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JERZY ŁOŚ POSITIONAL CALCULUS AND THE ORIGIN OF TEMPORAL LOGIC*

Abstract. Most accounts, including leading textbooks, credit Arthur Norman Prior with the invention of temporal (tense logic). However, (i) Jerzy Łoś delivered his version of temporal logic in 1947, several years before Prior; (ii) Henrk Hiż's review of Łoś's system in *Journal of Symbolic Logic* was published as early as 1951; (iii) there is evidence to the effect that, when constructing his tense calculi, Prior was aware of Łoś's system. Therefore, although Prior is certainly a key figure in the history tense logic, as well as modal logic in general, it should be accepted both in the literature that temporal logic was invented by Jerzy Łoś.

Keywords: positional logic; the realization operator $\mathcal{R}$; temporal logic

A significant number of books and papers, concerning the origin of temporal logic, have been published by prominent publishing houses and prestigious journals for twenty five years. In the vast majority of those works Arthur Norman Prior has been considered the inventor or the discoverer of temporal logic, whereas Jerzy Łoś is not even mentioned [cf., e.g., Øhrstrøm and Hasle, 1993, 1995, 2006a,b]. However, having recognized Prior's contribution to be crucial and irreplaceable, one should admit fairly that it is Łoś who invented the logic of time. That means particularly that (i) Łoś constructed, described and examined the first mature calculus of temporal logic and (ii) Prior was aware of and inspired by Łoś's ideas when beginning his own work in the field. The objective of this paper is to justify those claims.

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1. Łoś's master's thesis

Jerzy Łoś, a Polish philosopher, was born on March 22 in Lwów (today Lviv in Ukraine) and died of a cerebral stroke on June 1 1998 in Warsaw (he was baptized as Jerzy Maria Michał which is the Polish equivalent of George Mary Michael, and his surname 'Łoś' should be pronounced as something between 'wos' and 'wosh', because Polish 'ś' sounds similar to the opening 's' in English 'sure', while 'ł' is always equal to the 'w' in 'water').

Before the World War II he studied in Lwów: first medicine and then philosophy and chemistry. During the war he was an office worker in a sugar factory in Lublin (Poland). Straight after the war he completed his courses and took a master's degree in philosophy from the newly-established Maria Curie-Skłodowska University in Lublin.

Łoś's master's thesis was entitled "Analiza metodologiczna kanonów Milla" (A methodological analysis of Mill's canons) and was supervised by Jerzy Słupecki, an eminent disciple of Jan Łukasiewicz. An improved version of the thesis was entitled "Podstawy analizy metodologicznej kanonów Milla" [Łoś, 1947] (English version: "Foundations of the methodological analysis of Mill's canons" [Łoś, 1977]) and published in Polish (in 1948) in a predated volume of a local university journal. It should be emphasised that it is this very master's thesis which contains Łoś's temporal calculus and marks the origin of temporal logic. As early as 1951 Henryk Hiż published an English review of Łoś's work, thereby familiarising the global logical community with the newly-invented logic of time.

Having graduated in philosophy, Łoś became an assistant to Słupecki, however, as early as the end of 1947 Słupecki left Lublin for Wrocław (now situated in the west of Poland after the post-war redrawing of land boundaries) where he accepted the position of Chair of Mathematical Logic in the Institute of Mathematics, in the newly-established University of Wrocław. Together with Słupecki, Łoś moved to Wrocław as well. Having moved to Wrocław (and then to Toruń and Warsaw), Łoś abandoned his former interests and concentrated on metamathematics, algebra, and then applications of mathematics in economy and computer science. He was never to return to either temporal logic or philosophy of natural sciences. Logic owes many outstanding contributions to his move, such as Łoś's ultraproducts theorem, but the deserted logic of time became likely to shortly forget its inventor.

2. Łoś's language

The underlying logic of Łoś's calculus is Standard Logic with quantifiers ranging over any kind of variables (including propositional). It must be said that Łoś is not perfectly clear about the language and the logic he applies. He clearly uses all classical theorems and rules with respect to Boolean connectives and to quantifiers ranging over individual variables as well as propositional variables. However, it is not always clear in what context those operations are allowed. In what follows we attempt to reconstruct the exact, precise logic Łoś involves. The calculus Łoś uses as his underlying logic for his system seems intermediate between Classical Propositional Logic (shortly: **CPL**) and Leśniewski's full-blooded protothetics [cf. Słupecki, 1953]. Although Łoś speaks literally of the term variables without quantifying [cf. Łoś, 1947, p. 280 (303)],[1] the full theory of quantification is actually to be found here.

Begin with the language of **CPL**, containing propositional variables, parentheses and the connectives of negation '$\neg$', conjunction '$\wedge$', disjunction '$\vee$', conditional '$\rightarrow$' and equivalence '$\equiv$' (this is also the order of connectives in absence of parentheses, and any other one-place connectives will go first).

Enrich the language with the quantifiers, universal '$\forall$' and particular '$\exists$'. The quantifiers are classical with the qualification that they range over variables of any kind. It is a similar qualification to that which applies to second-order logic, but the propositional variables are also included. Well-known examples of theorems of the propositional calculus with quantifiers are the formulas '$\exists p\ p$' and '$\forall p \exists q (p \equiv q)$'. Actually, Leśniewski allows also connective variables. This would constitute the system **E** of elementary protothetics, but Łoś makes no use of them.

Such a language is to be further enriched with the full variety of first-order terms, including functions. As it has been already mentioned quantifiers range over term variables as well. The terms may designate instants of time as well as time intervals, so the first-order part of the language is actually many sorted [cf. Łoś, 1947, p. 279 (303)].

Finally, specific connectives are to be added: '$\mathcal{R}$' and 'δ'. The symbol '$\mathcal{R}$' is the connective of temporal realization. Using Łukasiewicz's Polish notation, Łoś was using the uppercase letter 'U' as a connective of

[1] We use the original version of Łoś's paper and translate it where required, but we also provide in parentheses all the page references to the published English translation.

temporal realization, to the effect that $\ulcorner \mathrm{U}\alpha\varphi \urcorner$ means that a formula φ occurs at the instant α (where α being a term). Following the common habit, initiated by Rescher, we use the sign '$\mathcal{R}$' instead. For example, we write '$\mathcal{R}_x(p \wedge q) \equiv (\mathcal{R}_x p \wedge \mathcal{R}_x q)$' instead Łoś's '$\mathrm{EU}x\mathrm{K}pq\mathrm{KU}xp\mathrm{U}xq$'.

The translation seems to be obvious. Aside from the connective '$\mathcal{R}$' an operator 'δ' is to be involved to the effect that $\ulcorner \delta(\alpha, \varepsilon) \urcorner$ refers to the instant following the instant α after an interval ε. For example, something like 'δ(September 12^{th} 1683, 3 days)' would be September 15^{th} 1683. Hence '$\mathcal{R}_{\delta(x,y)} p$' (or '$\mathcal{R}_{\delta xy} p$' for short) means that it occurs that p at an instant succeeding the instant x after a period y.

The symbols '$\mathcal{R}$' (originally 'U') and 'δ' are primitives of Łoś's calculus. There are also four predicates defined by the following abbreviations:

$$\ulcorner \varphi \cong \psi \urcorner \overset{\text{df}}{=} \ulcorner \forall x(\mathcal{R}_x \varphi \equiv \mathcal{R}_x \psi) \urcorner,$$

$$\ulcorner \alpha \simeq \beta \urcorner \overset{\text{df}}{=} \ulcorner \forall p(\mathcal{R}_\alpha p \equiv \mathcal{R}_\beta p) \urcorner,$$

$$\ulcorner \alpha \preccurlyeq_\varepsilon \beta \urcorner \overset{\text{df}}{=} \ulcorner \delta(\alpha, \varepsilon) \simeq \beta \urcorner,$$

$$\ulcorner \alpha \preccurlyeq \beta \urcorner \overset{\text{df}}{=} \ulcorner \exists \varepsilon\ \alpha \preccurlyeq_\varepsilon \beta \urcorner,$$

where 'ε' is any time interval term. Originally Łoś used the symbol 'I' instead of '$\cong$', the symbol 'ρ' instead of '$\simeq$', the symbol 'π' instead of '$\preccurlyeq$' and the symbol 'ν' instead of the indexed '$\preccurlyeq_\varepsilon$' [cf. Łoś, 1947, p. 281 (304)]. It is obvious that $\ulcorner \varphi \cong \psi \urcorner$ means that φ and ψ occur at exactly the same instants, $\ulcorner \alpha \simeq \beta \urcorner$ means that at the instants α and β there occur exactly the same formulas, $\ulcorner \alpha \preccurlyeq_\varepsilon \beta \urcorner$ means that the instant α is earlier than the instant β at the distance of the length ε, and $\ulcorner \alpha \preccurlyeq \beta \urcorner$ means that the instant α is not later than the instant β.

Łoś allows formulas free of the connective '$\mathcal{R}$', but no nested tokens of '$\mathcal{R}$'. For example, expressions '$p \vee q$ and '$\mathcal{R}_x(p \vee q)$' are formulas, unlike the expression '$\mathcal{R}_x \mathcal{R}_x(p \vee q)$'. However, all formulas may be freely transformed by means of classical connectives.

The following definition of the set of formulas may be reconstructed from Łoś's work given that he himself did not actually provide a definition:

(a) if φ is a formula of the Propositional Calculus with quantifiers, than it is also an atomic formula of temporal logic,
(b) if φ is a formula of the Propositional Calculus with quantifiers and τ is an instant term, than $\ulcorner \mathcal{R}_\tau \varphi \urcorner$ is an atomic formula of temporal logic,

(c) if τ is an instant term and ε is an interval variable, than $\delta(\tau, \varepsilon)$ is an instant term of temporal logic,

and the set of all formulas of the temporal logic is the smallest collection containing all the atomic formulas and closed the usual expression-forming operations of '$\neg$', '$\wedge$', '$\vee$', '$\rightarrow$', '$\equiv$', '$\forall$' and '$\exists$', provided the quantifiers range over all kinds of variables. The phrase 'with quantifiers' in the parentheses in point (b) should be probably deleted, because quantifiers never appear in the scope of the connective '$\mathcal{R}$' and nor are there axioms introducing them in such contexts. So, in the scope of the connective '$\mathcal{R}$' there appear only the formulas of pure **CPL** without quantifiers. However, Łoś never clearly makes these claims himself.

Now we introduce precise definitions of terms and formulas, which may be reconstructed from the above remarks. Firstly, we will use three sorts of variables::

- *propositional variables*: 'p', 'q', 'r', 'p_1', 'p_2', ...,
- *instant variables*: 'x', 'y', 'x_1', 'x_2', ...,
- *interval variables*: 'e', 'e_1', 'e_2', ...

The set of *instant terms* is the smallest set S satisfying the following conditions:

- all instant variables belong to S,
- if $\tau \in S$ and ε is an interval variable, then $\ulcorner\delta(\tau, \varepsilon)\urcorner \in S$.

Let $\mathrm{For}_{\mathbf{CPL}}$ be the set of all formulas of **CPL** (which we build in a standard way). *Temporal atomic formulas* are all expressions of the form $\ulcorner\mathcal{R}_\tau \varphi\urcorner$, where τ is an instant term and $\varphi \in \mathrm{For}_{\mathbf{CPL}}$. Finally, the set of *formulas* For is the smallest set satisfying the following conditions:

- $\mathrm{For}_{\mathbf{CPL}} \subseteq \mathrm{For}$,
- all temporal atomic formulas belong to For,
- if $\varphi \in \mathrm{For}$, then $\ulcorner\neg\varphi\urcorner \in \mathrm{For}$,
- if $\varphi, \psi \in \mathrm{For}$ and $\circ \in \{\wedge, \vee, \rightarrow, \equiv\}$, then $\ulcorner(\varphi \circ \psi)\urcorner \in \mathrm{For}$,
- if $\varphi \in \mathrm{For}$, $Q \in \{\forall, \exists\}$ and υ is a propositional, instant or interval variable, then $\ulcorner Q\upsilon\, \varphi\urcorner \in \mathrm{For}$.

Hence, it seems to be permissible to use the formulas of **CPL** inside the scope of the connective '$\mathcal{R}$'. Outside of the scope of this connective standard logic with quantifiers ranging over the three sorts of variables is allowed.

3. Axiomatics of Łoś's calculus

Łoś's calculus (**ŁC**) has been presented as an axiomatic system of the finite number — namely nine — of specific object-language axioms [cf. Łoś, 1947, pp. 280–281 (303–304)]. Of course, we also require classical theorems and inference rules concerning both the connectives and the quantifiers binding variables of all three sorts, i.e., propositional variables, instant variables and interval variables.

The first two axioms are well-known distribution laws and make the connective '$\mathcal{R}$' transparent to the connectives of **CPL**, provided we have other axioms as well as classical tautologies and rules of inference [see Theorem 2 and Jarmużek and Pietruszczak, 2004]:

$$\mathcal{R}_x \neg p \equiv \neg \mathcal{R}_x p, \tag{ax1}$$

$$\mathcal{R}_x (p \to q) \to (\mathcal{R}_x p \to \mathcal{R}_x q), \tag{ax2}$$

Three other axioms are counterparts of the well-known Gödelian rule of modal generalization, qualified to **CPL** analogously to the modal logic S0.5. As the formulas to appear in the scope of '$\mathcal{R}$' they are axioms of Łukasiewicz's version of **CPL**:

$$\mathcal{R}_x ((p \to q) \to ((q \to r) \to (p \to r))), \tag{ax3}$$

$$\mathcal{R}_x (p \to (\neg p \to q)), \tag{ax4}$$

$$\mathcal{R}_x ((\neg p \to p) \to p), \tag{ax5}$$

Axioms (ax2)–(ax5) allow us to derive all formulas of the form $\ulcorner \mathcal{R}_\tau \varphi \urcorner$, where τ is an instant term and φ is an instance of a theorem of **CPL** (see Theorem 1).

The next axiom says simply that any formula φ is a theorem, provided it holds at every instant: [cf. Łoś, 1947, p. 280 (304)].

$$\forall x\, \mathcal{R}_x p \ \to \ p. \tag{ax6}$$

The last three axioms are slightly more complicated:

$$\forall x \forall e \exists y \forall p (\mathcal{R}_{\delta(x,e)} p \equiv \mathcal{R}_y p), \tag{ax7}$$

$$\forall x \forall e \exists y \forall p (\mathcal{R}_{\delta(y,e)} p \equiv \mathcal{R}_x p), \tag{ax8}$$

$$\forall x \exists p \forall y (\mathcal{R}_y p \equiv \forall q (\mathcal{R}_x q \equiv \mathcal{R}_y q)). \tag{ax9}$$

They get clearer once one has transformed them by means of the definitions to the form:

$$\forall x \forall e \exists y (\delta(x, e) \simeq y), \tag{ax7$'$}$$

$$\forall x \forall e \exists y (\delta(y, e) \simeq x), \tag{ax8′}$$

$$\forall x \exists p \forall y (\mathcal{R}_y p \equiv x \simeq y). \tag{ax9′}$$

The formulas (ax7) and (ax7′) say that for any time instant x and any time interval e there exists a time instant y which is later than x by the interval ε. The formulas (ax8) and (ax8′) say that for any time instant x and any time interval e there exists a time instant y which is earlier than x by the interval ε. Łoś make this claim explicitly, however, it is that it is to be assumed that ε is always a non-zero interval. And this is indeed the case, for the axioms in question are to establish time as infinite. Finally, the formulas (ax9) and (ax9′) constitute the *Clock Axiom* to the (intended) effect that any time instant may be uniquely described by a temporal function [cf. Łoś, 1947, pp. 280–281 (304–305)]. The axiom is a very interesting anticipation of hybrid logics with propositions uniquely describing points.

Let us sketch proofs of some basic results we have mentioned, which went unproven by Łoś. We begin with the restricted Gödelian rule of modal generalization.

Theorem 1. *If φ is an instance of a theorem of* **CPL**, *then* $\ulcorner \mathcal{R}_\tau \varphi \urcorner$ *is a theorem of* **ŁC** *for any instant term τ.*

Proof. Focusing on axioms (ax3)–(ax5) you can see that the formulas within the scope of the connective '$\mathcal{R}$' constitute Łukasiewicz's complete axiomatization of **CPL**, provided we have the classical tautologies and rules of inference. Consider any instance φ of a theorem of **CPL** and let the sequence ψ_1, ψ_2, ..., ψ_n be the proof of φ in Łukasiewicz's system, i.e., $\varphi = \psi_n$. If $n = 1$, then φ is an instance of an axiom of Łukasiewicz's system and so $\ulcorner \mathcal{R}_x \varphi \urcorner$ is an instance of (ax3), or (ax4), or (ax5). Hence $\ulcorner \forall x\, \mathcal{R}_x \varphi \urcorner$ is a theorem of **ŁC**, by rules for '$\forall$' binding 'x'. Moreover, $\ulcorner \mathcal{R}_\tau \varphi \urcorner$ is a theorem of **ŁC**, since by $\ulcorner \forall x\, \mathcal{R}_x \varphi \to \mathcal{R}_\tau \varphi \urcorner$ we have a substitution of τ for 'x'.

Suppose the claim holds for any proof of the length of $n-1$ rows. If the row n is to be added by means of a tautology or a substitution (as in the case for $n = 1$), then $\ulcorner \mathcal{R}_\tau \psi_n \urcorner$ may be obtained from $\ulcorner \mathcal{R}_\tau \psi_{n-1} \urcorner$ by exactly the same tautology or substitution. Let the row n be added by means of modus ponens to some rows i, j such that $1 \leqslant i, j \leqslant n-1$, i.e., $\varphi_j = \ulcorner \psi_i \to \psi_n \urcorner$. Then (as in the case for $n = 1$) there are theorems of **ŁC**: $\ulcorner \mathcal{R}_\tau (\psi_i \to \psi_n) \urcorner$ and $\ulcorner \mathcal{R}_\tau \psi_i \urcorner$. Hence, by Modus Ponens and (ax2), the row $\ulcorner \mathcal{R}_\tau \psi_i \to \mathcal{R}_\tau \psi_n \urcorner$ may be added, and so also the row $\ulcorner \mathcal{R}_\alpha \psi_n \urcorner$. ⊣

Theorem 2 ([cf. Jarmużek and Pietruszczak, 2004]). *The connective '$\mathcal{R}$' is distributive over all classical connectives, i.e., we obtain the following theorems of* **ŁC**:

$$\mathcal{R}_x(p \to q) \equiv (\mathcal{R}_x p \to \mathcal{R}_x q), \tag{$\star$}$$

$$\mathcal{R}_x(\varphi \wedge \psi) \equiv (\mathcal{R}_x \varphi \wedge \mathcal{R}_x \psi), \tag{$\star\star$}$$

$$\mathcal{R}_x(\varphi \vee \psi) \equiv (\mathcal{R}_x \varphi \vee \mathcal{R}_x \psi),$$

$$\mathcal{R}_x(\varphi \equiv \psi) \equiv (\mathcal{R}_x \varphi \equiv \mathcal{R}_x \psi).$$

Proof. By Theorem 1 the following formulas are theorems of **ŁC**:

$$\mathcal{R}_x(q \to (p \to q)),$$

$$\mathcal{R}_x(\neg p \to (p \to q))$$

So, by (ax1), (ax2) and **CPL**, we have the following theorems of **ŁC**:

$$\mathcal{R}_x q \to \mathcal{R}_x(p \to q),$$

$$\mathcal{R}_x \neg p \to \mathcal{R}_x(p \to q),$$

$$\neg \mathcal{R}_x p \to \mathcal{R}_x(p \to q),$$

$$(\neg \mathcal{R}_x p \vee \mathcal{R}_x q) \to \mathcal{R}_x(p \to q).$$

So axiom (ax2) may be strengthened to ($\star$). Other laws of distribution are easily derivable from it. ⊣

Thus, as it has been already mentioned, the connective is transparent with respect to the classical connectives. It means that the minimal normal positional logic **MR**[2] is a proper part of Łoś's calculus **ŁC**, which is to be considered normal itself.

4. Consistency of ŁC

Łoś gave two proofs of the consistency of his calculus. First, the calculus has a simple model in the propositional calculus with quantifiers binding any kinds of variables, and the calculus is known to be consistent. To show this Łoś adds the formula:

$$\mathcal{R}_x p \equiv p \tag{$\dagger$}$$

to the propositional calculus with the quantifiers binding any kind of variables and first-order formulas. All axioms (ax1)–(ax9) are obviously provable in such a theory [cf. Łoś, 1947, p. 282 (306)]. In other words, if

[2] **MR** has been described and examined in [Jarmużek and Pietruszczak, 2004].

all occurrences of any expression '$\mathcal{R}_x$' are removed from axioms (ax1)–(ax9), then the axioms will change into theorems of the propositional calculus with quantifiers binding any variables.

The second proof is slightly more complicated, and yet quite instructive. Łoś shows a straight line to be a model of his calculus. We present a somewhat improved version of Łoś's model. So, let $\mathbb{T}$ ("time") be a straight line. Let then:

- for any propositional variable ξ, $V(\xi)$ be a subset of $\mathbb{T}$;
- for any instant variable α, $V(\alpha)$ belong to $\mathbb{T}$;
- for any interval variable ε, $V(\varepsilon)$ be a closed segment of $\mathbb{T}$.

By induction for any instant term of the form $\ulcorner\delta(\tau,\varepsilon)\urcorner$, $V(\delta(\tau,\varepsilon))$ is the unique member t of $\mathbb{T}$ such that $V(\varepsilon) = [V(\tau), t]$, which means, $\delta(\tau,\varepsilon)$ is interpreted as the unique point t of $\mathbb{T}$ such that the closed segment $V(\varepsilon)$ is equal to the closed segment from the point to which $V(\tau)$ refers to the point t.

Furthermore, for all formulas φ, ψ and any instant term τ we put:

$$V(\mathcal{R}_\tau\varphi) = \begin{cases} \mathbb{T} & \text{if } V(\tau) \in V(\varphi), \\ \varnothing & \text{if } V(\tau) \notin V(\varphi), \end{cases}$$

$$V(\neg\varphi) = \mathbb{T} \setminus V(\varphi),$$

$$V(\varphi \to \psi) = (\mathbb{T} \setminus V(\varphi)) \cup V(\psi),$$

$$V(\varphi \wedge \psi) = V(\varphi) \cap V(\psi),$$

$$V(\varphi \vee \psi) = V(\varphi) \cup V(\psi),$$

$$V(\varphi \equiv \psi) = (V(\varphi) \cap V(\psi)) \cup (\mathbb{T} \setminus (V(\varphi) \cup V(\psi))).$$

Finally, for any $\varphi \in \mathrm{For}$, any propositional variable ξ, any instant variable α and any interval variable ε let:

$$V(\forall\xi\,\varphi) = \begin{cases} \mathbb{T}, & V(\varphi(\xi'/\xi)) = \mathbb{T}, \text{ for any propositional variable } \xi' \\ \varnothing, & V(\varphi(\xi'/\xi)) = \varnothing, \text{ for some propositional variable } \xi', \end{cases}$$

$$V(\forall\alpha\,\varphi) = \begin{cases} \mathbb{T}, & V(\varphi(\alpha'/\alpha)) = \mathbb{T}, \text{ for any instant variable } \alpha', \\ \varnothing, & V(\varphi(\alpha'/\alpha)) = \varnothing, \text{ for some instant variable } \alpha', \end{cases}$$

$$V(\forall\varepsilon\,\varphi) = \begin{cases} \mathbb{T}, & V(\varphi(\varepsilon'/\varepsilon)) = \mathbb{T}, \text{ for any interval variable } \varepsilon', \\ \varnothing, & V(\varphi(\varepsilon'/\varepsilon)) = \varnothing, \text{ for some interval variable } \varepsilon'. \end{cases}$$

It is easy to observe that with respect to quantifiers Łoś covers only the cases of formulas built up from the subformulas of the form $\ulcorner\mathcal{R}_\alpha\varphi\urcorner$.

The cases of formulas of Propositional Calculus with quantifiers are not covered here. Consider for example $V(\varphi(\xi'/\xi)) \neq \varnothing$ for any ξ', but $V(\varphi(\xi'/\xi)) \neq \mathbb{T}$ for some ξ'. Of course, Łoś might have assumed that his calculus is an extension of Standard Logic, which is known to be consistent. And yet, Łoś's conditions may be extended to cover all the formulas of **ŁC**:

$$V(\forall\xi\,\varphi) = \textstyle\bigcap_{\xi'} V(\varphi(\xi'/\xi)),$$
$$V(\forall\alpha\,\varphi) = \textstyle\bigcap_{\alpha'} V(\varphi(\alpha'/\alpha)),$$
$$V(\forall\varepsilon\,\varphi) = \textstyle\bigcap_{\varepsilon'} V(\varphi(\varepsilon'/\varepsilon)),$$

for any propositional variables ξ, ξ', instant variables α, α' and interval variables ε, ε'. Łoś's conditions are special cases of these more general conditions.

One has to conjecture that formula φ is *true* in a model $\langle\mathbb{T}, V\rangle$ if and only if $V(\varphi) = \mathbb{T}$. A formula φ is *valid* if and only if it is true in all models (there is, however, no explicit mention of either condition).

Łoś claims that axioms (ax1)–(ax9) refer to the straight line $\mathbb{T}$ under that interpretation, i.e., $V(\varphi) = \mathbb{T}$, for every axiom φ of Łoś's calculus, and that the feature is invariant with respect to classical deductive rules. Furthermore, it is obvious that no contradictory formulas are both interpreted as the whole straight line $\mathbb{T}$. It follows that Łoś's calculus is consistent [cf. Łoś, 1947, pp. 283–284 (307–308)].

The claim seems a little problematic with respect to the Clock Axiom (ax9). Firstly, axiom (ax9) is clearly a constraint put on the set of models and formally does not have to be true in general. Secondly and more importantly for axiom (ax9) to be true it is necessary to regiment Łoś's model template. Axiom (ax9) could be valid in Łoś's line model only under some restrictions: either when there are uncountably many formulas or when there are only denumerably many instants in the straight line. The following theorem is likely to be true.

Theorem 3. *Axiom* (ax9) *is not valid in Łoś's line model unless there are either uncountably many formulas or there are only denumerably many instants in the straight line.*

This theorem follows from the fact that there are only denumerably many formulas and the formulas are finite. Hence, there are denumerably many formulas which are true at exactly one instant each. If the straight line is to be identified with the set of real numbers it is not possible to assign every instant with a formula true only at that very instant. One can remedy this either by identifying the straight line with the set of rational numbers rather than the real numbers or by modifying the language to the effect that there are uncountably many formulas.

It seems obvious that to avoid the problem uncovered by the above theorem the set of models should simply be restricted. One may simply assume that there is a sequence of indexed propositional variables such that $V(p_i) = \{V(x_i)\}$. However, in such a case there are only denumerably many instants. So, the straight line $\mathbb{T}$ is to be considered as the set of rational numbers rather than real numbers. Another option is to introduce an extra set of atomic formulas, say $\ulcorner \mathrm{N}(\alpha) \urcorner$, for any instant symbol α, to the effect that $V(\mathrm{N}(x)) = \{V(x)\}$, for any instant x. So, it is the case that $\mathcal{R}_x(\mathrm{N}(y))$ if and only if $x = y$ (or $x \simeq y$ in Łoś's language):

$$\mathcal{R}_y(\mathrm{N}(x)) \equiv x \simeq y. \tag{1}$$

If formulas $\ulcorner \mathrm{N}(\alpha) \urcorner$ were allowed in the object language, the formula (1) would serve as the Clock Axiom instead of the formula (ax9), as (ax9) follows from (1), but not conversely. It is now more evident how close Łoś's calculus gets to hybrid logics. Thus qualified, Łoś's calculus is in fact sound with respect to the $\mathbb{T}$ model, and so consistent as well. It is also clear that Łoś anticipated hybrid logics, as has been already mentioned.

Łoś did not provide nor even sketch a completeness result. He clearly focused on the question of consistency of the calculus itself as well as its extensions, which he called "applications". Furthermore it turns out that Łoś's calculus is not actually complete with respect to Łoś's model $\mathbb{T}$. For consider the formula:

$$\forall x \forall y (x \neq y \;\rightarrow\; \exists e (\delta(x, e) = y \;\vee\; \delta(y, e) = x)). \tag{2}$$

Since, as Łoś claims, $\mathbb{T}$ is a straight line, the formula (2) is obviously valid and yet it is not provable in Łoś's calculus.

Theorem 4. *The valid formula* (2) *is not provable in Łoś's calculus.*

Proof. The proof goes by interpretation. In every formula of **ŁC** read the term $\ulcorner \delta(\alpha, \varepsilon) \urcorner$ as α and $\ulcorner \mathcal{R}_\alpha \varphi \urcorner$ as φ (so ignore expressions $\ulcorner \mathcal{R}_\alpha \urcorner$). All the axioms of Łoś calculus become tautological under such an interpretation and the feature is invariant to inferences. However, the formula (2) becomes an invalid first-order one. $\dashv$

Łoś's calculus was developed as application-orientated and so purely formal questions have neither been properly posed nor answered.

5. Hiż's review

Reviewing in 1951 Łoś's work, Hiż ('ż' sounds exactly like 'g' in the word 'genre') described both the philosophical background and formal details of the calculus, and even a kind of improvement of its axiomatization. The objective is clear:

> The main purpose of this paper is to analyze Mill's canons as rules of operation for a part of the language of physics. To do it the author builds up an axiomatization of a fragment of the physical language.
> [Hiż, 1951, p. 58]

Hiż's review is really good. In barely two pages Hiż summarized all vital details of Łoś's calculus, such as its formulas, axioms and interpretations. Hiż even noticed that Łoś's original calculus did not exclude circular time:

> According to the author the axioms of the fragment of the physical language require that there be an infinite number of constants which can be substituted for the variables representing instants of time. To the reviewer this would seem to be true only if we exclude the possibility that, for some n_1, $\delta t_1 n_1$ is identical with t_1 — as can be done e.g. by adding the axiom "$\mathrm{C}\rho\delta t_1 n_1 t_2 \mathrm{N}\rho\, t_1 t_2$", where "$\rho\, t_1 t_2$" is defined, following Łoś, as "$\forall p_1 \mathrm{EU} t_1 p_1 \mathrm{U} t_2 p_1$". [Hiż, 1951, p. 59]

However, no matter how good Hiż's review is, it is an abbreviation. As far as we are aware Łoś's pioneering work was not even translated into English until 1977 [cf. Łoś, 1977]. To avoid circularity Hiż proposes another axiom

$$\delta(x, e) \simeq y \to x \not\simeq y,$$

which certainly can do the job, but at a price. The formula above excludes circularity but also any possibility of using zero-length intervals.

6. Formulas of the classical logic

The position and role of axiom (ax6) is of special interest. As we have already mentioned, by (ax6) Łoś means to accept as theorems all the formulas which hold at all instants. As Łukasiewicz was often doing, Łoś fails to clearly distinguish truth and validity. Obviously, by axiom (ax6), a formula $\varphi \in \mathrm{For}_{\mathbf{CPL}}$ is exclusively a theorem of **ŁC**, provided that the formula $\ulcorner \forall x\, \mathcal{R}_x \varphi \urcorner$ is too. And the latter holds if and only if φ is a theorem of **CPL**.

By assumption [cf. Łoś, 1947, p. 280 (303)], all theorems of **CPL** are theorems of **ŁC**. So no extra **CPL** formula is provable by axiom (ax6). As classical propositional calculus is strongly (syntactically) complete in the sense of Emil Post, the observation follows immediately from the claim of consistency of Łoś's system. An analogous claim is valid with respect to unsatisfiable formulas.

Theorem 5. *If φ or $\ulcorner \neg\varphi \urcorner$ is a theorem of* **CPL**, *then $\ulcorner \forall\alpha\, \mathcal{R}_\alpha \varphi \to \varphi \urcorner$ is provable without axiom* (ax6).

Proof. Suppose φ is a theorem of **CPL**. Then $\ulcorner p \to \varphi \urcorner$ is also a theorem of **CPL**. Hence $\ulcorner \forall\alpha \mathcal{R}_\alpha \varphi \to \varphi \urcorner$ is a substitution of a theorem of **CPL**, and so it is an axiom of **ŁC**.

Now suppose $\ulcorner \neg\varphi \urcorner$ is a theorem of **CPL**. Then, by (ax1)–(ax5) (see the proof of Theorem 1), the formula $\ulcorner \neg\mathcal{R}_\alpha \varphi \urcorner$ is a theorem of **ŁC**. As a substitution of the theorem $\ulcorner \neg p \to (p \to \varphi) \urcorner$ of **CPL** the formula $\ulcorner \neg\mathcal{R}_\alpha \varphi \to (\mathcal{R}_\alpha \varphi \to \varphi) \urcorner$ is also a theorem of **ŁC**. Hence $\ulcorner \mathcal{R}_\alpha \varphi \to \varphi \urcorner$ and $\ulcorner \exists\alpha(\mathcal{R}_\alpha \varphi \to \varphi) \urcorner$ are also theorems of **ŁC**. By the classical rules for quantifiers, so too is the formula $\ulcorner \forall\alpha \mathcal{R}_\alpha \varphi \to \varphi \urcorner$. ⊣

And yet, axiom (ax6) is perfectly independent. Hence, with this axiom, only certain relations holding between formulas containing Łoś's connective and formulas of **CPL** can be formalised. In this way the operation of consequence is influenced by (ax6).

Theorem 6. *Axiom* (ax6) *is independent of the other axioms.*

Proof. Let instant and interval variables be interpreted as integers, propositional variables as sets of integers. Let the connectives of **CPL** be interpreted as analogous operations on sets. Let $\ulcorner \mathcal{R}_\alpha \varphi \urcorner$ mean $\ulcorner \alpha \in \varphi \urcorner$ and $\ulcorner \delta(\alpha, \varepsilon) \urcorner$ mean $\ulcorner \alpha + \varepsilon \urcorner$. Under such an interpretation all axioms of Łoś's calculus, with the exception of (ax6), are true formulas of the arithmetic of integers. This kind of truth is invariant with respect to inference rules. It follows that axiom (ax6) is not derivable from the other axioms. ⊣

Hence, it would be possible to restrict the assumption of **CPL** to the formulas one obtains from tautologies by the substitution of $\ulcorner \mathcal{R}_\alpha \varphi \urcorner$ formulas for all propositional variables.

It was noted by Łoś that the converse of axiom (ax6), i.e., the following formula:

$$p \to \forall x\, \mathcal{R}_x p, \tag{‡}$$

is a consequence of the formula (†), and so, by the first proof of consistency of Łoś's calculus, if it is added to the calculus, it creates another consistent one [cf. Łoś, 1947, p. 282 (306)]. Therefore (ax6) may be replaced by the stronger formula

$$\forall x\, \mathcal{R}_x p \equiv p$$

and the calculus obtained that way remains consistent. And yet, if (‡) where a theorem, no formula would hold at some, but not all, instants. For consider the formulas:

$$\exists x\, \mathcal{R}_x p, \tag{3}$$

$$\exists x\, \mathcal{R}_x \neg p, \tag{4}$$

exemplified by such sentences as 'Sometimes it rains' and 'Sometimes it does not rain', respectively. They are not only clearly consistent, but the temporal logic is actually designed to deal with formulas of this kind. And yet, theorems of Łoś's calculus with addition of the formulas (‡), (3) and (4) create an inconsistent set. The contradiction arises in the following way. Notice that the formula '$\neg\forall x \neg \mathcal{R}_x p$' follows from (3), and by (ax1) so does '$\neg\forall x \mathcal{R}_x \neg p$'; and by (‡), so does '$\neg\neg p$', and so '$p$' itself. Notice also that the formula '$\neg\forall x \neg \mathcal{R}_x \neg p$' follows from (4), and by (ax1) so does '$\neg\forall x \mathcal{R}_x \neg\neg p$' and consequently '$\neg\forall x \mathcal{R}_x p$', and by the formula (‡) so does '$\neg p$'. For Łoś rejects for the formula (‡) for this reason. A question then arises of what ways there are to obtain such outcomes. It is called by Łoś *the question of applicability* [cf. Łoś, 1947, p. 283 (307)].

7. Applicability

The question of applicability is solved by Łoś by means of the following theorem of applicability. Let Φ be a set of formulas of the form $\ulcorner \mathcal{R}_\alpha \varphi \urcorner$. If there are no formulas φ, ψ and instant variables α, β such that $\ulcorner \mathcal{R}_\alpha \varphi \urcorner$,

$\ulcorner \mathcal{R}_\beta \psi \urcorner$ belong to Φ and $\ulcorner \neg\varphi \cong \psi \urcorner$ and $\ulcorner \alpha \simeq \beta \urcorner$ are theorems of **ŁC**, then the union of the set of theorems and the set Φ is consistent. Łoś does not deliver a proof of the theorem, but claims the proof to be quite easy to A question then arises of what ways there are to obtain such construct [cf. Łoś, 1947, p. 284 (308)].

Łoś's theorem needs a small improvement, for consider the two following versions of the set Φ:

(a) $\Phi = \{\mathcal{R}_x(p \wedge \neg p)\}$,
(b) $\Phi = \{\mathcal{R}_x p, \mathcal{R}_x(\neg p \wedge q)\}$.

They both meet Łoś's conditions, and yet create obviously inconsistent sets within Łoś's calculus. In (a) Φ has the only one formula, and obviously '$\neg(p \wedge \neg p) \cong (p \wedge \neg p)$' is not provable, as in any model $\langle \mathbb{T}, V \rangle$, the left side refers to $\mathbb{T}$, whereas the right side to $\varnothing$. In (b) too, neither '$\neg p \cong (\neg p \wedge q)$' nor '$\neg(\neg p \wedge q) \cong p$' is provable, as to belong to $V(\neg p \wedge q)$ it is compulsory to belong to $V(q)$ itself, which does not apply to $V(p)$. And yet, in both examples the formulas belonging to Φ allow immediately to infer a contradiction, by use of (ax1) and ($\star\star$).

To improve Łoś's applicability theorem it is sufficient to assume that φ is either an atomic formula or a negation of an atomic formula. Such a qualification causes no theoretical problem, since by means of the minimal normal positional logic **MR** — which is a proper part of Łoś's calculus [cf. Jarmużek and Pietruszczak, 2004] — all the formulas in question are effectively reducible to them. Furthermore it seems likely that this is exactly what Łoś had in mind. So, here is an improved version of applicability theorem.

Theorem 7. *Let Φ be a set of formulas of the form $\ulcorner \mathcal{R}_\tau \xi \urcorner$ or $\ulcorner \neg\mathcal{R}_\tau \xi \urcorner$, for any instant term τ and any propositional variable ξ. If there are no elements $\ulcorner \mathcal{R}_\tau \xi \urcorner$, $\ulcorner \neg\mathcal{R}_{\tau'} \xi \urcorner$ in Φ such that $\ulcorner \tau \simeq \tau' \urcorner$ is demonstrable, then the union of the set of theorems and the set Φ is consistent.*

Proof. There is a model $\langle \mathbb{T}, V \rangle$ of the set Φ united with the theorems of **ŁC**. To obtain such a model it is sufficient to find the sets referred to by formulas and instant terms. Let

$$V(\xi) = \{V(\tau) : \ulcorner \mathcal{R}_\tau \xi \urcorner \in \Phi\}$$

Every formula $\ulcorner \mathcal{R}_\tau \xi \urcorner$ of the set Φ is true in the model by definition. Every formula $\ulcorner \neg\mathcal{R}_\tau \xi \urcorner$ of the set Φ is true in the model, unless for some τ'

the formula $\ulcorner \mathcal{R}_{\tau'}\xi \urcorner$ is in Φ and $V(\tau) = V(\tau')$. But the condition $V(\tau) = V(\tau')$ is not a constraint on any mode unless the formula $\ulcorner \tau \simeq \tau' \urcorner$ is demonstrable. $\dashv$

8. Łoś and Prior

Prior's outstanding achievements in the field of temporal logic were inspired by three sources: the problem of future contingents and Łukasiewicz's many-valued logic; the medieval programme to construct the logic of the vernacular with its account of truth-values; and a small footnote on tense and modalities in a work by John Findlay. This is the standard story, based on Prior's texts and repeated by his followers. Our claim here is that there definitely was a vitally important fourth source: the work of Łoś. As we have said Łoś's pioneering work on temporal logic was published in Polish in 1948 and was summarised and reviewed in English by Hiż as early as 1951.

Prior's idea of tense logic appeared in 1953. This may be shown by the following two facts. Firstly, in the very year it was published, Prior's paper [1953] on Łukasiewicz's three-valued logic was also published. It shows that Prior was still interested in the many-valued programme although he was aware of its difficulties. Secondly, Prior's wife Mary remembered Prior waking her at night with Findlay's book in his hands and announcing the idea of tense logic. This took place in 1953 [cf. Øhrstrøm and Hasle, 2006a, pp. 414–415]. Furthermore, in *Past, Present and Future* Prior [1967, p. 212–213] explicitly acknowledged that he had known Łoś's work [1947] from Hiż's review [1951] and he was actually inspired by Łoś when beginning to work on his first full book on tense logic, i.e., *Time and Modality* [Prior, 1957]. In 1968 Łoś's work found its place in the bibliography of Prior's collected papers [cf. Prior, 1968, p. 161]. The only formal tool Łoś's work [1947] lacks is a way of representing tenses by means of modal connectives. Prior took this idea from Findlay.

It should therefore be agreed and acknowledged in the literature henceforth that it was Jerzy Łoś who invented and first explored temporal logic. Such an admission does in no way dethrone Prior's works on tense logic as classics of their kind. The key achievements in the field belong to Prior, Łoś having abandoned his investigations into temporal logic shortly after beginning them. Nevertheless, it is to Los that the credit is due for the first presentation of a formal logic of time in 1947. The more obvious Łoś's influence over Prior and the subsequent development of temporal logic becomes, the more mysterious failures to acknowledge L as the true founder of the logic of time will be.

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