Logic and Logical Philosophy
Volume 28 (2019), 409–425
DOI: 10.12775/LLP.2019.007

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RELEVANT CONNEXIVE LOGIC

Abstract. In this paper, a connexive extension of the relevance logic $\mathbf{R}_{\rightarrow}$ was presented. It is defined by means of a natural deduction system, and a deductively equivalent axiomatic system is presented too. The goal of such an extension is to produce a logic with stronger connection between the antecedent and the consequent of an implication.

Keywords: connexive extension of relevance logic; connexive logic; natural deduction; axiomatic system

1. Introduction

My aim in this paper is to combine two of the major revolts against the material implication as adequately capturing the use of the conditional in everyday (non-mathematical) reasoning, namely *relevance logic(s)* and *connexive logic(s)* (both briefly reviewed in Section 2). My aim strongly draws, and is driven by, the belief, now shared by many, that logical formulas have contents transcending their truth-value, a contents that an adequate logic should reflect. I regard the reduction of the contents of a formula merely to its truth-value, as done in classical logic, as an over-generalization of the meanings of natural language affirmative sentences.

I aim in this paper towards a connexive extension of the (implicational fragment of) one of the most typical relevance logics, namely $\mathbf{R}_{\rightarrow}$, the implicational fragment of $\mathbf{R}$ [1]. Such a combination should produce a logic endowed with a conditional reflecting both aspects of contents that underly the two combined logics.

The idea of combining relevance logics with connexive logics is not entirely new. See the Introduction section in [17] for some discussion of

Special Issue: Advances in Connexive Logic. Guest Editors: Hitoshi Omori and Heinrich Wansing

Published online January 31, 2019

such a combination of relevance logics and connexive logics. A connexive extension of a weak Relevance logic can be found in [11]. Also, in [10] McCall presents such a system, adopting the 'use tracing' technique of the ND-system of Anderson and Belnap [1], not specifically associated with $\mathbf{R}_{\to}$.

I refer to the resulting specific logic proposed in this paper as $\mathcal{L}_{rc}$, a logic of relevance *and* connexivity.

My definitional tool is a natural-deduction (ND) proof system $\mathcal{N}_{rc}$ (presented in Section 3), combining two methods adopted from the respective underlying families of logic:

- Providing rules for negating the conditional, adopted from the methodology of defining connexive logics, avoiding uniform I/E-rules for negation.
- Keeping track of the "*use*" of an assumption in a derivation as a necessary condition of its discharge by a rule application, adopted from the methodology of defining relevance logics, avoiding vacuous discharge.

Choosing ND as the definitional tool has the advantage of focusing on proofs from assumptions (derivations), and not merely on formal theorems (theses), as is the tendency in axiomatic, Hilbert-like presentations. See [15] for a general discussion of this difference, and [3] for a discussion of this difference in the context of relevance logics.

In a later stage, the ND-system $\mathcal{N}_{rc}$ is shown deductively equivalent to an axiomatic (Hilbert-like) presentation $\mathcal{H}_{rc}$. No model theory for $\mathcal{L}_{rc}$ is presented.

2. A brief review of connexive and relevance logics

For self-containment and definiteness, as well as for introducing some of the notation employed, I briefly delineate in this section the two families of logics underlying $\mathcal{L}_{rc}$: connexive logics and relevance logics. what is common to the two families is that their conditional is not truth-functional. Both presume some connection of contents between the antecedent and the consequent of a conditional.

2.1. Connexive logics

Let '$\to$' denote a generic conditional[1], '$\neg$' — a generic negation and 'φ' ranges over arbitrary formulas. The characteristics of connexive logics (cf. [17] for a general survey; see also [13]) are the following theorems — both are jointly known as *Aristotle's thesis* — which are not theorems of classical logic, and, hence, are not theorem of relevance logics, the latter being sub-classical:

$$\neg(\varphi \to \neg\varphi) \tag{A$_1$}$$

$$\neg(\neg\varphi \to \varphi) \tag{A$_2$}$$

In classical logic, $\varphi \supset \neg\varphi$ can certainly be true, and its truth is merely an opaque way of expressing that φ is false. Similarly, the truth of $\neg\varphi \supset \varphi$ implies that φ is true.

In [7], the other characteristic relationships

$$(\varphi \to \psi) \to \neg(\varphi \to \neg\psi) \tag{B$_1$}$$

$$(\varphi \to \neg\psi) \to \neg(\varphi \to \psi) \tag{B$_2$}$$

are attributed to the ancient philosopher and logician Boethius. For the history of connexive logics see [10].

2.2. Relevance logics

Relevance logics are sub-classical, rejecting some of the classical axioms that reflect the pure truth-functional conception of the material conditional, such as

$$\varphi \supset (\psi \supset \varphi) \qquad\qquad \varphi \supset (\neg\varphi \supset \psi)$$

(the "paradoxes" of the material implication), as well as the *explosion* and *implosion* of the classical consequence relation.

$$\varphi, \neg\varphi \models_{cl} \psi \qquad\qquad \Gamma \models_{cl} \varphi \vee \neg\varphi$$

whereby a contradiction entails *any* proposition, and a tautology is entailed by *any* collection of assumptions.

The common thread of the above rejected classical validities is the *lack of relevance* between:

1. The antecedent and the consequent of a conditional.
2. The assumptions and conclusion in an entailment.

[1] I will use 'conditional' and 'implication' as synonymous.

$$\frac{[\varphi]_i}{\varphi_i}\ (Ass),\ i \text{ fresh}$$

$$\frac{\begin{array}{c}[\varphi]_i\\ \vdots\\ \psi_{\alpha\cup\{i\}}\end{array}}{(\varphi\to\psi)_\alpha}\ (\to^+ I^i) \qquad \frac{(\varphi\to\psi)_\alpha \quad \varphi_\beta}{\psi_{\alpha\cup\beta}}\ (\to^+ E)$$

$$\frac{\begin{array}{c}[\varphi]_i\\ \vdots\\ (\sim\psi)_{\alpha\cup\{i\}}\end{array}}{(\sim(\varphi\to\psi))_\alpha}\ (\to^- I^i) \qquad \frac{(\sim(\varphi\to\psi))_\alpha \quad \varphi_\beta}{(\sim\psi)_{\alpha\cup\beta}}\ (\to^- E)$$

$$\frac{(\sim\sim\varphi)_\alpha}{\varphi_\alpha}\ (dne) \qquad \frac{\varphi_\alpha}{(\sim\sim\varphi)_\alpha}\ (dni)$$

Figure 1. $\mathcal{N}_{rc}$: the rules

There is a vast literature on relevance logics. A good starting point and a source of further references can be found in [17]. For an historical overview, see [14].

3. The natural-deduction system $\mathcal{N}_{rc}$

I consider here an implication-negation fragment. I use '$\to$' for implication and '$\sim$' for negation.

3.1. Defining $\mathcal{N}_{rc}$

The introduction and elimination rules (I/E rules) are presented in Figure 1. As usual, an assumption enclosed in square brackets indicates a discharged assumption. Its index (discharge index) appears as a superscript on the instance of the rule the application of which discharges this assumption. A characteristic feature of $\mathcal{N}_{rc}$ is that it has separate I/E-rules for positive occurrences of the conditional and and negative (i.e., negated) occurrences. Note that the non-vacuous discharge restriction applies both to the positive and negative I-rules for the conditional.

Derivations (tree-shaped, Prawitz's style), ranged over by $\mathcal{D}$, are defined inductively as usual, iterating rule applications starting from assumptions. Any interim occurrence of a formula φ in a derivation has a subscript α tracking the (open) assumptions this occurrence of φ has used, on which this occurrence of φ depends. See [3] for a critical discussion of ‘use’ in the literature of relevance logics.

Definition 3.1 (Use of assumptions). An assumption, say an instance of φ, is *used* in a derivation $\mathcal{D}$ when that instance of φ serves as a premise for the application of some rule within $\mathcal{D}$.

Use is *propagated* along a derivation according to the manipulation of indices of used assumption by the various rule applications along the derivation.

Notation. When this set α is a singleton the set brackets are omitted, and when α is empty the subscript itself is omitted.

By convention, I assume that in the $\mathcal{N}_{rc}$-derivations the open assumptions Γ are subscripted by 1, …, n, for some $n \geqslant 0$, and let $\hat{n} = \{1, \ldots, n\}$.

Remarks about the rules of $\mathcal{N}_{rc}$:

Ad (*Ass*): A similar way of formulating the introduction of an assumption is attributed by von Plato [12] to Gentzen. The rule was intended to make less awkward the derivation of $\varphi \supset \varphi$, and was later abandoned by Gentzen.

I revive this formulation here in order to make it explicit that φ, assumed in the premise, is also an explicit conclusion, *carrying the same index*! Thus, when this rule is applied, it uses the assumption in the premise, allowing later discharge of the latter. We thus get

$$\dfrac{\dfrac{[\varphi]_1}{\varphi_1}\,(Ass)}{\varphi \to \varphi}\,(\to^+ I^1) \tag{3.1}$$

To avoid notational clutter, I will omit applications of this rule in example derivations in which it has no beneficial contribution as it has in (3.1) above.

Regarding the freshness of i, strictly speaking, i is chosen as the *least* natural number not used as an index in the derivation at an earlier stage. I will usually ignore this strictness and just relate to i is being fresh, not used so far.

Ad ($\to^+ I$): This is the original I-rule of Anderson and Belnap [1] for introducing the relevant conditional. The assumption of the antecedent is only dischargeable if used during the sub-derivation of the consequent ψ from the assumed antecedent φ. This condition is enforced by requiring ψ to have an index containing i, the fresh index of the assumption.

Ad $(\rightarrow^+ E)$: This is a "relevantized" version of modus-ponens. The conclusion uses the union of the assumption sets used by both premises. Note that this rule, together with $(\rightarrow^- E)$, are rules in which use is *propagated*, the assumption set index of the conclusion growing.

Ad $(\rightarrow^- I)$: This rule endows the conditional '$\rightarrow$' its connexivity, by changing its falsification condition. I have originally introduced this rule in [6], without the tracing of indices needed for relevance, motivating it by certain uses of negated conditionals in natural language.

Note that as long as no additional connectives are included in the object language, there are three sources of $\sim\psi$, the negated conclusion of this rule:

1. A recursive application of the $(\rightarrow^- I)$ rule itself.
2. An application of $(\rightarrow^- E)$.
3. An application of (dni).

Ad $(\rightarrow^- E)$: This E-rule is naturally associated with the corresponding $(\rightarrow^- I)$ rule, retrieving the grounds for introduction of the negated conditional.

Ad (dni) and (dne): Note that both rules have no effect on the used assumptions index, merely preserving it.

Remark. In $\mathcal{N}_{rc}$, an assumption $[\varphi]_i$ can be used within $(\rightarrow^+ I)$ or $(\rightarrow^- I)$, before being discharged, in three ways:

1. As a premise of an application of the assumption rule (Ass).
2. As a minor premise of $(\rightarrow^+ E)$ or $(\rightarrow^- E)$; in this case, there is for some ξ one of the following two a sub-derivations:

$$\frac{[\varphi]_i \quad \overset{\vdots}{(\varphi \rightarrow \xi)_\beta}}{\xi_{\beta \cup i}}\,(\rightarrow^+ E) \qquad \frac{[\varphi]_i \quad \overset{\vdots}{\sim(\varphi \rightarrow \xi)_\beta}}{\sim\xi_{\beta \cup i}}\,(\rightarrow^- E)$$

   with $\beta \subseteq \alpha$ in both cases.
3. As a major premise of $(\rightarrow^+ E)$ or $(\rightarrow^- E)$; in this case, $\varphi = \chi \rightarrow \xi$ or $\varphi = \sim(\chi \rightarrow \xi)$ for some χ, ξ and there is one of the following two a sub-derivations:

$$\frac{[\chi \to \xi]_i \quad \begin{array}{c}\vdots\\ \chi_\beta\end{array}}{\xi_{\beta\cup i}}\,(\to^+ E) \qquad \frac{[\sim(\chi \to \xi)]_i \quad \begin{array}{c}\vdots\\ \chi_\beta\end{array}}{\sim\xi_{\beta\cup i}}\,(\to^- E)$$

with $\beta \subseteq \alpha$ in both cases.

Denote by $\vdash_{\mathcal{N}_{rc}} \Gamma : \varphi$ the derivability (deducibility) in $\mathcal{N}_{rc}$ of φ from the open assumptions Γ. When Γ is empty, φ is a thesis (formal theorem) of $\mathcal{L}_{rc}$.

Note that the rules $(\to^+ I/E)$ are the defining rules for $\mathbf{R}_\to$. The arguments for $\mathcal{N}_{rc}$ rendering $\mathcal{L}_{rc}$ a relevance logic are the same arguments put forward originally by Anderson and Belnap for $\mathbf{R}_\to$ being a relevance logic due to their use-tracking ND-system; I will not repeat them here.

Example 3.1. Below is a proof of one of the axioms of $\mathbf{R}_\to$ (see Section 4).

$$\vdash_{\mathcal{N}_{rc}} (\varphi \to \psi) \to ((\chi \to \varphi) \to (\chi \to \psi))$$

The derivation is

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{[\chi \to \varphi]_2 \quad [\chi]_1}{\varphi_{\{1,2\}}}\,(\to^+ E) \quad [\varphi \to \psi]_3}{\psi_{\{1,2,3\}}}\,(\to^+ E)}{(\chi \to \psi)_{\{2,3\}}}\,(\to I^1)}{((\chi \to \varphi) \to (\chi \to \psi))_3}\,(\to^+ I^2)}{(\varphi \to \psi) \to ((\chi \to \varphi) \to (\chi \to \psi))}\,(\to^+ I^3)$$

It easy to see that no assumption was vacuously discharged.

Example 3.2. Here is an example of a derivation of a non-axiom formal theorem of $\mathbf{R}_\to$ (due to Mordechai Wajsberg) in $\mathcal{N}_{rc}$.

$$\vdash_{\mathcal{N}_{rc}} ((\varphi \to \varphi) \to \psi) \to \psi$$

The derivation is

$$\cfrac{\cfrac{[(\varphi \to \varphi) \to \psi]_1 \quad \cfrac{\cfrac{[\varphi]_2}{\varphi_2}\,(Ass)}{\varphi \to \varphi}\,(\to^+ I^2)}{\psi_1}\,(\to^+ E)}{((\varphi \to \varphi) \to \psi) \to \psi}\,(\to^+ I^1)$$

The examples below establish the connexivity of $\mathcal{L}_{rc}$.

Example 3.3 (Aristotle's theses).

$$\dfrac{\dfrac{\dfrac{[\varphi]_1}{\varphi_1}\,(Ass)}{(\sim\sim\varphi)_1}\,(dni)}{\sim(\varphi\to\sim\varphi)}\,(\to^- I^1) \qquad\qquad \dfrac{\dfrac{[\sim\varphi]_1}{(\sim\varphi)_1}\,(Ass)}{\sim(\sim\varphi\to\varphi)}\,(\to^- I^1)$$

It is interesting to observe that Orlov (as reported by Došen in [4], discussed also in [14]), not aiming at connexivity, had a weaker version of this thesis as an axiom, fitting his conception of negation in relevance logic:

$$\varphi\to\neg(\varphi\to\neg\varphi)$$

a contraposition of a variant of *Reductio*:

$$(\varphi\to\neg\varphi)\to\neg\varphi$$

The other axiom added by Orlov for a characterisation of relevance logic negation is contraposition:

$$(\varphi\to\neg\psi)\to(\psi\to\neg\varphi)$$

Example 3.4 (Boethius' theses).

$$\dfrac{\dfrac{\dfrac{\dfrac{[\varphi\to\psi]_1\quad[\varphi]_2}{\psi_{\{1,2\}}}\,(\to^+ E)}{(\sim\sim\psi)_{\{1,2\}}}\,(dni)}{(\sim(\varphi\to\sim\psi))_1}\,(\to^- I^2)}{(\varphi\to\psi)\to\sim(\varphi\to\sim\psi)}\,(\to^+ I^1) \qquad \dfrac{\dfrac{\dfrac{[\varphi\to\sim\psi]_1\quad[\varphi]_2}{(\sim\psi)_{\{1,2\}}}\,(\to^+ E)}{(\sim(\varphi\to\psi))_1}\,(\to^- I^2)}{(\varphi\to\sim\psi)\to\sim(\varphi\to\psi)}\,(\to^+ I^1)$$

Example 3.5.

$$\vdash_{\mathcal{N}_{rc}}\sim(\varphi\to(\psi\to\chi)):\varphi\to(\psi\to\sim\chi)$$

The derivation is:

$$\dfrac{\dfrac{\dfrac{\dfrac{\sim(\varphi\to(\psi\to\chi))_1\quad[\varphi]_2}{\sim(\psi\to\chi)_{\{1,2\}}}\,(\to^- E)\qquad[\psi]_3}{\sim\chi_{\{1,2,3\}}}\,(\to^- E)}{(\psi\to\sim\chi))_{\{1,2\}}}\,(\to^+ I^3)}{(\varphi\to(\psi\to\sim\chi))_1}\,(\to^+ I^2)$$

This derivation naturally generalizes to show

$$\vdash_{\mathcal{N}_{rc}}\sim(\varphi_1\to(\cdots\to\varphi_n)\cdots):(\varphi_1\to(\cdots\to\sim\varphi_n)\cdots),\ n\geqslant 3$$

Example 3.6.

$$\vdash_{\mathcal{N}_{rc}} \varphi \to \psi, {\sim}(\psi \to \chi) : {\sim}(\varphi \to \chi)$$

the derivation is:

$$\dfrac{\dfrac{({\sim}(\psi \to \chi))_2 \qquad \dfrac{(\varphi \to \psi)_1 \quad [\varphi]_3}{\psi_{\{1,3\}}}\,(\to^+ E)}{({\sim}\chi)_{\{1,2,3\}}}\,(\to^- E)}{{\sim}(\varphi \to \chi)_{\{1,2\}}}\,(\to^- I^3)$$

Proposition 3.1 (Conservativity). *$\mathcal{N}_{rc}$ is conservative over $\mathbf{R}_\to$.*

Proof. Suppose $\vdash_{\mathcal{N}_{rc}} \Gamma : \chi$, where Γ and χ are negation free. The proof is by induction on the last $\mathcal{N}_{rc}$-rule applied in the derivation. For avoiding notational clutter, I ignore the use-tracking indices. Only rules which have a negation free conclusion and are not rules of $\mathbf{R}_\to$ need to be considered. This leaves only (*dne*). Consider now how the premise of (*dne*), namely ${\sim}{\sim}\chi$ was derived. Note that since Γ contains only $\sim$-free assumptions, the only way negation enters the derivation is by means of an (*dni*) application. Call the point where this happens the negation injection point.

There are two possibilities. 1. Via $(\to^- E)$: That is, ψ is ${\sim}\chi$, and the $(\to_- E)$-application looks like:

$$\dfrac{\dfrac{{\sim}(\varphi \to {\sim}\chi) \quad \varphi}{{\sim}{\sim}\chi}\,(\to^- E)}{\chi}\,(dne) \tag{3.2}$$

By the induction hypothesis, the minor premise φ is derivable in $\mathbf{R}_\to$. Assume w.l.o.g. that the major premise was introduced by the following sub-derivation, containing the negation injection point.

$$\dfrac{\dfrac{\begin{array}{c}[\varphi]_i\\ \mathcal{D}\\ \chi\end{array}}{{\sim}{\sim}\chi}\,(dni)}{{\sim}(\varphi \to {\sim}\chi)}\,(\to^- I^i) \tag{3.3}$$

But then, (3.3) can be replaced by

$$\dfrac{\begin{array}{c}[\varphi]_i\\ \mathcal{D}\\ \chi\end{array}}{\varphi \to \chi}\,(\to^+ I^i)$$

and (3.2) can be replaced by $(\to^+ E)$, an $\mathbf{R}_\to$-derivation, as required.

2. Via (*dni*): immediate. □

3.2. On the negated implication

So, how does the connexive-relevant negated implication differ from its counterpart in $\mathbf{R}_{\rightarrow,\neg}$? As a representative of the latter, I take the following two $(\neg I)$ and $(\neg E)$ rules from [8, Section 7.3]. One might use an explicit contradiction to avoid an appeal to $\bot$, which is not used in $\mathcal{N}_{rc}$. These are general rules for $\neg\varphi$, independent on the form of φ.

$$\frac{\begin{array}{c}[\varphi]_i\\ \vdots\\ \bot_{\alpha\cup i}\end{array}}{(\neg\varphi)_\alpha}\ (\neg I^i),\ i \text{ fresh} \qquad \frac{\varphi_\alpha \quad (\neg\varphi)_\beta}{\bot_{\alpha\cup\beta}}\ (\neg E)$$

Note the absence of $(\bot E)$ in order not to validate explosion.

When applied to derive a negated implication, we get

$$\frac{\begin{array}{c}[\varphi\rightarrow\psi]_i\\ \vdots\\ \bot_{\alpha\cup i}\end{array}}{(\neg(\varphi\rightarrow\psi))_\alpha}\ (\neg I^i) \qquad \frac{(\varphi\rightarrow\psi)_\alpha \quad (\neg(\varphi\rightarrow\psi))_\beta}{\bot_{\alpha\cup\beta}}\ (\neg E)$$

So, a negated implication means that assuming the (relevant) implication leads to a contradiction. Thus, there is *no* proof of ψ from an assumption φ using that assumption. This absence of proof may originate from several reasons; in particular, it leaves open the possibility that ψ is just *not relevant* to φ, and, hence, not relevantly provable from it.

On the other hand, the connexive-relevant rule $(\sim\rightarrow I)$ requires more! It requires that $\neg\psi$, hence ψ itself, *is* relevant to φ, as it requires a proof of ψ from an assumption $\neg\varphi$ *using the latter*!

Thus, the connexive-relevant implication has a stronger contents connection between the antecedent and a consequent of an implication than its relevant counterpart.

3.3. A natural deduction-theorem for $\mathcal{L}_{rc}$

The deduction meta-theorem (DT) is usually formulated for axiomatically defined logics. To recapitulate, let $\mathcal{H}$ be some generic axiomatic system over an object language containing a generic conditional '$\Rightarrow$'.

$$\vdash_{\mathcal{H}} \Gamma,\varphi : \psi \quad \text{iff} \quad \vdash_{\mathcal{H}} \Gamma : \varphi \Rightarrow \psi. \tag{DT}$$

According to Avron [2], the satisfaction of the condition of the deduction theorem, but formulated in terms of a consequence relation, constitutes the definition of a binary connective being an implication.

I will attend to the deduction-theorem for $\mathcal{L}_{rc}$ after the axiomatic definition of the latter, $\mathcal{H}_{rc}$, is presented in Section 4 and shown deductively equivalent to $\mathcal{N}_{rc}$.

However, one can conceive a version of the deduction-theorem applicable to natural-deduction systems instead of axiomatic ones. I refer to such version as a natural deduction-theorem (NDT). Such an (NDT) theorem is usually much easier to prove than its axiomatic counterpart because of the way conditionals of various kinds are introduced by natural-deduction I-rules.

Let $\mathcal{N}$ be some generic natural-deduction system over an object language containing a generic conditional '⇒'.

$$\vdash_{\mathcal{N}} \Gamma, \varphi : \psi \quad \text{iff} \quad \vdash_{\mathcal{N}} \Gamma : \varphi \Rightarrow \psi. \tag{NDT}$$

The formulation looks identical to the usual formulation of the deduction-theorem, but it pertains to ND-derivations and not to axiomatic derivations (from open assumptions).

I now formulate (NDT_{rc}), the *relevant-connexive* natural deduction-theorem for $\mathcal{N}_{rc}$. The formulation adapts the general case to the presence of assumption tracing indices in $\mathcal{N}_{rc}$.

Theorem 3.1 (Relevant-connexive natural deduction-theorem for $\mathcal{N}_{rc}$). *For any Γ, φ and ψ we have*:

Positive NDT_{rc}

$$i \notin \hat{n} \text{ and } \vdash_{\mathcal{N}_{rc}} \Gamma, \varphi_i : \psi_{\hat{n}\cup i} \text{ iff } \vdash_{\mathcal{N}_{rc}} \Gamma : (\varphi \to \psi)_{\hat{n}}. \tag{NDT^+_{rc}}$$

Negative NDT_{rc}

$$i \notin \hat{n} \text{ and } \vdash_{\mathcal{N}_{rc}} \Gamma, \varphi_i : (\sim\psi)_{\hat{n}\cup i} \text{ iff } \vdash_{\mathcal{N}_{rc}} \Gamma : (\sim(\varphi \to \psi))_{\hat{n}}. \tag{NDT^-_{rc}}$$

Proof. For positive NDT.

"Only if" Assuming $i \notin \hat{n}$ and $\vdash_{\mathcal{N}_{rc}} \Gamma, \varphi_i : \psi_{\hat{n}\cup i}$, just use $(\to^+ I)$.

"If" Assuming $\vdash_{\mathcal{N}_{rc}} (\varphi \to \psi)_{\hat{n}}$, we have

$$\dfrac{\Gamma : (\varphi \to \psi)_{\hat{n}} \qquad \dfrac{[\varphi]_i}{\varphi_i}\,(ass)}{\Gamma, \varphi_i : \psi_{\hat{n}\cup i}}\,(\to^+ E)$$

Also, $i \notin \hat{n}$ holds by the freshness of i.

For negative NDT.
"Only if" Assuming $i \notin \hat{n}$ and $\vdash_{\mathcal{N}_{rc}} \Gamma, \varphi_i : (\sim\psi)_{\hat{n}\cup i}$, just use $(\rightarrow^- I)$.
"If" Assuming $\vdash_{\mathcal{N}_{rc}} (\sim(\varphi \rightarrow \psi))_{\hat{n}}$, we have

$$\dfrac{\Gamma : (\sim(\varphi \rightarrow \psi))_{\hat{n}} \quad \dfrac{[\sim\varphi]_i}{(\sim\varphi)_i}\,(ass)}{\Gamma, \varphi_i : (\sim\psi)_{\hat{n}\cup i}}\,(\rightarrow^- E)$$

Again, $i \notin \hat{n}$ holds by the freshness of i. □

4. Axiomatic definition of $\mathcal{L}_{rc}$

4.1. Defining $\mathcal{H}_{rc}$

In this section, an axiomatic definition of $\mathcal{L}_{rc}$ by means of a Hilbert-system $\mathcal{H}_{rc}$ is presented and $\mathcal{H}_{rc}$ is shown to be deductively equivalent to $\mathcal{N}_{rc}$. Note that while in large parts of the literature axiomatic presentation are taken as a *definitional*, self-justifying presentation, I consider it as justified by the deductive equivalence to $\mathcal{N}_{rc}$, the latter being the definitional tool.

The development of the axiomatization is similar to that of Wansing's axiomatic definition of a connexive logic C [16], but instead of starting with the axioms of positive (propositional) intuitionistic logic, I start with the (positive) axioms of $\mathbf{R}_{\rightarrow}$, the positive implicative fragment of $\mathbf{R}$, to which negative axioms, inducing connexivity, are added.

Definition 4.1 (Axiomatic definition of $\mathcal{L}_{rc}$). The Hilbert-style axiomatic definition $\mathcal{H}_{rc}$ of $\mathcal{L}_{rc}$ is given by the following axiom schemes, divided into two groups — positive axioms[2] and negative axioms — with the single inference rule

$$\dfrac{\varphi \quad \varphi \rightarrow \psi}{\psi}\,(MP)$$

Positive axioms:

self implication: $\varphi \rightarrow \varphi$
prefixing: $(\varphi \rightarrow \psi) \rightarrow [(\chi \rightarrow \varphi) \rightarrow (\chi \rightarrow \psi)]$

[2] As for the implicational fragment of $\mathbf{R}$; see [5, Section 1.5], where alternative equivalent axiomatizations are discussed.

contraction: $[\varphi \to (\varphi \to \psi)] \to (\varphi \to \psi)$
permutation: $[\varphi \to (\psi \to \chi)] \to [\psi \to (\varphi \to \chi)]$

Negative axioms: First, note that because of the absence of conjunction from the object language, bi-implication '$\leftrightarrow$' cannot be directly defined. Below, I use axioms schemes with the notation $\varphi \leftrightarrow \psi$ as an abbreviation of pairs of axions schemes $\varphi \to \psi$ and $\psi \to \varphi$. Compare those negative axioms with the standard relevance logic negative axioms mentioned in Example 3.3.

double negation: $\sim\sim\varphi \leftrightarrow \varphi$
negating implication: $\sim(\varphi \to \psi) \leftrightarrow (\varphi \to \sim\psi)$

Suppose, first, that derivations were defined as usual for Hilbert-like systems[3], namely, sequences of formulas where each member is an assumption, or an axiom, or the result of an application of (MP) to two earlier members in the list. Denote by $\vdash_{\mathcal{H}_{rc}} \Gamma : \varphi$ the derivability in $\mathcal{H}_{rc}$ of φ from a multi-set of open assumptions Γ. Here Γ, φ and Γ_1, Γ_2 mean multi-set union. Again, when Γ is empty, φ is a thesis (formal theorem) of $\mathcal{L}_{rc}$. Clearly, this notion of derivation cannot be shown equivalent to $\mathcal{N}_{rc}$-derivations, as, in contrast to $\mathcal{N}_{rc}$-derivations, it ignores the relevance of assumptions to the derived conclusion.[4]

To make the two notions of derivations compatible for comparison, we redefine the notion of $\mathcal{H}_{rc}$-derivation by incorporating into it the tracing of used assumptions, as suggested in [5]. Thus, each assumption is flagged with a *fresh* flag, say i, and if an assumption occurs more than once in a derivation, it always occurs with the same index. Thus, assumptions can be still considered as forming a set. Furthermore, (MP) is modified to

$$\frac{\varphi_\alpha \quad \varphi \to \psi_\beta}{\psi_{\alpha\cup\beta}} \ (MP^r)$$

If, again by convention, we suppose that the assumptions Γ are flagged by $\{1 \cdots n\}$ (for some $n \geqslant 0$), then the conclusion has to be flagged by $\hat{n}$, having used all the assumptions, as in $\mathcal{N}_{rc}$. Denote this relevant deducibility notion by $\vdash^r_{\mathcal{H}_{rc}} \Gamma : \varphi_{\hat{n}}$.

[3] Referred to as 'protoproofs' in [2].

[4] See the discussion in [5, Section 2.1].

4.2. Deductive equivalence of $\mathcal{H}_{rc}$ and $\mathcal{N}_{rc}$

Theorem 4.1. *For every Γ and φ:*

$$\vdash^{r}_{\mathcal{H}_{rc}} \Gamma : \varphi_{\hat{n}} \quad \textit{iff} \quad \vdash_{\mathcal{N}_{rc}} \Gamma : \varphi_{\hat{n}}.$$

Proof. Without loss of generality, assume the open assumptions Γ are flagged for use with the same indices in both derivations. In $\mathcal{H}_{rc}$, axioms can be considered as flagged with $\emptyset$.

To show that $\vdash^{r}_{\mathcal{H}_{rc}} \Gamma : \varphi_{\hat{n}}$ implies $\vdash_{\mathcal{N}_{rc}} \Gamma : \varphi_{\hat{n}}$, I show that all the $\mathcal{H}_{rc}$ axioms are derivable (from no open assumptions) in $\mathcal{N}_{rc}$. The derivations for the positive axioms are rather standard and omitted (see, for example, Example 3.1). I show the derivations for the negative axioms.

Double negation:

$$\dfrac{\dfrac{\dfrac{[\varphi]_1}{\varphi_1}\,(Ass)}{(\sim\sim\varphi)_1}\,(dni)}{\varphi \to \sim\sim\varphi}\,(\to^{+}I^{1}) \qquad \dfrac{\dfrac{\dfrac{[\sim\sim\varphi]_1}{(\sim\sim\varphi)_1}\,(Ass)}{\varphi_1}\,(dne)}{\sim\sim\varphi \to \varphi}\,(\to^{+}I^{1})$$

Negating implication: The derivations are those of Boethius' theses in Example 3.4.

To show that $\vdash_{\mathcal{N}_{rc}} \Gamma : \varphi_{\hat{n}}$ implies $\vdash^{r}_{\mathcal{H}_{rc}} \Gamma : \varphi_{\hat{n}}$, assume $\vdash_{\mathcal{N}_{rc}} \Gamma : \varphi_{\hat{n}}$. The proof is by induction on the last rule applied in the $\mathcal{N}_{rc}$-derivation. Again, only the negative rules are of interest.

$(\to^{-}I)$: In this case, φ is $\sim(\psi \to \chi)$ for some ψ, χ. The premise of this application of $(\to^{-}I)$ is: $[\psi]_i \ \cdots \ (\sim\chi)_{\hat{n}\cup i}$. By the induction hypothesis on the premise, $\vdash^{r}_{\mathcal{H}_{cr}} \Gamma, \psi : (\sim\chi)_{\hat{n}\cup i}$. By the deduction-theorem, $\vdash^{r}_{\mathcal{H}_{cr}} \Gamma : (\psi \to \sim\chi)_{\hat{n}}$, and by the negating application axiom and (MP^{r}), $\vdash^{r}_{\mathcal{H}_{cr}} \Gamma : (\sim(\psi \to \chi))_{\hat{n}}$.

$(\to^{-}E)$: In this case, φ is $\sim\psi$, for some ψ, and the premises of the rule are $\sim(\chi \to \psi)_{\alpha}$ and χ_{β}, for some χ. By the induction hypothesis on the premises, (i) $\vdash^{r}_{\mathcal{H}_{rc}} \Gamma_1 : \sim(\chi \to \psi)_{\hat{n_1}}$ and (ii) $\vdash^{r}_{\mathcal{H}_{rc}} \Gamma_2 : \chi_{\hat{n_2}}$, where $\Gamma_1\Gamma_2 = \Gamma$ and $\hat{n_1} \cup \hat{n_2} = \hat{n}$. From (i) and the negating implication axiom, we get by (MP^{r}) (iii) $\vdash^{r}_{\mathcal{H}_{rc}} \Gamma : (\chi \to \sim\psi)_{\hat{n_2}}$. Finally, from (ii) and (iii) we get by (MP^{r}) $\vdash^{r}_{\mathcal{H}_{rc}} \Gamma : \sim\psi_{\hat{n}}$.

(dni) and (dne): Obvious and omitted. □

Corollary 4.1 (The relevant-connexive deduction-theorem).

$$i \notin \hat{n} \text{ and } \vdash^{r}_{\mathcal{H}_{rc}} \Gamma, \varphi_i : \psi_{\hat{n}\cup i} \quad \textit{iff} \quad \vdash^{r}_{\mathcal{H}_{rc}} \Gamma : (\varphi \to \psi)_{\hat{n}}, \qquad (\mathrm{DT}^{+}_{rc})$$

$$i \notin \hat{n} \text{ and } \vdash^{r}_{\mathcal{H}_{rc}} \Gamma, \varphi_i : \sim\psi_{\hat{n}\cup i} \quad \textit{iff} \quad \vdash^{r}_{\mathcal{H}_{rc}} \Gamma : \sim(\varphi \to \psi)_{\hat{n}}. \qquad (\mathrm{DT}^{-}_{rc})$$

Proof. First, for (DT^{+}_{rc}), assume (1) $\vdash^{r}_{\mathcal{H}_{rc}} \Gamma, \varphi_i : \psi_{\hat{n}\cup i}$ for some $i \notin \hat{n}$. By Theorem 4.1, (1) holds iff (2) $\vdash_{\mathcal{N}_{rc}} \Gamma, \varphi_i : \psi_{\hat{n}\cup i}$. By Theorem 3.1, (2) holds iff (3) $\vdash_{\mathcal{N}_{rc}} \Gamma : (\varphi \to \psi)_{\hat{n}}$. So, again by Theorem 4.1, (3) holds iff (4) $\vdash^{r}_{\mathcal{H}_{rc}} \Gamma : (\varphi \to \psi)_{\hat{n}}$, which establishes the result.

The proof of (DT^{-}_{rc}) is similar and omitted. □

5. Conclusion

In this paper, a connexive extension of the relevance logic $\mathbf{R}_{\to}$ was presented. It is defined by means of a natural deduction system, and a deductively equivalent axiomatic system is presented too. The goal of such an extension is to produce a logic with stronger connection between the antecedent and the consequent of an implication.

A natural question left untouched here is the presentation of a model theory for the combined logic, for which the proof-theory is sound and complete. Such a model theory, even if not intended to serve as semantics, is a useful tool for establishing non-derivability. At this stage, I have nothing to offer in this respect. Presumably, the Routley-Meyer ternary relation with the Routley star operator might be somehow "massaged" to obtain models for $\mathcal{L}_{rc}$. But, having been criticized as lacking a good explanatory power even for $\mathbf{R}_{\to,\neg}$ itself, it is not clear whether any useful insight about $\mathcal{L}_{rc}$ might be gained even if such an attempt succeeds.

An interesting question, deserving further research, is about the possibility of devising a connexive extension for a fuller relevance logic, including conjunction and disjunction. Following Avron [3], I believe that the *extensional* (boolean) conjunction and disjunction (known also as multiplicative connectives) do not fit relevance logic, and their *intensional* counterparts, fusion and fission (known also as additive connectives) are preferred. See [3] for an extensive discussion; see also Mares [9]. However, once the relevant implication and negation is replaced by a their connexive counterparts, the usual definitions of the intensional connectives cannot be applied anymore. For example, in $\mathbf{R}_{\to,\neg}$, one can define fusion by

$$\varphi \circ \psi =^{\mathrm{df}} \neg(\varphi \to \neg\psi)$$

Attempting to replace $\neg(\varphi \to \neg\psi)$ with $\sim(\varphi \to \sim\psi)$ (using the $\mathcal{L}_{rc}$ connectives) renders fusion identical to implication!

$$\sim(\varphi \to \sim\psi) \text{ iff } (\varphi \to \sim\sim\psi) \text{ iff } \varphi \to \psi$$

So, some more suitable definition is called for.

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