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BOCHVAR'S THREE-VALUED LOGIC AND LITERAL PARALOGICS: Their lattice and functional equivalence

Abstract. In the present paper, various features of the class of propositional literal paralogics are considered. Literal paralogics are logics in which the paraproperties such as paraconsistence, paracompleteness and paranormality, occur only at the level of literals; that is, formulas that are propositional letters or their iterated negations. We begin by analyzing Bochvar's three-valued nonsense logic $\mathbf{B_3}$, which includes two isomorphs of the propositional classical logic **CPC**. The combination of these two 'strong' isomorphs leads to the construction of two famous paralogics $\mathbf{P^1}$ and $\mathbf{I^1}$, which are *functionally equivalent*. Moreover, each of these logics is functionally equivalent to the fragment of logic $\mathbf{B_3}$ consisting of external formulas only. In conclusion, we structure a four-element lattice of three-valued paralogics with respect to the possession of paraproperties.

Keywords: Bochvar's logic $\mathbf{B_3}$; isomorphs, extended formulas; paraconsistent logics $\mathbf{P^1}$ and $\mathbf{P^1_2}$; paracomplete logics $\mathbf{I^1}$ and $\mathbf{I^1_2}$; paranormal logic $\mathbf{TK^1}$; strong and weak modus ponens; lattice of paralogics

1. Introduction

For almost a century, three-valued logics haved continued to spark genuine and ever-increasing interest. New logics in this class occasionally reveal some rather noteworthy properties. Indeed, the introduction of an additional truth-value to bivalent logic not only enables the simulation of classical logic but also allows us to simulate many properties unrelated to it, such as paraconsistency, paracompleteness, paranormality, maxi-

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mality, duality and others. Furthermore, different interpretations of the additional (third) truth-value allow for a distinction between classes of these logics. Among the latter, a special role is played by the class of nonsense logics, within which the additional truth-value is interpreted as "nonsense". The most interesting appears to be the Bochvar three-valued nonsense logic $\mathbf{B_3}$ [5]; its functional properties are determined by the union of different types of connectives — internal and external — and this fact accounts for $\mathbf{B_3}$ being "emergent" within a huge variety of three-valued logics.

In the present paper we examine the functional properties of three-valued logics, such as the functional inclusion of one logic into another, and the functional equivalence of logics. Analysis of three-valued logics based on these properties can lead to surprising results concerning the functional equivalence of logics having different axiomatizations and different meta-logical properties, as shown in the case of paraconsistent logic $\mathbf{P^1}$ [46] and paracomplete logic $\mathbf{I^1}$ [48]. In order to determine the functional relation among three-valued logics, some kind of systematization or even classification of different sets of their connectives is required. The necessary condition for such systematization consists in the existence of some basic structural principle that applies to the whole "three-valued logic" universe.

In [19, 20] the notions of "significance logic" and "nonsense logic" — the latter being just a special case of the former — are formally defined through algebraic semantic methods and by the introduction of the "truth-value type" notion. One classification of three-valued significance and nonsense logics is presented. The latter are divided into two subclasses — strong and weak nonsense logics.

In [3] A. Avron singles out the class of the so-called natural three-valued logics as extensions of Kleene's strong three-valued logic $\mathbf{K_3}$. Two of those extensions are functionally equivalent to the three-valued logic $\mathbf{Ł_3}$ of Łukasiewicz, and the remaining two are functionally equivalent to the three-valued paraconsistent logic of Batens $\mathbf{PI^s}$ [4]. However, it should be noted that the main object of our investigation, namely Bochvar's $\mathbf{B_3}$ nonsense logic, is excluded from the classification.

In [10] Cucci and Dubois studied relationships among three-valued functions, which in turn correspond to three-valued connectives. As a result the class containing 14 different implications and conjunctions, and 3 negations was singled out. The authors tried to draw a map of the relationships between conjunctions, negations and implications, which appear as extensions of their Boolean counterparts, and to clarify the connection between them through the use of truth-tables.

Even though all the aforementioned papers are interesting and valuable in their own right, without the analysis of different sets of connectives — functioning as different bases for one and the same logic — in terms of their functional equivalence, there is no firm guarantee that those sets, considered individually and separately, will not be seen as giving rise to different logics.

In what follows, we shall consider a completely different approach to the study of three-valued logics. It involves the partition of the set of three-valued functions under consideration into equivalence classes, with the subsequent definition of a lattice of such equivalence classes with respect to set inclusion. This way, all possible bases for a logic are brought together within some equivalence class, and the place of a specific logic in the lattice thereby obtained is uniquely determined. However, the implementation of this approach requires some initial "minimal" basis-forming set of connectives to be defined which is subsequently consistently expanded by the inclusion of other connectives (chosen according to specific criteria) as its new elements.

N. Tomova in [52] (see also [53, 54]) identified such a minimal basis-forming set with the set of Bochvar's internal connectives (which can be alternatively described as Kleene's week three-valued logic $\mathbf{K_3^w}$), and used the so-called natural implications in order to consistently expand it.

Prior to the proper introduction of the notion of natural implication, we state some basic definitions to which we refer throughout the paper.

2. Basic definitions

Let $\mathcal{L}$ be a sentential language, i.e., an algebra $\mathcal{L} = \langle \mathrm{For}, F_1, \ldots, F_m \rangle$ generated by a set of variables $\mathrm{Var} := \{p, q, r, p_1, p_2, \ldots\}$. The elements of For (*formulas*) are generated from the variables with the use of operations $F_1, \ldots, F_m$, representing sentential connectives.

Let $\mathcal{A} = \langle V, f_1, \ldots, f_m \rangle$ be an algebra similar to $\mathcal{L}$, where V is the set of truth-values and for every $1 \leqslant i \leqslant m$, f_i is a function from V into V with the same arity as F_i. A structure $\mathfrak{M} = \langle \mathcal{A}, D \rangle$ with $\mathcal{A}$ being an algebra similar to the propositional language $\mathcal{L}$ and $D \subseteq V$ a non-empty subset of the universe of $\mathcal{A}$ is called a *logical matrix* for $\mathcal{L}$. Elements of D are called *designated elements* of $\mathfrak{M}$.

Let F be the set of functions. The result of the *superposition* of functions $f_1, \ldots, f_k$ is the function obtained from $f_1, \ldots, f_k$ either (1) by substituting some of these functions for arguments of $f_1, \ldots, f_k$ or (2) by renaming arguments of $f_1, \ldots, f_k$ or by both (1) and (2). The closure $[F]$ of F is a set of all superpositions of elements in F.

Let F be a closed set of functions such that $F \subseteq P$. F is said to be *functionally precomplete* in P iff $[F] \neq P$ and for every function $f \in P$ such that $f \notin F$, $[F \cup \{f\}] = P$. Moreover, F is said to be *functionally complete* in P iff $[F] = P$.

Let $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ be logics with the set of function F_1 and F_2, respectively. We say that $\boldsymbol{L}_1$ is *functionally included* in $\boldsymbol{L}_2$ iff every function of F_1 can be defined by a superposition of functions of F_2. Moreover, $\boldsymbol{L}_1$ is *functionally equivalent* to $\boldsymbol{L}_2$ iff $\boldsymbol{L}_1$ is functionally included in $\boldsymbol{L}_2$ and logic $\boldsymbol{L}_2$ is functionally included in $\boldsymbol{L}_1$. Finally, $\boldsymbol{L}_1$ is a *fragment* of $\boldsymbol{L}_2$ iff $\boldsymbol{L}_1$ is functionally included in $\boldsymbol{L}_2$, but $\boldsymbol{L}_1$ is not functionally equivalent to $\boldsymbol{L}_2$, i.e., the opposite does not hold.

Moreover, $\boldsymbol{L}_2$ is said to be an *extension* of $\boldsymbol{L}_1$ iff F_2 is obtained by adding to F_1 a function which cannot be defined by a superposition of the functions of F_1.

Some fragment of a logic $\boldsymbol{L}$ is said to be an *isomorph of classical propositional logic* iff $\boldsymbol{L}$ has the classical set of tautologies and the classical consequence relation. Such isomorph is called a *strong isomorph*.

We will use the same symbols for both the propositional connective and the corresponding matrix function.

3. The notion of natural implication

Let V_3 be the set of truth-values $\{0, 1/2, 1\}$ and D be a set of designated values, such that $\emptyset \neq D \subseteq \{1, 1/2\}$. A function $\rightarrow$ from V_3 into V_3 with arity 2 is called *natural implication* iff it satisfies the following conditions [54]:

1. *C-extending*, i.e., it restrictions to the subset $\{0, 1\}$ of V_3 coincide with classical implication;
2. *normality* in the sense of Łukasiewicz-Tarski, i.e., for all $x, y \in V_3$: if $x \rightarrow y \in D$ and $x \in D$, then $y \in D$ (condition sufficient for the verification of modus ponens) [33, p. 134];
3. *consistency*, i.e., for all $x, y \in V_3$: if $x \leqslant y$ then $x \rightarrow y \in D$.

According to the definition of natural implication, there are 6 implications with $D = \{1\}$, and 24 implications with $D = \{1, {}^1\!/_2\}$.

The set of three-valued matrices, which are the extensions of Kleene's weak three-valued logic $\mathbf{K_3^w}$, is divided into 7 disjoint classes. Within our approach, these 7 classes can be represented by a lattice of 7 *basic* three-valued logics with respect to the relation of functional inclusion on the set of three-valued logics. As a result, we obtain 12 bases for the three-valued logic of Łukasiewicz **Ł$_3$**, 8 bases for the paraconsistent logic of Batens $\mathbf{PI^s}$, and 3 bases for Bochvar's nonsense logic $\mathbf{B_3}$. Henceforth we shall denote the class of logics obtainable from these three bases by $\mathcal{B}$. Later, we'll return to this class.

The logics from this class help us to establish the interesting relationship between the paralogics $\mathbf{P^1}$ and $\mathbf{I^1}$.

4. Bochvar's three-valued nonsense logic $\mathbf{B_3}$

Dmitry Anatol'evich Bochvar constructed the first nonsense logic $\mathbf{B_3}$ in 1938 (see [5][1]). The latter was developed in connection with the problem of logical antinomies and, in particular, Russell's paradox. Within Bochvar's system, the additional (third) truth-value of ${}^1\!/_2$ is interpreted as "nonsense".

Variables A, B, C with/without indices are used for formulas and variables x, y, z with/without indices are used for arbitrary truth-values.

4.1. Definition of $\mathbf{B_3}$

Logic $\mathbf{B_3}$ is presented by the following logical matrix:

$$\mathfrak{M}_3^{\mathrm{B}} = \langle \{0, {}^1\!/_2, 1\}, \sim, \vdash, \cap, \{1\} \rangle,$$

where $\{1\}$ is the set of designated values and $\sim$ (internal negation), $\vdash$ (external assertion[2]), $\cap$ (internal conjunction) are defined, respectively, by the truth-tables from Table 1.

[1] In his review of Bochvar's paper in *The Journal of Symbolic Logic*, 4, 2 (1939), pp. 98–99, A. Church presented an inaccurate axiomatization of Bochvar's system, which led Church to his conclusion about Russell's paradox being obtainable within it. Church himself spotted his mistake and published a correction in the same journal, vol. 5, no. 3 (1940), p. 119.

[2] A function f on $\{0, {}^1\!/_2, 1\}$ into $\{0, {}^1\!/_2, 1\}$ with arity n is called *external* iff for any values $x_1, \ldots, x_n$ we have either $f(x_1, \ldots, x_n) = 0$ or $f(x_1, \ldots, x_n) = 1$.

x	$\sim x$
1	0
$1/2$	$1/2$
0	1

x	$\vdash x$
1	1
$1/2$	0
0	0

$\cap$	1	$1/2$	0
1	1	$1/2$	0
$1/2$	$1/2$	$1/2$	$1/2$
0	0	$1/2$	0

Table 1. Truth-tables for $\sim$, $\vdash$, and $\cap$

Other internal connectives can be introduced by using $\sim$ and $\cap$:

$$
\begin{aligned}
A \cup B &:= \sim(\sim A \cap \sim B)\\
A \supset B &:= \sim A \cup B\\
A \equiv B &:= (A \supset B) \cap (B \supset A)
\end{aligned}
$$

One striking feature of the internal connectives is that the attribution of the value $1/2$ to at least one of its arguments suffices for the whole formula to assume the (same) value of $1/2$. This property comes as a consequence of the interpretation of the third truth-value as "nonsense". In other words, "nonsense" entails "nonsense". In the same year as Bochvar, Kleene [27] defined the same internal connectives, terming them "weak". Therefore, the logic with the set of connectives $\{\sim, \cap, \cup\}$ as defined above is the same as weak Kleene's logic $\mathbf{K_3^w}$.

4.2. Three-valued isomorphs of CPC

The connective of external assertion $\vdash$ plays an important role in $\mathbf{B_3}$. Here, we shall denote it as $\Box$, because its truth-table is the same as the one for the necessity operator in the three-valued modal logic of Łukasiewicz [32, p. 169]. By the use of external assertion we can define negation $\sim^\Box$, implication $\supset^\Box$, conjunction $\cap^\Box$, disjunction $\cup^\Box$, and equivalence $\equiv^\Box$ in the following manner:

$$
\begin{aligned}
\sim^\Box A &:= \sim\Box A \qquad\qquad \text{(this negation we denote as } \lceil A),\\
A \supset^\Box B &:= \Box A \supset \Box B,\\
A \cup^\Box B &:= \Box A \cup \Box B,\\
A \cap^\Box B &:= \Box A \cap \Box B,\\
A \equiv^\Box B &:= \Box A \equiv \Box B.
\end{aligned}
$$

These connectives are called *external* and the truth-tables for Bochvar's external connectives are in Table 2.

x	$\lceil x$
1	0
1/2	1
0	1

$\supset^{\Box}$	1	1/2	0
1	1	0	0
1/2	1	1	1
0	1	1	1

$\cup^{\Box}$	1	1/2	0
1	1	1	1
1/2	1	0	0
0	1	0	0

$\cap^{\Box}$	1	1/2	0
1	1	0	0
1/2	0	0	0
0	0	0	0

$\equiv^{\Box}$	1	1/2	0
1	1	0	0
1/2	0	1	1
0	0	1	1

Table 2. Truth-tables for Bochvar's external connectives

One important feature of these tables consists in the fact that the only possible truth-values for expressions obtained by the use of the aforementioned connectives are 0 and 1. In essence, Bochvar proposed a *translation* of internal connectives into external ones. We shall denote the logic based on these connectives as $\mathbf{B}_3^{\Box}$, i.e., the logic presented by the matrix $\mathfrak{M}_3^{\Box} = \langle\{0, 1/2, 1\}, \lceil, \supset^{\Box}, \cup^{\Box}, \cap^{\Box}, \equiv^{\Box}, \{1\}\rangle$, where $\{1\}$ is the set of designated values. So $\mathbf{B}_3^{\Box}$ is a fragment of $\mathbf{B_3}$ (see p. 210). This fragment proved to be quite peculiar. According to Bochvar's terminology, $\mathbf{B}_3^{\Box}$ is an isomorph of classical propositional logic **CPC**. Thus, logic $\mathbf{B_3}$ contains a fragment isomorphic with the classical two-valued system **CPC**.

Only N. Rescher [45, p. 31] took into consideration Bochvar's result — that is, the fact that logic $\mathbf{B_3}$ contains a fragment isomorphic with **CPC**. However, $\mathbf{B_3}$ contains another isomorph of **CPC**.

As mentioned above, the connective of external assertion $\vdash$ is $\Box$. Furthermore, the modal operator of possibility $\Diamond$ is standardly defined as follows: $\Diamond A := \sim \Box \sim A$. We shall define the *external* connectives $\sim^{\Diamond}$, $\supset^{\Diamond}$, $\cap^{\Diamond}$, $\cup^{\Diamond}$ and $\equiv^{\Diamond}$ in the same way we defined the external connectives in $\mathbf{B}_3^{\Box}$; that is, instead of the necessity operator $\Box$ we use the possibility operator $\Diamond$. As a result we get another translation of internal connectives to external ones. Let us now consider the truth-tables for negation $\sim^{\Diamond}$ (we shall denote it as $\rceil$) and implication $\supset^{\Diamond}$ as in Table 3.

We shall denote the logic based on these connectives as $\mathbf{B}_3^{\Diamond}$, i.e., the logic presented by the matrix $\mathfrak{M}_3^{\Diamond} = \langle\{0, 1/2, 1\}, \rceil, \supset^{\Diamond}, \cup^{\Diamond}, \cap^{\Diamond}, \equiv^{\Diamond}, \{1, 1/2\}\rangle$, where $\{1, 1/2\}$ is the set of designated values. $\mathbf{B}_3^{\Diamond}$ is also an isomorph of **CPC**.

x	$\rceil x$
1	0
$1/2$	0
0	1

$\supset^{\diamond}$	1	$1/2$	0
1	1	1	0
$1/2$	1	1	0
0	1	1	1

Table 3. Truth-tables for $\sim^{\diamond}$ and $\supset^{\diamond}$

Thus, Bochvar's three-valued logic $\mathbf{B_3}$ contains two fragments isomorphic with **CPC**. These isomorphs differ from each other, since the truth-value $1/2$ is identified with 0 in $\mathbf{B_3^{\Box}}$, and with 1 in $\mathbf{B_3^{\diamond}}$. Under this identification the connectives retain their classical properties, which ensures that all the axioms of **CPC** are verified. This way of proving the "equivalence" of $\mathbf{B_3^{\Box}}$ and **CPC** was proposed by Rescher [45, p. 32]. In [14] a strict proof of $\langle\{0, 1/2, 1\}, \lceil, \supset^{\Box}, \{1\}\rangle$ and $\langle\{0, 1/2, 1\}, \rceil, \supset^{\diamond}, \{1/2, 1\}\rangle$, being characteristic matrices for **CPC**, is given.

The understanding of the role of **CPC** isomorphs remains as yet incomplete.[3] The most interesting feature seems to be given by the result showing that some many-valued logic containing the isomorph **CPC**, can be axiomatized as the extension of **CPC** (cf. [26, p. 55]).

4.3. Axiomatization and algebraization of $\mathbf{B_3}$

$\mathbf{B_3}$ was first axiomatized by V. K. Finn in 1971; cf. [17].[4] The only connectives of Finn's calculus are $\sim$, $\cup$, $\cap$ and $\supset^{\Box}$ (it includes 23 axioms and 3 rules of inference).

Earlier, class $\mathcal{B}$ of Bochvar's three-valued logics was defined as the class consisting of three elements — three logics (i.e. the implicative extensions of Kleene's weak logic by the natural implications). The corresponding connective bases for the latter are: $\{\sim, \cup, \cap, \supset^{\Box}\}$, $\{\sim, \cup, \cap, \supset^{\diamond}\}$ and $\{\sim, \cup, \cap, \rightarrow_4\}$, where $\rightarrow_4$ is defined by the following truth-table:

$\rightarrow_4$	1	$1/2$	0
1	1	0	0
$1/2$	1	1	0
0	1	1	1

3 L. Yu. Devyatkin gives an account of general properties of three-valued isomorphs of **CPC** in his monograph [15].

4 One further calculus, equivalent to $\mathbf{B_3}$, is presented in [38].

From what has been said, it follows that the logics contained in $\mathcal{B}$ are pairwise functionally equivalent, and each of these systems is functionally equivalent to $\mathbf{B_3}$ [52].

It should be emphasized that not all the laws of classical propositional logic **CPC** are verified in the axiomatization proposed by Finn. For example, the law of contraposition $(p \supset^{\square} q) \supset^{\square} ({\sim} q \supset^{\square} {\sim} p)$ does not hold. Using the implication $\supset^{\diamond}$, we get the same result. However, keeping in mind that $\mathbf{B_3}$ contains the fragment isomorphic with **CPC**, it would be natural to give an axiomatization of **CPC** by the use of external formulas. The axiomatization of $\mathbf{B_3}$ can then be presented as an extension of **CPC**, which was done in [20, pp. 235–236] (using 29 axiom schemes and one inference rule, modus ponens).

In [17] the notion of Bochvar's three-valued algebra $\mathbf{B_3}$ is introduced. For more information, see [20]; there, the algebraic model of $\mathbf{B_3}$ is given in the following signature $\langle \cup, \cap, \sim, J_0, J_{1/2}, J_1, 0, 1\rangle$. Here $\langle \cup, \cap, \sim\rangle$ is De Morgan's distributive quasi-lattice (lattice without absorption laws), and the definitions of $J_i(x)$-operators are given at the end of Section 5.3. The class of all $\mathbf{B_3}$-algebras is a quasivariety but not a variety.

4.4. Functional properties of $\mathbf{B_3}$

Consider the matrix $\mathfrak{M}_3^{\text{Ł}} = \langle\{0, 1/2, 1\}, \sim, \to, \{1\}\rangle$ for the three-valued Łukasiewicz logic $\mathbf{Ł_3}$, where $\sim$ has been defined above and $\to$ is defined by the following truth-table:

$\to$	1	1/2	0
1	1	1/2	0
1/2	1	1	1/2
0	1	1	1

By the use of the primitive connectives the other connectives are introduced by definition in the following way:

$$A \vee B := (A \to B) \to B \qquad \text{disjunction, } \max(x, y),$$
$$A \wedge B := {\sim}({\sim} A \vee {\sim} B) \qquad \text{conjunction, } \min(x, y).$$

Note that the logic with the set of connectives $\{\sim, \vee, \wedge\}$ is nothing other than Kleene's strong regular logic $\mathbf{K_3}$ [28, §64].

In [50] V.I. Shestakov demonstrated that if in B_3 (for $\{\sim, \vdash, \cap\}$) internal conjunction $\cap$ is replaced by the strong Kleene's conjunction

$\wedge$, then $\mathbf{B_3}$ is changed into $\mathbf{Ł_3}$. This can be done in the following way: $A \to B := (\sim A \vee B) \vee (\downarrow A \wedge \downarrow B)$, where $\downarrow C := \sim(\vdash C \cup \vdash \sim C)$.

It should be noticed that the class of functions corresponding to $\mathbf{B_3}$ is functionally included in the class of functions corresponding to $\mathbf{Ł_3}$ but the converse is not true.

It should also be noticed that the functional properties of $\mathbf{Ł_3}$ make it functionally *precomplete* in $\mathbf{P_3}$, where $\mathbf{P_3}$ is a three-valued functionally complete logic of Post [41].

Another important result obtained by Finn in [18], concerns the determination of the functional completeness criterion for the class of functions B_3 corresponding to Bochvar's three-valued logic $\mathbf{B_3}$. Finn proves that B_3 has exactly 11 precomplete classes and that a set of functions F is complete in B_3 iff F is not included in any of these 11 precomplete classes.

4.5. Fragment of $\mathbf{B_3}$, consisting only of external formulas

In [49] Shestakov reduced the number of basic connectives of Bochvar's logic $\mathbf{B_3}$ to two $\{\psi, \vdash\}$, where ψ — by analogy with classical propositional logic — is antidisjunction (Peirce's arrow for internal connectives), and is defined as follows:

$$A \,\psi\, B := \sim A \cap \sim B\,.$$

Thus, all the internal connectives of $\mathbf{B_3}$ can be defined through $A \,\psi\, B$. Shestakov denoted the fragment $\mathbf{B_3}$, consisting only of internal connectives by $\mathbf{B}_0$.

In [51] Shestakov extracted the fragment of $\mathbf{B_3}$ which contains only external connectives (their truth-tables contain values 0 and 1 only), and denoted it as $\mathbf{B}_1$. In the same paper Peirce's arrow γ for external connectives (and Sheffer stroke λ) was introduced in the following way:

γ	1	1/2	0
1	0	0	0
1/2	0	0	1
0	0	1	1

By the use of this connective all of external connectives can be defined, including the connective $\sim^{\diamond}$, and all the connectives pertaining to the isomorph $\mathbf{B}_3^{\Box}$ (see Section 4.2). The latter allows for the following definition of γ [51, p. 105]:

$$A\,\gamma\,B := (\urcorner A \cap^{\Box} \ulcorner B) \cup^{\Box} (\ulcorner A \cap^{\Box} \urcorner B). \tag{Sh}$$

We shall refer to this notable formula in connection with the paralogics. Shestakov also discusses the relationship between $\mathbf{B}_0$ and $\mathbf{B}_1$, concluding that logic $\mathbf{B}_3$ is a union of disjoint logics $\mathbf{B}_0$ and $\mathbf{B}_1$. This implies that the set of all connectives of $\mathbf{B}_3$ cannot be presented in the form of the Sheffer stroke (Peirce's arrow).

The most important result of [51] is the proof — through the introduction of "canonical" (normal) forms — of the theorem on the functional completeness of the set of external connectives. Nonetheless, a stronger claim can be proved, which is that, from the aforementioned criterion concerning functional completeness for the class B_3 [18] (see Section 4.4), the criterion of functional completeness for the class of external functions B_1 follows: B_1 has 7 precomplete classes, and a set of functions F is complete in B_1 iff F is not included in any of these 7 precomplete classes.

5. Paralogics $\mathbf{P}^1$ and $\mathbf{I}^1$

Paraconsistent and paracomplete logics are areas where interest continues to grow. One reason for this may be due to their simplicity and to the wide range of their applications (in computer science, artificial intelligence, and other areas).

One crucial factor behind the development of paraconsistent logic is the belief that in certain circumstances we may find ourselves in a situation where our theory is inconsistent and yet we are required to draw inferences in a sensible fashion.

Let $\vDash$ be a consequence relation. We call $\vDash$ *explosive* iff $\{A, \neg A\} \vDash B$, for all A and B. Classical logic, and most standard 'non-classical' ones, such as intuitionist logic, many-valued logics $\mathbf{Ł}_3$ and $\mathbf{B}_3$, are explosive. Paraconsistent logic challenges this orthodoxy. A logic is said to be *paraconsistent* iff its logical consequence relation is not explosive [42]. Moreover, $\vDash$ is said to be *implosive* iff $B \vDash \{A, \neg A\}$, for all A and B. A logic is said to be *paracomplete* iff its logical consequence relation is not implosive.[5]

[5] The concept of paracompleteness was first introduced in [31].

CPC is neither paraconsistent nor paracomplete. Logics which are simultaneously paraconsistent and paracomplete are called *paranormal* logics (Miró Quesada's terminology). Below (Section 5.4.2), we present a remarkable example of the three-valued paranormal logic $\mathbf{TK^1}$, which is directly relevant to the Bochvarian class $\mathcal{B}$.

Different formal criteria may be used for the construction of paraconsistent logic (**PL**), but the "implicative-negative" criterion of S. Jaśkowski (who first considered this problem in 1948 (see [21])), best fits our scope: the logical system **PL** does not verify "the implicational law of over completeness": $A \to (\neg A \to B)$.[6] This criterion is best known as "the law of Duns Scotus".

5.1. Paraconsistent logic $\mathbf{P^1}$

At this point we shall consider one remarkable paraconsistent logic with unusual properties. Sette in 1973 [46] constructed the simplest possible paraconsistent logical calculus $\mathbf{P^1}$ with the following syntax:

- propositional variables: $p_1, p_2, \ldots, p_n$;
- logical connectives $\supset$ (implication) and $\neg$ (negation);
- auxiliary symbols (,).

Notions of well-formed formulas, atomic formulas, schemes of formulas, etc. are the usual ones (the same as in classical logic). Capital letters "A", "B", and "C" are used as metavariables over formulas.

The following schemes are axioms:

(P1) $A \supset (B \supset A)$
(P2) $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$
(P3) $(\neg A \supset \neg B) \supset ((\neg A \supset \neg\neg B) \supset A)$
(P4) $(\neg A \supset \neg\neg A) \supset A)$
(P5) $(A \supset B) \supset \neg\neg(A \supset B)$

and modus ponens, (MP) $A, A \supset B / B$ is the only rule of inference.

In [6] it is shown that axiom (P4) is not independent. Note that, if the law of contraposition $(\neg A \supset \neg B) \supset (B \supset A)$ is added to the axioms (P1) and (P2), then we get the axiomatization of **CPC** (cf. [33, p. 136]).

$\mathbf{P^1}$ is complete relative to the matrix $\langle\{0, 1/2, 1\}, \neg, \supset, \{1, 1/2\}\rangle$, where the connectives $\neg$ and $\supset$ are $\ulcorner$ and $\supset^\diamond$, respectively (see the corresponding truth-tables in Section 4.2). Other connectives are introduced by the following definitions:

[6] For a more detailed account see [23].

$$A \vee B := (A \supset \neg\neg A) \supset (\neg A \supset B)$$
$$A \wedge B := (((A \supset A) \supset A) \supset \neg((B \supset B) \supset B)) \supset \neg(A \supset \neg B)$$
$$A \equiv B := (A \supset B) \wedge (B \supset A)$$

These connectives are nothing but disjunction $\cup^{\diamond}$, conjunction $\cap^{\diamond}$, and equivalence $\equiv^{\diamond}$ from Bochvar's isomorph $\mathbf{B}_3^{\diamond}$ (see Section 4.2).

The definitions of connectives $\vee$ and $\wedge$ in $\mathbf{P^1}$ are further simplified in [22, p. 67]. This is done in the following way:

$$\rceil A := \lceil(\lceil A \supset A)$$
$$A \vee B := (\rceil A \supset B)$$
$$A \wedge B := \rceil(A \supset \rceil B)$$

Logic $\mathbf{P^1}$ has the following important properties:

(1) $\mathbf{P^1}$ is paraconsistent only for atomic formulas. It means that the law of Duns Scotus $A \supset (\neg A \supset B)$ is a $\mathbf{P^1}$-tautology only if A is not a propositional variable.
(2) $\mathbf{P^1}$ is *maximal* in the following sense: if A is a classical tautology not provable in $\mathbf{P^1}$, then by adding A to $\mathbf{P^1}$ as a new axiom schema, classical logic **CPC** is obtained [46, Proposition 11].
(3) $\mathbf{P^1}$ is algebraizable in the sense of Block and Pigozzi (see [29, 44]).
(4) $\mathbf{P^1}$ is the combination of logical operations from isomorphs $\mathbf{B}_3^{\square}$ and $\mathbf{B}_3^{\diamond}$, i.e., connectives of $\mathbf{P^1}$ are $\{\sim^{\square}, \supset^{\diamond}, \cap^{\diamond}, \cup^{\diamond}, \equiv^{\diamond}\}$ [25, p. 183].

5.2. Paracomplete logic $\mathbf{I^1}$

In [48] logic $\mathbf{I^1}$ (named "weakly-intuitionistic logic") was introduced as a dual of the paraconsistent calculus $\mathbf{P^1}$. Since $\mathbf{I^1}$ has a paracomplete character let's consider it as a paracomplete logic. By analogy with the paraconsistent logic it is convenient to use the following criterion for paracompleteness: a logic is *paracomplete* iff the law of Clavius, $(\neg A \supset A) \supset A$, is not valid in it (see e.g. [11]).

The calculus $\mathbf{I^1}$ is axiomatized by means of the following axiom schemes:

(I1) $A \supset (B \supset A)$,
(I2) $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$,
(I3) $(\neg\neg A \supset \neg B) \supset ((\neg\neg A \supset B) \supset \neg A)$,
(I4) $\neg\neg(A \supset B) \supset (A \supset B)$,

and modus ponens.

As shown in [48], $\mathbf{I^1}$ is complete with respect to the three-valued matrix $\langle\{0, 1/2, 1\}, \neg, \supset, \{1\}\rangle$, where $\neg$ and $\supset$ are $\rceil$ and $\supset^{\Box}$, respectively (see corresponding truth-tables in Section 4.2).

Conjunction $\wedge$ and disjunction $\vee$ can be defined in $\mathbf{I^1}$ as follows:

$$A \wedge B := \neg(((A \supset A) \supset A) \supset \neg((B \supset B) \supset B))$$
$$A \vee B := \neg((B \supset B) \supset B) \supset ((A \supset A) \supset A)$$

These connectives are the same as disjunction $\cup^{\Box}$ and conjunction $\cap^{\Box}$ from Bochvar's isomorph $\mathbf{B_3^{\Box}}$ (see Section 4.2). In $\mathbf{I^1}$ all axioms of the well-known Heyting system for intuitionistic logic are valid.

Logic $\mathbf{I^1}$ has the following important properties:

(1) $\mathbf{I^1}$ is paracomplete only for atomic formulas. It means that the law of Clavius, $(\neg A \supset A) \supset A$, is a $\mathbf{I^1}$-tautology only if A is not a propositional variable [11].
(2) $\mathbf{I^1}$ is *maximal* [48].
(3) $\mathbf{I^1}$ is algebraizable in the sense of Block and Pigozzi (see [48]).
(4) $\mathbf{I^1}$ is the combination of logical operations from isomorphs $\mathbf{B_3^{\Box}}$ and $\mathbf{B_3^{\Diamond}}$, i.e., connectives of $\mathbf{I^1}$ are $\sim^{\Diamond}$, $\supset^{\Box}$, $\cap^{\Box}$, $\cup^{\Box}$, $\equiv^{\Box}$ [25, p. 183].

It should be noted that systems similar to $\mathbf{P^1}$ and $\mathbf{I^1}$ have been studied from different perspectives. In [13] logic $\mathbf{P^1}$ is considered under the name of $\mathbf{F}$, as obtained from Łukasiewicz's three-valued logic $\mathbf{Ł_3}$. The logic $\mathbf{P^1}$, as extension of da Costa's paraconsistent logic $\mathbf{C}_1$, was found independently by C. Mortensen in 1979, who called it $\mathbf{C}_{0.1}$ (see [37, p. 299]). In [40] Popov draws attention to the fact that logic $\mathbf{P^1}$ (termed $\mathbf{I}_1$) is obtainable from Arruda's three-valued paraconsistent logic $\mathbf{V1}$ [2]. This is done by leaving in $\mathbf{V1}$ only the so called "Vasiliev's propositional letters", i.e., only atomic formulas. In [36] logic $\mathbf{P^1}$ is axiomatized by extension of da Costa's system $\mathbf{C}_n$. A new Hilbert-type axiomatization for $\mathbf{P^1}$ is presented in [12]. A system similar to $\mathbf{I^1}$ has also been studied in [31]. In [47], it was proved that calculus β_1 from [31] is equivalent to $\mathbf{I^1}$. A Gentzen-type sequent calculus for $\mathbf{I^1}$ (termed $\mathbf{I}_2$) is presented in [40]. A new Hilbert-type axiomatization for $\mathbf{I^1}$ can be found in [11].

5.3. Interconnections between $\mathbf{P^1}$ and $\mathbf{I^1}$

The year 2000 turned out to be very important: three papers (Carnielli [8], D'Ottaviano and Feitosa [16], and Karpenko [25]) were published independently of one other which started to look at the logics $\mathbf{P^1}$ and $\mathbf{I^1}$ both separately and as regards their interconnections.

In [8] the strong negation $\rceil A$ in $\mathbf{P^1}$ is defined in the following way (we use our own notation):

$$\rceil A := \lceil(\lceil A \supset^{\diamond} A)$$

In [16] the following definition is given:

$$\rceil A := A \supset^{\diamond} \lceil(A \supset^{\diamond} A)$$

Simpler definitions than those given by Sette [46] (see also [22]), of the connectives $\cup^{\diamond}$ and $\cap^{\diamond}$ in $\mathbf{I^1}$, can be found in [8] and [16]:

$$A \cup^{\diamond} B := (\rceil A \supset^{\diamond} B)$$

$$A \cap^{\diamond} B := \rceil(A \supset^{\diamond} \rceil B) \qquad \text{(see [8, p. 158])}$$

$$A \cap^{\diamond} B := \rceil(\rceil A \supset^{\diamond} \rceil B) \qquad \text{(see [16, p. 84])}$$

The definitions of connectives $\lceil$, $\cup^{\square}$, and $\cap^{\square}$ in $\mathbf{I^1}$ are also given in [8] and [16]. For example, $\lceil A := A \supset^{\square} \rceil A$ (see [8, p. 159]).

The purpose of the paper [8] is to offer semantic interpretations for finite-valued logics (including $\mathbf{P^1}$ and $\mathbf{I^1}$) that are both intuitively acceptable and simple to manipulate formally. In order to achieve the latter, Carnielli developed what he called a "possible translations semantics". Carnielli clarified that, in intuitive terms, translations can be seen as abstract forms of accessibility relations in the usual Kripke semantics, and the distinct three-valued logical systems as forms of possible worlds [8, p. 153].

In [16] a new type of translation between logics is introduced — the so-called "conservative translation". Specifically, two translations are defined: R — from $\mathbf{I^1}$ into $\mathbf{P^1}$, and S — from $\mathbf{P^1}$ into $\mathbf{I^1}$. The authors noticed there that Sette and Carnielli [48] "do not introduce any function either from $\mathbf{P^1}$ into $\mathbf{I^1}$, or from $\mathbf{I^1}$ into $\mathbf{P^1}$, which could explicate in terms of translations the meaning of the 'duality' between the two systems" (p. 90).[7]

The paper [25] is devoted to the clarification of the meaning of the duality between $\mathbf{P^1}$ and $\mathbf{I^1}$, through the combination of two three-valued isomorphs of **CPC**. From the fact that in $\mathbf{P^1}$ we can define

$$\vdash A := \lceil \lceil A \quad \text{and} \quad \rceil A := \lceil(\lceil A \supset^{\diamond} A)$$

[7] See the fundamental work of Brunner and Carnielli [7] about the duality of logics.

the following important conclusions are drawn [25, p. 185]:

(a) $\mathbf{P^1}$ contains fragments $\mathbf{B_3^\Box}$ and $\mathbf{B_3^\Diamond}$, isomorphic with **CPC**,
(b) $\mathbf{P^1}$ contains one fragment isomorphic with $\mathbf{I^1}$.

Using this line of reasoning, the following two further conclusions can be drawn:

(c) $\mathbf{I^1}$ contains fragments $\mathbf{B_3^\Box}$ and $\mathbf{B_3^\Diamond}$, isomorphic with **CPC**,
(d) $\mathbf{I^1}$ contains one fragment isomorphic with $\mathbf{P^1}$.

Obviously, this implies that logic $\mathbf{P^1}$ and $\mathbf{I^1}$ are functionally equivalent, i.e., connectives of $\mathbf{P^1}$ can be defined by the connectives of $\mathbf{I^1}$ and the connectives of $\mathbf{I^1}$ can be defined by the connectives of $\mathbf{P^1}$.

Moreover, since $\mathbf{P^1}$ (as $\mathbf{I^1}$) contains both the aforementioned isomorphs, connectives $\rceil$, $\lceil$, $\cap^\Box$, and $\cup^\Box$ are defined in it. Thus, by virtue of Shestakov's formula (Sh), $\mathbf{P^1}$ (as $\mathbf{I^1}$) is functionally equivalent to the fragment $\mathbf{B}_1$, which consists of the set of external formulas of Bochvar's logic $\mathbf{B_3}$. The same conclusion follows from papers [8] and [16], if we consider the logic $\mathbf{I^1}$ with negation $\lceil$. However, none of these authors, including the author of the paper [25], paid special attention to this conclusion. Only in [53, p. 75] was the functional equivalence of logics $\mathbf{P^1}$ and $\mathbf{I^1}$ proven, as a consequence of a more general result. In the same paper (p. 79) one version of Shestakov's formula (Sh) was rediscovered and, via it, the functional equivalence of $\mathbf{P^1}$ and $\mathbf{B}_1$ was proven.

However, the functional equivalence of $\mathbf{P^1}$ and $\mathbf{I^1}$ can be proven quite simply. We have already seen that the negations $\lceil$ and $\rceil$ can be mutually defined in both systems as follows:

$$\rceil A := \lceil(\lceil A \supset^\Diamond A) \quad \text{and} \quad \lceil A := A \supset^\Box \rceil A$$

The same can to be done for implications $\supset^\Diamond$ and $\supset^\Box$:

$$A \supset^\Diamond B := \lceil B \supset^\Box \lceil A \quad \text{and} \quad A \supset^\Box B := \rceil B \supset^\Diamond \rceil A$$

Thus, both $\mathbf{P^1}$ and $\mathbf{I^1}$ split into two isomorphs of **CPC**. Relative to the latter, the following fragment from [35, p. 14] on the application of possible-translations semantics to finite-valued logics should be noticed:

> Moreover, truth-functional finite-valued logics can themselves be split in terms of 2-valued logics, that is, fragments of classical logic [...], copies of classical logic can be combined into fragments of modal logics, and so on and so forth.

Also, it should be emphasized that the logic $\mathbf{P^1}$ ($\mathbf{I^1}$), and hence the logic of external connectives $\mathbf{B}_1$, can be axiomatized as a direct extension of **CPC**. This can be done by the use of the Anshakov-Rychkov method for construction of Hilbert-type calculi of finite-valued logics [1]. The axiomatization conditions are given as follows.

We say that the logic $\boldsymbol{L}_n$ is *truth-complete* iff all J-operators are functionally expressible in $\boldsymbol{L}_n$, where for all $i, x \in V_n \supseteq \{0,1\}$ we have:

$$J_i(x) = \begin{cases} 1, & \text{if } x = i \\ 0, & \text{if } x \neq i. \end{cases}$$

A logic $\boldsymbol{L}_n$ is said to be C-extending iff the binary operations $\vee$, $\wedge$, $\to$ and the unary operation $\neg$ (whose restrictions to the subset $\{0,1\}$ of V_n coincide with the classical logical operations of disjunction, conjunction, implication, and negation) can be functionally expressed in $\boldsymbol{L}_n$. Note that $\boldsymbol{L}_n$ coincides with **CPC** over the set $\{0,1\}$. In [1], general, *effective* methods for the construction of Hilbert-type calculi for any truth-complete C-extending logics are given.

It seems obvious, that logics $\mathbf{P^1}$ and $\mathbf{I^1}$ are C-extending, and also truth-complete. We will prove it for $\mathbf{I^1}$ by showing that for any $x \in \{0, {}^1/_2, 1\}$: $J_0(x) = \rceil(x)$, $J_{1/2}(x) = \rceil(x \cup^{\square} \rceil x)$ and $J_1(x) = \rceil(\rceil x \cup^{\square} J_{1/2}(x))$.

Notice that Arruda's logic **V3** [2, p. 16] — being itself nothing but $\mathbf{I^1}$ with all J-operators — is axiomatized in the same way.

5.4. Literal-paralogics

In a very interesting paper [30] Lewin and Mikenberg introduce a family of matrices that define logics in which paraconsisteny and/or paracompleteness occurs only at the level of literals, that is, formulas that are propositional letters (p, q, r, etc.) or their iterated negations.[8] Capital letters A, B, C, etc. are used as variables for complex formula, i.e., formulas that contain a binary connective and α, β, γ etc. as variables for general formulas.

Using this notation, we shall consider the definitions given in [30, p. 479]. Let V be a set of truth-values such that $\{0,1\} \subseteq V$ and $D \subseteq V$

[8] Let Fm be the set of formulas built in the usual recursive way from a denumerable set $\mathrm{Var} := \{p_1, p_2, \ldots\}$ of propositional variables and the connectives. The *literals* of Fm is the set Lit of all formulas of the form $\neg^k p$, where $\neg^0 p = p$ and $\neg^{k+1} p = \neg(\neg^k p)$, for any $p \in \mathrm{Var}$ (see e.g. [30, p. 479]).

such that $1 \in D$ and $0 \notin D$. Let $\sim\colon V \to V$ be a function such that $\sim 1 = 0$ and $\sim 0 = 1$. They define *the literal-paraconsistent-paracomplete matrix* (or *LPP-matrix*) $\langle V, D, \sim\rangle$ along with the following operations for all $x, y \in V$:

$$x \vee y := \begin{cases} 1 & \text{if } x \in D \text{ or } y \in D \\ 0 & \text{otherwise} \end{cases}$$

$$x \wedge y := \begin{cases} 1 & \text{if } x \in D \text{ and } y \in D \\ 0 & \text{otherwise} \end{cases}$$

$$x \to y := \begin{cases} 1 & \text{if } x \notin D \text{ or } y \in D \\ 0 & \text{otherwise} \end{cases}$$

An LPP-matrix is *paraconsistent* iff for some $x \in V$ both $x \in D$ and $\sim x \in D$; it is *paracomplete* iff for some $x \in V$ both $x \notin D$ and $\sim x \notin D$. The matrix $\langle\{1, 0\}, \{1\}, \sim\rangle$, where $\sim 0 = 1$ and $\sim 1 = 0$, which defines **CPC**, is neither paraconsistent nor paracomplete.

Lewin and Mikenberg define a sound and complete deductive system for the logic defined by the class of all LPP-matrices $\langle V, D, \sim\rangle$, with no conditions on V, D or $\sim$. This system is called *literal-paraconsistent-paracomplete logic* (**LPPL**) with modus ponens (MP) as its only rule and the following axioms:

(A1) $\alpha \to (\beta \to \alpha)$.
(A2) $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$.
(A3) $(\alpha \wedge \beta) \to \alpha$.
(A4) $(\alpha \wedge \beta) \to \beta$.
(A5) $(\alpha \to \beta) \to ((\alpha \to \gamma) \to (\alpha \to (\beta \wedge \gamma)))$.
(A6) $\alpha \to (\alpha \vee \beta)$.
(A7) $\beta \to (\alpha \vee \beta)$.
(A8) $(\alpha \to \gamma) \to ((\beta \to \gamma) \to ((\alpha \wedge \beta) \to \gamma))$.
(A9) Axiom for negation: $(\sim A \to \sim B) \to (B \to A)$, where A and B are complex formulas.

Lewin and Mikenberg observe that this system was proposed by Puga and da Costa in [43] as axiomatization of the imaginary logic of Vasil'ev. In [30] the class of three-valued matrices $\langle V_3, D, \sim\rangle$ is examined, where $V_3 = \{0, 1/2, 1\}$. There are three possible functions $\sim$, namely:

- either $\sim_1 1/2 = 1/2$ (i.e., $\sim_1 = \sim$),
- or $\sim_2 1/2 = 1$ (i.e., $\sim_2 = \lceil$),
- or $\sim_3 1/2 = 0$ (i.e., $\sim_3 = \rceil$).

There are two possible sets of designated values, namely, $D_1 = \{1\}$ and $D_2 = \{1/2, 1\}$. Therefore, we obtain the following six combinations:

- $\mathfrak{M}^3_{1,1} = \langle V_3, D_1, \sim_1\rangle$,
- $\mathfrak{M}^3_{1,3} = \langle V_3, D_1, \sim_3\rangle$,
- $\mathfrak{M}^3_{2,1} = \langle V_3, D_2, \sim_1\rangle$,
- $\mathfrak{M}^3_{2,2} = \langle V_3, D_2, \sim_2\rangle$,
- $\mathfrak{M}^3_{1,2} = \langle V_3, D_1, \sim_2\rangle$,
- $\mathfrak{M}^3_{2,3} = \langle V_3, D_2, \sim_3\rangle$.

Lewin and Mikenberg consider only the reduced matrices (see the definition on p. 482), i.e., the first four of the above. In the reduced matrices there is one single element for each negation type present in $\langle V, D, \sim\rangle$. The last two matrices are the Bochvarian isomorphs $\mathbf{B}_3^{\Box}$ and $\mathbf{B}_3^{\Diamond}$ of **CPC**.

In fact, the structuring method for LPP-matrices presents itself as a more strict exposition of the generation method for isomorphs (by translation of the intermediate truth values into 1 or 0) and their successive combination.

5.4.1. Axiomatizations and a lattice of expansions of LPPL

The axiomatizations of the logics determined by the first four matrices [30] are rather interesting. Now, for any formula α we put

$$\alpha^\circ := \sim(\alpha \wedge \sim \alpha)$$
$$\alpha^\bullet := \alpha \vee \sim \alpha$$

The matrix $\mathfrak{M}^3_{2,2} = \langle V_3, D_2, \sim_2\rangle$ determines the following axiomatization of the logic $\mathbf{S}_{2,2}$ (the system thus obtained is Sette's $\mathbf{P^1}$). The axioms are

($A_{2,2}$.1) the axioms of **LPPL**
($A_{2,2}$.2) $\beta^\circ \to ((\alpha \to \beta) \to ((\alpha \to \sim_2 \beta) \to \sim_2 \alpha))$
($A_{2,2}$.3) $(\sim_2 \alpha)^\circ$

MP is the only rule of inference.

The following is an axiomatization for the logic $\mathbf{S}_{1,3}$ defined by the matrix $\mathfrak{M}^3_{1,3} = \langle V_3, D_1, \sim_3\rangle$. This system is the logic $\mathbf{I^1}$. The axioms are:

($A_{1,3}$.1) the axioms of **LPPL**
($A_{1,3}$.2) $\alpha^\bullet \to ((\alpha \to \beta) \to ((\alpha \to \sim_3 \beta) \to \sim_3 \alpha))$
($A_{1,3}$.3) $(\sim_3 \alpha)^\bullet$

MP is the only rule of inference.

Matrix $\mathfrak{M}^3_{2,1} = \langle V_3, D_2, \sim_1\rangle$ determines one axiomatization of the logic $\mathbf{S}_{2,1}$. The latter appears in [9, p. 62] under the name of $\mathbf{P^2}$ (in our notation $\mathbf{P^1_2}$) as obtained from $\mathbf{P^1}$ through the replacement of the negation $\sim_2$ with the negation $\sim_1$. Moreover, one axiomatization of this logic and the proof of its completeness are presented in [36]. There, Marcos defines (p. 64):

$$\sim_2 \alpha := \alpha \supset^{\diamond} \sim_1 \alpha$$

The axioms are:

($\mathrm{A}_{2,1}$.1) the axioms of **LPPL**
($\mathrm{A}_{2,1}$.2) $\beta^{\circ} \to ((\alpha \to \beta) \to ((\alpha \to \sim_1 \beta) \to \sim_1 \alpha))$
($\mathrm{A}_{2,1}$.3) $\alpha \leftrightarrow \sim_1 \sim_1 \alpha$

MP is the only rule of inference. Or $\mathbf{P^1_2}$ is $\mathbf{P^1} + \alpha \leftrightarrow \sim_1 \sim_1 \alpha$.

Finally, matrix $\mathfrak{M}^3_{1,1} = \langle V_3, D_1, \sim_1\rangle$ determines one axiomatization of the logic $\mathbf{S}_{1,1}$. The latter first appeared in [39] under the name **LAP**, where it was presented in the form of Hilbert and sequent calculi. In [36, p. 66] this system appeared under the name $\mathbf{I^2}$ (in our notation $\mathbf{I^1_2}$). Note that here,

$$\sim_3 \alpha := \sim_1(\sim_1 \alpha \supset^{\square} \alpha).$$

The axioms are:

($\mathrm{A}_{1,1}$.1) the axioms of **LPPL**
($\mathrm{A}_{1,1}$.2) $\alpha^{\bullet} \to ((\alpha \to \beta) \to ((\alpha \to \sim_1 \beta) \to \sim_1 \alpha))$
($\mathrm{A}_{1,1}$.3) $\beta \leftrightarrow \sim_1 \sim_1 \beta$

MP is the only rule of inference. Or $\mathbf{I^1_2}$ is $\mathbf{I^1} + \beta \leftrightarrow \sim_1 \sim_1 \beta$.

In the same paper [30] Lewin and Mikenberg give conditions for the maximality of these logics.

In what follows we construct a lattice of axiomatic extensions of propositional logic **LPPL**. Let $\mathbf{LPPL}_{\sim_1}$ be $\mathbf{LPPL} + \alpha \leftrightarrow \sim_1 \sim_1 \alpha$. We obtain the eight-element Boolean lattice from Figure 1. However, since (1) $\mathbf{P^1}$, $\mathbf{I^1}$ and $\mathbf{I^1P^1}$ are functionally equivalent, and (2) $\mathbf{P^1_2}$, $\mathbf{I^1_2}$ and $\mathbf{I^1_2}, \mathbf{P^1_2}$ are also functionally equivalent, the eight-element Boolean lattice given in Figure 1 collapses in the four-element Boolean lattice shown in Figure 2.

5.4.2. Lattice of paralogics

The four-valued paranormal logic $\mathbf{S}^4$, whose matrix is the smallest one being both paraconsistent and paracomplete, is presented in [30, p. 487].

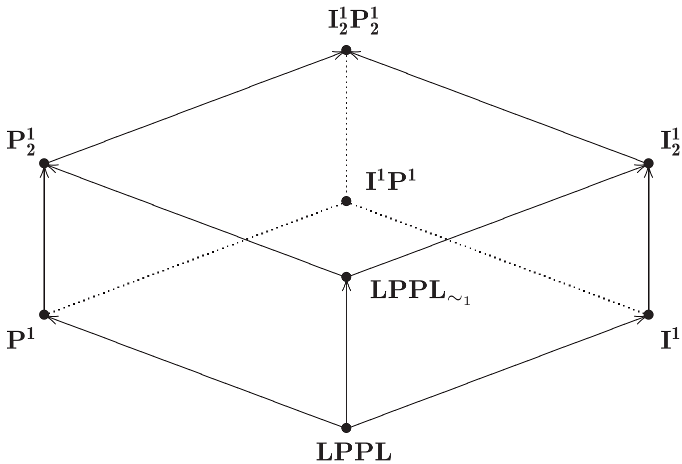

Figure 1. The eight-element Boolean lattice

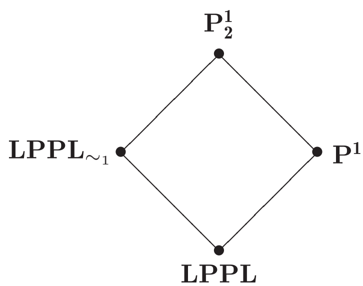

Figure 2. The four-element Boolean lattice

Nonetheless, we can single out a *three-valued* paranormal logic. Recall that the class $\mathcal{B}$ of Bochvar's three-valued logics includes three logics that are implicative extensions of Kleene's weak logic obtained by natural implications. One of them is the logic with natural implication $\rightarrow_4$.

Let's consider the matrix $\mathfrak{M}_3 = \langle\{0, {}^1\!/_2, 1\}, \sim, \rightarrow_4, \{1\}\rangle$. Disjunction $\vee$ and conjunction $\wedge$ are defined in the following way:

$$A \vee B := {\sim}A \rightarrow_4 B$$
$$A \wedge B := {\sim}({\sim}\, A \vee {\sim}\, B)$$

We shall call the logic associated with the above matrix $\mathbf{TK^1}$. One can easily prove that $\mathbf{TK^1}$ is paranormal, paraconsistent and paracom-

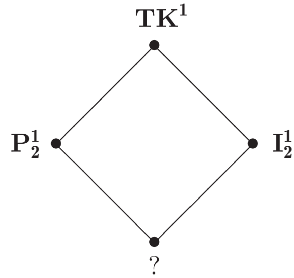

Figure 3. The missing logic in four-element Boolean lattice

plete. However, unlike $\mathbf{S}^4$, $\mathbf{TK^1}$ is not a maximal paranormal fragment of classical logic **CPC**. Even though the law of contraposition turns to be valid in $\mathbf{TK^1}$, the law of affirming consequent $A \to_4 (B \to_4 A)$ and the law of permutation $(A \to_4 (B \to_4 C)) \to_4 (B \to_4 (A \to_4 C))$ are not tautologies of $\mathbf{TK^1}$.[9]

As a result, all three natural implications from the logic of Bochvarian class $\mathcal{B}$ with the negation $\sim$ define three paralogics: paraconsistent logic $\mathbf{P^1_2}$ with connectives $\{\sim, \supset^\diamond\}$, paracomplete logic $\mathbf{I^1_2}$ with connectives $\{\sim, \supset^\square\}$ and paranormal logic $\mathbf{TK^1}$ with the following connectives $\{\sim, \to_4\}$, which is paraconsistent and paracomplete.

The existence of yet another logic, being neither paraconsistent nor paracomplete, would allow us to construct a lattice of logics with respect to the possession of one of the paraproperties (see Figure 3). Indeed, the missing logic can be constructed, and in order to do so it is necessary to generalize the concept of natural implication.

6. Strong and weak modus ponens: the final lattice

In an attempt to generalize the notion of natural implication, at least two different approaches may be undertaken. First, we can remove the requirement for the implication $\to$ to be C-extending. This immediately leads to the classes of logics and to the lattice of these classes, where the

[9] In the usual way, by using truth-tables to prove the independence of axioms, it can be shown that the addition of the axiom $(A \to_4 (B \to_4 C)) \to_4 (B \to_4 (A \to_4 C))$ to $\mathbf{TK^1}$ as a tautology does not change $\mathbf{TK^1}$ to **CPC**, because $A \to_4 (B \to_4 A)$ remains an independent axiom (see [24, p. 256, matrix 5]), and therefore it is still not verified in $\mathbf{TK^1}$.

supremum is a class consisting of logics which are functionally equivalent to Post's three-valued logic of $\mathbf{P_3}$. This new lattice cannot be constructed without an appropriate computer program.

The second approach consists in loosening the restriction of normality — in the sense of Łukasiewicz-Tarski [33, p. 134] — for the logical matrix. The latter condition appears as sufficient for the verification of modus ponens that preserves the designated value.

In [45, p. 70] Rescher pointed out the need "to distinguish between two ways in which a modus ponens principle can be operative in a system of many-valued logic". There are two different formulations:

(s) a stronger condition: whenever A and $A \to B$ both assume designated truth values, then B must also assume the designated value;
(w) a weaker condition: whenever A and $A \to B$ are both tautologies, then B must also be a tautology.

The symbolic formulation of rule modus ponens, corresponding to (s) and (w) can be represented as follows for any matrix $\mathfrak{M}$ (cf. [55]):

(s) $\forall v \in \mathrm{Val}(\mathfrak{M})$: if $v(A) \in D$ and $v(A \to B) \in D$, then $v(B) \in D$;
(w) if $\forall v \in \mathrm{Val}(\mathfrak{M})\ v(A) \in D$ and $\forall v \in \mathrm{Val}(\mathfrak{M})\ v(A \to B) \in D$, then $\forall v \in \mathrm{Val}(\mathfrak{M})\ v(B) \in D$.

where $\mathrm{Val}(\mathfrak{M})$ is the set of all valuations in $\mathfrak{M}$.

In the case of the matrix of two-valued propositional logic **CPC** and material implication both of the above formulations are true. Of course, (s) logically entails (w), for any matrix and any connective $\to$. However, where three-valued logics are concerned, the opposite does not hold.

In [55, 56] through the consideration of the example of a three-valued logic, it is proven that the difference between forms (s) and (w) of the modus ponens principle is of a fundamental nature. As a consequence, we end up having 18 three-valued logical matrices with implications satisfying conditions 1–3 from Section 3, where condition 2 on the normality of a logical matrix is replaced by the requirement for modus ponens to be a tautology-preserving rule (weak formulation of modus ponens). As a result, the set of three-valued logics — extensions of Kleene's weak three-valued logic $\mathbf{K_3^w}$ by natural implication — is divided into 10 disjoint classes. The most interesting finding concerns the enlargement of the class of Bochvarian logics $\mathcal{B}$ with one more logic (we denote it as $\mathbf{TK^2}$) having a new implication $\to_{29}$.

Let us now consider the logic $\mathbf{TK^2}$ determined by the matrix

$$\mathfrak{M} = \langle \{0, 1/2, 1\}, \sim, \to_{29}, \{1, 1/2\} \rangle,$$

where $\{1, 1/2\}$ is the set of designated values and $\to_{29}$ is defined by the following truth-table:

$\to_{29}$	1	1/2	0
1	1	1	0
1/2	1	1	1
0	1	1	1

Disjunction $\vee$ and conjunction $\wedge$ are defined in the following way:

$$A \vee B := \sim A \to_{29} B$$

$$A \wedge B := \sim(\sim A \vee \sim B)$$

It is easy to see that the law of Duns Scotus and the law of Clavius $A \to_{29} (\sim A \to_{29} B)$ and $(\sim A \to_{29} A) \to_{29} A$ are verified on the atomic level.

Thus, the infimum of the lattice in Figure 3 turns to be the logic $\mathbf{TK^2}$ which is neither paraconsistent nor paracomplete. As a result, the final lattice of paralogics (let us denote it by $\mathcal{TK}$) is presented in Figure 4.

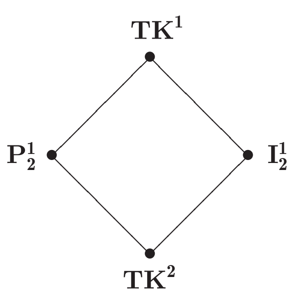

Figure 4. The lattice $\mathcal{TK}$

Theorem 1. *The logics $\mathbf{P^1_2}$, $\mathbf{I^1_2}$, $\mathbf{TK^1}$, and $\mathbf{TK^2}$ are pairwise functionally equivalent.*

Proof. The functional equivalence of logics $\mathbf{P^1_2}$ and $\mathbf{I^1_2}$, with connectives $\{\sim, \to_7\}$ and $\{\sim, \to_5\}$ respectively, where $\to_7$ is $\supset^{\diamond}$ and $\to_5$ is $\supset^{\square}$, follows from the definitions:

(d1) $A \to_5 B := \sim B \to_7 \sim A$
(d2) $A \to_7 B := \sim B \to_5 \sim A$

The functional equivalence of the logics $\mathbf{I}^1_2$ and $\mathbf{TK}^2$ with connectives $\{\sim, \to_5\}$ and $\{\sim, \to_{29}\}$, respectively, follows from the definitions:

(d3) $A \to_5 B := A \to_{29} \sim(B \to_{29} \sim A)$
(d4) $A \to_{29} B := \sim(A \to_5 B) \to_5 (\sim B \to_5 \sim A)$

From (d1), (d2), (d3), (d4) and the transitivity of the relation of functional equivalence it follows that the logics $\mathbf{TK}^2$ and $\mathbf{P}^1_2$ are functionally equivalent.

At this point, it suffices to prove the functional equivalence of the logics $\mathbf{P}^1_2$ and $\mathbf{TK}^1$ with connectives $\{\sim, \to_7\}$ and $\{\sim, \to_4\}$, respectively. This follows from the following definitions:

(d5) $A \to_7 B := \sim(A \to_4 B) \to_4 \sim(B \to_4 \sim A)$,
(d6) $A \to_4 B := \sim((\sim B \to_7 \sim A) \to_7 \sim(A \to_7 B))$.

From (d1), (d2), (d5), (d6) and the transitivity of the relation of functional equivalence it follows that the logics $\mathbf{TK}^2$ and $\mathbf{I}^1_2$ are also functionally equivalent. $\square$

THEOREM 2. *Let $\mathbf{B}^{\sim}_1$ be the class of all external formulas of Bochvar's three-valued logic $\mathbf{B_3}$. Let this class be defined by the Peirce's arrow γ (see Section 4.5) and extended by the connective $\sim$. Then the logic $\mathbf{I}^1_2$ with connectives $\{\sim, \supset^{\square}\}$ and the logic $\mathbf{B}^{\sim}_1$ with connectives $\{\sim, \gamma\}$ are functionally equivalent.*

PROOF. It is obvious that the connective $\supset^{\square}$ can be defined by Peirce's arrow γ. Moreover, we put:

$$\begin{aligned}
\rceil A &:= \sim(\sim A \supset^{\square} A)\\
\lceil A &:= A \supset^{\square} \sim A\\
A \cup^{\square} B &:= \lceil A \supset^{\square} B\\
A \cap^{\square} B &:= \lceil(A \supset^{\square} \lceil B)\\
A \equiv^{\square} B &:= (\rceil A \cap^{\square} \lceil B) \cup^{\square} (\lceil A \cap^{\square} \rceil B) \qquad \text{(Sh)}
\end{aligned}$$
$\square$

COROLLARY 1. *The logics $\mathbf{P}^1_2$, $\mathbf{I}^1_2$, $\mathbf{TK}^1$, and $\mathbf{TK}^2$ are functionally equivalent to $\mathbf{B}^{\sim}_1$.*

COROLLARY 2. *The logics $\mathbf{P}^1_2$, $\mathbf{I}^1_2$, $\mathbf{TK}^1$, and $\mathbf{TK}^2$ are axiomatized as an extension of classical logic **CPC** by the Anshakov-Rychkov's method.*[10]

[10] See the end of Section 5.3.

In conclusion, we would like to refer to a class appearing in [18, §8], in the list of the eleven precomplete classes of B_3, namely $B_1^{\sim}$. The functional precompleteness of the class $B_1^{\sim}$ in B_3 follows from Finn's Theorem [18] by the criteria of functional completeness of the class of functions of B_3. This means that the class of functions $B_1^{\sim}$ is the *minimal* class that generates the lattice $\mathcal{TK}$.

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