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ULTRAPRODUCT FOR QUANTUM STRUCTURES

Abstract. Quantum Kripke frames are certain quantum structures recently introduced by Zhong. He has defined certain properties such as Existence of Approximation and Superposition for these structures. In this paper, we define the ultraproduct for the family of quantum Kripke frames and show that the aforementioned properties are invariant under ultraproduct. In this way we prove that the ultraproduct of each family of quantum Kripke frames is also a quantum Kripke frame. We also show the same results for other related quantum structures.

Keywords: Quantum Kripke frame; state space; ultraproduct

1. Introduction

Quantum logic was born in 1936 in a joint paper by John von Neumann and Garrett Birkhoff [3]. The aim of their work was to find the logical structure of quantum mechanics. The logic of quantum experimental propositions is different from classical logic. The differences are related to the properties which connect meet and join in classical logic, especially the distributive law. They gave an example to show that this law may fail for propositions related to quantum mechanics. They replaced the distributive law with a weak version of it which is called modular law. In [11], Piron defined a special kind of lattice later called a Piron lattice and proved a representation theorem for them via generalized Hilbert spaces. The quantum dynamic frame is the other important kind of quantum structure which was proposed in [1]. In [2], a categorical duality between quantum dynamic frames and Piron lattices was shown. In a

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quantum dynamic frame, each state corresponds to a one-dimentional subspace of a fixed Hilbert space and the transitions between states are modelled by binary relations.

A Kripke frame is a set equipped with a binary relation (to be referred to later as the "non-orthogonality relation") on it. In [12], Zhong introduced special Kripke frames to study quantum phenomena. He chose the non-orthogonality relation as a primitive notion. Then he defined the orthogonality relation and indistinguishability relation in terms of the non-orthogonality relation. Next, by defining notions such as orthocomplement and bi-orthogonally closed subsets, he introduced five kinds of special Kripke frames relating to quantum logic: state space, geometric frame, complete geometric frame, quasi-quantum Kripke frame and quantum Kripke frame. He showed that there is a categorical duality between quantum dynamic frames and quantum Kripke frames.

In 2017, Zhong published [13] in which he showed that geometric frames, irreducible geometric frames, complete geometric frames and quantum Kripke frames correspond to pure orthogeometries (or, equivalently, projective geometries with pure polarities), irreducible pure orthogeometries, Hilbertian geometries and irreducible Hilbertian geometries, respectively. The discovery of these correspondences raises interesting research topics and will enrich the study of quantum logic.

The simplicity of these structures and their power to model quantum systems are the main reasons to research these structures. The main goal of this paper is to study the ultraproduct for these Kripke frames. The books [4] and [5] are standard references for this model theoretic construction for Kripke frames in the context of modal logic with additional pointers to how they are to be constructed for special modal logics. [7] and [10] are also useful.

The structure of the paper is as follows.

In Section 2, we recall the definition of the properties reflexivity, symmetry, separation, AL, AH, A and superposition, (see [1, 12]). Next, the definition of state spaces, quantum Kripke frames and other structures related to these properties are reviewed.

In Section 3, we recall the definition of the ultraproduct of a family of Kripke frames. The obvious result is that this ultraproduct preserves the properties of symmetry, reflexivity, separation, AL, AH and superposition since they are first-order definable properties.

In Section 4, the main result is that the ultraproduct of each family of quantum Kripke frames is a quantum Kripke frame. To do this, we show that the property A is preserved under ultraproduct. Finally, we state Łós's theorem for this product. For more details regarding ultrafilters and ultraproducts, [4, 5, 6, 9] are useful references.

2. Some background

In this section we review the definitions of special Kripke frames such as a quantum Kripke frame and a state space. We also recall some of their basic properties. The main reference is [12].

DEFINITION 2.1 (Kripke frame). A Kripke frame is a tuple $F = (\Sigma, \to)$ such that Σ is a non-empty set (of states) and $\to \subseteq \Sigma \times \Sigma$ is a binary relation.

Below we give some terminology and notations concerning a Kripke frame $F = (\Sigma, \to)$, (see [1] and [12]).

- Non-orthogonality relation: if $(s, t) \in \to$, we write $s \to t$ and we say that s and t are non-orthogonal.
- Orthogonality relation: if $s \nrightarrow t$, i.e., $s \to t$ does not hold, then we say that s and t are orthogonal and write $s \perp t$.
- For each $P \subseteq \Sigma$, we define the orthocomplement of P as $P^{\perp} := \{t \in \Sigma : \forall s \in P, s \nrightarrow t\}$.
- The set $\mathbf{L}_F = \{P \subseteq \Sigma : P = (P^{\perp})^{\perp}\}$ is the set of all bi-orthogonally closed subsets of Σ.
- Two states $s, t \in \Sigma$ are indistinguishable with respect to $P \subseteq \Sigma$, denoted by $s \approx_P t$, if $s \to x$ if and only if $t \to x$ for every $x \in P$.
- The state $t \in \Sigma$ is an approximation of $s \in \Sigma$ in $P \subseteq \Sigma$, if $t \in P$ and $s \approx_P t$.

Remark 2.2. Here are some obvious properties of the indistinguishability relation, see [12, 2.1.2].

- For each $P \subseteq \Sigma$, $\approx_P$ is an equivalence relation on Σ.
- We have $\approx_{\emptyset} = \Sigma \times \Sigma$.
- If $P \subseteq Q \subseteq \Sigma$ then $\approx_Q \subseteq \approx_P$.

Here are some conditions on Kripke frames studied in [12] that we will be concerned with throughout this paper.

DEFINITION 2.3. Let $F = (\Sigma, \to)$ be a Kripke frame. Some properties F may have are as follows.

- *Reflexivity*: for each $s \in \Sigma$, $s \to s$.
- *Symmetry*: for each $s, t \in \Sigma$, $s \to t$ implies $t \to s$.
- *Separation*: for each $s, t \in \Sigma$, if $s \neq t$ then there exists a $w \in \Sigma$ such that $w \to s$ and $w \nrightarrow t$.
- *Existence of Approximation for Lines* (AL): for any $s, t \in \Sigma$, if $w \in \Sigma \setminus \{s,t\}^{\perp}$, then there is a w' which is an approximation of w in $\{s,t\}^{\perp\perp}$, i.e., $w' \in P = \{s,t\}^{\perp\perp}$ and $w \approx_P w'$.
- *Existence of Approximation for Hyperplanes* (AH): for any $s \in \Sigma$, if $w \in \Sigma \setminus \{s\}^{\perp\perp}$, then there is a w' which is an approximation of w in $\{s\}^{\perp}$, i.e., $w' \in P = \{s\}^{\perp}$ and $w \approx_P w'$.
- *Existence of Approximation* (A): for any $P \subseteq \Sigma$, if $P = P^{\perp\perp}$ and $w \in \Sigma \setminus P^{\perp}$ then there is a w' which is an approximation of w in P, i.e., $w' \in P$ and $w \approx_P w'$.
- *Superposition*: for each $s, t \in \Sigma$, there exists a $w \in \Sigma$ such that $w \to s$ and $w \to t$.

Below, some special Kripke frames satisfying certain conditions mentioned above are defined. Some of them were first introduced and studied in [12].

DEFINITION 2.4 (State space). A Kripke frame $F = (\Sigma, \to)$ which satisfies Reflexivity, Symmetry, and Separation is called a state space.

DEFINITION 2.5. Let $F = (\Sigma, \to)$ be a Kripke frame. Some important special Kripke frames are the following.

- A geometric frame is a state space satisfying properties AL and AH.
- A complete geometric frame is a state space satisfying property A.
- A quasi-quantum Kripke frame is a state space satisfying properties AL, AH and superposition.
- A quantum Kripke frame is a state space satisfying properties A and superposition.

We have the following two facts (see propositions 2.2.1 and 2.2.4 in [12]).

PROPOSITION 2.6. *For each state space $F = (\Sigma, \to)$, we have the following properties.*

- *For each $A, B \subseteq \Sigma$, if $A \subseteq B$ then $B^{\perp} \subseteq A^{\perp}$.*
- *For each $A \subseteq \Sigma$, the set $A^{\perp}$ is bi-orthogonally closed.*
- *For each $A \subseteq \Sigma$, $A \subseteq A^{\perp\perp}$.*
- *For each $A \subseteq \Sigma$, $A \cap A^{\perp} = \emptyset$.*

PROPOSITION 2.7. *In every state space $F = (\Sigma, \to)$, the property A implies both properties AL and AH. In the other words, every complete geometric frame is a geometric frame, and every quantum Kripke frame is a quasi-quantum Kripke frame.*

In the rest of this section, we recall the definition of a Quantum Dynamic Frame (**QDF**) and its relation to the definition of a quantum Kripke frame (see [1, 2]).

DEFINITION 2.8 (Dynamic frame). A dynamic frame is an ordered triple $F = (\Sigma, \mathbf{L}, \{\xrightarrow{P?}\}_{P\in\mathbf{L}})$ where

(1) Σ is a set (interpreted as the set of states),
(2) $\mathbf{L} \subseteq \mathbf{P}(\Sigma)$ (the power set of Σ) and for each $P \in \mathbf{L}$ (called a testable property), $\xrightarrow{P?} \subseteq \Sigma \times \Sigma$ is a binary relation.

Let $F = (\Sigma, \mathbf{L}, \{\xrightarrow{P?}\}_{P\in\mathbf{L}})$ be a dynamic frame. To define **QDF**, we need the following definitions and notations.

- Non-orthogonality relation $\to$: $s \to t$ if and only if there exists a $P \in L$ such that $s \xrightarrow{P?} t$.
- Orthogonality relation $\perp$: We have $\perp = (\Sigma \times \Sigma) \setminus \to$. If $(s, t) \in \perp$, we write $s \perp t$ or $s \not\to t$.
- For each $A \subseteq \Sigma$, the orthocomplementation of A is defined as $A^\perp = \{s \in \Sigma : s \perp t, \forall t \in A\}$.
- Bi-orthogonal closure of A is defined as $(A^\perp)^\perp = A^{\perp\perp}$. If $A = A^{\perp\perp}$, we say that A is bi-orthogonally closed.

DEFINITION 2.9 ([2, Definition 2.7]). A **QDF** is a dynamic frame F which satisfies the following conditions. For any $P \in L$ and $s, t, v, w \in \Sigma$:

(1) Closure under arbitrary conjunction: if $\mathbf{L}' \subseteq \mathbf{L}$ then $\bigcap \mathbf{L}' \in \mathbf{L}$.
(2) Atomicity: states are testable, i.e., for each state $s \in \Sigma$, $\{s\} \in \mathbf{L}$.
(3) Closure under orthocomplementation: if $P \in \mathbf{L}$ then $P^\perp \in \mathbf{L}$.
(4) Adequacy: if $s \in P$ then $s \xrightarrow{P?} s$.
(5) Repeatability: any property holds after it has been successfully tested, i.e., if $s \xrightarrow{P?} t$ then $t \in P$.
(6) Proper superposition: every two states can be properly superposed into a new state, i.e., for each $s, t \in \Sigma$ there exists $w \in \Sigma$ such that $s \to w \to t$.

(7) Self-Adjointness: if $s \xrightarrow{P?} w \to t$, then there exists some element $v \in \Sigma$ such that $t \xrightarrow{P?} v \to s$.
(8) Covering law: if $s \xrightarrow{P?} w$ and $t \in P$ such that $w \neq t$, then there exists some $v \in P$ such that $t \to v \nrightarrow s$.

The following theorem, which gives a close relationship between quantum Kripke frames and quantum dynamic frames, is proved in [12, 2.7.25]

THEOREM 2.10. *We have the following.*

- *For every quantum Kripke frame $F = (\Sigma, \to)$, $X(F) = (\Sigma, \mathbf{L}_F, \{\xrightarrow{P?}\}_{P \in \mathbf{L}_F})$ is a quantum dynamic frame where $\mathbf{L}_F$ is the set of all bi-orthogonally closed subsets of Σ and for every $P \in \mathbf{L}_F$, $\xrightarrow{P?} \subseteq \Sigma \times \Sigma$ is such that for any $s, t \in \Sigma$, $s \xrightarrow{P?} t$ if and only if $t \in P$ and $s \approx_P t$.*
- *For each quantum dynamic frame $F = (\Sigma, \mathbf{L}, \{\xrightarrow{P?}\}_{P \in \mathbf{L}})$, the structure $Y(F) = (\Sigma, \to)$ is a quantum Kripke frame, where $\to$ is a binary relation on Σ defined by: $s \to t$ if and only if $s \xrightarrow{P?} t$, for some $P \in \mathbf{L}$.*

3. Ultraproduct for quantum structures

In this section we review the definition of ultraproduct for Kripke structures and show that preserves symmetry, reflexivity, separation, AL, AH and superposition.

DEFINITION 3.1 (Ultraproduct of a family of Kripke frames).
Let $\{F_i = (\Sigma_i, \to_i)\}_{i \in I}$ be a family of Kripke frames, $\mathcal{U}$ be an ultrafilter over I (a set of indexes), and $\prod_{i \in I} \Sigma_i = \{f : I \to \bigcup \Sigma_i | \ f(i) \in \Sigma_i\}$ be the Cartesian product of $\{\Sigma_i\}_{i \in I}$. For each two functions f and g, we say that f and g are $\mathcal{U}$-equivalent if $\{i \in I : f(i) = g(i)\} \in \mathcal{U}$. The ultraproduct $\prod_{\mathcal{U}} F_i$ of $\{F_i\}_{i \in I}$ modulo $\mathcal{U}$ is the frame $\prod_{\mathcal{U}} F_i = (\Sigma_{\mathcal{U}}, \to)$ such that

- The universe $\Sigma_{\mathcal{U}}$ of $\prod_{\mathcal{U}} F_i$ is the set $\Sigma_{\mathcal{U}} = \prod_{\mathcal{U}} \Sigma_i = \{\lceil f \rceil : f \in \prod_{i \in I} \Sigma_i\}$, i.e., the set of $\mathcal{U}$- equivalence classes.
- $\to$ on $\Sigma_{\mathcal{U}}$ is defined by: $\lceil f \rceil \to \lceil g \rceil$ if and only if $\{i \in I : f(i) \to_i g(i)\} \in \mathcal{U}$.

Note. For each family $\{P_i \subseteq \Sigma_i : i \in I\}$, we define $\prod_{\mathcal{U}} P_i := \{\lceil g \rceil : \{i \in I : g(i) \in P_i\} \in \mathcal{U}\}$. We show that these sets are bi-orthogonally closed (Corollary 3.3). This result shows how we can reach a bi-orthogonally closed set in the ultraproduct frame $\prod_{\mathcal{U}} F_i = (\Sigma_{\mathcal{U}}, \to)$ from a family of bi-orthogonally closed sets $\{P_i \in \mathbf{L}_{F_i}\}_{i \in I}$.

We define orthogonality relation and bi-orthogonal closure of a subset of $\Sigma_{\mathcal{U}}$ as follows:

- Orthogonality relation: $\lceil f \rceil \perp \lceil g \rceil$ if and only if $\{i \in I : f(i) \perp_i g(i)\} \in \mathcal{U}$.
  We shall use henceforth the same symbol $\perp$ for showing the orthogonality relation between two states in all models F_i and $\prod_{\mathcal{U}} F_i$.
- The bi-orthogonal closure of each subset $S^{\mathcal{U}} \subseteq \Sigma_{\mathcal{U}}$ is defined as $(S^{\mathcal{U}})^{\perp\perp} = ((S^{\mathcal{U}})^{\perp})^{\perp}$ where

$$(S^{\mathcal{U}})^{\perp} = \{\lceil f \rceil : \forall \lceil g \rceil \in S^{\mathcal{U}}, \lceil f \rceil \perp \lceil g \rceil\}.$$

By $\mathbf{L}_F^{\mathcal{U}}$, we denote the set of all bi-orthogonally closed subsets of $\prod_{\mathcal{U}} F_i$.

LEMMA 3.2. *Let $P_i \subseteq \Sigma_i$, for each $i \in I$. Then $(\prod_{\mathcal{U}} P_i)^{\perp} = \prod_{\mathcal{U}}(P_i^{\perp})$.*

PROOF. First we show that $(\prod_{\mathcal{U}} P_i)^{\perp} \subseteq \prod_{\mathcal{U}}(P_i^{\perp})$. Let $\lceil f \rceil \in (\prod_{\mathcal{U}} P_i)^{\perp}$.

If $\lceil f \rceil \notin \prod_{\mathcal{U}}(P_i^{\perp})$ then $A = \{i \in I : f(i) \in P_i^{\perp}\}^c \in \mathcal{U}$. So for each $i \in A$, there exists an $r_i \in P_i$ such that $f(i) \to_i r_i$. Let $g : I \to \bigcup \Sigma_i$ be a function such that $g(i) = r_i$ for each $i \in A$. Therefore there exists a $\lceil g \rceil \in \prod_{\mathcal{U}} P_i$ such that $\{i \in I : f(i) \to_i g(i)\} \in \mathcal{U}$. This is a contradiction. Therefore $\lceil f \rceil \in \prod_{\mathcal{U}}(P_i^{\perp})$.

Now we show that $\prod_{\mathcal{U}}(P_i^{\perp}) \subseteq (\prod_{\mathcal{U}} P_i)^{\perp}$. Let $\lceil f \rceil \in \prod_{\mathcal{U}}(P_i^{\perp})$ and $\lceil g \rceil \in \prod_{\mathcal{U}} P_i$.

So $A = \{i \in I : f(i) \in P_i^{\perp}\} \in \mathcal{U}$ and $B = \{i \in I : g(i) \in P_i\} \in \mathcal{U}$. Therefore for each $i \in A \cap B$, $f(i) \not\to_i g(i)$. So $\lceil f \rceil \not\to \lceil g \rceil$. Therefore $\lceil f \rceil \in (\prod_{\mathcal{U}} P_i)^{\perp}$. ⊣

COROLLARY 3.3. *For each family of bi-orthogonally closed sets $\{P_i \in \mathbf{L}_{F_i}\}_{i \in I}$, the set $\prod_{\mathcal{U}} P_i$ is a bi-orthogonally closed set (so $\prod_{\mathcal{U}} P_i$ belongs to $\mathbf{L}_F^{\mathcal{U}}$).*

Kripke frames can be viewed as first-order relational structures with one binary relation. The conditions reflexivity, symmetry, separation, AL, AH and superposition are also first-order properties and hence by Łós's theorem, they are preserved under ultraproduct (see, e.g., [6, 9]). So we have the following theorem immediately. We will only give a proof of one part of the theorem directly.

THEOREM 3.4. *Let $\{F_i = (\Sigma_i, \to_i)\}_{i\in I}$ be a family of Kripke frames and $\prod_{\mathcal{U}} F_i = (\Sigma_{\mathcal{U}}, \to)$ be the ultraproduct of them. Then*

(1) *If for each $i \in I$, F_i is reflexive, then $\prod_{\mathcal{U}} F_i$ is reflexive.*
(2) *If for each $i \in I$, F_i is symmetric, then $\prod_{\mathcal{U}} F_i$ is symmetric.*
(3) *If for each $i \in I$, F_i is separated, then $\prod_{\mathcal{U}} F_i$ is separated.*
(4) *If for each $i \in I$, F_i has the property AL, then $\prod_{\mathcal{U}} F_i$ has the property AL.*
(5) *If for each $i \in I$, F_i has the property AH, then $\prod_{\mathcal{U}} F_i$ has the property AH.*
(6) *If for each $i \in I$, F_i has the property of superposition, then $\prod_{\mathcal{U}} F_i$ has the property of superposition.*

PROOF. We just give the proof of part 4. Let $\lceil f\rceil, \lceil g\rceil \in \Sigma_{\mathcal{U}}$ and $\lceil h\rceil \notin \{\lceil f\rceil, \lceil g\rceil\}^{\perp}$. So $\lceil h\rceil \notin \{\lceil f\rceil\}^{\perp}$ or $\lceil h\rceil \notin \{\lceil g\rceil\}^{\perp}$. If $\lceil h\rceil \notin \{\lceil f\rceil\}^{\perp}$ then $A = \{i \in I : h(i) \to_i f(i)\} \in \mathcal{U}$. For each $i \in A$, $h(i) \notin \{f(i), g(i)\}^{\perp}$ and Kripke frame F_i that has the property AL there exists an $r_i \in \{f(i), g(i)\}^{\perp\perp}$ such that for each $s_i \in \{f(i), g(i)\}^{\perp\perp}$, $h(i) \to s_i$ if and only if $r_i \to s_i$. We define the function $r : I \to \bigcup \Sigma_i$ such that for each $i \in A$, $r(i) = r_i$. For each $\lceil t\rceil \in \{\lceil f\rceil, \lceil g\rceil\}^{\perp}$, we have $\{i \in I : t(i) \in \{f(i), g(i)\}^{\perp}\} \in \mathcal{U}$. Since $\{i \in I : r(i) \in \{f(i), g(i)\}^{\perp\perp}\} \in \mathcal{U}$, we can conclude that $\lceil r\rceil \perp \lceil t\rceil$. So $\lceil r\rceil \in \{\lceil f\rceil, \lceil g\rceil\}^{\perp\perp}$. Now we show that for each $\lceil s\rceil \in \{\lceil f\rceil, \lceil g\rceil\}^{\perp\perp}$, $\lceil r\rceil \to \lceil s\rceil$ if and only if $\lceil h\rceil \to \lceil s\rceil$.

If $\lceil s\rceil \in \{\lceil f\rceil, \lceil g\rceil\}^{\perp\perp}$, then $C = \{i \in I : s(i) \in \{f(i), g(i)\}^{\perp\perp}\} \in \mathcal{U}$.

If $\lceil r\rceil \to \lceil s\rceil$, then $B = \{i \in I : r(i) \to_i s(i)\} \in \mathcal{U}$. So for each $i \in A \cap B \cap C$, $h(i) \to_i s(i)$. So $\lceil h\rceil \to \lceil s\rceil$. In the same way we can see that if $\lceil h\rceil \to \lceil s\rceil$, then $\lceil r\rceil \to \lceil s\rceil$. $\dashv$

COROLLARY 3.5. *We have the following results.*

(1) *If for each $i \in I$, F_i is a state space, then $\prod_{\mathcal{U}} F_i$ is a state space.*
(2) *If for each $i \in I$, F_i is a geometric frame, then $\prod_{\mathcal{U}} F_i$ is a geometric frame.*
(3) *If for each $i \in I$, F_i is a complete geometric frame, then $\prod_{\mathcal{U}} F_i$ is a complete geometric frame.*
(4) *If for each $i \in I$, F_i is a quasi-quantum Kripke frame, then $\prod_{\mathcal{U}} F_i$ is a quasi-quantum Kripke frame.*

4. Ultraproduct of Quantum Kripke frames

In this section we show that the ultraproduct of each family of quantum Kripke frames is a quantum Kripke frame. To do this, using Theorem 3.4, it is enough to show that the property A is preserved under ultraproduct. First we need some preliminaries from [8].

Let $\perp$ be a binary relation on a non-empty set S. For $A \subseteq S$, we define $A^{\perp} = \{y \in S : x \perp y, \text{ for all } x \in A\}$. If $A = \{x\}$, then we write $x^{\perp}$ for $A^{\perp}$.

DEFINITION 4.1. Let S be non-empty set and $\perp$ be a symmetric relation on S. The relational structure $(S, \perp)$ is an orthogonality space if satisfies the following conditions:

(1) For each $x \in S$, $x \perp x$ implies $x \perp y$, for every $y \in S$,
(2) For each $x, y \in S$, if $y \notin S^{\perp}$ and $x^{\perp} \subseteq y^{\perp}$, then $x^{\perp} = y^{\perp}$.

DEFINITION 4.2. Let S be non-empty set and $\perp$ be a symmetric relation on S.

- By $L(S, \perp)$, we denote the set of all bi-orthogonally closed subsets of S, i.e., we have $L(S, \perp) = \{A \subseteq S : A = A^{\perp\perp}\}$.
- By $K(S, \perp)$, we denote the set of all linear subsets of S, i.e., we have $K(S, \perp) = \{A \subseteq S : \forall x, y \in A;\ \{x, y\}^{\perp\perp} \subseteq A\}$.

The following theorem is immediate from Proposition 2.2.1 of [12].

THEOREM 4.3. *Let* $(\Sigma, \rightarrow)$ *be a state space. Then* $(\Sigma, \perp)$ *is an orthogonality space.*

DEFINITION 4.4. Let $\mathbf{L} = (L, \leq)$ be a partially ordered set with a top I and a bottom O. An orthocomplementation on $\mathbf{L}$ is a function $(.)' \colon L \rightarrow L$ such that the following conditions hold:

(i) for every $P \in L$, $P \vee P' = I$ and $P \wedge P' = O$;
(ii) for every $P, Q \in L$, $P \leq Q$ implies that $Q' \leq P'$;
(iii) for every $P \in L$, $P = P''$.

DEFINITION 4.5. An orthocomplemented lattice is an algebra $\mathbf{L} = (L, \wedge, \vee, ', 0, 1)$ such that $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ is a bounded lattice and $'$ is an orthocomplementation.

Here, we bring Proposition 2.2.3 of [12].

PROPOSITION 4.6. *Let $F = (\Sigma, \rightarrow)$ be a state space. Then the set $\mathbf{L}_F$ of all bi-orthogonally closed subsets of F forms a complete atomistic orthocomplemented lattice with $\subseteq$ as the partial order and $\sim(.)\colon \mathbf{L}_F \rightarrow \mathbf{L}_F$ as the orthocomplementation. In particular,*

(1) *For every $\{P_i : i \in I\} \subseteq \mathbf{L}_F$, $\bigcap_{i\in I} P_i$ is bi-orthogonally closed and is the greatest lower bound, or the meet, of $\{P_i : i \in I\}$.*
(2) *For each $s \in \Sigma$, $\{s\}$ is bi-orthogonally closed, and is the atom of this lattice.*
(3) *For $\{P_i : i \in I\} \subseteq \mathbf{L}_F$, $\bigvee_{i\in I} P_i := \bigcap\{Q \in \mathbf{L}_F : P_i \subseteq Q$ for each $i \in I\}$ is bi-orthogonally closed and is the least upper bound, or the join, of $\{P_i : i \in I\}$.*
(4) *For every $P \in \mathbf{L}_F$, $P = \bigvee\{\{s\} \in \mathbf{L}_F : s \in P\}$.*
(5) *For each $P \in \mathbf{L}_F$, $\sim\sim P = P$.*
(6) *For any $P, Q \in \mathbf{L}_F$, $P \subseteq Q$ implies that $\sim Q \subseteq \sim P$.*
(7) *For each $P \in \mathbf{L}_F$, $P \wedge \sim P = \emptyset$ and $P \vee \sim P = \Sigma$.*
(8) *De Morgan's laws hold, i.e., $\bigcap_{i\in I} \sim P_i = \sim \bigvee_{i\in I} P_i$ and $\bigvee_{i\in I} \sim P_i = \sim \bigcap_{i\in I} P_i$ for every $\{P_i : i \in I\} \subseteq \mathbf{L}_F$.*

In the rest of this section we assume that $\{F_i = (\Sigma_i, \rightarrow_i)\}_{i\in I}$ is a family of quantum Kripke frames and $\prod_{\mathcal{U}} F_i = (\Sigma_{\mathcal{U}}, \rightarrow)$ is its ultraproduct with respect to an ultrafilter $\mathcal{U}$. Since this product is a state space (Corollary 3.5), we have the following result.

COROLLARY 4.7. *The lattice $(\mathbf{L}_F^{\mathcal{U}}, \wedge, \vee, ^{\perp}, \emptyset, \Sigma_{\mathcal{U}})$ where $P \wedge Q := P \cap Q$, $P \vee Q := (P \cup Q)^{\perp\perp}$, for $P, Q \in \mathbf{L}_F^{\mathcal{U}}$, is orthocomplemented and satisfies all the properties mentioned in Proposition 4.6.*

Note. In [8], the authors mentioned the following two conditions.

($*$) For each $x \in S$, $x \not\perp x$.
($**$) If $x, y \in S$, then $x^{\perp} \subseteq y^{\perp}$ implies $x = y$.

The following theorem is immediate from Proposition 2.2.1 of [12].

THEOREM 4.8. *Let $(\Sigma, \rightarrow)$ be a state space. Then $(\Sigma, \perp)$ satisfies conditions ($*$) and ($**$).*

We now state some parts of the Theorem 2.2 of [8] that we will need.

THEOREM 4.9. *Let $(S, \perp)$ be an orthogonality space satisfying ($*$) and ($**$). Then the following conditions are equivalent.*

(i) *$L(S,\perp)$ satisfies the covering law, i.e., for each $y \in L(S,\perp)$ and atom $x \in L(S,\perp)$, if $x \wedge y = \emptyset$ then $x \vee y$ covers y which means that for each $z \in L(S,\perp)$ such that $y \subseteq z \subseteq x \vee y$, we have $y = z$ or $x \vee y = z$. If x covers $\emptyset$, we call it an atom.*
(ii) *$(S,\perp)$ satisfies the condition of 3-minimal dependence, i.e., for each $x, x_1, x_2, x_3 \in S$ such that $x \in \{x_1, x_2, x_3\}^{\perp\perp}$ and for each $u, v \in \{x_1, x_2, x_3\}$, if $x \notin \{u, v\}^{\perp\perp}$ then $\{x, x_1\}^{\perp\perp} \cap \{x_2, x_3\}^{\perp\perp} \neq \emptyset$.*
(iii) *$K(S,\perp)$ is modular, i.e., for each $x, y, z \in K(S,\perp)$, if $z \subseteq y$ then $(x \wedge y) \vee z = (x \vee z) \wedge y$.*

As mentioned before, by $\mathbf{L}_F^{\mathcal{U}}$, we mean the set of all bi-orthogonally closed subsets of $\Sigma_{\mathcal{U}}$. Let $\mathbf{K}_F^{\mathcal{U}}$ be the set of linear subsets of $\Sigma_{\mathcal{U}}$, i.e., $\mathbf{K}_F^{\mathcal{U}} = \{A \subseteq \Sigma_{\mathcal{U}} : \forall \lceil f \rceil, \lceil g \rceil \in A;\ \{\lceil f \rceil, \lceil g \rceil\}^{\perp\perp} \subseteq A\}$. By using Theorems 4.3, 4.8 and 4.9, we have the following result.

PROPOSITION 4.10. *The following conditions are equivalent.*

(i) *$\mathbf{L}_F^{\mathcal{U}}$ satisfies the covering law.*
(ii) *$\prod_{\mathcal{U}} F_i$ (with the relation $\perp$) satisfies the condition of 3-minimal dependence.*
(iii) *$\mathbf{K}_F^{\mathcal{U}}$ is modular.*

PROPOSITION 4.11. *The structure $\prod_{\mathcal{U}} F_i$ satisfies the condition of 3-minimal dependence.*

PROOF. Let $\lceil f \rceil, \lceil f_1 \rceil, \lceil f_2 \rceil, \lceil f_3 \rceil \in \Sigma_{\mathcal{U}}$, $\lceil f \rceil \in \{\lceil f_1 \rceil, \lceil f_2 \rceil, \lceil f_3 \rceil\}^{\perp\perp}$ and for each $\lceil g \rceil, \lceil h \rceil \in \{\lceil f_1 \rceil, \lceil f_2 \rceil, \lceil f_3 \rceil\}$, we have $\lceil f \rceil \notin \{\lceil g \rceil, \lceil h \rceil\}^{\perp\perp}$. Then

- $\alpha = \{i \in I : f(i) \in \{f_1(i), f_2(i), f_3(i)\}^{\perp\perp}\} \in \mathcal{U}$.
- $\beta = \{i \in I : f(i) \notin \{f_1(i), f_2(i)\}^{\perp\perp}\} \in \mathcal{U}$.
- $\gamma = \{i \in I : f(i) \notin \{f_2(i), f_3(i)\}^{\perp\perp}\} \in \mathcal{U}$.
- $\delta = \{i \in I : f(i) \notin \{f_1(i), f_3(i)\}^{\perp\perp}\} \in \mathcal{U}$.

Let $i \in \alpha \cap \beta \cap \gamma \cap \delta$ be arbitrary. Since the frame $F_i = (\Sigma_i, \to_i)$ is a quantum Kripke frame so by Theorem 2.10, the set of all bi-orthogonally closed subsets of Σ_i, i.e., $\mathbf{L}_{F_i}$, satisfies the covering law. By using Theorems 4.3, 4.8 and 4.9, we conclude that each $F_i = (\Sigma_i, \to_i)$ satisfies the condition of 3-minimal dependence. So there exists $a_i \in \{f(i), f_1(i)\}^{\perp\perp} \cap \{f_2(i), f_3(i)\}^{\perp\perp}$. We define function $a \colon I \to \bigcup \Sigma_i$ such that for each $i \in \alpha \cap \beta \cap \gamma \cap \delta$, $a(i) = a_i$.

Since $\alpha \cap \beta \cap \gamma \cap \delta \subseteq \{i \in I : a_i \in \{f(i), f_1(i)\}^{\perp\perp} \cap \{f_2(i), f_3(i)\}^{\perp\perp}\}$, we have $\lceil a \rceil \in \{\lceil f \rceil, \lceil f_1 \rceil\}^{\perp\perp} \cap \{\lceil f_2 \rceil, \lceil f_3 \rceil\}^{\perp\perp}$. So $\{\lceil f \rceil, \lceil f_1 \rceil\}^{\perp\perp} \cap$

$\{\lceil f_2\rceil, \lceil f_3\rceil\}^{\perp\perp} \neq \emptyset$. Therefore $\prod_{\mathcal{U}} F_i$ satisfies the condition of 3-minimal dependence. $\dashv$

THEOREM 4.12. *The orthocomplemented lattice $\mathbf{L}_F^{\mathcal{U}}$ is orthomodular, i.e., for each $P, Q, R \in \mathbf{L}_F^{\mathcal{U}}$, if $P \subseteq Q$ then $P \vee (P^{\perp} \wedge Q) = Q$.*

PROOF. By using Proposition 4.10 and Proposition 4.11, we conclude that $\mathbf{L}_F^{\mathcal{U}}$ satisfies the covering law or equivalently that $\mathbf{K}_F^{\mathcal{U}}$ is modular. Therefore $\mathbf{L}_F^{\mathcal{U}}$ is modular. So for each $P, Q, R \in \mathbf{L}_F^{\mathcal{U}}$, if $P \subseteq Q$ then $P \vee (R \wedge Q) = (P \vee R) \wedge Q$. Let $R = P^{\perp}$. Since $P^{\perp} \in \mathbf{L}_F^{\mathcal{U}}$, we have

$$P \vee (P^{\perp} \wedge Q) = (P \vee P^{\perp}) \wedge Q = \Sigma_{\mathcal{U}} \wedge Q = Q$$

So the orthocomplemented lattice $\mathbf{L}_F^{\mathcal{U}}$ is orthomodular. $\dashv$

COROLLARY 4.13. *The frame $\prod_{\mathcal{U}} F_i = (\Sigma_{\mathcal{U}}, \to)$ satisfies the following two conditions.*

- *Exchange property: for each $E \subseteq \Sigma_{\mathcal{U}}$ and $\lceil f\rceil, \lceil g\rceil \in \Sigma_{\mathcal{U}}$ if $\lceil f\rceil \in (E \vee \{\lceil g\rceil\}) \setminus E^{\perp\perp}$, then we have $\lceil g\rceil \in (E \vee \{\lceil f\rceil\})$.*
- *Straightening property: for each $E \subseteq \Sigma_{\mathcal{U}}$ and $\lceil f\rceil \in \Sigma_{\mathcal{U}}$ if $\lceil f\rceil \notin E^{\perp\perp}$, there exists a $\lceil g\rceil \in E^{\perp}$ such that $\lceil f\rceil \in (E \vee \{\lceil g\rceil\})$.*

PROOF. First we establish the exchange property. Let $E \subseteq \Sigma_{\mathcal{U}}$ and $\lceil f\rceil, \lceil g\rceil \in \Sigma_{\mathcal{U}}$ be arbitrary. If $\lceil f\rceil \in (E \vee \{\lceil g\rceil\}) \setminus E^{\perp\perp}$, then we show that $\lceil g\rceil \in (E \vee \{\lceil f\rceil\})$. If $\lceil g\rceil \in E^{\perp\perp}$, since $E^{\perp\perp} \subseteq (E \vee \{\lceil f\rceil\})$, we have $\lceil g\rceil \in (E \vee \{\lceil f\rceil\})$. If $\lceil g\rceil \notin E^{\perp\perp}$, then $\{\lceil g\rceil\} \cap E^{\perp\perp} = \emptyset$. So by Corollary 4.7 and the covering law for $\mathbf{L}_F^{\mathcal{U}}$, $(E^{\perp\perp} \vee \{\lceil g\rceil\})$ covers $E^{\perp\perp}$. Since $\lceil f\rceil \in (E \vee \{\lceil g\rceil\})$, so $(E \cup \{\lceil f\rceil\}) \subseteq (E \vee \{\lceil g\rceil\}) \subseteq (E^{\perp\perp} \vee \{\lceil g\rceil\})$. So $(E \vee \{\lceil f\rceil\}) \subseteq (E^{\perp\perp} \vee \{\lceil g\rceil\})$. Therefore we have $E^{\perp\perp} \subseteq (E \vee \{\lceil f\rceil\}) \subseteq (E^{\perp\perp} \vee \{\lceil g\rceil\})$. Since $\lceil f\rceil \notin E^{\perp\perp}$, we conclude that $(E \vee \{\lceil f\rceil\}) = (E^{\perp\perp} \vee \{\lceil g\rceil\})$. So $\lceil g\rceil \in (E \vee \{\lceil f\rceil\})$.

For the straightening property, let $E \subseteq \Sigma_{\mathcal{U}}$ and $\lceil f\rceil \in \Sigma_{\mathcal{U}}$ be arbitrary such that $\lceil f\rceil \notin E^{\perp\perp}$. By the orthomodularity of $\mathbf{L}_F^{\mathcal{U}}$, since $E^{\perp\perp} \subseteq (E \vee \{\lceil f\rceil\})$ we conclude that $E^{\perp\perp} \vee (E^{\perp} \cap (E \vee \{\lceil f\rceil\})) = (E \vee \{\lceil f\rceil\})$. If $E^{\perp} \cap (E \vee \{\lceil f\rceil\}) = \emptyset$, then $E^{\perp\perp} = (E \vee \{\lceil f\rceil\})$. It is impossible since $\lceil f\rceil \notin E^{\perp\perp}$. So there exists $\lceil g\rceil \in E^{\perp} \cap (E \vee \{\lceil f\rceil\})$. Since $\lceil f\rceil \notin E^{\perp\perp}$, by Corollary 4.7 and the covering law for $\mathbf{L}_F^{\mathcal{U}}$, we conclude that $(E^{\perp\perp} \vee \{\lceil f\rceil\})$ covers $E^{\perp\perp}$. Also we know that $\lceil g\rceil \in (E \vee \{\lceil f\rceil\})$ and $(E \vee \{\lceil f\rceil\}) \subseteq (E^{\perp\perp} \vee \{\lceil f\rceil\})$. So $E^{\perp\perp} \subseteq (E \vee \{\lceil g\rceil\}) \subseteq (E^{\perp\perp} \vee \{\lceil f\rceil\})$. Since $\lceil g\rceil \in E^{\perp}$, so $\lceil g\rceil \notin E^{\perp\perp}$.

Therefore $(E \vee \{\lceil g\rceil\}) = (E^{\perp\perp} \vee \{\lceil f\rceil\})$. So $\lceil f\rceil \in (E \vee \{\lceil g\rceil\})$. Therefore there exists a $\lceil g\rceil \in E^{\perp}$ such that $\lceil f\rceil \in (E \vee \{\lceil g\rceil\})$. $\dashv$

Finally, by using the exchange and straightening properties, we show that the ultraproduct frame satisfies the property A.

THEOREM 4.14. *Let* $\{F_i = (\Sigma_i, \to_i)\}_{i\in I}$ *be a family of quantum Kripke frames. Then* $\prod_{\mathcal{U}} F_i$ *has the property A.*

PROOF. By Corollary 4.13, $\prod_{\mathcal{U}} F_i$ satisfies the exchange and straightening properties. Now let P be a bi-orthogonally closed set and $\lceil f \rceil \notin P^\perp$ be arbitrary. If we define $E^\perp = P$, then $\lceil f \rceil \notin E^{\perp\perp}$.

By the straightening property, there exists $\lceil g \rceil \in P$ such that $\lceil f \rceil \in (E \vee \{\lceil g \rceil\})$. We show that $\lceil f \rceil \approx_P \lceil g \rceil$. Let $\lceil r \rceil \in P = E^\perp$ be arbitrary. We show $\lceil g \rceil \perp \lceil r \rceil$ if and only if $\lceil f \rceil \perp \lceil r \rceil$. Let $\lceil g \rceil \perp \lceil r \rceil$ and $\lceil f \rceil \not\perp \lceil r \rceil$. Since $\lceil f \rceil \in (E \vee \{\lceil g \rceil\})$ we have $(E \cup \{\lceil g \rceil\})^\perp \subseteq \{\lceil f \rceil\}^\perp$. So $\lceil r \rceil \notin (E \cup \{\lceil g \rceil\})^\perp$. This is a contradiction because we assumed that $\lceil r \rceil \in (E \cup \{\lceil g \rceil\})^\perp$. So $\lceil f \rceil \perp \lceil r \rceil$.

Now, let $\lceil f \rceil \perp \lceil r \rceil$ and $\lceil g \rceil \not\perp \lceil r \rceil$. Since $\lceil f \rceil \in (E \vee \{\lceil g \rceil\}) \setminus E^{\perp\perp}$, by the exchange property we have $\lceil g \rceil \in (E \vee \{\lceil f \rceil\})$. So $(E \cup \{\lceil f \rceil\})^\perp \subseteq \{\lceil g \rceil\}^\perp$. So $\lceil r \rceil \notin (E \cup \{\lceil f \rceil\})^\perp$. This is a contradiction since $\lceil r \rceil \in (E \cup \{\lceil f \rceil\})^\perp$. So $\lceil g \rceil \perp \lceil r \rceil$. ⊣

COROLLARY 4.15. *If for each* $i \in I$, F_i *is a quantum Kripke frame, then* $\prod_{\mathcal{U}} F_i$ *is a quantum Kripke frame.*

In the rest of this section, we use a modal language equipped with conjunction and negation and a unary modal operator $\Box$ which denotes the non-orthogonality relation in quantum Kripke frames. One can define a Kripke model by adding an interpretation function $\|.\|$ to a Kripke frame $F = (\Sigma, \to)$. The same can be done to the other quantum structures studied in this paper. The last result of this section is Łós's theorem for the ultraproduct of a family of quantum Kripke frames. The proof is similar to the one in the standard reference books on modal logic and is omitted.

DEFINITION 4.16. Let $\{M_i = (\Sigma_i, \to_i, \|.\|_i)\}_{i\in I}$ be a family of quantum Kripke models. We add an interpretation function $\|.\|_{\mathcal{U}}$ to the Kripke frame $\prod_{\mathcal{U}} F_i = (\Sigma_{\mathcal{U}}, \to)$ as follows.

For each atomic proposition p,

$$\lceil f \rceil \in \|p\|_{\mathcal{U}} \Leftrightarrow \{i \in I : f(i) \in \|p\|_i\} \in \mathcal{U}$$

For negation, conjunction and unary modal operator $\Box$, the definition is as usual. We denote this new model by $\prod_{\mathcal{U}} M_i = (\Sigma_{\mathcal{U}}, \to, \|.\|_{\mathcal{U}})$.

Note. For convenience, we use the notation $M, s \vDash \varphi$ instead of $s \in \|\varphi\|$ in the following.

Theorem 4.17 (Łoś's Theorem). *Let $\prod_{\mathcal{U}} M_i = (\Sigma_{\mathcal{U}}, \to, \|.\|_{\mathcal{U}})$ be the ultraproduct of a family of quantum Kripke models. For each formula φ, $\prod_{\mathcal{U}} M_i, \lceil f \rceil \vDash \varphi$ if and only if $\{i \in I : M_i, f(i) \vDash \varphi\} \in \mathcal{U}$.*

Corollary 4.18. *Let $\prod_{\mathcal{U}} M$ be the ultrapower of a of quantum Kripke model M. For each formula φ, $M, w \vDash \varphi$ if and only if $\prod_{\mathcal{U}} M, \lceil f_w \rceil \vDash \varphi$. Here f_w is the constant function such that for each $i \in I$, $f_w(i) = w$.*

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