Logic and Logical Philosophy
Volume 25 (2016), 3–33
DOI: 10.12775/LLP.2016.002

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THE LATTICE OF BELNAPIAN MODAL LOGICS: Special Extensions and Counterparts

Abstract. Let K be the least normal modal logic and BK its Belnapian version, which enriches K with 'strong negation'. We carry out a systematic study of the lattice of logics containing BK based on:

- introducing the classes (or rather sublattices) of so-called explosive, complete and classical Belnapian modal logics;
- assigning to every normal modal logic three special conservative extensions in these classes;
- associating with every Belnapian modal logic its explosive, complete and classical counterparts.

We investigate the relationships between special extensions and counterparts, provide certain handy characterisations and suggest a useful decomposition of the lattice of logics containing BK.

Keywords: algebraic logic; paraconsistent logic; many-valued modal logic; strong negation

Introduction

The study of lattices of paraconsistent constructive logics with 'strong negation' presented in [22] shows that such lattices have nice internal structure, and one can often transfer certain fundamental results from the well-understood lattice of intermediate logics to them. We wish to

* The first author acknowledges the support of the Russian Foundation for Basic Research, project No. 15-07-03410-a.

** The second author acknowledges the support of the Alexander von Humboldt Foundation.

Received July 26, 2015. *Revised* February 25, 2016. *Published online* February 25, 2016

study paraconsistent modal logics with 'strong negation' in a similar way. Namely, in this article we start investigating the lattice of Belnapian modal logics, i.e. those containing the four-valued version BK of the least normal modal logic K, in a systematic manner.

The logic BK was introduced in [25], and it can be viewed as conservatively extending K by adding 'strong negation', denoted by $\sim$, which allows both 'gaps' (incomplete information) and 'gluts' (inconsistent information). Further — BK may be seen as enriching Belnap-Dunn useful four-valued logic [3, 4, 8], which coincides with *first-degree entailment*, written FDE for short. More precisely, the smallest normal modal enrichment of FDE is known as $\mathsf{K}_{\mathrm{FDE}}$ [26, 27], and following [25] one can obtain BK from $\mathsf{K}_{\mathrm{FDE}}$ by adding the absurdity constant $(\bot)$ and the material implication $(\rightarrow)$. So in [25] the four-valued matrix $\mathtt{BD4}$ for Belnap-Dunn logic was augmented with $\rightarrow$ and $\bot$. The resulting matrix $\mathtt{BD4}^{\bot}_{\rightarrow}$ determines the extension $\mathsf{N4}^{\bot}_{p}$ of the logic $\mathsf{N4}^{\bot}$ by Peirce's law, i.e.

$$\mathsf{N4}^{\bot}_{p} \;:=\; \mathsf{N4}^{\bot} + \{((p \rightarrow q) \rightarrow p) \rightarrow p\};$$

see [21].[1] Naturally there exist alternative ways of adding a conditional to $\mathtt{BD4}$. E.g., some relevant logicians take the connective $\Rightarrow$, suggested by R. Brady in [7], to be the most natural truth-functional conditional associated with FDE (cf. [17, 31]). Following Brady, let $\mathsf{BN4}$ denote the logic determined by the matrix obtained by augmenting $\mathtt{BD4}$ with $\Rightarrow$. In effect, the connectives $\rightarrow$ and $\Rightarrow$ coincide with the weak implication and the strong implication defined on $\mathtt{BD4}$ by O. Arieli and A. Avron [2]. It was proved in [2] that $\rightarrow$ and $\Rightarrow$ are interdefinable modulo the language $\{\vee, \wedge, \sim\}$ of $\mathtt{BD4}$ as follows:

$$\begin{aligned} x \Rightarrow y \;&:=\; (x \rightarrow y) \wedge (\sim y \rightarrow \sim x), \\ x \rightarrow y \;&:=\; (x \Rightarrow (x \Rightarrow y)) \vee y. \end{aligned}$$

Hence $\mathsf{BN4}$ turns out to be definitially equivalent to the $\bot$-free fragment of $\mathsf{N4}^{\bot}_{p}$. Then in [11], L. Goble introduced a modal system extending $\mathsf{BN4}$. Notice that although this system and BK were originally described in different languages, [11] and [25] treat the modal operators in exactly the same way. Consequently Goble's logic will be definitially equivalent to the $\bot$-free fragment of BK. Yet another modal system was intro-

[1] Here $\mathsf{N4}^{\bot}$ is a version of Nelson's paraconsistent logic $\mathsf{N4}$ [1] augmented with the absurdity constant $\bot$ (see also [21]). Actually, the lattice of $\mathsf{N4}^{\bot}$-extensions has a more regular structure than that for $\mathsf{N4}$.

duced by A. Jung and U. Rivieccio in [13]. They took the logic of bilattices $\mathsf{GBL}_{\supset}$ from [2], which is a conservative enrichment of $\mathsf{N4}_p^{\perp}$, as the non-modal basis of their system.[2] More importantly, the modal operators in [13] behave differently than in [25], so in particular, the $\{\vee, \wedge, \rightarrow, \sim, \perp, \Box\}$-fragment of Jung-Rivieccio modal logic turns out to be essentially different from that of BK.

Moreover, it was proved in [25] that the Belnapian version $\mathsf{BS4}$ of the modal logic $\mathsf{S4}$ and its three-valued extension $\mathsf{B3S4}$ are closely related to Nelson's constructive logics $\mathsf{N3}$ (from [18, 36]) and $\mathsf{N4}^{\perp}$. Namely, $\mathsf{N3}$ and $\mathsf{N4}^{\perp}$ can be faithfully embedded into $\mathsf{B3S4}$ and $\mathsf{BS4}$ respectively — this is done not by using the method of introducing new propositional letters (first exploited in [36] and [12]), but by providing an analogue of Gödel-McKinsey-Tarski translation of intuitionistic logic into $\mathsf{S4}$. In particular, the weak implications of $\mathsf{N3}$ and $\mathsf{N4}^{\perp}$ are definable in terms of modalities, as a kind of 'strict implication' in the sense of C. I. Lewis. Now since different interesting results for intermediate logics were obtained by transferring suitable theorems for $\mathsf{S4}$-extensions, one expects that various results for Belnapian modal logics extending $\mathsf{B3S4}$ and $\mathsf{BS4}$ can be transferred to $\mathsf{N3}$- and $\mathsf{N4}^{\perp}$-extensions in an effective way. This also motivates our study.

Actually, we should note the similarity between algebraic semantics for extensions of Nelson's constructive logics and that for Belnapian modal logics: both can be characterised using so-called *twist-structures* (the term is due to M. Kracht [14]) — where any twist-structure is an algebra defined on the direct power of the universe of another algebra, but the new operations are not componentwise, they are somehow 'twisted'. In effect, for the first time such structures (over Heyting algebras, and with the property that the meet of the components of every element is zero) were introduced independently by M. Fidel [9] and D. Vakarelov [35] as a presentation of N-lattices [29], to provide an algebraic semantics for Nelson's explosive logic $\mathsf{N3}$. Then in [19] it was proved that Nelson's paraconsistent logic $\mathsf{N4}$ can be characterised using twist-structures over implicative lattices, and that the abstract closure of the class of these structures forms a variety.[3] Other examples of application of twist-structures may be found in [28] and [23].

[2] In [2], $\supset$ denotes the weak implication.

[3] Here by the abstract closure of a given class $\mathcal{K}$ of algebras we mean the collection of all isomorphic copies of algebras in $\mathcal{K}$.

Later, in [25], twist-structures over modal algebras were defined. And it was proved in [24] that the abstract closure of the class of all such structures, which coincides with the collection of so-called BK-*lattices*, is a variety, and that the lattice of its subvarieties and the lattice of BK-extensions are dually isomorphic. In effect, BK turns out to be algebraizable (in the sense of [6]), with equivalent algebraic semantics given by the BK-latices. It was also established in [24] that every twist-structure for BK is uniquely determined by three invariants, namely its underlying modal algebra, a suitable $\Box$-filter and a suitable $\Diamond$-ideal — this may be compared with the representation for $\mathsf{N3}$ in [30] and its generalisations to $\mathsf{N4}$ and $\mathsf{N4}^{\perp}$ in [20, 21].[4]

In view of the aforementioned similarity between algebraic semantics of $\mathsf{N4}^{\perp}$ and that of BK we will take a route reminiscent of that in [21]. In the case of $\mathsf{N4}^{\perp}$-extensions it was essential to isolate the subclasses of so-called explosive and normal logics — these arise naturally when collapsing one or other invariant (apart from the underlying algebra, of course) in the representation of twist-structures over Heyting algebras.[5] So if an $\mathsf{N4}^{\perp}$-extension belongs to both classes, then its semantics boils down to some collection of Heyting algebras — hence it was proved that the lattice of such extensions is isomorphic to that of superintuitionistic logics (in fact, by mapping each such $\mathsf{N4}^{\perp}$-extension to its $\sim$-free fragment), cf. [21]. Also, it was important to introduce so-called special conservative extensions for superintuitionistic logics and counterparts for $\mathsf{N4}^{\perp}$-extensions, and to study them (see [21] for details). In effect all this turned out to be very useful in obtaining transfer results from the superintuitionistic logics to the $\mathsf{N4}^{\perp}$-extensions. E.g., by a remarkable theorem of L. Maksimova [16], there exist exactly seven non-trivial superintuitionistic logics with Craig interpolation property — hence, by a transfer theorem from [22], one can get exactly twenty eight special $\mathsf{N4}^{\perp}$-extensions sharing the same property. Turning to BK, here is our plan:

- We introduce the classes — or rather sublattices — of *explosive*, *complete* and *classical* Belnapian modal logics (see [33]). Then we prove that these classes are semantically characterised by collapsing one or other invariant — or even both — in the representation of twist-structures over modal algebras.

[4] While [20] deals with $\mathsf{N4}$, [21] adapts the main results of [20] to the logic $\mathsf{N4}^{\perp}$.

[5] In the case of Belnapian modal logics, 'normal' will correspond to 'complete'.

- We show that for any normal modal logic L, the intersection of the class of its conservative BK-extensions with each of the classes from above is an interval. Moreover, the endpoints of these intervals correspond to the four *special conservative extensions* of L, which are the most plausible candidates for transferring certain fundamental properties from the lattice of normal modal logics (cf. [21, 22, 33]).
- We associate with each BK-extension its *counterparts* in the classes of explosive, complete and classical logics, and define embeddings of these counterparts into the original logic.

More generally, the article studies the relationships between special extensions and counterparts, provides some handy characterisations and offers a useful decomposition of the lattice of logics containing BK.

The rest of the article is organised as follows. Section 1 consists of preliminary material on BK and the lattice of its extensions, including algebraic semantics and a rather detailed survey of related results. In particular, we define the variety of BK-lattices, which provides an adequate algebraic semantics for BK in the sense of [6], and recall that any BK-lattice $\mathcal{A}$ is completely determined by its underlying modal algebra $\mathcal{A}_{\bowtie}$ and two special invariants, namely the $\Box$-filter $\nabla_l(\mathcal{A})$ on $\mathcal{A}_{\bowtie}$ and the $\Diamond$-ideal $\Delta_l(\mathcal{A})$ on $\mathcal{A}_{\bowtie}$ (see [24, 32]).

In Section 2, the sublattices of explosive, complete and classical BK-extensions are introduced. We prove that a BK-extension L is explosive (normal) iff $\Delta_l(\mathcal{A})$ (respectively $\nabla_l(\mathcal{A})$) is trivial for all BK-lattices $\mathcal{A}$ satisfying L, and L is classical iff it is both explosive and normal. Then for every normal modal logic L we define the four special BK-extensions $\eta(L)$, $\eta^3(L)$, $\eta^\circ(L)$ and $\eta^{\mathsf{c}}(L)$, characterise them semantically, and also show how they can be embedded into L. Moreover, we prove that:

- the Belnapian logics $\eta(L)$ and $\eta^{\mathsf{c}}(L)$ are the least and greatest conservative enrichments of L in the lattice of BK-extensions;
- $\eta^3(L)$ and $\eta^\circ(L)$ are the least explosive and least complete conservative enrichments of L respectively.

This means that the set of all conservative enrichments of L, as well as its intersections with the classes of explosive and complete BK-extensions, is an interval, and that $\eta^{\mathsf{c}}(L)$ is the unique classical BK-extension of L (Proposition 2.12; see also [33]). Next we obtain a characterization of special logics in terms of admissible rules (Proposition 2.13). We use it to prove that the mappings η, η^3 and η° are lattice monomorphisms commuting with infinite meets and joins, whereas η^{c} is an isomorphism between the lattice of normal modal logics and the lattice of classical BK-extensions (Proposition 2.15; see also [33]).

Section 3 investigates the connections between the lattice of Belnapian modal logics and its subclasses of explosive, complete and classical BK-extensions (in a manner similar to Section 2). For every BK-extension L we define its explosive, normal and classical counterparts as the least L-extensions in the respective classes of logics. Then we provide a semantic characterisation of the counterparts of L, and prove that they can be embedded into L. Finally, we study the classes $\mathrm{Spec}\,(L_1, L_2)$ of Belnapian modal logics, where the explosive and complete counterparts of any $L \in \mathrm{Spec}\,(L_1, L_2)$ coincide with L_1 and L_2 respectively. Here we prove that all such classes are intervals, and describe the endpoints of these intervals.

We conclude with a few comments on future research (Section 4).

1. Preliminaries

1.1. Belnapian modal logics

In [25] the logic BK and its extensions were introduced in the language

$$\mathcal{L} \;:=\; \{\vee, \wedge, \to, \bot, \sim, \Box, \Diamond\}$$

where $\sim$ stands for 'strong negation'. Let $\mathrm{For}(\mathcal{L})$ denote the set of all $\mathcal{L}$-formulas; and similarly for other languages. By an *$\mathcal{L}$-logic* we mean a collection of $\mathcal{L}$-formulas closed under the *substitution rule*, *modus ponens* and the *monotonicity rules* for $\Box$ and $\Diamond$, i.e. under

$$\frac{\varphi\,(p_1, \ldots, p_n)}{\varphi\,(\psi_1, \ldots, \psi_n)}, \quad \frac{\varphi \quad \varphi \to \psi}{\psi}, \quad \frac{\varphi \to \psi}{\Box\varphi \to \Box\psi} \quad \text{and} \quad \frac{\varphi \to \psi}{\Diamond\varphi \to \Diamond\psi}.$$

Further — for any $X, Y \subseteq \mathrm{For}(\mathcal{L})$ we take

$$X + Y \;:=\; \text{the intersection of all } \mathcal{L}\text{-logics containing } X \cup Y.$$

Denote by $\mathcal{E}L$ the set of all $\mathcal{L}$-logics extending L. One readily verifies that $\mathcal{E}L$ with operations $\cap$ and $+$ is a lattice, in which the lattice ordering coincides with the inclusion relation.

For convenience we shall also use certain abbreviations:

$$\neg\,\varphi \;:=\; \varphi \to \bot, \quad \varphi \leftrightarrow \psi \;:=\; (\varphi \to \psi) \wedge (\psi \to \varphi)$$
$$\text{and} \quad \varphi \Leftrightarrow \psi \;:=\; (\varphi \leftrightarrow \psi) \wedge (\sim\varphi \leftrightarrow \sim\psi).$$

DEFINITION 1.1. BK is the least $\mathcal{L}$-logic containing the following axioms:

1. all tautologies of the classical propositional logic stated in the language $\{\vee, \wedge, \rightarrow, \bot\}$;
2. the five strong negation axioms

$$\begin{array}{ll} \sim(p \wedge q) \leftrightarrow (\sim p \vee \sim q), & \sim(p \rightarrow q) \leftrightarrow (p \wedge \sim q), \\ \sim(p \vee q) \leftrightarrow (\sim p \wedge \sim q), & \sim\sim p \leftrightarrow p \quad \text{and} \quad \sim\bot; \end{array}$$

3. the two K axioms

$$(\Box p \wedge \Box q) \rightarrow \Box(p \wedge q) \qquad \text{and} \qquad \Box(p \rightarrow p);$$

4. the four modal interaction axioms

$$\begin{array}{ll} \neg\Box p \leftrightarrow \Diamond\neg p, & \Box p \Leftrightarrow \sim\Diamond\sim p, \\ \neg\Diamond p \leftrightarrow \Box\neg p, & \Diamond p \Leftrightarrow \sim\Box\sim p. \end{array}$$

We shall be concerned with the sublattices of $\mathcal{E}\mathsf{BK}$ corresponding to

$$\mathsf{B3K} := \mathsf{BK} + \{\sim p \rightarrow (p \rightarrow q)\}, \quad \mathsf{BK}^{\circ} := \mathsf{BK} + \{p \vee \sim p\}$$
$$\text{and} \quad \mathsf{B3K}^{\circ} := \mathsf{BK} + \{\sim p \rightarrow (p \rightarrow q), p \vee \sim p\}.$$

With every $\mathcal{L}$-logic L we associate two consequence relations, $\vdash_L$ (*local*) and $\vdash_L^*$ (*global*), as follows. For any $\Gamma \cup \{\varphi\} \subseteq \mathrm{For}(\mathcal{L})$:

- $\Gamma \vdash_L \varphi$ iff φ can be obtained from $L \cup \Gamma$ by *modus ponens* only;
- $\Gamma \vdash_L^* \varphi$ iff φ can be obtained from $L \cup \Gamma$ by *modus ponens* and the monotonicity rules for $\Box$ and $\Diamond$.

In particular, it was proved in [25, Sections 4–5] that the relations $\vdash_{\mathsf{BK}}$ and $\vdash_{\mathsf{B3K}}$ are strongly complete w.r.t. suitable classes of Kripke frames — with Belnapian valuations employed — and $\vdash_{\mathsf{BK}}^*$ and $\vdash_{\mathsf{B3K}}^*$ are strongly complete w.r.t. suitable classes of twist-structures over modal algebras.

As expected, BK shares some interesting features with the constructive Nelson's logic: $\leftrightarrow$ does not have the congruence property w.r.t. $\sim$, but only for all the other connectives; while $\Leftrightarrow$ in effect has the congruence property w.r.t. each connective in $\mathcal{L}$. More precisely, as was observed earlier in [25], although BK is not closed under the ordinary replacement rule, every L from $\mathcal{E}\mathsf{BK}$ will be closed under the *positive replacement rule* and the *weak replacement rule*, i.e. under

$$\frac{\varphi \leftrightarrow \psi}{\gamma(\varphi) \leftrightarrow \gamma(\psi)}\ (\mathrm{PR}) \qquad \text{and} \qquad \frac{\varphi \Leftrightarrow \psi}{\chi(\varphi) \leftrightarrow \chi(\psi)}\ (\mathrm{WR}).$$

where γ does not contain $\sim$. Note that (WR) is easily seen to be equivalent to the *replacement rule for the strong equivalence*, viz. to

$$\frac{\varphi \Leftrightarrow \psi}{\chi(\varphi) \Leftrightarrow \chi(\psi)}.$$

An $\mathcal{L}$-formula φ is said to be a *negation normal form* (*nnf* for short) iff all occurrences of $\sim$ in φ immediately precede propositional variables. It is straightforward to define a translation which maps each $\mathcal{L}$-formula φ into an nnf $\overline{\varphi}$ such that $\varphi \leftrightarrow \overline{\varphi} \in \mathsf{BK}$ — by exploiting the axioms (2) and (4), and the rule (PR).

We now recall some algebraic terminology. Throughout this text we use $\mathcal{A}$, $\mathcal{B}$ and the like to stand for algebras, reserving the corresponding uppercase italic letters A, B, etc. for their respective domains.

Given a propositional language, an expression of the form $\varphi = \psi$ where φ and ψ are formulas of the language is called an *identity*. Further, for an algebra $\mathcal{A}$ (of the same language) we say that

$$\varphi = \psi \ \textit{ holds in } \mathcal{A} \quad \text{iff} \quad v(\varphi) = v(\psi) \text{ for each } \mathcal{A}\text{-valuation } v.$$

Denote by $\mathrm{Eq}(\mathcal{A})$ the set of all identities which hold in $\mathcal{A}$.

By a *modal algebra* (see, e.g., [15]) we understand an algebra of the form $\langle A, \vee, \wedge, \neg, \Box \rangle$ where $\langle A, \vee, \wedge, \neg \rangle$ is a Boolean algebra and the operation $\Box$ satisfies the following conditions:

- $\Box(a \wedge b) = \Box a \wedge \Box b$ for any $\{a, b\} \subseteq A$;
- $\Box 1 = 1$ where 1 is the greatest element w.r.t. the usual lattice ordering $\leqslant$ on the Boolean algebra.

Alternatively, a modal algebra may be defined as a Boolean algebra with $\Diamond$ instead of $\Box$, such that:

- $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$ for any two elements of it;
- $\Diamond 0 = 0$ where 0 is the least element of it.

Actually we can switch between $\Box$ and $\Diamond$ by using

$$\Diamond a \;=\; \neg \Box \neg a \quad \text{and} \quad \Box a \;=\; \neg \Diamond \neg a,$$

and moreover, take $a \to b := \neg a \vee b$ and $\bot := 0$ if needed.

Next, by a *De Morgan algebra* we mean an $\langle A, \vee, \wedge, \sim, \bot, \top \rangle$, for which $\langle A, \vee, \wedge, \bot, \top \rangle$ is a bounded distributive lattice, and in which the identities $\sim\sim p = p$ and $\sim(p \wedge q) = \sim p \vee \sim q$ hold. In effect, one easily checks that $\sim(p \vee q) = \sim p \wedge \sim q$ and $\sim\bot = \top$ also hold in any De Morgan algebra.

By an *Ockham lattice* (consult [5, 34]) we understand a bounded distributive lattice $\langle A, \vee, \wedge, \neg, \bot, \top \rangle$ in which the following identities hold:

$$\neg (p \wedge q) = \neg p \vee \neg q, \quad \neg (p \vee q) = \neg p \wedge \neg q, \quad \neg \bot = \top \text{ and } \neg \top = \bot$$

(but not necessarily $\neg \neg p = p$, of course).

Let $\mathcal{B} = \langle B, \vee, \wedge, \neg, \Box \rangle$ be a modal algebra. Call a non-empty subset S of B a $\Box$-*filter* ($\Diamond$-*ideal*) on $\mathcal{B}$ iff S is a lattice filter (ideal) on $\langle B, \vee, \wedge \rangle$ and for every $b \in S$ we have $\Box b \in S$ (respectively $\Diamond b \in S$). Further — as can be readily verified, the family of all $\Box$-filters and that of all $\Diamond$-ideals form two lattices, which we denote by $\mathscr{F}^{\Box}(\mathcal{B})$ and $\mathscr{I}^{\Diamond}(\mathcal{B})$ respectively. They play an important role in the algebraic semantics for BK; cf. [24, Section 6] and [32, Section 3]. It is well-known that $\mathscr{F}^{\Box}(\mathcal{B})$ and $\mathscr{I}^{\Diamond}(\mathcal{B})$ are isomorphic to the lattice of congruences on $\mathcal{B}$ (see, e.g., [15, Theorem 4.1.10]).

Definition 1.2 (see [25]). For a modal algebra $\mathcal{B} = \langle B, \vee, \wedge, \neg, \Box \rangle$, by the *full twist-structure* over $\mathcal{B}$ we mean the algebra

$$\mathcal{B}^{\bowtie} = \langle B \times B; \vee, \wedge, \rightarrow, \bot, \sim, \Box, \Diamond \rangle$$

whose operations are given by

$$(a, b) \vee (c, d) := (a \vee c, b \wedge d), \quad (a, b) \wedge (c, d) := (a \wedge c, b \vee d),$$
$$(a, b) \rightarrow (c, d) := (\neg a \vee c, a \wedge d), \quad \bot := (0, 1), \quad \sim (a, b) := (b, a),$$
$$\Box (a, b) := (\Box a, \Diamond b) \quad \text{and} \quad \Diamond (a, b) := (\Diamond a, \Box b).$$

A *twist-structure* over $\mathcal{B}$ is a subalgebra $\mathcal{A}$ of $\mathcal{B}^{\bowtie}$ such that $\pi_1(A) = B$ (we use π_i to denote the i-th projection function, for $i \in \{1, 2\}$). Let $S^{\bowtie}(\mathcal{B})$ be the collection of all twist-structures over $\mathcal{B}$.

Given a twist-structure $\mathcal{A}$ (over some modal algebra) and a set $\Gamma \cup \{\varphi\}$ of $\mathcal{L}$-formulas, we write $\Gamma \models_{\mathcal{A}} \varphi$ iff for any $\mathcal{A}$-valuation v, if $\pi_1(v(\psi)) = 1$ for each $\psi \in \Gamma$, then $\pi_1(v(\varphi)) = 1$. Now let

$$\Gamma \models^{\bowtie}_{\mathsf{BK}} \varphi \quad \text{iff} \quad \Gamma \models_{\mathcal{A}} \varphi \text{ for all twist-structures } \mathcal{A}.$$

We can also define $\models^{\bowtie}_{\mathsf{B3K}}$ by restricting ourselves to $\mathcal{A}$'s such that $a \wedge b = 0$ for any $(a, b) \in A$.

Theorem 1.3 ([25]). *For every* $\Gamma \cup \{\varphi\} \subseteq \mathrm{For}(\mathcal{L})$ *with* Γ *non-trivial w.r.t.* BK, *we have*

$$\Gamma \vdash^{*}_{\mathsf{BK}} \varphi \iff \Gamma \models^{\bowtie}_{\mathsf{BK}} \varphi.$$

And similarly for B3K.

To axiomatise the abstract closure of the class of all twist-structures, it may be convenient to pass to the language

$$\mathcal{L}' := \{\vee, \wedge, \neg, \bot, \top, \sim, \Box\},$$

as was done in [24]. Clearly we can switch between $\mathcal{L}$ and $\mathcal{L}'$ by using

$$\neg(a,b) = (a,b) \to \bot \quad \text{and} \quad \top = \sim\bot,$$
$$(a,b) \to (c,d) = \neg(a,b) \vee (c,d) \quad \text{and} \quad \Diamond(a,b) = \sim\Box\sim(a,b).$$

We shall work in $\mathcal{L}$, so in the next section certain results of [24, 32] will be adapted to this language. In particular, when dealing with BK-lattices, $\neg a$, $\top$, etc. merely abbreviate $a \to \bot$, $\sim\bot$, etc.

1.2. Belnapian modal algebras

Now we turn to the lattice-theoretic description of twist-structures.

DEFINITION 1.4 (see [24]). A BK-*lattice* (or *Belnapian modal algebra*) is an algebra of the form $\langle A, \vee, \wedge, \to, \bot, \sim, \Box, \Diamond\rangle$ such that:

1. $\langle A, \vee, \wedge, \sim, \bot, \top\rangle$ is a De Morgan algebra;
2. $\langle A, \vee, \wedge, \neg, \bot, \top\rangle$ is an Ockham lattice;
3. $\neg a \wedge \neg\neg a = \bot$ and $\neg a \vee \neg\neg a = \top$ for each $a \in A$;
4. $\sim\neg a = \neg\neg a$ for each $a \in A$;
5. $\neg\Box a = \Diamond\neg a$ and $\neg\Diamond a = \Box\neg a$ for each $a \in A$;
6. $\Box\top = \top$ and $\Box(\neg a \wedge \neg b) = \Box\neg a \wedge \Box\neg b$ for any $\{a, b\} \subseteq A$;
7. for any $\{a, b\} \subseteq A$, if $\neg a = \neg b$ and $\neg\sim a = \neg\sim b$, then $a = b$.

Denote by $\mathcal{V}_{\mathsf{BK}}$ the class of all BK-lattices.

As was observed in [24], twist-structures over modal algebras belong to $\mathcal{V}_{\mathsf{BK}}$, and moreover it was proved that every BK-lattice can be transformed into such a structure. Given $\mathcal{A} \in \mathcal{V}_{\mathsf{BK}}$, for each $a \in A$, take

$$e_{\bowtie}(a) := \neg\neg a \quad \text{and} \quad \iota^{\bowtie}(a) := (e_{\bowtie}(a), e_{\bowtie}(\sim a)).$$

In effect, $e_{\bowtie}(A)$ turns out to be the domain of a modal algebra, called the *underlying modal algebra* of $\mathcal{A}$, and we get a suitable structure via $\iota^{\bowtie}$.

PROPOSITION 1.5 ([24]). *Let $\mathcal{A}$ be a* BK-*lattice. Then:*

- *the set $e_{\bowtie}(A)$ is closed under the operations $\vee$, $\wedge$, $\neg$ and $\Box$ of $\mathcal{A}$, hence $\mathcal{A}_{\bowtie} := \langle e_{\bowtie}(A), \vee, \wedge, \neg, \Box\rangle$ is a modal algebra;*

- *the mapping $\iota^{\bowtie}$ is an embedding of $\mathcal{A}$ into $(\mathcal{A}_{\bowtie})^{\bowtie}$, whence $\mathcal{A}$ is isomorphic to the twist-structure $\iota^{\bowtie}(\mathcal{A})$ over $\mathcal{A}_{\bowtie}$.*

Notice that if $\mathcal{A} \in S^{\bowtie}(\mathcal{B})$, then $e_{\bowtie}(A) = \{(a, \neg a) \mid a \in B\}$, and thus π_1 is an isomorphism from $\mathcal{A}_{\bowtie}$ onto $\mathcal{B}$.

Next we revise Theorem 1.3. Given a BK-lattice $\mathcal{A}$, take

$$\mathcal{D}^{\mathcal{A}} := \{a \in A \mid \neg a = \bot\}.$$

For a set $\Gamma \cup \{\varphi\}$ of $\mathcal{L}$-formulas, we write $\Gamma \models_{\langle \mathcal{A}, \mathcal{D}^{\mathcal{A}} \rangle} \varphi$ iff for each $\mathcal{A}$-valuation v, if $v(\psi) \in \mathcal{D}^{\mathcal{A}}$ for every $\psi \in \Gamma$, then $v(\varphi) \in \mathcal{D}^{\mathcal{A}}$. In particular, when $\mathcal{A} \in S^{\bowtie}(\mathcal{B})$, we easily get $\mathcal{D}^{\mathcal{A}} = \{(1, b) \mid b \in B\}$, whence $\Gamma \models_{\langle \mathcal{A}, \mathcal{D}^{\mathcal{A}} \rangle} \varphi$ turns out to be equivalent to $\Gamma \models_{\mathcal{A}} \varphi$ (cf. [24]). This quickly leads to

THEOREM 1.6 ([24]). *For every $\Gamma \cup \{\varphi\} \subseteq \mathrm{For}(\mathcal{L})$ with Γ non-trivial w.r.t.* BK, *we have*

$$\Gamma \vdash^{*}_{\mathsf{BK}} \varphi \quad \Longleftrightarrow \quad \Gamma \models_{\langle \mathcal{A}, \mathcal{D}^{\mathcal{A}} \rangle} \varphi \text{ for all } \mathcal{A} \in \mathcal{V}_{\mathsf{BK}}.$$

And similarly for B3K, *in which case we restrict ourselves to* BK*-lattices $\mathcal{A}$ such that $\neg(a \wedge {\sim} a) = \top$ for any $a \in A$.*

As was also proved in [24], the quasi-identity of Item 7 in Definition 1.4 can be replaced by identities. Consequently

THEOREM 1.7 ([24]). *$\mathcal{V}_{\mathsf{BK}}$ is a variety.*

As was shown in [24, Section 5], $\mathcal{V}_{\mathsf{BK}}$ in fact gives us an *equivalent algebraic semantics* for the global consequence relation $\vdash^{*}_{\mathsf{BK}}$, using $\neg p = \bot$ and $p \Leftrightarrow q$ as the *defining equation* and the *equivalence formula*, respectively — see [6] for the correspongingdefinitions. From this we deduce, using results from [6, Section 4], that the lattice of extensions of $\vdash^{*}_{\mathsf{BK}}$ is, in effect, dually isomorphic to that of sub-quasi-varieties of $\mathcal{V}_{\mathsf{BK}}$. In the axiomatic case, viz. for elements of $\mathcal{E}\mathsf{BK}$, the intended isomorphism works as follows. For every collection L of $\mathcal{L}$-formulas and every family V of BK-lattices, we take

$$\begin{aligned} \mathbb{V}(L) &:= \{\mathcal{A} \in \mathcal{V}_{\mathsf{BK}} \mid \neg\varphi = \bot \in \mathrm{Eq}(\mathcal{A}) \text{ for all } \varphi \in L\}, \\ \mathbb{L}(V) &:= \{\varphi \in \mathrm{For}(\mathcal{L}) \mid \neg\varphi = \bot \in \mathrm{Eq}(\mathcal{A}) \text{ for all } \mathcal{A} \in V\}. \end{aligned}$$

Notice that given $\mathcal{A} \in \mathcal{V}_{\mathsf{BK}}$ and $\varphi \in \mathrm{For}(\mathcal{L})$, we shall occasionally write $\mathcal{A} \models \varphi$ instead of $\neg\varphi = \bot \in \mathrm{Eq}(\mathcal{A})$, without danger of confusion; and similarly for classes of BK-lattices and sets of $\mathcal{L}$-formulas.[6]

THEOREM 1.8 ([24]). *$\mathbb{V}$ and $\mathbb{L}$ define two mutually inverse dual isomorphisms between $\mathcal{E}\mathsf{BK}$ and the lattice of subvarieties of $\mathcal{V}_{\mathsf{BK}}$.*

One very useful property deserves mention here.

PROPOSITION 1.9 ([32]). *The lattice $\mathcal{E}\mathsf{BK}$ is distributive.*

Given a twist-structure $\mathcal{A}$ over a modal algebra $\mathcal{B}$, take

$$\nabla(\mathcal{A}) \;:=\; \{a \vee b \mid (a,b) \in A\} \quad \text{and} \quad \Delta(\mathcal{A}) \;:=\; \{a \wedge b \mid (a,b) \in A\}.$$

The importance of these sets will become clear in a moment.

PROPOSITION 1.10 ([24]). *For each $\mathcal{A} \in S^{\bowtie}(\mathcal{B})$ we have*

$$\nabla(\mathcal{A}) \;=\; \{a \mid (a,0) \in A\} \quad \text{and} \quad \Delta(\mathcal{A}) \;=\; \{a \mid (1,a) \in A\},$$

whence $\nabla(\mathcal{A}) \in \mathscr{F}^{\Box}(\mathcal{B})$ and $\Delta(\mathcal{A}) \in \mathscr{I}^{\Diamond}(\mathcal{B})$; moreover

$$A \;=\; \{(a,b) \in B \times B \mid a \vee b \in \nabla(\mathcal{A}) \text{ and } a \wedge b \in \Delta(\mathcal{A})\}.$$

On the other hand, if $\nabla \in \mathscr{F}^{\Box}(\mathcal{B})$ and $\Delta \in \mathscr{I}^{\Diamond}(\mathcal{B})$, then

$$\{(a,b) \in B \times B \mid a \vee b \in \nabla \text{ and } a \wedge b \in \Delta\}$$

is closed under all twist-operations; in effect for $\mathcal{A} \in S^{\bowtie}(\mathcal{B})$ having this set as domain, $\nabla(\mathcal{A}) = \nabla$ and $\Delta(\mathcal{A}) = \Delta$.

Consequently $\mathcal{A}$ is uniquely determined by the triple $(\mathcal{B}, \nabla(\mathcal{A}), \Delta(\mathcal{A}))$ which we write as $\mathcal{A} = \mathrm{Tw}(\mathcal{B}, \nabla(\mathcal{A}), \Delta(\mathcal{A}))$.

A version for $\mathcal{V}_{\mathsf{BK}}$ was provided in [32]. Given a BK-lattice $\mathcal{A}$, take

$$\begin{aligned} \nabla_l(\mathcal{A}) &:= e_{\bowtie}(\{a \vee {\sim} a \mid a \in A\}), \\ \Delta_l(\mathcal{A}) &:= e_{\bowtie}(\{a \wedge {\sim} a \mid a \in A\}). \end{aligned}$$

(actually, $e_{\bowtie}$ plays the role of π_1 in the case of twist-structures).

PROPOSITION 1.11 ([32]). *Let $\mathcal{A} \in \mathcal{V}_{\mathsf{BK}}$. Then $\nabla_l(\mathcal{A}) \in \mathscr{F}^{\Box}(\mathcal{A}_{\bowtie})$ and $\Delta_l(\mathcal{A}) \in \mathscr{I}^{\Diamond}(\mathcal{A}_{\bowtie})$. Furthermore, $\iota^{\bowtie}(\mathcal{A}) = \mathrm{Tw}(\mathcal{A}_{\bowtie}, \nabla_l(\mathcal{A}), \Delta_l(\mathcal{A}))$.*

[6] However, for a modal algebra $\mathcal{B}$ and a formula ψ of its language, $\mathcal{B} \models \psi$ means that $\psi = 1$ holds in $\mathcal{B}$, i.e. $\psi = 1$ belongs to $\mathrm{Eq}(\mathcal{B})$.

2. Special Extensions

Logics from $\mathcal{E}\mathsf{B3K}$, $\mathcal{E}\mathsf{BK}^\circ$ and $\mathcal{E}\mathsf{B3K}^\circ$ are respectively called *explosive*, *complete* and *classical*. Obviously $\mathcal{E}\mathsf{B3K} \cap \mathcal{E}\mathsf{BK}^\circ = \mathcal{E}\mathsf{B3K}^\circ$.

PROPOSITION 2.1. 1. $\mathbb{V}(\mathsf{B3K})$ *coincides with* $\{\mathcal{A} \in \mathcal{V}_{\mathsf{BK}} \mid \Delta_l(\mathcal{A}) = \{\bot\}\}$.
2. $\mathbb{V}(\mathsf{BK}^\circ)$ *coincides with* $\{\mathcal{A} \in \mathcal{V}_{\mathsf{BK}} \mid \nabla_l(\mathcal{A}) = \{\top\}\}$.
3. $\mathbb{V}(\mathsf{B3K}^\circ)$ *coincides with* $\{\mathcal{A} \in \mathcal{V}_{\mathsf{BK}} \mid \Delta_l(\mathcal{A}) = \{\bot\}$ *and* $\nabla_l(\mathcal{A}) = \{\top\}\}$.

PROOF. Notice that by Proposition 1.11, for each $\mathcal{A} \in \mathcal{V}_{\mathsf{BK}}$ we have:

$$\begin{aligned}
\Delta_l(\mathcal{A}) &= \{\bot\} &\iff& \Delta(\iota^{\bowtie}(\mathcal{A})) = \{\bot_{\mathcal{A}}\} = \{0_{\mathcal{A}_{\bowtie}}\};\\
\nabla_l(\mathcal{A}) &= \{\top\} &\iff& \nabla(\iota^{\bowtie}(\mathcal{A})) = \{\top_{\mathcal{A}}\} = \{1_{\mathcal{A}_{\bowtie}}\}.
\end{aligned}$$

So it suffices to consider only $\mathcal{A} \in S^{\bowtie}(\mathcal{B})$ where $\mathcal{B}$ is a modal algebra.

Ad 1. We need to show that

$$\neg(\sim p \to (p \to q)) = \bot \in \mathrm{Eq}(\mathcal{A}) \iff \Delta(\mathcal{A}) = \{0\}.$$

Let (a, b) and (c, d) be elements of A. Then

$$\begin{aligned}
\neg(\sim(a, b) \to ((a, b) \to (c, d))) &= \neg((b, a) \to (\neg a \vee c, a \wedge d)) = \\
\neg(\neg b \vee \neg a \vee c, b \wedge a \wedge d) &= (b \wedge a \wedge \neg c, \neg b \vee \neg a \vee c),
\end{aligned}$$

which equals $(0, 1)$ iff $a \wedge b \wedge \neg c = 0$. Hence

$$\mathcal{A} \models \sim p \to (p \to q) \iff a \wedge b \wedge \neg c = 0 \text{ for all } (a, b) \in A \text{ and } c \in B,$$

i.e. iff $a \wedge b = 0$ for any $(a, b) \in A$.

Ad 2. Now we need to show that

$$\neg(p \vee \sim p) = \bot \in \mathrm{Eq}(\mathcal{A}) \iff \nabla(\mathcal{A}) = \{1\}.$$

Let (a, b) be an element of A. Then

$$\neg((a, b) \vee \sim(a, b)) = \neg(a \vee b, b \wedge a) = (\neg a \wedge \neg b, a \vee b),$$

which equals $(0, 1)$ iff $a \vee b = 1$. The rest is trivial.

Ad 3. It follows immediately from (1) and (2). ⊣

COROLLARY 2.2. *Let $L \in \mathcal{E}\mathsf{BK}$. The following equivalences hold:*

1. $\mathsf{B3K} \subseteq L$ *iff for every* $\mathcal{A} \in \mathbb{V}(L)$, $\Delta_l(\mathcal{A}) = \{\bot\}$;
2. $\mathsf{BK}^\circ \subseteq L$ *iff for every* $\mathcal{A} \in \mathbb{V}(L)$, $\nabla_l(\mathcal{A}) = \{\top\}$;
3. $\mathsf{B3K}^\circ \subseteq L$ *iff for every* $\mathcal{A} \in \mathbb{V}(L)$, $\Delta_l(\mathcal{A}) = \{\bot\}$ *and* $\nabla_l(\mathcal{A}) = \{\top\}$.

PROOF. By Theorem 1.8, for any $\{L_1, L_2\} \subset \mathcal{E}\mathsf{BK}$,

$$L_1 \subseteq L_2 \iff \mathbb{V}(L_1) \supseteq \mathbb{V}(L_2).$$

It remains to apply the previous proposition. ⊣

Recall that $\mathcal{E}\mathsf{K}$ is the lattice of normal modal logics in the language

$$\mathcal{L}^- = \{\vee, \wedge, \rightarrow, \bot, \Box, \Diamond\},$$

with least element K. Given a class $\mathcal{K}$ of modal algebras, define

$$\mathrm{L}(\mathcal{K}) := \{\varphi \in \mathrm{For}(\mathcal{L}^-) \mid \mathcal{K} \models \varphi\}.$$

Clearly there are natural ways to get elements of $\mathcal{E}\mathsf{BK}$ from elements of $\mathcal{E}\mathsf{K}$. For each $L \in \mathcal{E}\mathsf{K}$, let

$$\eta(L) := \mathsf{BK} + L, \quad \eta^3(L) := \mathsf{B3K} + L,$$
$$\eta^\circ(L) := \mathsf{BK}^\circ + L \quad \text{and} \quad \eta^{\mathsf{c}}(L) := \mathsf{B3K}^\circ + L.$$

Belnapian logics from $\eta(\mathcal{E}\mathsf{K})$, $\eta^3(\mathcal{E}\mathsf{K})$, $\eta^\circ(\mathcal{E}\mathsf{K})$ and $\eta^{\mathsf{c}}(\mathcal{E}\mathsf{K})$ are respectively called *special*, *special explosive*, *special complete* and *classical*. Indeed — as we shall see — these four play an important role in our study, because:

- $\eta(L)$ and $\eta^{\mathsf{c}}(L)$ are respectively the least and greatest conservative extensions of $L \in \mathcal{E}\mathsf{K}$ in the lattice $\mathcal{E}\mathsf{BK}$;
- at the same time $\eta^3(L)$ and $\eta^\circ(L)$ are the least conservative extensions of L in $\mathcal{E}\mathsf{B3K}$ and $\mathcal{E}\mathsf{BK}^\circ$ respectively.

But before going on to that, we need to look at $\mathcal{L}^-$-fragments of Belnapian modal logics. Given $L \in \mathcal{E}\mathsf{BK}$, define $\sigma(L)$ to be $L \cap \mathrm{For}(\mathcal{L}^-)$.

LEMMA 2.3. *For every* BK*-lattice* $\mathcal{A}$ *and every* $\mathcal{L}^-$*-formula* φ,

$$\mathcal{A} \models \varphi \iff \mathcal{A}_{\bowtie} \models \varphi.$$

PROOF. Since any BK-lattice $\mathcal{A}$ is isomorphic to the twist-structure $\iota^{\bowtie}(\mathcal{A})$ over $\mathcal{A}_{\bowtie}$, it suffices to show that for all $\mathcal{A} \in S^{\bowtie}(\mathcal{B})$,

$$\neg\varphi = \bot \in \mathrm{Eq}(\mathcal{A}) \iff \varphi = 1 \in \mathrm{Eq}(\mathcal{B}).$$

Suppose $\mathcal{B} \models \varphi = 1$. Each $\mathcal{A}$-valuation v induces the $\mathcal{B}$-valuation

$$v' : p \mapsto \pi_1(v(p)).$$

Certainly $v'(\varphi) = 1$. By Definition 1.2 it follows that $v(\neg\varphi) = (0, 1)$.

Suppose that $\mathcal{B} \not\models \varphi = 1$ — let v' be a $\mathcal{B}$-valuation for which $v'(\varphi) \neq 1$. Clearly there exists an $\mathcal{A}$-valuation v such that

$$v'(p) = \pi_1(v(p)) \text{ for all variables } p.$$

Then using Definition 1.2 we obtain $v(\neg\varphi) \neq (0, 1)$. ⊣

In other words, $\sigma(L)$ is completely determined by the underlying modal algebras of L-models. More precisely, given $\mathcal{K} \subseteq \mathcal{V}_{\mathsf{BK}}$, define

$$\mathcal{K}_{\bowtie} := \{\mathcal{A}_{\bowtie} \mid \mathcal{A} \in \mathcal{K}\}.$$

The previous lemma trivially implies

PROPOSITION 2.4. *Let L be a logic in $\mathcal{E}\mathsf{BK}$ and $\mathcal{K}$ a class of BK-lattices. If $L = \mathbb{L}(\mathcal{K})$, then $\sigma(L) = \mathrm{L}(\mathcal{K}_{\bowtie})$.*

We now turn to special Belnapian modal logics.

PROPOSITION 2.5. *For any $L \in \mathcal{E}\mathsf{K}$ and $\mathcal{A} \in \mathcal{V}_{\mathsf{BK}}$, the following hold:*

1. $\mathcal{A} \models \eta(L)$ *iff* $\mathcal{A}_{\bowtie} \models L$;
2. $\mathcal{A} \models \eta^3(L)$ *iff* $\mathcal{A}_{\bowtie} \models L$ *and* $\Delta_l(\mathcal{A}) = \{\bot\}$;
3. $\mathcal{A} \models \eta^{\circ}(L)$ *iff* $\mathcal{A}_{\bowtie} \models L$ *and* $\nabla_l(\mathcal{A}) = \{\top\}$;
4. $\mathcal{A} \models \eta^{\mathsf{c}}(L)$ *iff* $\mathcal{A}_{\bowtie} \models L$, $\Delta_l(\mathcal{A}) = \{\bot\}$ *and* $\nabla_l(\mathcal{A}) = \{\top\}$.

PROOF. *Ad* 1. This is by Lemma 2.3 — remembering that $\mathcal{V}_{\mathsf{BK}} \models \mathsf{BK}$.

Ad 2. Suppose $\mathcal{A} \models \eta^3(L)$. Then $\mathcal{A}_{\bowtie} \models L$ by Lemma 2.3. Moreover, since $\mathcal{A} \in \mathbb{V}(\mathsf{B3K})$, by Proposition 2.1 we get $\Delta_l(\mathcal{A}) = \{\bot\}$.

Suppose $\mathcal{A}_{\bowtie} \models L$ and $\Delta_l(\mathcal{A}) = \{\bot\}$. Then $\mathcal{A} \models L$ by Lemma 2.3. Also by Proposition 2.1 we get $\mathcal{A} \in \mathbb{V}(\mathsf{B3K})$, i.e. $\mathcal{A} \models \mathsf{B3K}$. Hence $\mathcal{A} \models \eta^3(L)$.

Ad 3. Similar to 2.

Ad 4. It follows immediately from 2 and 3. ⊣

With each modal algebra $\mathcal{B}$ we associate three twist-structures:

$$\mathcal{B}_3^{\bowtie} := \mathrm{Tw}(\mathcal{B}, B, \{0\}), \quad \mathcal{B}_\circ^{\bowtie} := \mathrm{Tw}(\mathcal{B}, \{1\}, B)$$
$$\text{and} \quad \mathcal{B}_\mathsf{c}^{\bowtie} := \mathrm{Tw}(\mathcal{B}, \{1\}, \{0\}),$$

i.e. the twist-structures over $\mathcal{B}$ with domains

$$B_3^{\bowtie} := \{(a,b) \in B^2 \mid a \wedge b = 0\}, \quad B_\circ^{\bowtie} := \{(a,b) \in B^2 \mid a \vee b = 1\}$$
$$\text{and} \quad B_\mathsf{c}^{\bowtie} := B_3^{\bowtie} \cap B_\circ^{\bowtie} = \{(a, \neg a) \mid a \in B\},$$

respectively. Using Proposition 2.1 it is easy to prove

PROPOSITION 2.6. *Let $\mathcal{A} \in S^{\bowtie}(\mathcal{B})$. We have the following:*

1. $\mathcal{B}_\mathsf{c}^{\bowtie} = \mathcal{A}_{\bowtie} \leq \mathcal{A} \leq \mathcal{B}^{\bowtie}$.
2. $\mathcal{A} \models \mathsf{B3K}$ *iff* $\mathcal{A} \leq \mathcal{B}_3^{\bowtie}$;
3. $\mathcal{A} \models \mathsf{BK}^\circ$ *iff* $\mathcal{A} \leq \mathcal{B}_\circ^{\bowtie}$;
4. $\mathcal{A} \models \mathsf{B3K}^\circ$ *iff* $\mathcal{A} = \mathcal{B}_\mathsf{c}^{\bowtie}$.

By analogy with the case of BK-extensions, we call a BK-lattice *special*, *special explosive*, *special complete* or *classical* iff it is isomorphic to a twist-structure of the form $\mathcal{B}^{\bowtie}$, $\mathcal{B}_3^{\bowtie}$, $\mathcal{B}_\circ^{\bowtie}$ or $\mathcal{B}_\mathsf{c}^{\bowtie}$ respectively.

PROPOSITION 2.7. *For each $L \in \mathcal{E}\mathsf{K}$, the corresponding logics $\eta(L)$, $\eta^3(L)$, $\eta^\circ(L)$ and $\eta^\mathsf{c}(L)$ are conservative extensions of L, i.e. belong to $\sigma^{-1}(L)$.*

PROOF. Evidently $L \subseteq \eta(L) \cap \mathrm{For}(\mathcal{L}^-)$. Furthermore,

$$\eta(L) \subseteq \eta^3(L) \cap \eta^\circ(L) \quad \text{and} \quad \eta^3(L) + \eta^\circ(L) = \eta^\mathsf{c}(L),$$

so we only need to check that for every $\varphi \in \mathrm{For}(\mathcal{L}^-)$,

$$\varphi \notin L \implies \varphi \notin \eta^\mathsf{c}(L).$$

Suppose $\varphi \in \mathrm{For}(\mathcal{L}^-) \setminus L$. Hence there exists a modal algebra $\mathcal{B}$ such that $\mathcal{B} \models L$ and $\mathcal{B} \not\models \varphi$. Now since $(\mathcal{B}_\mathsf{c}^{\bowtie})_{\bowtie}$ is isomorphic to $\mathcal{B}$, we have $\mathcal{B}_\mathsf{c}^{\bowtie} \not\models \varphi$ by Lemma 2.3, and $\mathcal{B}_\mathsf{c}^{\bowtie} \models \eta^\mathsf{c}(L)$ by Proposition 2.5. Thus $\varphi \notin \eta^\mathsf{c}(L)$. ⊣

Given an $\mathcal{L}$-formula $\varphi(p_1, ..., p_n)$, we can write its nnf $\overline{\varphi}(p_1, ..., p_n)$ as

$$\varphi'(p_1, \ldots, p_n, {\sim} p_1, \ldots, {\sim} p_n)$$

where $\varphi' \in \mathrm{For}(\mathcal{L}^-)$. Then we define:

$$\begin{aligned}
\varphi_{\bowtie} &:= \varphi'(p_1, \ldots, p_n, p_{n+1}, \ldots, p_{2n});\\
\varphi_{\bowtie}^{3} &:= \varphi'(p_1, \ldots, p_n, p_{n+1} \wedge \neg p_1, \ldots, p_{2n} \wedge \neg p_n);\\
\varphi_{\bowtie}^{\circ} &:= \varphi'(p_1, \ldots, p_n, p_{n+1} \vee \neg p_1, \ldots, p_{2n} \vee \neg p_n);\\
\varphi_{\bowtie}^{\mathsf{c}} &:= \varphi'(p_1, \ldots, p_n, \neg p_1, \ldots, \neg p_n).
\end{aligned}$$

Recall that whenever φ is an nnf, it coincides with $\overline{\varphi}$ by construction.

Proposition 2.8. *For any $\mathcal{L}$-formula $\varphi(p_1, \ldots, p_n)$ and any modal algebra $\mathcal{B}$, the following equivalences hold*:

1. $\mathcal{B}^{\bowtie} \models \varphi$ *iff* $\mathcal{B} \models \varphi_{\bowtie}$;
2. $\mathcal{B}_3^{\bowtie} \models \varphi$ *iff* $\mathcal{B} \models \varphi_{\bowtie}^3$;
3. $\mathcal{B}_{\circ}^{\bowtie} \models \varphi$ *iff* $\mathcal{B} \models \varphi_{\bowtie}^{\circ}$;
4. $\mathcal{B}_{\mathsf{c}}^{\bowtie} \models \varphi$ *iff* $\mathcal{B} \models \varphi_{\bowtie}^{\mathsf{c}}$.

Proof. Without loss of generality we may assume that φ is a nnf.

Ad 1. Suppose $\mathcal{B}^{\bowtie} \not\models \varphi$ — whence there exists a $\mathcal{B}^{\bowtie}$-valuation v for which $\pi_1(v(\varphi)) \neq 1$. Let $v_{\bowtie}$ be a $\mathcal{B}$-valuation such that for all $i \in \{1, \ldots, n\}$,

$$v_{\bowtie}(p_i) = \pi_1(v(p_i)) \quad \text{and} \quad v_{\bowtie}(p_{n+i}) = \pi_2(v(p_i)),$$

so in particular, $v_{\bowtie}(p_{n+i}) = \pi_1(v({\sim} p_i))$. Then, as can be readily verified, we have $v_{\bowtie}(\varphi_{\bowtie}) = \pi_1(v(\varphi)) \neq 1$. Thus $\mathcal{B} \not\models \varphi_{\bowtie}$.

Conversely, suppose $v(\varphi_{\bowtie}) \neq 1$ for some $\mathcal{B}$-valuation v. And let $v^{\bowtie}$ be a $\mathcal{B}^{\bowtie}$-valuation such that for all $i \in \{1, \ldots, n\}$,

$$v^{\bowtie}(p_i) = (v(p_i), v(p_{n+i})),$$

and so $\pi_1(v^{\bowtie}({\sim} p_i)) = v(p_{n+i})$. It is now straightforward to check that $\pi_1(v^{\bowtie}(\varphi)) = v(\varphi_{\bowtie}) \neq 1$. Hence $\mathcal{B}^{\bowtie} \not\models \varphi$.

Ad 2. Suppose there exists a $\mathcal{B}_3^{\bowtie}$-valuation v for which $\pi_1(v(\varphi)) \neq 1$. Let $v_{\bowtie}$ be as in (1). Since v is a $\mathcal{B}_3^{\bowtie}$-valuation, we have

$$\pi_1(v(p)) \wedge \pi_1(v({\sim} p)) = \pi_1(v(p)) \wedge \pi_2(v(p)) = 0,$$

i.e. $\pi_1(v({\sim} p)) \leqslant \pi_1(v(\neg p))$. Thus for every $i \in \{1, \ldots, n\}$,

$$\pi_1(v({\sim} p_i)) = \pi_1(v({\sim} p_i)) \wedge \pi_1(v(\neg p_i)) = v_{\bowtie}(p_{n+i} \wedge \neg p_i).$$

Furthermore, one can easily verify that $v_{\bowtie}(\varphi_{\bowtie}^3) = \pi_1(v(\varphi)) \neq 1$.

Conversely, suppose $v\left(\varphi^3_{\bowtie}\right) \neq 1$ for some $\mathcal{B}$-valuation v. Then let $v^{\bowtie}_3$ be a $\mathcal{B}^{\bowtie}$-valuation such that for all $i \in \{1, \ldots, n\}$,

$$v^{\bowtie}_3(p_i) \;=\; (v(p_i), v(p_{n+i}) \wedge \neg\, v(p_i)),$$

so in particular, $\pi_1(v^{\bowtie}_3(p_i)) \wedge \pi_2(v^{\bowtie}_3(p_i)) = 0$. Hence $v^{\bowtie}_3$ is a $\mathcal{B}^{\bowtie}_3$-valuation. It is straightforward to verify now that $\pi_1(v^{\bowtie}_3(\varphi)) = v\left(\varphi^3_{\bowtie}\right) \neq 1$.

Analogous arguments apply to (3) and (4). ⊣

COROLLARY 2.9. *For any $\varphi \in \mathrm{For}(\mathcal{L})$ and $L \in \mathcal{E}\mathsf{K}$, the following hold:*

1. $\varphi \in \eta(L)$ *iff* $\varphi_{\bowtie} \in L$;
2. $\varphi \in \eta^3(L)$ *iff* $\varphi^3_{\bowtie} \in L$;
3. $\varphi \in \eta^{\circ}(L)$ *iff* $\varphi^{\circ}_{\bowtie} \in L$;
4. $\varphi \in \eta^{\mathsf{c}}(L)$ *iff* $\varphi^{\mathsf{c}}_{\bowtie} \in L$.

PROOF. *Ad* 1. Suppose $\varphi \in \eta(L)$. Then for every modal algebra $\mathcal{B}$,

$$\mathcal{B} \models L \quad \overset{\text{Prop. 2.5}}{\Longrightarrow} \quad \mathcal{B}^{\bowtie} \models \eta(L) \quad \Longrightarrow \quad \mathcal{B}^{\bowtie} \models \varphi \quad \overset{\text{Prop. 2.8}}{\Longrightarrow} \quad \mathcal{B} \models \varphi_{\bowtie}.$$

Consequently $\varphi_{\bowtie}$ holds in all models of L, i.e. belongs to L.

Suppose $\varphi_{\bowtie} \in L$. Then for each BK-lattice $\mathcal{A}$,

$$\begin{aligned} \mathcal{A} \models \eta(L) \quad &\overset{\text{Prop. 2.5}}{\Longrightarrow} \quad \mathcal{A}_{\bowtie} \models L \quad \Longrightarrow \quad \mathcal{A}_{\bowtie} \models \varphi_{\bowtie} \\ &\overset{\text{Prop. 2.8}}{\Longrightarrow} \quad (\mathcal{A}_{\bowtie})^{\bowtie} \models \varphi \quad \Longrightarrow \quad \mathcal{A} \models \varphi, \end{aligned}$$

where the last implication is because $\mathcal{A}$ is isomorphic to a subalgebra of $(\mathcal{A}_{\bowtie})^{\bowtie}$, namely to $\iota^{\bowtie}(\mathcal{A})$. Thus φ belongs to $\eta(L)$.

Ad 2. Suppose $\varphi \in \eta^3(L)$. Then for every modal algebra $\mathcal{B}$,

$$\mathcal{B} \models L \quad \overset{\text{Prop. 2.5}}{\Longrightarrow} \quad \mathcal{B}^{\bowtie}_3 \models \eta^3(L) \quad \Longrightarrow \quad \mathcal{B}^{\bowtie}_3 \models \varphi \quad \overset{\text{Prop. 2.8}}{\Longrightarrow} \quad \mathcal{B} \models \varphi^3_{\bowtie}.$$

Consequently $\varphi^3_{\bowtie}$ holds in all models of L.

Suppose $\varphi^3_{\bowtie} \in L$. Then for each BK-lattice $\mathcal{A}$,

$$\begin{aligned} \mathcal{A} \models \eta^3(L) \quad &\overset{\text{Prop. 2.5}}{\Longrightarrow} \quad \mathcal{A}_{\bowtie} \models L \quad \Longrightarrow \quad \mathcal{A}_{\bowtie} \models \varphi^3_{\bowtie} \\ &\overset{\text{Prop. 2.8}}{\Longrightarrow} \quad (\mathcal{A}_{\bowtie})^{\bowtie}_3 \models \varphi \quad \overset{\text{Prop. 2.6}}{\Longrightarrow} \quad \mathcal{A} \models \varphi \end{aligned}$$

(for the last implication, notice that by Proposition 2.6, $\iota^{\bowtie}(\mathcal{A})$ is a subalgebra of $(\mathcal{A}_{\bowtie})^{\bowtie}_3$ whenever $\mathcal{A} \models \mathsf{B3K}$). Thus φ belongs to $\eta^3(L)$.

Analogous arguments apply to (3) and (4). ⊣

Given a class $\mathcal{K}$ of modal algebras, we define

$$\begin{aligned} \mathcal{K}^{\bowtie} &:= \{\mathcal{A}^{\bowtie} \mid \mathcal{A} \in \mathcal{K}\}, & \mathcal{K}_3^{\bowtie} &:= \{\mathcal{A}_3^{\bowtie} \mid \mathcal{A} \in \mathcal{K}\}, \\ \mathcal{K}_\circ^{\bowtie} &:= \{\mathcal{A}_\circ^{\bowtie} \mid \mathcal{A} \in \mathcal{K}\}, & \mathcal{K}_{\mathsf{c}}^{\bowtie} &:= \{\mathcal{A}_{\mathsf{c}}^{\bowtie} \mid \mathcal{A} \in \mathcal{K}\}. \end{aligned}$$

It is now easy to get results reminiscent of Proposition 2.4.

PROPOSITION 2.10. *Let L be a logic in $\mathcal{E}\mathsf{K}$ and $\mathcal{K}$ a class of modal algebras such that $L = \mathrm{L}(\mathcal{K})$. Then*

$$\begin{aligned} \eta(L) &= \mathbb{L}(\mathcal{K}^{\bowtie}), & \eta^3(L) &= \mathbb{L}(\mathcal{K}_3^{\bowtie}), \\ \eta^\circ(L) &= \mathbb{L}(\mathcal{K}_\circ^{\bowtie}), & \eta^{\mathsf{c}}(L) &= \mathbb{L}(\mathcal{K}_{\mathsf{c}}^{\bowtie}). \end{aligned}$$

PROOF. We shall only consider $\eta^\circ(L) = \mathbb{L}(\mathcal{K}_\circ^{\bowtie})$. Perfectly analogous arguments work for the other three equalities.

By Proposition 2.5, $\mathcal{K} \models L$ implies $\mathcal{K}_\circ^{\bowtie} \models \eta^\circ(L)$ — and so the inclusion $\eta^\circ(L) \subseteq \mathbb{L}(\mathcal{K}_\circ^{\bowtie})$ follows.

In the opposite direction, suppose $\varphi \notin \eta^\circ(L)$. By Corollary 2.9 we have $\varphi_{\bowtie}^\circ \notin L$, i.e. there exists $\mathcal{B} \in \mathcal{K}$ such that $\mathcal{B} \not\models \varphi_{\bowtie}^\circ$ — and hence $\mathcal{B}_\circ^{\bowtie} \not\models \varphi$ by Proposition 2.8. Thus $\varphi \notin \mathbb{L}(\mathcal{K}_\circ^{\bowtie})$. ⊣

Actually, we have shown that a logic in $\mathcal{E}\mathsf{BK}$ has the form $\eta(L)$, $\eta^3(L)$, $\eta^\circ(L)$ or $\eta^{\mathsf{c}}(L)$ iff it can be characterised by a family of twist-structures of the appropriate form (or their isomorphic copies). More precisely:

COROLLARY 2.11. *Let $L \in \mathcal{E}\mathsf{BK}$. The following equivalences hold*:

1. *L is special iff $L = \mathbb{L}(\mathcal{K})$ for some class $\mathcal{K}$ of special* BK*-lattices.*
2. *L is special explosive iff $L = \mathbb{L}(\mathcal{K})$ for some class $\mathcal{K}$ of special explosive* BK*-lattices.*
3. *L is special complete iff $L = \mathbb{L}(\mathcal{K})$ for some class $\mathcal{K}$ of special complete* BK*-lattices.*
4. *L is classical iff $L = \mathbb{L}(\mathcal{K})$ for some class $\mathcal{K}$ of classical* BK*-lattices.*

PROOF. From left to right, apply Proposition 2.10.

For right to left we shall only consider (2), because almost the same argument works for the other cases. Assume that $L = \mathbb{L}(\mathcal{K})$ for some class $\mathcal{K}$ of special explosive BK-lattices. Then, as can be easily verified, L coincides with $\mathbb{L}\left((\mathcal{K}_{\bowtie})_3^{\bowtie}\right)$. Thus $L = \eta^3(\sigma(L))$ by Propositions 2.4 and 2.10. ⊣

We are now ready to demonstrate the role $\eta(L)$, $\eta^3(L)$, $\eta^\circ(L)$ and $\eta^{\mathsf{c}}(L)$ play in the study of conservative extensions of L.

Proposition 2.12. *For each $L \in \mathcal{E}\mathsf{K}$, the following hold*:

1. $\sigma^{-1}(L) = [\eta(L), \eta^{\mathsf{c}}(L)]$;
2. $\sigma^{-1}(L) \cap \mathcal{E}\mathsf{B3K} = [\eta^3(L), \eta^{\mathsf{c}}(L)]$;
3. $\sigma^{-1}(L) \cap \mathcal{E}\mathsf{BK}^\circ = [\eta^\circ(L), \eta^{\mathsf{c}}(L)]$;
4. $\sigma^{-1}(L) \cap \mathcal{E}\mathsf{B3K}^\circ = \{\eta^{\mathsf{c}}(L)\}$.

Proof. Obviously we deal with intervals: for any $\{L_1, L_2, L_3\} \subseteq \mathcal{E}\mathsf{BK}$,

$$L_1 \subseteq L_2 \subseteq L_3 \text{ and } \{L_1, L_3\} \subseteq \sigma^{-1}(L) \implies L_2 \in \sigma^{-1}(L).$$

Moreover $\{\eta(L), \eta^3(L), \eta^\circ(L), \eta^{\mathsf{c}}(L)\} \subseteq \sigma^{-1}(L)$ by Proposition 2.7.

Ad 1. Clearly $[\eta(L), \eta^{\mathsf{c}}(L)] \subseteq \sigma^{-1}(L)$, thus we need to show the converse. Let $L' \in \mathcal{E}\mathsf{BK}$ be such that $\sigma(L') = L$. For each BK-lattice $\mathcal{A}$,

$$\mathcal{A} \models L' \overset{\text{Lemma 2.3}}{\implies} \mathcal{A}_{\bowtie} \models L \overset{\text{Prop. 2.5}}{\implies} \mathcal{A} \models \eta(L).$$

This gives $\eta(L) \subseteq L'$, hence $\eta(L)$ is the infimum of $\sigma^{-1}(L)$ in $\mathcal{E}\mathsf{BK}$. On the other hand, for every BK-lattice $\mathcal{A}$, $(\mathcal{A}_{\bowtie})^{\bowtie}_{\mathsf{c}}$ is a subalgebra of $\iota^{\bowtie}(\mathcal{A})$ by Proposition 2.6, and consequently is embeddable in $\mathcal{A}$, so

$$\mathcal{A} \models L' \implies (\mathcal{A}_{\bowtie})^{\bowtie}_{\mathsf{c}} \models L'.$$

Take $\mathcal{K} = \{\mathcal{A}_{\bowtie} \mid \mathcal{A} \in \mathcal{V}_{\mathsf{BK}} \text{ and } \mathcal{A} \models L'\}$. Clearly $\mathcal{K}^{\bowtie}_{\mathsf{c}} \models L'$, i.e. $L' \subseteq \mathbb{L}(\mathcal{K}^{\bowtie}_{\mathsf{c}})$. By Proposition 2.4, $L = \mathrm{L}(\mathcal{K})$ — which in turn implies $\eta^{\mathsf{c}}(L) = \mathbb{L}(\mathcal{K}^{\bowtie}_{\mathsf{c}})$ by Proposition 2.10. This gives $L' \subseteq \eta^{\mathsf{c}}(L)$, hence $\eta^{\mathsf{c}}(L)$ is the supremum.

It is now straightforward to verify (2), (3) and (4). $\dashv$

Corollary 2.9 readily implies the admissibility of certain rules in certain logics. E.g. for every $\mathcal{L}$-formula we have

$$\varphi \in \eta(L) \implies \varphi_{\bowtie} \in \eta(L),$$

i.e. the rule $\varphi / \varphi_{\bowtie}$ is admissible in $\eta(L)$. As we shall shortly see, such rules can be used to characterise $\eta(\mathcal{E}\mathsf{K})$, $\eta^3(\mathcal{E}\mathsf{K})$, $\eta^\circ(\mathcal{E}\mathsf{K})$ and $\eta^{\mathsf{c}}(\mathcal{E}\mathsf{K})$.[7]

Proposition 2.13. *Let $L \in \mathcal{E}\mathsf{BK}$. The following equivalences hold*:

1. *L is special iff the rule $\varphi / \varphi_{\bowtie}$ is admissible in L.*
2. *L is special explosive iff the rule $\varphi / \varphi^3_{\bowtie}$ is admissible in L.*

[7] For the first time this kind of characterisation appeared in [14], where it was obtained for special extensions of Nelson's explosive logic.

3. *L is special complete iff the rule $\varphi/\varphi^{\circ}_{\bowtie}$ is admissible in L.*
4. *L is classical iff the rule $\varphi/\varphi^{\mathsf{c}}_{\bowtie}$ is admissible in L.*

Proof. Before presenting the main proof, we establish

Lemma 2.14. 1. *The rule $\varphi_{\bowtie}/\varphi$ is admissible in every $L \in \mathcal{E}\mathsf{BK}$.*
2. *The rule $\varphi^{3}_{\bowtie}/\varphi$ is admissible in every $L \in \mathcal{E}\mathsf{B3K}$.*
3. *The rule $\varphi^{\circ}_{\bowtie}/\varphi$ is admissible in every $L \in \mathcal{E}\mathsf{BK}^{\circ}$.*
4. *The rule $\varphi^{\mathsf{c}}_{\bowtie}/\varphi$ is admissible in every $L \in \mathcal{E}\mathsf{B3K}^{\circ}$.*

Proof. We shall only consider (3). Similar arguments apply to (1), (2) and (4), and are omitted.

Ad 3. Fix $L \in \mathcal{E}\mathsf{BK}^{\circ}$. Let $\mathcal{A} \in \mathcal{V}_{\mathsf{BK}}$ be such that $\mathcal{A} \models L$. We have

$$\mathcal{A} \models \varphi^{\circ}_{\bowtie} \overset{\text{Lemma 2.3}}{\Longrightarrow} \mathcal{A}_{\bowtie} \models \varphi^{\circ}_{\bowtie} \overset{\text{Prop. 2.8}}{\Longrightarrow} (\mathcal{A}_{\bowtie})^{\bowtie}_{\circ} \models \varphi \overset{\text{Prop. 2.8}}{\Longrightarrow} \mathcal{A} \models \varphi$$

(for the last implication, notice that by Proposition 2.6, $\iota^{\bowtie}(\mathcal{A})$ is a subalgebra of $(\mathcal{A}_{\bowtie})^{\circ}_{3}$ whenever $\mathcal{A} \models \mathsf{BK}^{\circ}$). So if $\varphi^{\circ}_{\bowtie} \in L$, then $\varphi \in L$. ⊣

As for the proposition, we shall only consider (2). In fact perfectly analogous arguments work for (1), (3) and (4).

By Corollary 2.9 the rule $\varphi/\varphi^{3}_{\bowtie}$ is admissible in each $L \in \eta^{3}(\mathcal{E}\mathsf{K})$.

Now suppose $\varphi/\varphi^{3}_{\bowtie}$ is admissible in $L \in \mathcal{E}\mathsf{BK}$. Then, using Lemma 2.14 and Corollary 2.9, we obtain that for every $\mathcal{L}$-formula φ,

$$\varphi \in L \iff \varphi^{3}_{\bowtie} \in L \iff \varphi^{3}_{\bowtie} \in \sigma(L) \iff \varphi \in \eta^{3}(\sigma(L)).$$

Thus $L = \eta^{3}(\sigma(L))$, so L is special explosive. ⊣

Proposition 2.15. 1. *σ is a lattice epimorphism from $\mathcal{E}\mathsf{BK}$ onto $\mathcal{E}\mathsf{K}$ commuting with infinite meets and joins.*
2. *η, η^{3} and η° are lattice monomorphism from $\mathcal{E}\mathsf{K}$ to $\mathcal{E}\mathsf{BK}$, $\mathcal{E}\mathsf{B3K}$ and $\mathcal{E}\mathsf{B3K}^{\circ}$ respectively, which commute with infinite meets and joins.*
3. *η^{c} is a lattice isomorphism between $\mathcal{E}\mathsf{K}$ and $\mathcal{E}\mathsf{B3K}^{\circ}$.*

Proof. *Ad* 1. By Proposition 2.7, σ is onto $\mathcal{E}\mathsf{K}$. Obviously σ commutes with infinite meets. We now turn to infinite joins. Let $\{L_i \mid i \in I\} \subseteq \mathcal{E}\mathsf{BK}$ where I is a non-empty set. Evidently

$$\textstyle\sum_{i\in I} \sigma(L_i) \subseteq \sigma\left(\sum_{i\in I} L_i\right);$$

thus it suffices to establish the reverse inclusion. Note that by Proposition 2.12, $L_i \subseteq \eta^{\mathsf{c}}(\sigma(L_i))$. So for each modal algebra $\mathcal{B}$,

$$\mathcal{B} \models \textstyle\sum_{i \in I} \sigma(L_i) \quad \Longrightarrow \quad \mathcal{B} \models \sigma(L_i) \text{ for all } i \in I$$
$$\overset{\text{Prop. 2.5}}{\Longrightarrow} \quad \mathcal{B}^{\bowtie}_{\mathsf{c}} \models \eta^{\mathsf{c}}(\sigma(L_i)) \text{ for all } i \in I \quad \Longrightarrow \quad \mathcal{B}^{\bowtie}_{\mathsf{c}} \models L_i \text{ for all } i \in I$$
$$\Longrightarrow \quad \mathcal{B}^{\bowtie}_{\mathsf{c}} \models \textstyle\sum_{i \in I} L_i \quad \overset{\text{Lemma 2.3}}{\Longrightarrow} \quad \mathcal{B} \models \sigma\left(\textstyle\sum_{i \in I} L_i\right).$$

This gives the desired equality.

Ad 2. Clearly η, η^3 and η° are one-one, by Proposition 2.7; and they commute with infinite joins, by definition. Turning to infinite meets — we shall now only consider η. Actually, perfectly analogous arguments work for the other mappings. Let $\{L_i \mid i \in I\} \subseteq \mathcal{E}\mathsf{K}$ where I is a nonempty set. Take

$$L^* \;:=\; \textstyle\bigcap_{i \in I} L_i \quad \text{and} \quad L' \;:=\; \textstyle\bigcap_{i \in I} \eta(L_i).$$

Observe that since $L_i = \sigma(\eta(L_i))$ by Proposition 2.7, we have

$$L^* \;=\; \textstyle\bigcap_{i \in I} \sigma(\eta(L_i)) \;=\; \sigma(L').$$

Further, by Proposition 2.13 the rule $\varphi / \varphi_{\bowtie}$ is admissible in any $\eta(L_i)$, and hence in L'. Thus L' is special, by the same proposition. Consequently

$$L' \;=\; \eta(\sigma(L')) \;=\; \eta(L^*)$$

(remember Proposition 2.7 to get the first equality).

Ad 3. Using Corollary 2.11 and Proposition 2.6 (for both see (4)), we can easily show that η^{c} is onto $\mathcal{E}\mathsf{B3K}^\circ$. The rest follows as in (2). $\dashv$

In particular, the lattice of all Belnapian modal logics can be viewed as a union of pairwise disjoint intervals of the form $\sigma^{-1}(L)$, viz.

$$\mathcal{E}\mathsf{BK} \;=\; \textstyle\bigcup_{L \in \mathcal{E}\mathsf{K}} [\eta(L), \eta^{\mathsf{c}}(L)];$$

and $\mathcal{E}\mathsf{BK}$ contains an isomorphic copy of $\mathcal{E}\mathsf{K}$, via η^{c}.

3. Counterparts

For convenience we introduce the following notation:

$$\mathsf{Exp} \;:=\; \mathcal{E}\mathsf{B3K}, \quad \mathsf{Com} \;:=\; \mathcal{E}\mathsf{BK}^\circ, \quad \mathsf{Clas} \;:=\; \mathcal{E}\mathsf{B3K}^\circ$$
$$\text{and} \quad \mathsf{Gen} \;:=\; \mathcal{E}\mathsf{BK} \setminus (\mathcal{E}\mathsf{B3K} \cup \mathcal{E}\mathsf{BK}^\circ).$$

Remark. $\mathsf{Clas} = \mathsf{Exp} \cap \mathsf{Com} = \eta^{\mathsf{c}}(\mathcal{E}\mathsf{K})$ by Proposition 2.15(3). As expected, Belnapian logics from Exp, Com and Clas are said to be *explosive*, *complete* and *classical* respectively. Finally by *logics of general form* we mean those which belong to Gen.

As we shall see, $\mathcal{E}\mathsf{BK}$ can be decomposed into classes of such logics in a way very similar to two decompositions that appeared in [22] — one of the lattice of extensions of Nelson's paraconsistent logic $\mathsf{N4}^{\perp}$, with constant $\perp$, into subclasses of explosive logics, normal logics and those of general form, and one of the lattice of nontrivial extensions of Johansson's minimal logic into classes of intermediate, negative and properly paraconsistent logics.

In the case of $\mathcal{E}\mathsf{BK}$ we shall exploit the mappings $(\cdot)_{\mathtt{exp}} : \mathcal{E}\mathsf{BK} \to \mathcal{E}\mathsf{B3K}$, $(\cdot)_{\mathtt{com}} : \mathcal{E}\mathsf{BK} \to \mathcal{E}\mathsf{BK}^{\circ}$ and $(\cdot)_{\mathtt{cl}} : \mathcal{E}\mathsf{BK} \to \mathcal{E}\mathsf{B3K}^{\circ}$ given by

$$L_{\mathtt{exp}} := L + \mathsf{B3K}, \quad L_{\mathtt{com}} := L + \mathsf{BK}^{\circ} \quad \text{and} \quad L_{\mathtt{cl}} := L + \mathsf{B3K}^{\circ},$$

where L ranges over elements of $\mathcal{E}\mathsf{BK}$. Obviously $L_{\mathtt{cl}} = L_{\mathtt{exp}} + L_{\mathtt{com}}$.

PROPOSITION 3.1. *$(\cdot)_{\mathtt{exp}}$, $(\cdot)_{\mathtt{com}}$, and $(\cdot)_{\mathtt{cl}}$ are lattice epimorphisms.*

PROOF. By Proposition 1.9, they are homomorphisms. The rest is easy. ⊣

For each L in $\mathcal{E}\mathsf{BK}$, the logics $L_{\mathtt{exp}}$, $L_{\mathtt{com}}$, and $L_{\mathtt{cl}}$ are respectively called the *explosive*, *complete* and *classical counterparts of* L. E.g.

$$\mathsf{BK}_{\mathtt{exp}} = \mathsf{B3K}, \quad \mathsf{BK}_{\mathtt{com}} = \mathsf{BK}^{\circ} \quad \text{and} \quad \mathsf{BK}_{\mathtt{cl}} = \mathsf{B3K}^{\circ}.$$

We collect some basic facts about counterparts in

PROPOSITION 3.2. *For any $L \in \mathcal{E}\mathsf{BK}$, the following hold:*

1. $L \in \mathsf{Exp}$ *iff* $L = L_{\mathtt{exp}}$ *iff* $L_{\mathtt{com}} = L_{\mathtt{cl}}$;
2. $L \in \mathsf{Com}$ *iff* $L = L_{\mathtt{com}}$ *iff* $L_{\mathtt{exp}} = L_{\mathtt{cl}}$;
3. $L \in \mathsf{Clas}$ *iff* $L = L_{\mathtt{cl}}$ *iff* $L_{\mathtt{exp}} = L_{\mathtt{com}}$;
4. $\sigma(L) = \sigma(L_{\mathtt{exp}}) = \sigma(L_{\mathtt{com}}) = \sigma(L_{\mathtt{cl}})$;
5. $L_{\mathtt{cl}} = \eta^{\mathsf{c}}(\sigma(L))$.

PROOF. *Ad* 1. Clearly since $L_{\mathtt{exp}}$ is the least logic in Exp which extends L, and $\mathsf{B3K} + \mathsf{BK}^{\circ} = \mathsf{B3K}^{\circ}$, we have

$$L \in \mathsf{Exp} \ \Leftrightarrow \ L = L_{\mathtt{exp}} \quad \text{and} \quad L = L_{\mathtt{exp}} \ \Rightarrow \ L_{\mathtt{com}} = L_{\mathtt{cl}}.$$

Hence it remains to show that $L_{\mathtt{com}} = L_{\mathtt{cl}}$ implies $L \in \mathsf{Exp}$. Assume $\mathsf{B3K} \not\subseteq L$. So by Corollary 2.2 (keeping in mind Proposition 1.5) there exists some twist-structure $\mathcal{A} = \mathrm{Tw}\,(\mathcal{B}, \nabla, \Delta)$ with $\Delta \neq \{0\}$ such that $\mathcal{A} \models L$. We take $\mathcal{A}'$ to be $\mathrm{Tw}\,(\mathcal{B}, \{1\}, \Delta)$. Obviously $\mathcal{A}'$, being a subalgebra of $\mathcal{A}$, also satisfies L. Then by Proposition 2.1, $\mathcal{A}' \models L_{\mathtt{com}}$ but $\mathcal{A}' \not\models L_{\mathtt{cl}}$.

Ad 2. Similar to (1).

Ad 3. This follows easily from (1) and (2).

Ad 4. Remember that $\mathsf{B3K}$, BK° and $\mathsf{B3K}^{\circ}$ are conservative extensions of K, by Proposition 2.7; and σ is a homomorphism, by Proposition 2.15(1).

Ad 5. Clearly $L_{\mathtt{cl}}$ and $\eta^{\mathsf{c}}\sigma\,(L)$ belong to $\mathsf{B3K}^{\circ}$. Furthermore, their images under σ coincide, by (4). Observe that by Proposition 2.12(4), σ restricted to $\mathsf{B3K}^{\circ}$ is an isomorphism, which implies the desired equality. ⊣

Here is a simple semantic characterisation of counterparts.

Proposition 3.3. *For any $L \in \mathcal{E}\mathsf{BK}$ and $\mathcal{A} \in \mathcal{V}_{\mathsf{BK}}$, the following hold*:

1. $\mathcal{A} \models L_{\mathtt{exp}}$ *iff* $\mathcal{A} \models L$ *and* $\Delta_l\,(\mathcal{A}) = \{\bot\}$.
2. $\mathcal{A} \models L_{\mathtt{com}}$ *iff* $\mathcal{A} \models L$ *and* $\nabla_l\,(\mathcal{A}) = \{\top\}$.
3. $\mathcal{A} \models L_{\mathtt{cl}}$ *iff* $\mathcal{A} \models L$, $\Delta_l\,(\mathcal{A}) = \{\bot\}$ *and* $\nabla_l\,(\mathcal{A}) = \{\top\}$.

Proof. Immediate from Proposition 2.1. ⊣

Given $\varphi\,(p_1, \ldots, p_n) \in \mathrm{For}(\mathcal{L})$, we represent its nnf $\overline{\varphi}\,(p_1, \ldots, p_n)$ as

$$\varphi'\,(p_1, \ldots, p_n, \sim p_1, \ldots, \sim p_n)$$

with $\varphi' \in \mathrm{For}(\mathcal{L}^-)$, and then define:

$$\begin{aligned} \varphi_{\mathtt{exp}} &:= \varphi'\,(p_1, \ldots, p_n, \sim p_1 \wedge \neg p_1, \ldots, \sim p_n \wedge \neg p_n); \\ \varphi_{\mathtt{com}} &:= \varphi'\,(p_1, \ldots, p_n, \sim p_1 \vee \neg p_1, \ldots, \sim p_n \vee \neg p_n); \\ \varphi_{\mathtt{cl}} &:= \varphi'\,(p_1, \ldots, p_n, \neg p_1, \ldots, \neg p_n). \end{aligned}$$

With every twist-structure $\mathcal{A} = \mathrm{Tw}\,(\mathcal{B}, \nabla, \Delta)$ (where $\mathcal{B}$ stands for some modal algebra, as before) we associate three substructures:

$$\mathcal{A}_{\mathtt{exp}} := \mathrm{Tw}\,(\mathcal{B}, \nabla, \{0\}), \quad \mathcal{A}_{\mathtt{com}} := \mathrm{Tw}\,(\mathcal{B}, \{1\}, \Delta)$$
$$\text{and} \quad \mathcal{A}_{\mathtt{cl}} := \mathrm{Tw}\,(\mathcal{B}, \{1\}, \{0\}).$$

Clearly, according to Proposition 3.3, if $\mathcal{A}$ is a model of L, then $\mathcal{A}_{\mathtt{exp}}$, $\mathcal{A}_{\mathtt{com}}$ and $\mathcal{A}_{\mathtt{cl}}$ are models of the corresponding counterparts of L. Also the truth of formulas on $\mathcal{A}_{\mathtt{exp}}$, $\mathcal{A}_{\mathtt{com}}$ and $\mathcal{A}_{\mathtt{cl}}$ can be naturally simulated in $\mathcal{A}$ via the above translations.

PROPOSITION 3.4. *For each $\mathcal{L}$-formula $\varphi(p_1,\dots,p_n)$ and each twist-structure $\mathcal{A}$ over some modal algebra, the following equivalences hold*:

1. $\mathcal{A} \models \varphi_{\mathtt{exp}}$ *iff* $\mathcal{A}_{\mathtt{exp}} \models \varphi$;
2. $\mathcal{A} \models \varphi_{\mathtt{com}}$ *iff* $\mathcal{A}_{\mathtt{com}} \models \varphi$;
3. $\mathcal{A} \models \varphi_{\mathtt{cl}}$ *iff* $\mathcal{A}_{\mathtt{cl}} \models \varphi$.

PROOF. Without loss of generality we may assume that φ is a nnf.

Ad 1. Suppose that $\mathcal{A} \models \varphi_{\mathtt{exp}}$. Let v be an $\mathcal{A}_{\mathtt{exp}}$-valuation. Then (remembering Proposition 1.10) for all $i \in \{1,\dots,n\}$ we have

$$\pi_1(v(p_i)) \wedge \pi_1(v(\sim p_i)) \;=\; \pi_1(v(p_i)) \wedge \pi_2(v(p_i)) \;=\; 0,$$

i.e. $\pi_1(v(\sim p_i)) \leq \pi_1(v(\neg p_i))$. Thus for every $i \in \{1,\dots,n\}$,

$$\pi_1(v(\sim p_i)) \;=\; \pi_1(v(\sim p_i)) \wedge \pi_1(v(\neg p_i)) \;=\; \pi_1(v(\sim p_i \wedge \neg p_i)).$$

Moreover, one readily verifies that $\pi_1(v(\varphi))$ equals $\pi_1(v(\varphi_{\mathtt{exp}}))$, which must be 1, because v is also an $\mathcal{A}$-valuation.

Conversely, suppose $\mathcal{A}_{\mathtt{exp}} \models \varphi$. Let v be an $\mathcal{A}$-valuation. Then consider v' such that for all $i \in \{1,\dots,n\}$,

$$v'(p_i) \;:=\; (\pi_1(v(p_i)), \pi_2(v(p_i)) \wedge \neg\pi_1(v(p_i))).$$

Obviously $\pi_1(v'(p_i)) \wedge \pi_2(v'(p_i)) = 0$, hence v' is an $\mathcal{A}_{\mathtt{exp}}$-valuation, so we have $\pi_1(v'(\varphi)) = 1$. By construction, for every $i \in \{1,\dots,n\}$,

$$\pi_1(v(\sim p_i \wedge \neg p_i)) \;=\; \pi_1(v(\sim p_i)) \wedge \neg\pi_1(v(p_i)) \;=\; \pi_1(v'(\sim p_i)).$$

Now one easily checks that $\pi_1(v(\varphi_{\mathtt{exp}}))$ equals $\pi_1(v'(\varphi))$, i.e. must be 1.

Analogous arguments apply to (2) and (3). ⊣

We are ready to embed $L_{\mathtt{exp}}$, $L_{\mathtt{com}}$ and $L_{\mathtt{cl}}$ into L.

PROPOSITION 3.5. *For any $\varphi \in \mathrm{For}(\mathcal{L})$ and $L \in \mathcal{E}\mathsf{BK}$, the following hold*:

1. $\varphi \in L_{\mathtt{exp}}$ *iff* $\varphi_{\mathtt{exp}} \in L$;
2. $\varphi \in L_{\mathtt{com}}$ *iff* $\varphi_{\mathtt{com}} \in L$;
3. $\varphi \in L_{\mathtt{cl}}$ *iff* $\varphi_{\mathtt{cl}} \in L$.

PROOF. *Ad* 1. Suppose $\varphi \in L_{\mathtt{exp}}$. Then for each twist-structure $\mathcal{A}$,

$$\mathcal{A} \models L \overset{\text{Prop. 3.3}}{\Longrightarrow} \mathcal{A}_{\mathtt{exp}} \models L_{\mathtt{exp}} \Longrightarrow \mathcal{A}_{\mathtt{exp}} \models \varphi \overset{\text{Prop. 3.4}}{\Longrightarrow} \mathcal{A} \models \varphi_{\mathtt{exp}}.$$

So — remembering Proposition 1.5 — $\varphi_{\mathtt{exp}}$ holds in all models of L.

Suppose $\varphi_{\mathtt{exp}} \in L$. Then for every twist-structure $\mathcal{A}$,

$$\mathcal{A} \models L_{\mathtt{exp}} \quad \overset{\text{Prop. 3.3}}{\Longrightarrow} \quad \mathcal{A} \models L \text{ and } \Delta(\mathcal{A}) = \{0\}$$
$$\Longrightarrow \quad \mathcal{A} \models \varphi_{\mathtt{exp}} \text{ and } \mathcal{A} = \mathcal{A}_{\mathtt{exp}} \quad \overset{\text{Prop. 3.4}}{\Longrightarrow} \quad \mathcal{A} \models \varphi.$$

Consequently φ belongs to $L_{\mathtt{exp}}$.

Analogous arguments apply to (2) and (3). ⊣

Given a class $\mathcal{K}$ of twist-structures over modal algebras, we define

$$\mathcal{K}_{\mathtt{exp}} := \{\mathcal{A}_{\mathtt{exp}} \mid \mathcal{A} \in \mathcal{K}\}, \quad \mathcal{K}_{\mathtt{com}} := \{\mathcal{A}_{\mathtt{com}} \mid \mathcal{A} \in \mathcal{K}\}$$
$$\text{and} \quad \mathcal{K}_{\mathtt{cl}} := \{\mathcal{A}_{\mathtt{cl}} \mid \mathcal{A} \in \mathcal{K}\}.$$

Obviously $\mathcal{K}_{\mathtt{cl}} = \mathcal{K}_{\mathtt{exp}} \cap \mathcal{K}_{\mathtt{com}}$. Now for counterparts we get

PROPOSITION 3.6. *Let L be a logic in $\mathcal{E}\mathsf{BK}$, and $\mathcal{K}$ a class of twist-structures over modal algebras such that $L = \mathbb{L}(\mathcal{K})$. Then*

$$L_{\mathtt{exp}} = \mathbb{L}(\mathcal{K}_{\mathtt{exp}}), \quad L_{\mathtt{com}} = \mathbb{L}(\mathcal{K}_{\mathtt{com}}) \quad \textit{and} \quad L_{\mathtt{cl}} = \mathbb{L}(\mathcal{K}_{\mathtt{cl}}).$$

PROOF. We shall only consider $L_{\mathtt{com}} = \mathbb{L}(\mathcal{K}_{\mathtt{com}})$, because perfectly analogous arguments work for the other two equalities.

Since $\mathcal{K}_{\mathtt{com}}$ consists of substructures of $\mathcal{K}$, we have $\mathcal{K}_{\mathtt{com}} \models L$, and hence $\mathcal{K}_{\mathtt{com}} \models L_{\mathtt{com}}$ by Proposition 3.3. So the inclusion $L_{\mathtt{com}} \subseteq \mathbb{L}(\mathcal{K}_{\mathtt{com}})$ follows.

In the opposite direction, suppose $\varphi \notin L_{\mathtt{exp}}$. Thus $\varphi_{\mathtt{exp}} \notin L$ by Proposition 3.5, i.e. there exists $\mathcal{A} \in \mathcal{K}$ such that $\mathcal{A} \not\models \varphi_{\mathtt{exp}}$, which is clearly equivalent to $\mathcal{A}_{\mathtt{exp}} \not\models \varphi$ by Proposition 3.4. Consequently $\varphi \notin \mathbb{L}(\mathcal{K}_{\mathtt{com}})$. ⊣

One nice feature of our framework deserves mention here.

PROPOSITION 3.7. *For every $L_1 \in \mathsf{Exp}$ and every $L_2 \in \mathsf{Com}$,*

$$\sigma(L_1) = \sigma(L_2) \quad \Longleftrightarrow \quad \begin{array}{c} \textit{there exists } L \in \mathcal{E}\mathsf{BK} \textit{ such that} \\ L_{\mathtt{exp}} = L_1 \textit{ and } L_{\mathtt{com}} = L_2. \end{array}$$

PROOF. Suppose $\sigma(L_1) = \sigma(L_2)$. Observe that by Proposition 3.2(5),

$$(L_1)_{\mathtt{cl}} = \eta^{\mathtt{c}}(\sigma(L_1)) = \eta^{\mathtt{c}}(\sigma(L_2)) = (L_2)_{\mathtt{cl}}.$$

Let L be $L_1 \cap L_2$. Then using Proposition 3.1, we obtain

$$L_{\mathtt{exp}} = (L_1)_{\mathtt{exp}} \cap (L_2)_{\mathtt{exp}} = L_1 \cap (L_2)_{\mathtt{cl}} = L_1 \cap (L_1)_{\mathtt{cl}} = L_1,$$

$$L_{\mathtt{com}} \;=\; (L_1)_{\mathtt{com}} \cap (L_2)_{\mathtt{com}} \;=\; (L_1)_{\mathtt{cl}} \cap L_2 \;=\; (L_2)_{\mathtt{cl}} \cap L_2 \;=\; L_2.$$

(bearing in mind that $L' \subseteq (L')_{\mathtt{cl}}$ for all logics L' in $\mathcal{E}\mathsf{BK}$).

The converse is immediate from Proposition 3.2(4). ⊣

For any $L_1 \in \mathsf{Exp}$ and $L_2 \in \mathsf{Com}$ satisfying $\sigma(L_1) = \sigma(L_2)$, we consider the special family

$$\mathrm{Spec}(L_1, L_2) \;:=\; \{L \in \mathcal{E}\mathsf{BK} \mid L_{\mathtt{exp}} = L_1 \text{ and } L_{\mathtt{com}} = L_2\}$$

and the distinguished logic

$$L_1 * L_2 \;:=\; \mathsf{BK} + (\{\varphi_{\mathtt{exp}} \mid \varphi \in L_1\} \cup \{\varphi_{\mathtt{com}} \mid \varphi \in L_2\}).$$

A simple semantic characterisation of $L_1 * L_2$ is given by

PROPOSITION 3.8. *Suppose $L_1 \in \mathsf{Exp}$ and $L_2 \in \mathsf{Com}$ are such that $\sigma(L_1) = \sigma(L_2)$. Then for each twist-structure $\mathcal{A}$,*

$$\mathcal{A} \models L_1 * L_2 \quad\Longleftrightarrow\quad \mathcal{A}_{\mathtt{exp}} \models L_1 \text{ and } \mathcal{A}_{\mathtt{exp}} \models L_2.$$

PROOF. This follows easily from Proposition 3.4. ⊣

Here are some basic facts about families of the form $\mathrm{Spec}(L_1, L_2)$.

PROPOSITION 3.9. *Suppose $L_1 \in \mathsf{Exp}$ and $L_2 \in \mathsf{Com}$ are such that $\sigma(L_1) = \sigma(L_2)$. Then we have the following:*

1. $\mathrm{Spec}(L_1, L_2) \subseteq \sigma^{-1}(L)$ *where L denotes* $\sigma(L_1)$;
2. *if* $L_2 \in \mathsf{Exp}$, *then* $\mathrm{Spec}(L_1, L_2) = \{L_1\}$;
3. *if* $L_1 \in \mathsf{Com}$, *then* $\mathrm{Spec}(L_1, L_2) = \{L_2\}$.

PROOF. *Ad* 1. Let $L' \in \mathrm{Spec}(L_1, L_2)$. So by Proposition 3.2(4) we get $\sigma(L') = \sigma((L')_{\mathtt{exp}}) = \sigma(L_1) = L$, as desired.

Ad 2. Assume that L_2 is in Exp, and hence in Clas. Let $L' \in \mathrm{Spec}(L_1, L_2)$. Note that $\sigma(L') = \sigma(L_2)$ by (1). Then using Proposition 3.2, we obtain

$$(L')_{\mathtt{com}} \;=\; L_2 \;=\; (L_2)_{\mathtt{cl}} \;=\; \eta^{\mathsf{c}}(\sigma(L_2)) \;=\; \eta^{\mathsf{c}}(\sigma(L')) \;=\; (L')_{\mathtt{cl}},$$

i.e. $L' \in \mathsf{Exp}$. Thus $L' = (L')_{\mathtt{exp}} = L_1$.

Ad 3. Similar to (2). ⊣

Further, each family of the form $\mathrm{Spec}(L_1, L_2)$ turns out to be an interval in the lattice $\mathcal{E}\mathsf{BK}$, with endpoints looking quite natural.

PROPOSITION 3.10. *For all* $L_1 \in \mathsf{Exp}$ *and* $L_2 \in \mathsf{Com}$ *which satisfy* $\sigma(L_1) = \sigma(L_2)$, *we have* $\mathrm{Spec}(L_1, L_2) = [L_1 * L_2, L_1 \cap L_2]$.

PROOF. Obviously, for any $\{L^\dagger, L^\ddagger\} \subseteq \mathrm{Spec}(L_1, L_2)$ and $L' \in [L^\dagger, L^\ddagger]$,

$$(L^\dagger)_{\mathtt{exp}} \subseteq (L')_{\mathtt{exp}} \subseteq (L^\ddagger)_{\mathtt{exp}} \quad \text{and} \quad (L^\dagger)_{\mathtt{com}} \subseteq (L')_{\mathtt{com}} \subseteq (L^\ddagger)_{\mathtt{com}},$$

whence $(L')_{\mathtt{exp}} = L_1$ and $(L')_{\mathtt{com}} = L_2$, i.e. $L' \in \mathrm{Spec}(L_1, L_2)$. In words, we deal with intervals.

By definition $L \subseteq L_{\mathtt{exp}} \cap L_{\mathtt{com}}$ for each $L \in \mathcal{E}\mathsf{BK}$. So $L \subseteq L_1 \cap L_2$ for all $L \in \mathrm{Spec}(L_1, L_2)$. Furthermore, as we saw already in the proof of Proposition 3.7, $L_1 \cap L_2$ belongs to $\mathrm{Spec}(L_1, L_2)$, and so it is the supremum.

On the other hand, for every $L \in \mathrm{Spec}(L_1, L_2)$ we have

$$\{\varphi_{\mathtt{exp}} \mid \varphi \in L_1\} \cup \{\varphi_{\mathtt{com}} \mid \varphi \in L_2\} \subseteq L$$

by Proposition 3.5, and thus $L_1 * L_2 \subseteq L$. To show that $L_1 * L_2$ belongs to $\mathrm{Spec}(L_1, L_2)$, fix some $L \in \mathrm{Spec}(L_1, L_2)$, e.g. $L_1 \cap L_2$. Then

$$(L_1 * L_2)_{\mathtt{exp}} \subseteq L_{\mathtt{exp}} = L_1 \quad \text{and} \quad (L_1 * L_2)_{\mathtt{com}} \subseteq L_{\mathtt{com}} = L_2.$$

And it is straightforward to verify that the reverse inclusions hold—using Proposition 3.5 again. Hence $L_1 * L_2$ is indeed the infimum. $\dashv$

We finish this section with an open question:

- *Is it true that* $L_1 * L_2 \neq L_1 \cap L_2$ *if neither* L_1 *nor* L_2 *is classical?*

4. Conclusion

In effect, we have done all the work needed for providing so-called 'transfer results' from $\mathcal{E}\mathsf{K}$ (and its distinguished sublatices, like $\mathcal{E}\mathsf{K4}$ or $\mathcal{E}\mathsf{S4}$) to $\mathcal{E}\mathsf{BK}$ (and its respective sublatices, like $\mathcal{E}\eta(\mathsf{K4})$ or $\mathcal{E}\eta(\mathsf{S4})$). To be more precise, in our subsequent articles we plan to transfer general results on tabularity, pretabularity, interpolation and definability properties (the reader might consult [10] for details).

Another natural direction of research concerns the connections between $\mathcal{E}\mathsf{N4}^\perp$ and $\mathcal{E}\eta(\mathsf{S4})$. We would want to first define Belnapian modal companions for each logic in $\mathcal{E}\mathsf{N4}^\perp$, and then to investigate relationships between the properties of $\mathsf{N4}^\perp$-extensions and those of their companions.

Acknowledgements. We would like to thank an anonymous referee for valuable comments on an earlier version of the manuscript.

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