, then, will also be a counterexample to S_0, which is what we were looking for. Thus, for any sequent S, either it has a proof or it has a counterexample.

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