Logic and **L**ogical **P**hilosophy
Volume 26 (2017), 19–61
DOI: 10.12775/LLP.2016.017

Gerard Allwein, William L. Harrison, Thomas Reynolds

DISTRIBUTED RELATION LOGIC

Abstract. We extend the relational algebra of Chin and Tarski so that it is multisorted or, as we prefer, *typed*. Each type supports a *local* Boolean algebra outfitted with a converse operator. From Lyndon, we know that relation algebras cannot be represented as proper relation algebras where a proper relation algebra has binary relations as elements and the algebra is singly-typed. Here, the intensional conjunction, which was to represent relational composition in Chin and Tarski, spans three different local algebras, thus the term *distributed* in the title. Since we do not rely on proper relation algebras, we are free to re-express the algebras as typed. In doing so, we allow many different intensional conjunction operators.

We construct a typed logic over these algebras, also known as *heterogeneous algebras* of Birkhoff and Lipson. The logic can be seen as a form of relevance logic with a classical negation connective where the Routley-Meyer star operator is reified as a converse connective in the logic. Relevance logic itself is not typed but our work shows how it can be made so. Some of the properties of classical relevance logic are weakened from Routley-Meyer's version which is too strong for a logic over relation algebras.

Keywords: relation algebra; multisorted algebra; distributed logic; Kripke frames; Kripke models

1. Introduction

We present a typed logic over a hypergraph where each node is a local logic and each edge connects three nodes, which we will call *cliques*, deviating slightly from the use of that term in graph theory. Cliques are not directed in the sense of considering the first two elements of the

Received December 4, 2015. *Revised* July 11, 2016. *Published online* July 19, 2016

clique as inputs or arguments and the third is the output or result. Connectives defined using cliques are considered directed. The reason for the difference is that we may use a single clique to define several different connectives. The connectives, under interpretation, become morphisms of a multicategory. The term *distributed* refers to the logic being distributed among several types which are *local* classical propositional logics with an added converse connective. This results in a hypergraph of local logics with distributed *fusion* or intensional conjunction and two entailment connectives providing the connections among the local logics.

The logic is shown to be sound and complete with respect to an appropriate class of heterogeneous, i.e., typed, algebras and a class of typed Kripke models. The algebras and Kripke frames naturally form multicategories and so the logic has been augmented to specify the additional categorical structure. The algebras then form an essentially algebraic category. The three place Kripke relation is similar to the Kripke relation used in relevance logics except that it is now typed, or rather is a morphism in a multicategory. The modeling extends that in [2] by including the typing schemes.

The fusion operator, $\circ$, of relevance logic's algebraic models is relation algebra's relational composition. Two entailment connectives, $-\!\!\triangleright$ and $\triangleleft\!\!-$, can be defined [16]. Relevance and relation logics' entailment connectives are "residuated" with, or adjoint to, $\circ$. The relation logic is a classical relevance logic without a permutation axiom; the permutation axiom expresses the commutation of fusion. The relation logic also lacks contraction of fusion, and the logic has a classical negation as opposed to relevance logic's usual De Morgan negation. The lack of permutation results in there being two entailment arrows. The logic also includes a converse, $\breve{\ }$, connective for each type. The Routley-Meyer [18] star operator is similar to the converse operator of relation algebras, but converse is a weakened form of the star operator, this latter is cited in [18]. The residuation axiom of [9] is much clearer when expressed using relevance logic's entailment arrows. Residuation is also brought up in [12] and [14].

The logic, through the use of defined the entailment connectives, becomes easier to understand in light of relevance logic's entailment connectives. The main difference is that the De Morgan negation of relevance logic is now defined in terms of Boolean negation and converse. Relation logic thus allows the definition of right and left entailments similar to relevance logic (lacking permutation). The use of these defined connectives simplifies an axiom of relation algebra which expresses residuation although in a somewhat opaque form. Residuation of entailment with fusion is easier to understand.

The relation logic we present is considered a class of relation logics. The distribution structure is parametric for the logic in the sense that the distribution structure (and hence the types) are chosen based upon an application area such as System-on-a-Chip (SoC) architectures. A typical SoC will contain one or more digital processors, some input-output circuitry, some on-board memory, etc. These SoCs appear in just about all new devices. New automobiles will contain several; the Internet-of-Things uses SoCs for control and communication.

SoC architectures have several subcomponents. Each subcomponent of an SoC is an entire world unto itself; each subcomponent typically forms a state machine. As logicians, we know state machines as generators of Kripke frames. We are developing distributed logics to take advantage of the information available in the distributed structure. The relationships among subcomponents are modeled in terms of Kripke relations. These distributed relations are not next-state relations such as arise in individual subcomponents but rather co-occurrence relations. They relate states in different subcomponents that obtain simultaneously or are induced by subcomponents controlling others; the subcomponents, being hardware, all run in parallel.

In a previous paper [4], we showed how to abstract over SoCs using two-place relations. The resulting single-place connectives were distributed in the sense of taking formulas in one local logic to formulas in another. These were modal connectives because the relations were two-placed and the connectives single place. However, not all relations are so simple. In this current paper, we extend our work to three place relations which are used to interpret two-place distributed connectives. Each distributed connective takes as its arguments formulas from two local logics and results in a formula in a third local logic. One could easily add in the modal connectives from [4]. Indeed, most applications of this logic would require both kinds of distributed connectives. We chose to abstract relation algebras because they have a well-developed theory and can be considered a paradigm case. It is convenient that they are very close to the algebras used in relevance logic. Slight modifications to the work in this paper can be used to define distributed relevance logics. We now have a new collection of tools for proving properties about SoC architectures.

2. Distributed Relation Logic (DRel)

A relation logic over the algebraic models of Tarski's relation algebras is a version of classical relevance logic. The logic is centered around classical logic as a base just as relation algebra is centered around a Boolean algebra. Fusion (relation composition) is represented in the logic using the symbol, $\circ$, and converse using $\breve{\ }$. In distributed relation logic, the intuition behind $\circ$ is not the same as it is for a singly-typed relation logic, where it is interpreted as relation composition in the latter. Rather, since the composition may have distinct input and output types, it is a generalized or *distributed* semigroup connective. Of course, one could collapse the input and output types to be all the same type to recover the intuition. However, the logic would then lose its distributed expressibility. The intuitive picture is as follows:

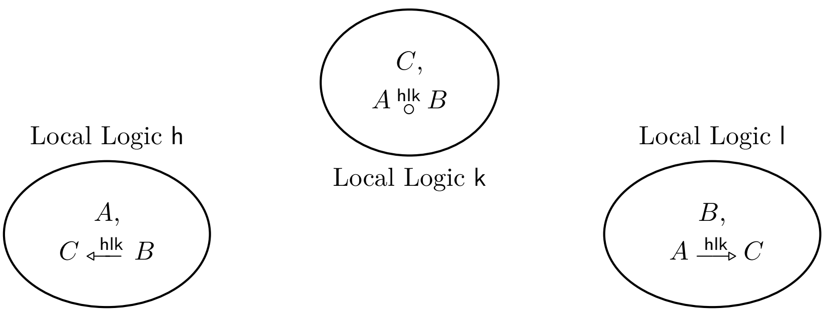


Figure 1. Distributing the two-place connectives

using the following convention

Connective	Type h	Type l	Type k
$\overset{\mathsf{hlk}}{\circ}$	A	B	$A \overset{\mathsf{hlk}}{\circ} B$
$\xrightarrow{\mathsf{hlk}}$	A	$A \xrightarrow{\mathsf{hlk}} C$	C
$\xleftarrow{\mathsf{hlk}}$	$C \xleftarrow{\mathsf{hlk}} B$	B	C

Note the positions of the input and output types of each operator shift depending upon the operator. This matches the corresponding position of the formula containing the operator in the semantic definitions of interpreting relations:

- $z \models^{\mathsf{k}} A \overset{\mathsf{hlk}}{\circ} B$ iff $\exists x, y(x \models^{\mathsf{h}} A$ and $y \models^{\mathsf{l}} B$ and $\mathcal{R}^{\mathsf{hlk}}xyz)$,
- $y \models^{\mathsf{l}} A \xrightarrow{\mathsf{hlk}} C$ iff $\forall x, z(x \models^{\mathsf{h}} A$ and $\mathcal{R}^{\mathsf{hlk}}xyz$ implies $z \models^{\mathsf{k}} C)$,
- $x \models^{\mathsf{h}} C \xleftarrow{\mathsf{hlk}} B$ iff $\forall y, z(\mathcal{R}^{\mathsf{hlk}}xyz$ and $y \models^{\mathsf{l}} B$ implies $z \models^{\mathsf{k}} C)$.

Notice also that the $\models$ sign must be typed as indicated. When a formula will work for any collection of types, we tend to leave the types off. So in classical logic, $\vdash A$ implies $\vdash T \supset A$; the typing structure could be used to display this as $\overset{\mathsf{h}}{\vdash} A$ implies $\overset{\mathsf{h}}{\vdash} T \supset A$ but we frequently elide the extra "noise" of the types where the types are generic or clear from the context.

The extensional $\supset$ connective is the usual classical entailment that associates to the left. $\wedge$ and $\vee$ are classical logic's extensional conjunction and disjunction. $\neg$ is classical logic's negation. The $\neg$ and $^\smile$ connectives bind the tightest, $\circ$ binds less strongly than $\neg$. And $\supset$, $\wedge$, and $\vee$ bind least tightest of all. The Boolean connectives $\vee$ and $\wedge$ are derived in the usual way from $\supset$ and $\neg$, we assume a constant T (extensional truth), and the constant F (extensional falsity), defined as $\neg T$. $A \equiv B$ is short hand for $A \supset B$ and $B \supset A$. The constant t (intensional truth) corresponds to a monoid identity and in general, $t \neq T$. t cannot be an identity for a distributed semigroup connective unless all the types are the same because otherwise it is possible to show a contradiction.

There is a delicate point to be made about working in the typed structure. Considering Axiom B4 below (the change in font for the type indices is explained in the next section)

$$(A \overset{\mathsf{hlk}}{\circ} B)^\smile \overset{\mathsf{k}}{\equiv} B^\smile \overset{\mathsf{lhk}}{\circ} A^\smile, \text{ for any } A \in \mathsf{h},\ B \in \mathsf{l},$$

the connectives $\overset{\mathsf{hlk}}{\circ}$ and $\overset{\mathsf{lhk}}{\circ}$ are two different connectives. In the untyped system, the axiom

$$(A \circ B)^\smile \equiv B^\smile \circ A^\smile$$

refers to the same connective on both sides of the equivalence sign. This axiom could be expressed with two connectives as in

$$(A \circ B)^\smile \equiv B^\smile \breve{\circ} A^\smile$$

and adding another axiom

$$(A \breve{\circ} B)^\smile \equiv B^\smile \circ A^\smile.$$

This breaks the intuitive semantics of treating $\circ$ as relation composition. However, the semantics presented in this paper is not that semantics. The way to see that the two connectives are clearly two distinct connectives is to look at their semantics using a single Kripke frame. The two connectives have the semantics

$$z \models A \circ B \text{ iff } \forall x, y(x \models A \text{ and } y \models B \text{ and } \mathcal{R}xyz),$$
$$z \models A \breve{\circ} B \text{ iff } \forall x, y(x \models A \text{ and } y \models B \text{ and } \mathcal{R}yxz).$$

So $\breve{\circ}$ is a "twisted" version of $\circ$. The two proposed axioms state the relationship between the two connectives.

It is now a short step to give the connectives typing information. If each argument position had its own type, then we could type these connectives as

$$\circ\colon \mathsf{h} \times \mathsf{l} \to \mathsf{k}, \qquad \breve{\circ}\colon \mathsf{l} \times \mathsf{h} \to \mathsf{k}.$$

To keep the typing from becoming too abstract, we display these as

$$\overset{\mathsf{hlk}}{\circ}\colon \mathsf{h} \times \mathsf{l} \to \mathsf{k}, \qquad \overset{\mathsf{lhk}}{\circ}\colon \mathsf{l} \times \mathsf{h} \to \mathsf{k}.$$

Notice that if we collapse the two argument types together, say to h, then the typing information is not sufficient to distinguish two different connectives. However, if the two argument types are different, then the typing information suffices to disambiguate the two connectives. In the sequel, we use the most general typing of the connectives and rarely collapse types. However, in any particular distributed system, this is unlikely to be the case and some other notational convention must be adopted so as not to collapse connectives inadvertently.

All of the connectives used in the axioms that use the same types but with different ordering of the types are interpreted by different three place relations. The relations are related themselves via a $\breve{}$ operation applied to the points of the Kripke frame used in the interpretation. As an example, the axiom

$$(A \overset{\mathsf{hlk}}{\circ} B)^{\smile} \overset{\mathsf{k}}{\equiv} B^{\smile} \overset{\mathsf{lhk}}{\circ} A^{\smile}$$

requires the following condition

$$\mathcal{R}^{\mathsf{hlk}}xyz \text{ iff } \mathcal{R}^{\mathsf{lhk}}y^{\smile}x^{\smile}z.$$

Technically, these are two different relations, however they are related by a switch in the types of the first two positions and the use of $\breve{}$. If one wanted to collapse the argument types to a common type yet still continue to have two connectives, one needs two different relation symbols, say, $\mathcal{R}^{\mathsf{hhk}}$ and $\mathcal{S}^{\mathsf{hhk}}$. So their types are the same but the two relations will interpret respectively two different generalized semigroup connectives. The axiom with collapsed types could then need to be stated as

$$(A \overset{\mathsf{hhk}}{\circ} B)^{\smile} \overset{\mathsf{k}}{\equiv} B^{\smile} \overset{\mathsf{hhk}}{\bullet} A^{\smile},$$

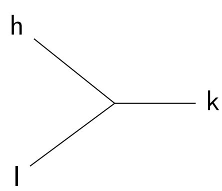
where it is clear there are two connectives involved which cannot be disambiguated via the typing information alone.

In axioms and rules below, the typing information should be considered as variables. Depending upon the application, the graph of logics and the connectives are chosen. So one includes as many instances of $(A \overset{\mathsf{hlk}}{\circ} B)^{\smile} \overset{\mathsf{k}}{\equiv} B^{\smile} \overset{\mathsf{lhk}}{\circ} A^{\smile}$ as one needs changing the types to reflect the necessary connectives for the application at hand. In our parlance, one adds as many cliques as is necessary to specify the distribution structure. In this sense, the axioms and rules below give only schemes. The situation is very similar to an "axiom" $A \supset A$ from classical logic. This is really only an axiom scheme where the A is a metalinguistic variable. One is supposed to fill in particular object language proposition symbols in any application.

Classical propositional logic is not axiomatized here. You may pick your favorite axiomatization although it is easier to think of just using all the classical truth functional theorems. When a proof requires classical propositional logic's axioms and rules, and a step should be achievable with someone who has experience with classical propositional logic, we frequently elide citing classical logic. Many of the proofs are contained in the Appendix.

2.1. Axioms and Rules

Distribution Structure. The distribution structure is a hypergraph of a collection of nodes with certain tuples identified as cliques, each clique has three elements, say, hlk. In diagram form:



We need a way of abstracting over cliques in a notationally convenient way. We use a string such as hlk to refer to an arbitrary clique. For any one clique hlk, we assume h refers to a node h and similarly for the rest. When we wish to restrict reference to a clique such that the clique has all the same members, we use hhh. We will also have need to abstract over the members of an arbitrary clique, we use the variables h, l, k to range over an arbitrary clique hlk with the restriction that the variables refer to pairwise distinct positions in hlk. $h, k, l \Subset$ hlk denotes this. h can take on any of the values h, k, and l and similarly with l and k, respecting the pairwise distinct restriction. When abstracting over the clique hhh, the variables h, l, and k are still respecting the condition of no two referring to the same position in the clique. The locution $A \in \mathsf{h}$ means that the formula A is a member of the local logic at node h.

Alternatively, one could require that the set of cliques, $\mathfrak{C}$, be *full* in the sense that if $\mathsf{hlk} \in \mathfrak{C}$, then $\mathsf{lkh}, \mathsf{khl}, \mathsf{lhk}, \mathsf{hkl}, \mathsf{klh} \in \mathfrak{C}$. That is, the set of cliques contains all permutations. However, this seems to complicate the exposition of the logic with no increase in clarity.

The Logic DRel

Axiom Scheme: Group G (Directed Hypergraph).

G1. A set $\mathfrak{G}$ of nodes

G2. A set $\mathfrak{C}$ of cliques

G3. A set $\{\overset{\mathsf{hlk}}{\circ}, \overset{\mathsf{lkh}}{\circ}, \overset{\mathsf{khl}}{\circ}, \overset{\mathsf{lhk}}{\circ}, \overset{\mathsf{hkl}}{\circ}, \overset{\mathsf{klh}}{\circ}\}$ of distributed semigroup connectives for each $\mathsf{hlk} \in \mathfrak{C}$

Axiom Scheme: Group A. For each node h in $\mathfrak{G}$,

A1. A local classical logic

A2. $(A \vee B)^{\smile} \overset{\mathsf{h}}{\supset} A^{\smile} \vee B^{\smile}$

A3. $A^{\smile\smile} \overset{\mathsf{h}}{\equiv} A$

where $A^{\smile\smile} \overset{\mathsf{h}}{\equiv} A$ stands for $A^{\smile\smile} \overset{\mathsf{h}}{\supset} A$ and $A \overset{\mathsf{h}}{\supset} A^{\smile\smile}$.

Axiom Scheme: Group B. For $A \in \mathsf{h}$, $B, \hat{B} \in \mathsf{l}$, and $C \in \mathsf{k}$:

B1. $((A \overset{\mathsf{hlm}}{\circ} B) \overset{\mathsf{mko}}{\circ} C) \overset{\mathsf{o}}{\equiv} (A \overset{\mathsf{hno}}{\circ} (B \overset{\mathsf{lkn}}{\circ} C))$ $\quad \mathsf{hlm}, \mathsf{mko}, \mathsf{hno}, \mathsf{lkn} \in \mathfrak{C}$

B2. $A \overset{\mathsf{hhh}}{\circ} t \overset{\mathsf{h}}{\equiv} A$ $\quad \mathsf{hhh} \in \mathfrak{C}$

B3. $A \overset{hlk}{\circ} (B \vee \hat{B}) \overset{k}{\supset} (A \overset{hlk}{\circ} B) \vee (A \overset{hlk}{\circ} \hat{B})$ $\quad h, k, l \in \mathsf{hlk} \in \mathfrak{C}$

B4. $(A \overset{hlk}{\circ} B)^{\smile} \overset{k}{\equiv} B^{\smile} \overset{lhk}{\circ} A^{\smile}$ $\quad h, l, k \in \mathsf{hlk} \in \mathfrak{C}$

B5. $A^{\smile} \overset{hlk}{\circ} \neg(A \overset{hkl}{\circ} C) \overset{k}{\supset} \neg C$ $\quad h, l, k \in \mathsf{hlk} \in \mathfrak{C}$

The Group B axioms do not specify that any clique actually be in $\mathfrak{C}$. Rather, if the clique does exist in $\mathfrak{C}$, then the necessary axioms govern the logic's behavior. In particular, there is not necessarily a clique $\mathsf{hhh} \in \mathfrak{C}$ for every h. Also, the use of variables over a clique generates several axioms. That is, for Axiom B4, there are the axioms

$$(A \overset{\mathsf{hlk}}{\circ} B)^{\smile} \overset{\mathsf{k}}{\equiv} B^{\smile} \overset{\mathsf{lhk}}{\circ} A^{\smile} \qquad (B \overset{\mathsf{lhk}}{\circ} A)^{\smile} \overset{\mathsf{k}}{\equiv} A^{\smile} \overset{\mathsf{hlk}}{\circ} B^{\smile}$$

$$(C \overset{\mathsf{klh}}{\circ} B)^{\smile} \overset{\mathsf{h}}{\equiv} B^{\smile} \overset{\mathsf{lkh}}{\circ} C^{\smile} \qquad (B \overset{\mathsf{lkh}}{\circ} C)^{\smile} \overset{\mathsf{h}}{\equiv} C^{\smile} \overset{\mathsf{klh}}{\circ} B^{\smile}$$

$$(C \overset{\mathsf{khl}}{\circ} A)^{\smile} \overset{\mathsf{l}}{\equiv} A^{\smile} \overset{\mathsf{hkl}}{\circ} C^{\smile} \qquad\qquad (A \overset{\mathsf{hkl}}{\circ} C)^{\smile} \overset{\mathsf{l}}{\equiv} C^{\smile} \overset{\mathsf{khl}}{\circ} A^{\smile}$$

The last three axioms are derivable from the first three axioms and vice versa.

The Group C axioms specify multicategory structure:

Axiom Scheme: Group C. For $\mathsf{h_1h_2h_3},\ \mathsf{l_1l_2l_3},\ \mathsf{h_3l_3o} \in \mathfrak{C}$

C1. $$\overset{\mathsf{h_3l_3o}}{\circ}(\overset{\mathsf{h_1h_2h_3}}{\circ} \times \overset{\mathsf{l_1l_2l_3}}{\circ})\langle\langle A, \hat{A}\rangle, \langle B, \hat{B}\rangle\rangle \overset{\mathsf{o}}{\equiv} \left[\overset{\mathsf{h_3l_3o}}{\circ} \cdot (\overset{\mathsf{h_1h_2h_3}}{\circ} \times \overset{\mathsf{l_1l_2l_3}}{\circ})\right]\langle\langle A, \hat{A}\rangle, \langle B, \hat{B}\rangle\rangle$$

where the formulas are in prefix form to make the axiom easier to state. This will be explained further in section 2.3.

C2. $[\iota]\, A \overset{\mathsf{h}}{\equiv} A$

where $[\iota]$ is a modal connective that is akin to a no-op in computer assembly code. It does nothing except force the models, as multicategories, to have identities.

Deduction Rules

$$\frac{A \qquad A \overset{\mathsf{h}}{\supset} B}{B}\ \text{Modus Ponens} \qquad\qquad \frac{A \overset{\mathsf{h}}{\supset} B}{A^{\smile} \overset{\mathsf{h}}{\supset} B^{\smile}}\ {}^{\smile}\ \text{Monotonicity}$$

$$\frac{B \overset{l}{\supset} \hat{B} \quad A \in h, \quad h, l, k \Subset \mathsf{hlk} \in \mathfrak{C}}{A \overset{hlk}{\circ} B \overset{k}{\supset} A \overset{hlk}{\circ} \hat{B}}\ \text{Left} \circ \text{Monotonicity}$$

LEMMA 2.1.1. *The rule*

$$\frac{A \overset{h}{\supset} \hat{A}, \quad B \in l, \quad h, l, k \Subset \mathsf{hlk} \in \mathfrak{C}}{A \overset{hlk}{\circ} B \overset{\mathsf{k}}{\supset} \hat{A} \overset{hlk}{\circ} B}\ \text{Right} \circ \text{Monotonicity}$$

is derivable.

The axioms and rules have been set up to keep track of the types automatically. In a sparse distribution structure where only some connectives are present, one needs to be sure of the availability of a connective to know whether the formulas in the axioms of a proof can be generated. The rules Modus Ponens and $\smile$ Monotonicity are not at issue because they do not involve any distributed connectives, the local connectives are all assumed present. The only rule at issue is Left $\circ$ Monotonicity. This rule will correctly use the required types. As in the axioms, this rule is actually six rules for all the instantiations of h, l, and k.

Theorem 2.1.2 (Replacement Theorem). *The connectives preserve equivalence, i.e., for* $\bullet \in \{\wedge, \vee, \circ\}$ *for a generic typed* $\circ$,

$$\frac{A \equiv B}{A \bullet C \equiv B \bullet C} \qquad \frac{A \equiv B}{A^{\smile} \equiv B^{\smile}} \qquad \frac{A \equiv B}{\neg A \equiv \neg B}$$

Proof. Replacement Theorem holds for classical propositional logic. The two connectives we must be concerned with are $^{\smile}$ and $\circ$. The Rule $^{\smile}$ Monotonicity clearly causes $^{\smile}$ to preserve $\equiv$. Let $h, l, k \mathrel{\underline{\in}} \mathsf{hlk} \in \mathfrak{C}$, and assume $B \overset{l}{\equiv} \hat{B}$ and $A \in h$. Hence $B \overset{l}{\supset} \hat{B}$ and $A \overset{hlk}{\circ} B \overset{k}{\supset} A \overset{hlk}{\circ} \hat{B}$. Similarly, $\hat{B} \overset{l}{\supset} B$ and $A \overset{hlk}{\circ} \hat{B} \overset{k}{\supset} A \overset{hlk}{\circ} B$. Consequently, $A \overset{hlk}{\circ} B \overset{k}{\equiv} \hat{A} \overset{hlk}{\circ} B$. An analogous argument shows that $A \overset{h}{\equiv} \hat{A}$ and $B \in l$ implies $A \overset{hlk}{\circ} B \overset{k}{\equiv} A \overset{hlk}{\circ} \hat{B}$. So, the rule Left $\circ$ Monotonicity and Lemma 2.1.1 guarantee that $\circ$ preserves equivalence. □

Theorem 2.1.3 (Congruence Theorem). $\equiv$ *is a congruence relation.*

The proof is the fact $\equiv$ is reflexive, symmetric, transitive, and preserves substitution (through the use of meta-linguistic variables in the axioms and rules). $\equiv$ satisfies Replacement from Theorem 2.1.2 and the fact that no extensional formula of the form $A \supset B$ spans more than a single type, i.e., must be of the form $A \overset{\mathsf{h}}{\supset} B$.

2.2. Alternate Axioms and Rules

The following theorem is useful:

Theorem 2.2.1. $(\neg A)^{\smile} \equiv \neg(A^{\smile})$.

Thus we rarely distinguish between either sides of the equivalence and use $\neg A^{\smile}$. De Morgan negation is defined:

$$\sim A \overset{\text{def}}{\equiv} \neg A^{\smile}.$$

An alternate axiom for B5 is

B5$^{\smile}$. $\neg(C \overset{khl}{\circ} A) \overset{lhk}{\circ} A^{\smile} \overset{k}{\supset} \neg C$

Theorem 2.2.2. *Axioms* B5 *and* B5$^{\smile}$ *are inter-derivable by Axiom* B4.

Axiom B5 appears ungainly, and Chin and Tarski [9] also thought so:

> Among the postulates just listed, B5 has a somewhat more involved structure and a less clear algebraic content than the remaining postulates.

The axiom will become quite a bit clearer using the right residual of $\circ$. The definitions for the residuals are:

DEFINITION 2.2.3. For $h, l, k \Subset \mathsf{hlk} \in \mathfrak{C}$

Formula	De Morgan Encoding	Classical Encoding	Name
$A \xrightarrow{hlk} C$	$\sim(\sim C \overset{khl}{\circ} A)$	$\neg(A^{\smile} \overset{hkl}{\circ} \neg C)$	right residual
$C \xleftarrow{hlk} B$	$\sim(B \overset{lkh}{\circ} \sim C)$	$\neg(\neg C \overset{klh}{\circ} B^{\smile})$	left residual

Axiom B5 may also be replaced with either $\rightarrow$ Residuation or $\leftarrow$ Residuation via the following two-way rules:

Rule Residuation

$$\frac{A \overset{hlk}{\circ} B \overset{k}{\supset} C}{B \overset{l}{\supset} A \xrightarrow{hlk} C} \rightarrow \text{Residuation} \qquad \frac{A \overset{hlk}{\circ} B \overset{k}{\supset} C}{A \overset{h}{\supset} C \xleftarrow{hlk} B} \leftarrow \text{Residuation}$$

THEOREM 2.2.4. *Rule $\rightarrow$ Residuation is derivable from Axiom B5, and Rule $\leftarrow$ Residuation is derivable from Axiom B5$^{\smile}$.*

Given the monotonic properties of $\circ$, Rule $\rightarrow$ Residuation can be replaced with two axioms

B6. $A \overset{hlk}{\circ} (A \xrightarrow{hlk} C) \overset{k}{\supset} C$ B7. $B \overset{l}{\supset} A \xrightarrow{hlk} (A \overset{hlk}{\circ} B)$

and Rule $\leftarrow$ Residuation can be replaced with,

B8. $(C \xleftarrow{hlk} B) \overset{hlk}{\circ} B \overset{k}{\supset} C$ B9. $A \overset{h}{\supset} (A \overset{hlk}{\circ} B) \xleftarrow{hlk} B$

Axiom B6 is the encoded form of Axiom B5. Considering the proof of Theorem 2.2.4, it is clear these axiom sets are not as finely chopped as they could be. We only need the bottom to top of the Rule $\rightarrow$ Residuation to generate Axiom B6. From that we can generate the top to bottom of Rule $\rightarrow$ Residuation. That allows us to generate Axiom B7.

Similar statements hold for Axiom B5$^{\smile}$ and the Rule $\leftarrow$ Residuation. Since Axiom B5$^{\smile}$ can be generated from Axiom B5 and vice versa, one only needs one of Axiom B5$^{\smile}$, Axiom B5, and one of the bottom to top of Rules $\rightarrow$ Residuation and $\leftarrow$ Residuation.

COROLLARY 2.2.5. *Axioms B5$^{\smile}$ and B5, and each of the bottom to top of rules $\rightarrow$ Residuation and $\leftarrow$ Residuation are all inter-derivable.*

The following formulas show the relationships between $^{\smile}$ and $\multimap$ and between $^{\smile}$ and $\leftarrow$. They reveal that contraposition using the De Morgan negation and the residuals is "mediated" by $^{\smile}$ in DRel:

D1. $(A \xrightarrow{hlk} {\sim} C)^{\smile} \overset{l}{\supset} C^{\smile} \xrightarrow{klh} {\sim} A^{\smile}$

D2. $({\sim} C \xleftarrow{hlk} B)^{\smile} \overset{h}{\supset} {\sim} B^{\smile} \xleftarrow{hkl} C^{\smile}$

2.3. Multicategories

The algebraic and relational models will be multicategories, so the mechanics must be provided for in the logic. We assume the syntax of the categorical product of categories construction where $\vec{X}$ and $\vec{Y}$ are objects of a product of categories and the morphisms need to be combined as products to handle the prefixing notation. Let X, $\hat{X}$, Y, and $\hat{Y}$ be objects in a category:

$$f : X \to Y,\ g : \hat{X} \to \hat{Y} \text{ iff } f \times g : X \times \hat{X} \to Y \times \hat{Y}.$$

Given $\overset{\mathsf{h_3l_3o}}{\circ} : \mathsf{h}_3 \times \mathsf{l}_3 \to \mathsf{o}$, and $\overset{\mathsf{h_1h_2h_3}}{\circ} : \mathsf{h}_1 \times \mathsf{h}_2 \to \mathsf{h}_3$ and $\overset{\mathsf{l_1l_2l_3}}{\circ} : \mathsf{l}_1 \times \mathsf{l}_2 \to \mathsf{l}_3$, we need to generate

$$\overset{\mathsf{h_3l_3o}}{\circ} \cdot \left(\overset{\mathsf{h_1h_2h_3}}{\circ} \times \overset{\mathsf{l_1l_2l_3}}{\circ}\right) : (\mathsf{h}_1 \times \mathsf{h}_2) \times (\mathsf{l}_1 \times \mathsf{l}_2) \to \mathsf{o}\,.$$

where the $\cdot$ is the multicategory composition operator; note that o is a type, not an connective in $\mathsf{h_3l_3o}$. We will also need formulas in prefix form instead of their usual infix. We repeat the axiom for the reader:

C1. $\overset{\mathsf{h_3l_3o}}{\circ}\left(\overset{\mathsf{h_1h_2h_3}}{\circ} \times \overset{\mathsf{l_1l_2l_3}}{\circ}\right)\langle\langle A, \hat{A}\rangle, \langle B, \hat{B}\rangle\rangle \circeq$
$\left[\overset{\mathsf{h_3l_3o}}{\circ} \cdot \left(\overset{\mathsf{h_1h_2h_3}}{\circ} \times \overset{\mathsf{l_1l_2l_3}}{\circ}\right)\right]\langle\langle A, \hat{A}\rangle, \langle B, \hat{B}\rangle\rangle$

for $\mathsf{h_1h_2h_3}$, $\mathsf{l_1l_2l_3}$, $\mathsf{h_3l_3o} \in \mathfrak{C}$.

The order of application for the left side formula is to the right, i.e.,

$$\overset{\mathsf{h_3l_3o}}{\circ}\left(\overset{\mathsf{h_1h_2h_3}}{\circ} \times \overset{\mathsf{l_1l_2l_3}}{\circ}\right)\langle\langle A, \hat{A}\rangle, \langle B, \hat{B}\rangle\rangle \overset{\text{def}}{=} \overset{\mathsf{h_3l_3o}}{\circ}\left[\left(\overset{\mathsf{h_1h_2h_3}}{\circ} \times \overset{\mathsf{l_1l_2l_3}}{\circ}\right)\langle\langle A, \hat{A}\rangle, \langle B, \hat{B}\rangle\rangle\right]$$

The prefixing and reordering are contained in the following definition:

DEFINITION 2.3.1. Base Case: $\circ\langle A, B\rangle \overset{\text{def}}{=} A \circ B$.

Induction steps:

$$\frac{\Gamma\left(\langle \vec{\circ}_1 \vec{A}, \vec{\circ}_2 \vec{B}\rangle\right)}{\Gamma\left((\vec{\circ}_1 \times \vec{\circ}_2)\langle \vec{A}, \vec{B}\rangle\right)}\ \textit{App Reorder}$$

$$\frac{\Gamma\Big((\vec{\circ}_2\vec{\circ}_i) \times (\vec{\circ}_3\vec{\circ}_j)\Big)}{\Gamma\Big((\vec{\circ}_2 \times \vec{\circ}_3) \cdot (\vec{\circ}_i \times \vec{\circ}_j)\Big)} \; \textit{Arr Reorder}$$

where Γ is some outside context; also $\vec{\circ}_i$ and $\vec{\circ}_j$ means that $\vec{\circ}_i$ and $\vec{\circ}_j$ may be (possibly) nested sequences of $\circ$, e.g., $\circ\langle\circ,\circ\rangle$ (properly typed, of course). $\vec{A}$ and $\vec{B}$ are formulas possibly in prefix notation and possibly with nested sequences of formulas.

Theorem 2.3.2. *Letting*

$$\vec{X} \stackrel{\text{def}}{=} \langle\langle A, \hat{A}\rangle, \langle B, \hat{B}\rangle\rangle, \quad \vec{Y} \stackrel{\text{def}}{=} \langle\langle C, \hat{C}\rangle, \langle D, \hat{D}\rangle\rangle,$$

the following equivalence is provable:

$$\Big[{}^{\mathsf{n}_1}\underset{\circ}{\mathsf{n}_2}{}^{\mathsf{o}} \cdot \big[\big({}^{\mathsf{h}_3}\underset{\circ}{\mathsf{l}_3}{}^{\mathsf{n}_1} \times {}^{\mathsf{k}_3}\underset{\circ}{\mathsf{m}_3}{}^{\mathsf{n}_2}\big) \cdot \big(\big({}^{\mathsf{h}_1}\underset{\circ}{\mathsf{h}_2}{}^{\mathsf{h}_3} \times {}^{\mathsf{l}_1}\underset{\circ}{\mathsf{l}_2}{}^{\mathsf{l}_3}\big) \times \big({}^{\mathsf{k}_1}\underset{\circ}{\mathsf{k}_2}{}^{\mathsf{k}_3} \times {}^{\mathsf{m}_1}\underset{\circ}{\mathsf{m}_2}{}^{\mathsf{m}_3}\big)\big)\big]\Big]\langle\vec{X}, \vec{Y}\rangle \stackrel{\circ}{\equiv}$$

$$\Big[\big[{}^{\mathsf{n}_1}\underset{\circ}{\mathsf{n}_2}{}^{\mathsf{o}} \cdot \big({}^{\mathsf{h}_3}\underset{\circ}{\mathsf{l}_3}{}^{\mathsf{n}_1} \times {}^{\mathsf{k}_3}\underset{\circ}{\mathsf{m}_3}{}^{\mathsf{n}_2}\big)\big] \cdot \big(\big({}^{\mathsf{h}_1}\underset{\circ}{\mathsf{h}_2}{}^{\mathsf{h}_3} \times {}^{\mathsf{l}_1}\underset{\circ}{\mathsf{l}_2}{}^{\mathsf{l}_3}\big) \times \big({}^{\mathsf{k}_1}\underset{\circ}{\mathsf{k}_2}{}^{\mathsf{k}_3} \times {}^{\mathsf{m}_1}\underset{\circ}{\mathsf{m}_2}{}^{\mathsf{m}_3}\big)\big)\Big]\langle\vec{X}, \vec{Y}\rangle$$

$$\mathsf{h}_1\mathsf{h}_2\mathsf{h}_3,\ \mathsf{l}_1\mathsf{l}_2\mathsf{l}_3,\ \mathsf{k}_1\mathsf{k}_2\mathsf{k}_3,\ \mathsf{m}_1\mathsf{m}_2\mathsf{m}_3,\ \mathsf{h}_3\mathsf{l}_3\mathsf{n}_1,\ \mathsf{k}_3\mathsf{m}_3\mathsf{n}_2,\ \mathsf{n}_1\mathsf{n}_2\mathsf{o}, \in \mathfrak{C}$$

This theorem is easier to grasp in its graphical form on Figure 2.

2.4. DRel Relationship to Relevance Logic

Untyped (or using a single type) DRel is a relative of a form of classical relevance logic reported in [5, 18] with emphasis on the latter reference. One difference is that Routley-Meyer's star operator for the logic must be weakened a bit for it to act as converse operator when the elements of the logic are interpreted as relations. In particular, Routley and Meyer would have the following axiom

$$(A \rightarrow B) \rightarrow (A^* \rightarrow B^*).$$

They also have an extension which most nearly matches our formulas for contraposition, namely

$$(A \rightarrow B) \rightarrow (\neg B^* \rightarrow \neg A^*).$$

Using De Morgan negation, this is

$$(A \rightarrow B) \rightarrow ({\sim} B \rightarrow {\sim} A).$$

Given the period two nature of the De Morgan negation, this can be recoded as

$$(A \rightarrow {\sim} B) \rightarrow (B \rightarrow {\sim} A).$$

This is nearly formula D1 without the mediation provided by converse.

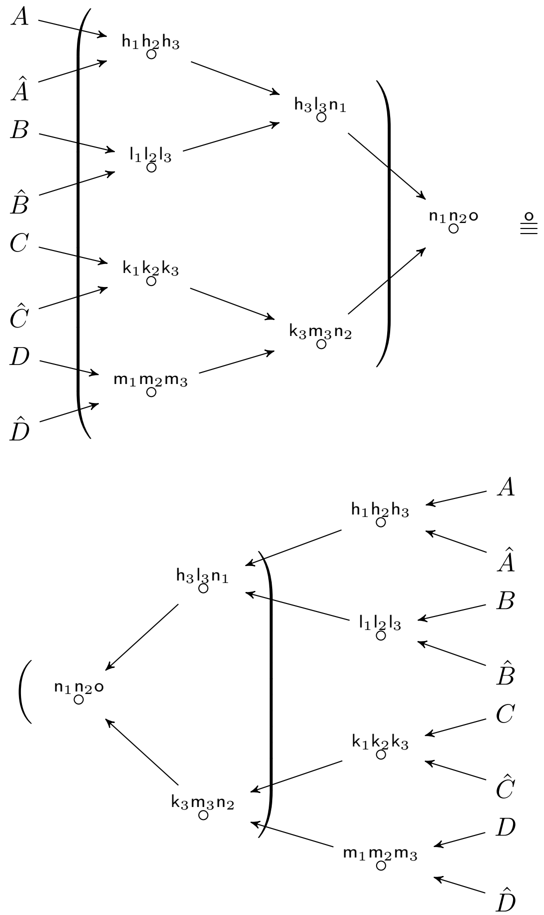


Figure 2. Multicategory associativity

Since $A \circ B$ is equivalent to $B \circ A$ in classical relevance logic, the cliques are reduced to two place cliques instead of three. The typing structure is further reduced to a single node if a t identity is included for distributed $\circ$ connectives.

DRel is almost a classical relevance logic system except that $\overset{\mathsf{h}}{\vdash} t$ is not provable. The relationship between the relevant like entailment $\multimap$ and $\supset$ is shown in Theorem 2.4.1:

Theorem 2.4.1. $\overset{\mathsf{h}}{\vdash} A \xrightarrow{\mathsf{hhh}} \hat{A}$ *implies* $\overset{\mathsf{h}}{\vdash} A \overset{\mathsf{h}}{\supset} \hat{A}$,

3. Algebras and Frames

The algebras and frames are typed. The algebras are known as heterogeneous algebras [7] where every "sort" is a type and each algebra forms an essentially algebraic category [1]. The distributed semigroup operators in the algebras are morphisms in a multicategory; we outfit every type with special *no-op* modal operators as identity morphisms. Each type in an algebra is a local algebra that will be used to interpret a local logic. Similarly, the three place Kripke relations on the frames are morphisms in a multicategory with (binary) diagonal relations as the identity morphisms. Each type in a frame is a local frame that will be used to interpret a local logic.

The use of multicategories in this paper is similar to the distributed modal logic [4] case where categories, not multicategories, are used. The reason to use category theory is that it is a convenient theory of types and provides a way of expressing morphisms. The reason to lift morphisms into the logic is that then morphisms can be given properties by logical axioms. For example, in distributed modal logic, the addition of two axioms can force the morphisms to be functions. In this paper, we are not using any of the deeper constructions of category theory, they are merely convenient structures to hold the information over which we wish to abstract.

3.1. Algebras

The Boolean join $\vee$ symbol will be used in place of relation algebra's typical $+$ symbol. The reason is that $+$ is used for an intensional disjunction in relevance logic, not extensional disjunction which uses $\vee$.

We rely on heterogeneous (multityped) algebras [7] for the free algebra construction. The categorical version is most easily accessible in [1] who attribute the multityped (non-categorical) case to [7]. We repeat the original definition of [7]:

Definition 3.1.1 (Birkhoff and Lipson [7]). A heterogeneous algebra is a system $A = [\mathscr{L}, F]$ in which

1. $\mathscr{L} = \{S_i\}$ is a family of non-void sets S_i of different types of elements, each called a *phylum* of the algebra A. The phyla S_i are indexed by some set I; i.e., $S_i \in \mathscr{L}$ for $i \in I$ (or are called by appropriate names).

2. $F = \{f_\alpha\}$ is a set of finitary operations, where each f_α is a mapping

$$f_\alpha \colon S_{i(1,\alpha)} \times S_{i(2,\alpha)} \times \cdots \times S_{i(n(\alpha),\alpha)} \to S_{p(\alpha)}$$

for some non-negative integer $n(\alpha)$, function $i_\alpha : j \to i(j,\alpha)$ from $n(\alpha) = \{1, 2, \ldots, n(\alpha)\}$ to I, and $p(\alpha) \in I$. The operations f_α are indexed by some set Ω; i.e., $f_\alpha \in F$ for $\alpha \in \Omega$ (or are called by appropriate names).

A generic *distributed semigroup operator* has the typing structure $\mathsf{h} \times \mathsf{l} \to \mathsf{k}$. Our version of heterogenous algebras is contained in the following definition:

Definition 3.1.2. A Distributed Relation Algebra (DRAlg) contains a distribution structure $\mathfrak{G}$ of nodes called *types* and *operators* with types being collected into a set $\mathfrak{C}$ of three-element cliques. A type for h is a *converse algebra* $(D_\mathsf{h}, \wedge, \vee, \neg, \bot_\mathsf{h}, \top_\mathsf{h}, \breve{\ })$ (the *local converse algebra at* h) LCAlg where $(D_\mathsf{h}, \wedge, \vee, \neg, \bot_\mathsf{h}, \top_\mathsf{h})$ is a Boolean lattice, and the $\breve{\ }$ operator is *converse*. $\bot_\mathsf{h}$ and $\top_\mathsf{h}$ are the bottom and top of the lattice respectively. The *distributed operators* are of the form $\overset{\mathsf{hlk}}{\circ} : \mathsf{h} \times \mathsf{l} \to \mathsf{k}$ where we use the indices h, l, and k to refer the carrier sets, D_h, D_l, and D_k of the types. The phyla of heterogeneous algebras are the carrier sets. The distribution structure of a DRAlg is specified with

GA1. A set $\mathfrak{G}$ of nodes

GA2. A set $\mathfrak{C}$ of cliques

GA3. A set $\{\overset{\mathsf{hlk}}{\circ}, \overset{\mathsf{lkh}}{\circ}, \overset{\mathsf{khl}}{\circ}, \overset{\mathsf{lhk}}{\circ}, \overset{\mathsf{hkl}}{\circ}, \overset{\mathsf{klh}}{\circ}\}$ of distributed semigroup operators for each $\mathsf{hlk} \in \mathfrak{C}$

The original axioms from Chin and Tarski and Ng [9, 15] are first and a DRAlg respects the axioms below are second.

M1. $(B, \vee, \neg)$ is a Boolean algebra
$(D_\mathsf{h}, \vee, \neg, \breve{\ })$ is a LCAlg for each sort h

M2. $(a \circ b) \circ c = a \circ (b \circ c)$
$((a \overset{\mathsf{hlm}}{\circ} b) \overset{\mathsf{mko}}{\circ} c) \overset{\mathsf{o}}{=} (a \overset{\mathsf{hno}}{\circ} (b \overset{\mathsf{lkn}}{\circ} c))$ $\mathsf{hlm}, \mathsf{mko}, \mathsf{hno}, \mathsf{lkn} \in \mathfrak{C}$

M3. $(a \vee b) \circ c = (a \circ c) \vee (b \circ c)$
$(a \vee b) \overset{hlk}{\circ} c \overset{k}{=} (a \overset{hlk}{\circ} c) \vee (b \overset{hlk}{\circ} c) \quad h, l, k \in \mathsf{hlk} \in \mathfrak{C}$

M4. $a \circ 1 = a$ for any $a \in A$
$a \overset{\mathsf{hhh}}{\circ} 1_\mathsf{h} \overset{\mathsf{h}}{=} a, \quad \mathsf{hhh} \in \mathfrak{C}$

M5. $a^{\smile\smile} = a$ for any $a \in A$

$a^{\smile\smile} \overset{\mathsf{h}}{=} a$

M6. $(a \vee b)^{\smile} = a^{\smile} \vee b^{\smile}$ for any $a, b \in A$

$(a \vee b)^{\smile} \overset{\mathsf{h}}{=} a^{\smile} \vee b^{\smile}$

M7. $(a \circ b)^{\smile} = b^{\smile} \circ a^{\smile}$, for any $a, b \in A$

$(a \overset{hlk}{\circ} b)^{\smile} \overset{k}{=} b^{\smile} \overset{lhk}{\circ} a^{\smile} \qquad h, l, k ⋹ \mathsf{hlk} \in \mathfrak{C}$

M8. $(a^{\smile} \circ \neg(a \circ c)) \vee \neg c = \neg c$ for any $a, c \in A$

$(a^{\smile} \overset{hlk}{\circ} \neg(a \overset{hkl}{\circ} c)) \vee \neg c \overset{k}{=} \neg c \qquad h, l, k ⋹ \mathsf{hlk} \in \mathfrak{C}$

Let

$$\vec{p} \overset{\text{def}}{=} \langle\langle a, \hat{a}\rangle, \langle b, \hat{b}\rangle\rangle, \qquad \vec{q} \overset{\text{def}}{=} \langle\langle c, \hat{c}\rangle, \langle d, \hat{d}\rangle\rangle.$$

Then the multicategory axioms are:

M9. $$\left[\overset{\mathsf{n_1n_2o}}{\circ} \cdot \left[\left(\overset{\mathsf{h_3l_3n_1}}{\circ} \times \overset{\mathsf{k_3m_3n_2}}{\circ}\right) \cdot \left(\left(\overset{\mathsf{h_1h_2h_3}}{\circ} \times \overset{\mathsf{l_1l_2l_3}}{\circ}\right) \times \left(\overset{\mathsf{k_1k_2k_3}}{\circ} \times \overset{\mathsf{m_1m_2m_3}}{\circ}\right)\right)\right]\right] \langle\vec{p}, \vec{q}\rangle \overset{\mathsf{o}}{=}$$
$$\left[\left[\overset{\mathsf{n_1n_2o}}{\circ} \cdot \left(\overset{\mathsf{h_3l_3n_1}}{\circ} \times \overset{\mathsf{k_3m_3n_2}}{\circ}\right)\right] \cdot \left(\left(\overset{\mathsf{h_1h_2h_3}}{\circ} \times \overset{\mathsf{l_1l_2l_3}}{\circ}\right) \times \left(\overset{\mathsf{k_1k_2k_3}}{\circ} \times \overset{\mathsf{m_1m_2m_3}}{\circ}\right)\right)\right] \langle\vec{p}, \vec{q}\rangle$$

where $\mathsf{h_1h_2h_3}$, $\mathsf{l_1l_2l_3}$, $\mathsf{k_1k_2k_3}$, $\mathsf{m_1m_2m_3}$, $\mathsf{h_3l_3n_1}$, $\mathsf{k_3m_3n_2}$, $\mathsf{n_1n_2o}, \in \mathfrak{C}$;

M10. $[\iota]\, a \overset{\mathsf{h}}{=} a$

Considering Axiom M2, there are the two cliques on the left of the $\overset{\mathsf{o}}{=}$, hlm and mko. There are two cliques on the right, hno and lkn. The equation does not force any cliques to exist. It is tempting to think that the left side cliques force the right side cliques to exist. This is incorrect. Rather, if the two formulas on either side of the equality can be formed, then the equality says they are equal. It is easier to think of using a word algebra and then dividing out by the equation. The equation says that if the two sides exist, then they must be equal. It does not force any clique to exist.

Axiom M7 is really six axioms when all the elements of the clique hlk are considered:

M7a. $(a \overset{\mathsf{hlk}}{\circ} b)^{\smile} \overset{\mathsf{k}}{=} b^{\smile} \overset{\mathsf{lhk}}{\circ} a^{\smile}$

M7b. $(a \overset{\mathsf{hkl}}{\circ} c)^{\smile} \overset{\mathsf{l}}{=} c^{\smile} \overset{\mathsf{khl}}{\circ} a^{\smile}$

M7c. $(c \overset{\mathsf{klh}}{\circ} b)^{\smile} \overset{\mathsf{h}}{=} b^{\smile} \overset{\mathsf{lkh}}{\circ} c^{\smile}$

The missing three can be generated by these and the use of ${}^{\smile}$. Axiom M8 has an equivalent partner due to ${}^{\smile}$:

LEMMA 3.1.3. *The following is equivalent to Axiom* M8:

M8$^{\smile}$. $(\neg(c \overset{khl}{\circ} a) \overset{lhk}{\circ} a^{\smile}) \vee \neg c \overset{k}{=} \neg c \qquad h, k, l ⋹ \mathsf{hlk} \in \mathfrak{C}$

The De Morgan negation $\sim a$ is defined as $\neg a^{\smile}$ (that $\neg$ and $^{\smile}$ commute is well known in relation algebra). Similar to the syntactic logic case, we make use of the following encodings for $h, k, l \mathrel{\in\!\!\!\!\!\shortmid} \mathsf{hlk} \in \mathfrak{C}$:

Formula	De Morgan Encoding	Classical Encoding	Name
$a \xrightarrow{hlk} c$	$\sim(\sim c \overset{khl}{\circ} a)$	$\neg(a^{\smile} \overset{hkl}{\circ} \neg c)$	right residual
$c \xleftarrow{hlk} b$	$\sim(b \overset{lkh}{\circ} \sim c)$	$\neg(\neg c \overset{klh}{\circ} b^{\smile})$	left residual

Similar to the logic, we use $a \in \mathsf{h}$ to denote $a \in D_{\mathsf{h}}$. The tonicity properties of the distributed operators are as follows:

Lemma 3.1.4. *$\circ$ is monotone on the left and the right, i.e., for $h, k, l \mathrel{\in\!\!\!\!\!\shortmid} \mathsf{hlk} \in \mathfrak{C}$,*

$$\frac{b \leq_l \hat{b}, \quad a \in h}{a \overset{hlk}{\circ} b \leq_k a \overset{hlk}{\circ} \hat{b}} \ \circ \text{ Left Monotone}$$

$$\frac{a \leq_h \hat{a}, \quad b \in l}{a \overset{hlk}{\circ} b \leq_k \hat{a} \overset{hlk}{\circ} b} \ \circ \text{ Right Monotone}$$

The tonicities of $\xrightarrow{hlk}$ and $\xleftarrow{hlk}$ are contained in the following derived rules:

$$\frac{a \leq_h \hat{a}, \quad c \in k}{\hat{a} \xrightarrow{hlk} c \leq_l a \xrightarrow{hlk} c} \ \rightarrow \text{ Left Antitone}$$

$$\frac{c \leq_k \hat{c}, \quad a \in h}{a \xrightarrow{hlk} c \leq_l a \xrightarrow{hlk} \hat{c}} \ \rightarrow \text{ Right Monotone}$$

$$\frac{c \leq_k \hat{c}, \quad b \in l}{c \xleftarrow{hlk} b \leq_h \hat{c} \xleftarrow{hlk} b} \ \leftarrow \text{ Left Monotone}$$

$$\frac{b \leq_l \hat{b}, \quad c \in k}{c \xleftarrow{hlk} \hat{b} \leq_h c \xleftarrow{hlk} b} \ \leftarrow \text{ Right Antitone}$$

Following the logic, we can recast residuation as

$$b \leq_l a \xrightarrow{hlk} c \text{ iff } a \overset{hlk}{\circ} b \leq_k c \text{ iff } a \leq_h c \xleftarrow{hlk} b.$$

The algebraic axioms for residuation are equivalent to the following, for $h, k, l \mathrel{\in\!\!\!\!\!\shortmid} \mathsf{hlk} \in \mathfrak{C}$:

N8. $a \overset{hlk}{\circ} (a \xrightarrow{hlk} c) \leq_k c$

N8$^{\smile}$. $(c \xleftarrow{lhk} a) \overset{lhk}{\circ} a \leq_k c$

N8$^{\neg}$. $c \leq_k a \xrightarrow{hkl} (a \overset{hkl}{\circ} c)$

N8$^{\smile\neg}$. $c \leq_k (c \overset{khl}{\circ} a) \xleftarrow{khl} a$

The indexing on the cliques has been changed from the logic counterparts of these axioms in order to simplify proofs. Since the h, k, l are variables over the clique hlk, there is no appreciable difference.

THEOREM 3.1.5. *Axioms* M8, M8$^{\smile}$, N8, N8$^{\neg}$, N8$^{\smile}$, *and* N8$^{\smile\neg}$ *are all equivalent.*

The normality of the distributed semigroup operators is used in the representation theorem presented later:

THEOREM 3.1.6. *The distributed semigroup operators are normal: for* $h, l, k \Subset \mathsf{hlk} \in \mathfrak{C}$, $\overset{hlk}{\circ}$ *distributes over* $\vee$ *from both sides and*

$$a \overset{hlk}{\circ} \bot_l \overset{k}{=} \bot_k = \bot_l \overset{lhk}{\circ} a$$

3.2. Frames

The frames use three-place relations like relevance logic. However, now the relations must be typed. The relations will be morphisms in a multicategory with diagonal relations as the identity morphisms of the category. Following the logic, a type for us will be a node in the underlying graph of a category. We will not use the term "object" but rather the term "node".

DEFINITION 3.2.1. A *local frame* at a node h is a structure $\mathcal{H} = (\mathsf{H}, \mathbb{H}, \mathring{\mathsf{H}}, {}^{\smile})$ such that H is a collection of *points* (also called *worlds*), $\mathring{\mathsf{H}} \subseteq \mathsf{H}$ and $\mathring{\mathsf{H}} \in \mathbb{H}$ where $\mathring{\mathsf{H}}$ is a collection of "zero" worlds, and $\mathbb{H}$ is a subset of the power set of H required to be closed under the Boolean operations and under the operation ${}^{\smile} : \mathsf{H} \to \mathsf{H}$ extended to sets in $\mathbb{H}$ by:

$$C^{\smile} \overset{\text{def}}{=} \{x^{\smile} \mid x \in C\}.$$

It is allowable for $\mathring{\mathsf{H}} = \emptyset$, this occurs if $\mathsf{hhh} \not\Subset \mathfrak{C}$. The distributed relations will be used to interpret the distributed connectives $\circ$ and the defined distributed connectives $\multimap$ and $\multimapinv$.

Each node in a distributed relation logic's distribution structure has a local logic associated with it. Semantically, that local logic must have a local frame associated with it. We use the locution $x \in \mathsf{h}$ to indicate that $x \in \mathsf{H}$ where $\mathcal{H} = (\mathsf{H}, \mathbb{H}, \mathring{\mathsf{H}}, {}^{\smile})$ is the local frame for node h in a graph of nodes.

DEFINITION 3.2.2. Let $\mathcal{H}$, $\mathcal{L}$, and $\mathcal{K}$ be local frames. A *distributed relation* $\mathcal{R}^{\mathsf{hlk}} \colon \mathsf{h} \times \mathsf{l} \to \mathsf{k}$ as a multicategory morphism is a subset $\mathcal{R}^{\mathsf{hlk}} \subseteq \mathsf{H} \times \mathsf{L} \times \mathsf{K}$.

We collect local frames together into a multicategory whose structure is given by a graph $\mathfrak{G}$ and collection of cliques $\mathfrak{C}$:

Definition 3.2.3. A *distributed relation frame*, $\mathcal{DF}$, has a local frame for every node in $\mathfrak{G}$, the underlying graph of the category. The distributed relations are specified by the collection of cliques $\mathfrak{C}$ (see Frame Condition FG3 below). There is a diagonal relation $\mathcal{I}^{\mathsf{hh}}$ for every node h. A distributed relation frame must satisfy the following conditions:

Frame Conditions G

FG1. A collection of nodes $\mathfrak{G}$

FG2. A set $\mathfrak{C}$ of cliques

FG3. A set $\{\mathcal{R}^{\mathsf{hlk}}, \mathcal{R}^{\mathsf{lkh}}, \mathcal{R}^{\mathsf{khl}}, \mathcal{R}^{\mathsf{lhk}}, \mathcal{R}^{\mathsf{hkl}}, \mathcal{R}^{\mathsf{klh}}\}$ of distributed relations for each $\mathsf{hlk} \in \mathfrak{C}$

Frame Conditions A

FA1. A local frame for each node in $\mathfrak{G}$

FA2. $\breve{}\,: \mathsf{h} \to \mathsf{h}$ is a function on H

FA3. $x^{\breve{}\,\breve{}} = x$

Frame Conditions B. For $x \in \mathsf{h}$, $y \in \mathsf{l}$, $u \in \mathsf{k}$, $v \in \mathsf{o}$, and $z \in \mathsf{m}$:

FB1. $\exists z \in \mathsf{m}(\mathcal{R}^{\mathsf{hlm}}xyz$ and $\mathcal{R}^{\mathsf{mko}}zuv)$ iff
$\exists w \in \mathsf{n}(\mathcal{R}^{\mathsf{hno}}xwv$ and $\mathcal{R}^{\mathsf{lkn}}yuw)$ $\quad \mathsf{hlm}, \mathsf{mko}, \mathsf{hno}, \mathsf{lkn} \in \mathfrak{C}$

FB2. For all $z \in \mathring{\mathsf{H}}$, $\mathcal{R}^{\mathsf{hhh}}xzy$ implies $x = y$ $\quad \mathsf{hhh} \in \mathfrak{C}$

FB3. $\mathcal{R}^{hlk}$ is a three-place relation $\quad h, k, l \Subset \mathsf{hlk} \in \mathfrak{C}$

FB4. $\mathcal{R}^{hlk}xyz$ iff $\mathcal{R}^{lhk}y^{\breve{}}x^{\breve{}}z^{\breve{}}$ $\quad h, k, l \Subset \mathsf{hlk} \in \mathfrak{C}$

FB5. $\mathcal{R}^{hlk}xyz$ iff $\mathcal{R}^{hkl}x^{\breve{}}zy$ $\quad h, k, l \Subset \mathsf{hlk} \in \mathfrak{C}$

A defined permutation can be had by combining the effects of FB4 and FB5.

$\mathcal{R}^{hlk}xyz$ iff $\mathcal{R}^{klh}zy^{\breve{}}x$

The conditions for a multicategory with the three place relations and the diagonal relation as morphisms are

FC1. Composition is associative:

$$\mathcal{R}^{\mathsf{n_1n_2o}} \cdot \left[(\mathcal{R}^{\mathsf{h_3l_3n_1}} \times \mathcal{R}^{\mathsf{l_3m_3n_2}}) \cdot ((\mathcal{R}^{\mathsf{h_1h_2h_3}} \times \mathcal{R}^{\mathsf{l_1l_2l_3}}) \times (\mathcal{R}^{\mathsf{k_1k_2k_3}} \times \mathcal{R}^{\mathsf{m_1m_2m_3}}))\right] \text{ iff}$$
$$\left[\mathcal{R}^{\mathsf{n_1n_2o}} \cdot (\mathcal{R}^{\mathsf{h_3l_3n_1}} \times \mathcal{R}^{\mathsf{l_3m_3n_2}})\right] \cdot ((\mathcal{R}^{\mathsf{h_1h_2h_3}} \times \mathcal{R}^{\mathsf{l_1l_2l_3}}) \times (\mathcal{R}^{\mathsf{k_1k_2k_3}} \times \mathcal{R}^{\mathsf{m_1m_2m_3}}))$$

FC2. $\mathcal{I}^{\mathsf{hh}}xy$ iff $x = y$

Note that · in FC1 is multicategory composition and not relational composition; the latter would have its arguments reversed.

Lemma 3.2.4. *The distributed relation frames are multicategories where the non-identity morphisms are three-place relations, and the diagonal relations are the identity morphisms.*

The proof is just to observe that the diagonal relations satisfy the conditions for identity morphisms and composition of the three place relations shown in the following diagram is associative:

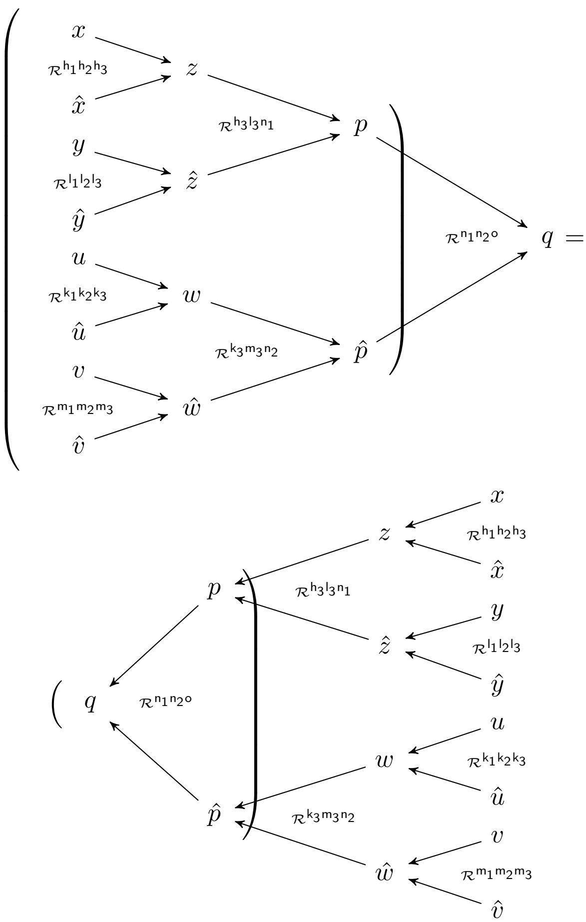


Figure 3. Relation multicategory associativity

3.3. Soundness

Algebraic soundness follows from the Lindenbaum-Tarski heterogeneous algebra (LTA) being free for the class of appropriate algebras for DRel, also termed a DRAlg-free algebra. Note this is not to say that the LTA is a free DRAlg algebra. Free algebras need not be a member of the class of algebras for which they are free. It turns out that the LTA is actually a free DRAlg, but that is fact is used for completeness.

Definition 3.3.1. A DRAlg *appropriate* for a distributed logic DRel is a local converse algebra for each node of the distribution structure, *distributed operators* $\overset{\mathsf{hlk}}{\circ}, \overset{\mathsf{lkh}}{\circ}, \overset{\mathsf{khl}}{\circ}, \overset{\mathsf{lhk}}{\circ}, \overset{\mathsf{hkl}}{\circ}, \overset{\mathsf{klh}}{\circ}$ for each $\mathsf{hlk} \in \mathfrak{C}$ of DRel, and a modal operator $[\iota]$ for every node h. The DRAlg is a multicategory with the morphisms being the distributed and modal operators.

Definition 3.3.2. Let $\mathcal{DA}$ be a DRAlg. An interpretation, $[\![\cdots]\!]$, is a mapping of the propositional variables of DRel into $\mathcal{DA}$ such that for each node h, $[\![T]\!] = \top_{\mathsf{h}}$ and $[\![F]\!] = \bot_{\mathsf{h}}$. If t appears in a local logic at h, then $[\![t]\!] = 1_{\mathsf{h}}$. An interpretation can be extended in the obvious way to preserve the connectives, i.e.,

$$
\begin{array}{ll}
[\![A \vee B]\!] \overset{\text{def}}{=} [\![A]\!] \vee [\![B]\!] & [\![A \wedge B]\!] \overset{\text{def}}{=} [\![A]\!] \wedge [\![B]\!] \\
[\![A \supset B]\!] \overset{\text{def}}{=} \neg[\![A]\!] \vee [\![B]\!] & [\![\neg A]\!] \overset{\text{def}}{=} \neg[\![A]\!] \\
[\![A \overset{\mathsf{hlk}}{\circ} B]\!] \overset{\text{def}}{=} [\![A]\!] \overset{\mathsf{hlk}}{\circ} [\![B]\!] & [\![A^{\smile}]\!] \overset{\text{def}}{=} [\![A]\!]^{\smile}
\end{array}
$$

A DRel sentence A is *true* under an interpretation $[\![\cdots]\!]$ iff $\top_{\mathsf{h}} \leq_{\mathsf{h}} [\![A]\!]$. A sentence is *valid* in $\mathcal{DA}$ iff it is true under all interpretations and it is *DRAlg-valid* iff it is valid in all appropriate DRAlg $\mathcal{DA}$.

A useful feature of Boolean lattices is the following Lemma:

Lemma 3.3.3 (Residuation for Boolean lattices).

$$a \wedge b \leq_{\mathsf{h}} c \text{ iff } a \leq_{\mathsf{h}} b \overset{\mathsf{h}}{\supset} c\,.$$

$A \overset{\mathsf{h}}{\supset} B$ is true in the interpretation $[\![\cdots]\!]$ iff $\top_{\mathsf{h}} \leq_{\mathsf{h}} [\![A \overset{\mathsf{h}}{\supset} B]\!]$ iff $\top_{\mathsf{h}} \leq_{\mathsf{h}} [\![A]\!] \overset{\mathsf{h}}{\supset} [\![B]\!]$. By Lemma 3.3.3, this latter is true iff $\top_{\mathsf{h}} \wedge [\![A]\!] \leq_{\mathsf{h}} [\![B]\!]$ iff $[\![A]\!] \leq_{\mathsf{h}} [\![B]\!]$. Hence $A \overset{\mathsf{h}}{\supset} B$ is true in the interpretation $[\![\cdots]\!]$ iff $[\![A]\!] \leq_{\mathsf{h}} [\![B]\!]$.

From the Replacement Theorem 2.1.2, all the connectives respect bi-equivalence. The LTA is defined in the usual way:

DEFINITION 3.3.4. The elements of the carrier sets are $[A] \overset{\mathsf{h}}{\equiv} \{B \mid A \overset{\mathsf{h}}{\equiv} B\}$ for $A, B \in \mathsf{h}$. The operators are defined inductively: $[A]\bullet[B] = [A\bullet B]$ for $\bullet \in \{\wedge, \vee, \supset\}$, $\neg[A] = [\neg A]$, $[A]^{\smile} = [A^{\smile}]$, and $[A] \overset{\mathsf{hlk}}{\circ} [B] = [A \overset{\mathsf{hlk}}{\circ} B]$ for all $\mathsf{hlk} \in \mathfrak{C}$.

LEMMA 3.3.5. *The* LTA *satisfies all the properties required for an appropriate* DRAlg *algebra under the interpretation* $[\ldots]$.

PROOF SKETCH. All logical axioms easily map to the algebraic axioms under the canonical interpretation $[\cdots]$. The following is an example proving Axiom B5 is true:

1	$[A^{\smile} \overset{hlk}{\circ} \neg(A \overset{hkl}{\circ} B)] \leq_k [\neg B]$	Identity
2	$[A]^{\smile} \overset{hlk}{\circ} \neg([A] \overset{hkl}{\circ} [B]) \leq_k \neg[B]$	Def. 3.3.4
3	$[A]^{\smile} \overset{hlk}{\circ} \neg([A] \overset{hkl}{\circ} [B]) \vee \neg[B] \overset{k}{\equiv} \neg[B]$	Lattice properties

The demonstration that each of the rules of inference preserves truth is also routine. The Rule Left $\circ$ Monotonicity is an example. Assume $B \overset{l}{\supset} \hat{B}$, $A \in h$, and $\overset{hlk}{\circ} : h \times l \to k$, then $[B] \leq_l [\hat{B}]$. Therefore $[B] \vee [\hat{B}] \overset{l}{\equiv} [\hat{B}]$. Applying $\overset{hlk}{\circ}$ to both sides yields $[A] \overset{hlk}{\circ} ([B] \vee [\hat{B}]) \overset{k}{\equiv} [A] \overset{hlk}{\circ} [\hat{B}]$. From Theorem 3.1.6, $([A] \overset{hlk}{\circ} [B]) \vee ([A] \overset{hlk}{\circ} [\hat{B}]) \overset{k}{\equiv} [A] \overset{hlk}{\circ} [\hat{B}]$ and from lattice properties, $[A] \overset{hlk}{\circ} [B] \leq_k [A] \overset{hlk}{\circ} [\hat{B}]$.

Theorem 2.3.2 shows that multicategory associativity holds and Axiom C2 provides the category theory identities. □

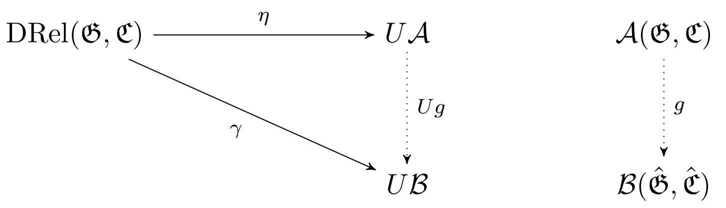
The following lemma allows the transfer of provability of a sentence A in the logic to the condition $\top \leq [\![A]\!]$ for an arbitrary interpretation in a DRAlg.

LEMMA 3.3.6. $\vdash_{\mathsf{h}} A$ *iff* $\vdash_{\mathsf{h}} T \supset A$.

PROOF. For any h in a distribution structure for a DRel, assume $\vdash_{\mathsf{h}} A$. From the axioms, $\vdash_{\mathsf{h}} A \supset (T \supset A)$. From the assumption $\vdash_{\mathsf{h}} A$, and modus ponens, $\vdash_{\mathsf{h}} T \supset A$. For the other half, assume $\vdash_{\mathsf{h}} T \supset A$, then $\vdash_{\mathsf{h}} T$ is provable, and so $\vdash_{\mathsf{h}} A$ follows. □

In the freeness diagram below, the free algebra is $\mathcal{A}$. The algebra $\mathcal{B}$ is some other appropriate distributed algebra, and γ is any interpretation, $U\mathcal{A}$ is the forgetful functor U (from algebras to sets) applied to the algebra $\mathcal{A}$ and returns the carrier sets (types) of $\mathcal{A}$, and similarly for $U\mathcal{B}$. Ug is the underlying set function of the unique homomorphism g

such that the left hand diagram commutes. η injects the atoms of the logic into the proper types of the category $U\mathcal{A}$.



Each node of the distribution structure representing a distinct local logic must be mapped to a distinct local algebra. This informal way of restricting interpretations is the result of treating the hypergraph as not defining everything in a distributed relation logic, but the alternative would make the logic impenetrable. That classes of heterogeneous algebras have free algebras, in our case termed DRAlg-free algebras, is a theorem of [7]. Thus if a formula is provable in DRel, it is DRAlg-valid; this is stated formally as:

Theorem 3.3.7. DRel *is sound with respect to* DRAlg.

The proof is a recognition of the fact that the LTA for a DRel is free for the class of algebras appropriate for the DRel. A provable sentence in the logic is a true expression in the LTA, i.e., $\vdash A$ implies $[\top] \leq [A]$. Any interpretation of the sentence into an algebra in the class factors uniquely into the map from the logic into the LTA and a unique map to the algebra carrying the original interpretation.

Next, we show the soundness for evaluations in DRFrame. Similarly to the algebraic case, each node representing a distinct local logic must be mapped to a distinct frame object in any interpretation. The convention is that the relations that use upper case script letters with appellation hlk, i.e., $\mathcal{R}^{\mathsf{hlk}}$, will interpret $\overset{\mathsf{hlk}}{\circ}$ connectives that use the corresponding types. Each distributed frame interpreting a DRel logic will have conditions matching the axioms.

We assume an initial valuation $[\![\ldots]\!]$ of DRel on all the propositional variables yielding a collection of points in a frame for each variable. The double turnstile $\models$ for the DRel evaluation will mean the set theoretic "element of". Hence $x \overset{\mathsf{h}}{\models} A$ is defined as $x \in_{\mathsf{h}} [\![A]\!]$ where with $[\![\ldots]\!]$, we overload the meaning of these brackets to carry the recursion necessary to extend the interpretation to all nodes h, i.e., $\overset{\mathsf{h}}{\models}$.

DEFINITION 3.3.8. Let $\mathcal{H} = (\mathsf{H}, \mathbb{H}, \mathring{\mathsf{H}}, {}^{\smile})$ be a local frame. $\mathcal{H}^* = (\mathbb{H}, \cap, \cup, \neg, \emptyset, \mathsf{H}, {}^{\smile})$ is the local algebra of sets (of points) at h where $\mathbb{H}$ is a collection of sets closed under $\cap$, $\cup$, set complement $\neg$, and converse ${}^{\smile}$, a bottom element $\emptyset$, and a top element H (the set of all points at h). ${}^{\smile}$ is defined as

$$A^{\smile} = \{x^{\smile} \mid x \in A\},$$

LEMMA 3.3.9. *For $\mathcal{H}$ a local DRFrame, $\mathcal{H}^*$ is a local converse algebra of sets.*

PROOF. Converse clearly satisfies $(A \cup B)^{\smile} \stackrel{\mathsf{h}}{=} A^{\smile} \cup B^{\smile}$ and $A^{\smile\smile} \stackrel{\mathsf{h}}{=} A$. The rest of the proof follows from [13, 20]. See also [8, 19, 21]. □

DEFINITION 3.3.10. For $\mathcal{DF}$ a DRFrame multicategory with graph $\mathfrak{G}$ and clique set $\mathfrak{C}$, the algebraic multicategory $\mathcal{DF}^*$ has an LCAlg algebra of sets for every node $\mathsf{h} \in \mathfrak{G}$, and morphisms defined as

$$[\iota]\, A = \{x \mid \mathcal{I}^{\mathsf{hh}} xx\},$$
$$A \overset{\mathsf{hlk}}{\circ} B = \{z \mid \exists x, y(x \in_{\mathsf{h}} A \text{ and } y \in_{\mathsf{l}} B \text{ and } \mathcal{R}^{\mathsf{hlk}} xyz)\}, \quad \mathsf{hlk} \in \mathfrak{C}.$$

THEOREM 3.3.11. *For $\mathcal{DF}$ a DRFrame multicategory with clique set $\mathfrak{C}$, $\mathcal{DF}^*$ is a DRAlg multicategory.*

PROOF. The axioms are sound. The following is an example using M2: $A^{\smile} \overset{hlk}{\circ} \neg(A \overset{hkl}{\circ} C) \subseteq_k \neg C$.

1 $z \stackrel{k}{\models} A^{\smile} \overset{hlk}{\circ} \neg(A \overset{hkl}{\circ} C)$ . . . . Assume
2 $\mathcal{R}^{hlk} xyz$ and $x \stackrel{h}{\models} A^{\smile}$ and . . . . . . Def. of $\models$ for some x, y,
$y \stackrel{l}{\models} \neg(A \overset{hkl}{\circ} C)$ . . . . line 1
3 $y \stackrel{l}{\not\models} A \overset{hkl}{\circ} C$ . . . . Def. of $\models$, line 2
4 $\neg\mathcal{R}^{hkl} uvy$ or $u \stackrel{h}{\not\models} A$ or $v \stackrel{k}{\not\models} C$ . . . Def. of $\models$ for all u, v, line 3
5 $\neg\mathcal{R}^{hlk} u^{\smile} yv$ or $u \stackrel{h}{\not\models} A$ or $v \stackrel{k}{\not\models} C$. Frame Condition FB5, line 4
6 $\neg\mathcal{R}^{hlk} u^{\smile} yv$ or $u^{\smile} \stackrel{h}{\not\models} A^{\smile}$ or $v \stackrel{k}{\not\models} C$ . . . . Def. $\models$, FA3, line 5
7 $\mathcal{R}^{hlk} u^{\smile} yv$ and $u^{\smile} \stackrel{h}{\models} A^{\smile}$ implies $v \stackrel{k}{\not\models} C$ . . Classical logic, line 6
8 $\mathcal{R}^{hlk} xyz$ and $x \stackrel{h}{\models} A^{\smile}$ . . . . Classical logic, line 2
9 $\mathcal{R}^{hlk} x^{\smile\smile} yz$ and $x^{\smile\smile} \stackrel{h}{\models} A^{\smile}$ implies $z \stackrel{k}{\not\models} C$ $x^{\smile}$ for u, z for v, line 7
10 $\mathcal{R}^{hlk} xyz$ and $x \stackrel{h}{\models} A^{\smile}$ implies $z \stackrel{k}{\not\models} C$ Frame Condition FA3, line 9
11 $z \stackrel{k}{\not\models} C$ . . . . Modus Ponens, lines 8,10
12 $z \stackrel{k}{\models} \neg C$ . . . . Def. $\models$, line 11

For $\mathsf{h_1h_2h_3},\ \mathsf{l_1l_2l_3},\ \mathsf{h_3l_3o} \in \mathfrak{C}$, the composition axiom:

$$\text{C1. } {}^{\mathsf{h_3}\underset{\circ}{\mathsf{l_3}}\mathsf{o}}({}^{\mathsf{h_1}\underset{\circ}{\mathsf{h_2}}\mathsf{h_3}} \times {}^{\mathsf{l_1}\underset{\circ}{\mathsf{l_2}}\mathsf{l_3}})\langle\langle A, \hat{A}\rangle, \langle B, \hat{B}\rangle\rangle \overset{\circ}{\equiv} \big[{}^{\mathsf{h_3}\underset{\circ}{\mathsf{l_3}}\mathsf{o}} \cdot ({}^{\mathsf{h_1}\underset{\circ}{\mathsf{h_2}}\mathsf{h_3}} \times {}^{\mathsf{l_1}\underset{\circ}{\mathsf{l_2}}\mathsf{l_3}})\big]\langle\langle A, \hat{A}\rangle, \langle B, \hat{B}\rangle\rangle$$

is sound. In relevance logic, there is a useful syntactic device for expressing relational composition:

$$\mathcal{R}^2(u\hat{u})(vz) \text{ iff } \exists x(\mathcal{R}u\hat{u}x \text{ and } \mathcal{R}xvz).$$

We can utilize something similar:

$$\mathcal{R}^2(u\hat{u})(v\hat{v})z\colon \mathsf{h_1h_2h_3} \times \mathsf{l_1l_2l_3} \to \mathsf{h_3l_3o} \text{ iff } \exists x, y(\mathcal{R}^{\mathsf{h_1h_2h_3}}u\hat{u}x \text{ and } \mathcal{R}^{\mathsf{l_1l_2l_3}}v\hat{v}y \text{ and } \mathcal{R}^{\mathsf{h_3l_3o}}xyz).$$

Now we can show the soundness of the axiom over the frames:

1	$z \overset{\circ}{\models} {}^{\mathsf{h_3}\underset{\circ}{\mathsf{l_3}}\mathsf{o}}({}^{\mathsf{h_1}\underset{\circ}{\mathsf{h_2}}\mathsf{h_3}} \times {}^{\mathsf{l_1}\underset{\circ}{\mathsf{l_2}}\mathsf{l_3}})\langle\langle A, \hat{A}\rangle, \langle B, \hat{B}\rangle\rangle$	assumption
2	$\mathcal{R}^{\mathsf{h_3l_3o}}xyz$ and $\langle x, y\rangle \models {}^{\mathsf{h_1}\underset{\circ}{\mathsf{h_2}}\mathsf{h_3}} \times {}^{\mathsf{l_1}\underset{\circ}{\mathsf{l_2}}\mathsf{l_3}}\langle\langle A, \hat{A}\rangle, \langle B, \hat{B}\rangle\rangle$	for some x, y
3	$\mathcal{R}^{\mathsf{h_1h_2h_3}}u\hat{u}x$ and $u \overset{\mathsf{h_1}}{\models} A$ and $\hat{u} \overset{\mathsf{h_2}}{\models} \hat{A}$ and $\mathcal{R}^{\mathsf{l_1l_2l_3}}v\hat{v}y$ and $v \overset{\mathsf{l_1}}{\models} B$ and $\hat{v} \overset{\mathsf{l_2}}{\models} \hat{B}$	def. ${}^{\mathsf{h_1}\underset{\circ}{\mathsf{h_2}}\mathsf{h_3}} \times {}^{\mathsf{l_1}\underset{\circ}{\mathsf{l_2}}\mathsf{l_3}}$, line 2 for some $u, \hat{u}, v, \hat{v}$
4	$\mathcal{R}^{\mathsf{h_3l_3o}}xyz$ and $\mathcal{R}^{\mathsf{h_1h_2h_3}}u\hat{u}x$ and $\mathcal{R}^{\mathsf{l_1l_2l_3}}v\hat{v}y$	classical logic, lines 2,3
5	$\mathcal{R}^2(u\hat{u})(v\hat{v})z$ and $\langle u, \hat{u}\rangle \overset{\mathsf{h_1} \times \mathsf{h_2}}{\models} \langle A, \hat{A}\rangle$ and $\langle v, \hat{v}\rangle \overset{\mathsf{l_1} \times \mathsf{l_2}}{\models} \langle B, \hat{B}\rangle$	classical logic, lines 3,4
6	$z \overset{\circ}{\models} \big[{}^{\mathsf{h_3}\underset{\circ}{\mathsf{l_3}}\mathsf{o}} \cdot ({}^{\mathsf{h_1}\underset{\circ}{\mathsf{h_2}}\mathsf{h_3}} \times {}^{\mathsf{l_1}\underset{\circ}{\mathsf{l_2}}\mathsf{l_3}})\big]\langle\langle A, \hat{A}\rangle, \langle B, \hat{B}\rangle\rangle$	def. ${}^{\mathsf{h_3}\underset{\circ}{\mathsf{l_3}}\mathsf{o}}$, line 5

The proof reads easily from bottom to top and adjusting the justifications on the proof lines. □

DEFINITION 3.3.12. A *distributed relation model* for a distributed relation logic has local frames in a DRFrame multicategory and has a valuation, called a *local model*, for every local frame. A valuation specifies a collection of points in the local frame where the atomic propositions of the appropriate type are true. If the local logic contains t, then the local algebra of sets contains a non-empty $\overset{\circ}{\mathsf{H}}$. The following table then extends the DRel evaluation scheme taken from an interpretation defined on the DRAlg of sets that the DRFrame provides:

Derivation of the Kripke Semantics

$x \overset{\mathsf{h}}{\models} A$	iff	$x \in_{\mathsf{h}} [\![A]\!]$ for A a propositional variable		
$x \overset{\mathsf{h}}{\models} T$	iff	$x \in_{\mathsf{h}} [\![T]\!]$	iff	$x \in_{\mathsf{h}} \mathsf{H}$
$x \overset{\mathsf{h}}{\models} t$	iff	$x \in_{\mathsf{h}} [\![t]\!]$	iff	$x \in_{\mathsf{h}} \mathring{\mathsf{H}}$
$x \overset{\mathsf{h}}{\models} \neg A$	iff	$x \in_{\mathsf{h}} [\![\neg A]\!]$	iff	$x \notin_{\mathsf{h}} [\![A]\!]$
			iff	$x \overset{\mathsf{h}}{\not\models} A$
$x \overset{\mathsf{h}}{\models} A^{\smile}$	iff	$x \in_{\mathsf{h}} [\![A^{\smile}]\!]$	iff	$x \in_{\mathsf{h}} [\![A]\!]^{\smile}$
			iff	$x^{\smile} \overset{\mathsf{h}}{\models} A$
$x \overset{\mathsf{h}}{\models} A \overset{\mathsf{h}}{\supset} B$	iff	$x \in_{\mathsf{h}} [\![A \overset{\mathsf{h}}{\supset} B]\!]$	iff	$x \in_{\mathsf{h}} \neg[\![A]\!] \cup [\![B]\!]$
			iff	$x \overset{\mathsf{h}}{\not\models} A$ or $x \overset{\mathsf{h}}{\models} B$
$z \overset{\mathsf{k}}{\models} A \overset{\mathsf{hlk}}{\circ} B$	iff	$z \in_{\mathsf{k}} [\![A \overset{\mathsf{hlk}}{\circ} B]\!]$	iff	$z \in_{\mathsf{k}} [\![A]\!] \overset{\mathsf{hlk}}{\circ} [\![B]\!]$
			iff	$\exists x, y(\mathcal{R}^{\mathsf{hlk}}xyz$ and $x \in_{\mathsf{h}} [\![A]\!]$ and $y \in_{\mathsf{l}} [\![B]\!])$
			iff	$\exists x, y(\mathcal{R}^{\mathsf{hlk}}xyz$ and $x \overset{\mathsf{h}}{\models} [\![A]\!]$ and $y \overset{\mathsf{l}}{\models} [\![B]\!])$

Notice that the following chain of iffs is valid: $\overset{\mathsf{h}}{\vdash} A$ iff $\overset{\mathsf{h}}{\vdash} T \overset{\mathsf{h}}{\supset} A$ iff $[\![T]\!] \subseteq_{\mathsf{h}} [\![A]\!]$ iff for all $x \in \mathsf{H}$, $x \overset{\mathsf{h}}{\models} T$. Now we can state soundness:

Theorem 3.3.13. DRel *is sound with respect to the DRFrame multicategories.*

Proof. All that needs to be shown for soundness with respect to a Kripke semantics, once it is known that DRAlg is sound with respect DRel, is that every DRFrame yields a DRAlg of sets and that the DRel interpretation conditions arise directly from the definitions yielding this DRAlg of sets. □

3.4. Completeness

From Birkhoff [6], a *variety of algebras* is any class of algebras which is closed under homomorphic images, subalgebras, and products. Varieties always have free algebras and these free algebras reside in the very same varietal class. The free algebra can be formed by the usual Lindenbaum-Tarski construction on the logic for which the class provides the algebraic models. That is, the set of well-formed formulas is "divided" by bi-implication yielding a carrier set of equivalence classes, and operations on those classes are defined via the equivalence class representatives. This carries over into heterogeneous algebras [7] or essentially algebraic theories [1]. The Lindenbaum-Tarski algebra LTA for DRel is a member of DRAlg from Lemma 3.3.5.

Theorem 3.4.1. (1) DRel *is complete with respect to DRAlg.* (2) DRel *is complete over the category of DRFrame.*

Proof. (1) $[\ldots]$ is the LTA interpretation. Completeness with respect to DRAlg follows via a contraposition argument: assume $\not\vdash^{\mathsf{h}} A$, then by the definition of $\leq_{\mathsf{h}}$ in the LTA, $\top \not\leq_{\mathsf{h}} [A]$. Therefore, for all DRAlg and for all interpretations, $[\![\ldots]\!]$, $\top \leq_{\mathsf{h}} [\![A]\!]$ implies $\vdash^{\mathsf{h}} A$.

Note that no formula with $\overset{\mathsf{h}}{\supset}$ spans types, i.e., there are no legal formulas of the form $A \overset{\mathsf{h}}{\supset} B$ where A and B in different local logics.

(2) Completeness over DRFrame follows via a contraposition argument using the LTA provided by a DRAlg and the fact that a representation theorem can be shown. The representation map, β, takes a DRAlg into a DRAlg of sets via a DRFrame that is generated directly from the LTA. This representation map is shown to be a 1–1 homomorphism. Then one argues as follows: suppose that $\not\vdash^{\mathsf{h}} C$, then $\top \not\leq_{\mathsf{h}} [C]$ where $[\ldots]$ denotes the equivalence class of C in the LTA. Since the representation map, β, is 1–1 and a homomorphism, then $\beta\top \not\subseteq_{\mathsf{h}} \beta[C]$. By construction, the map $[\ldots]$ composed with β is itself a valuation and hence by the valuation conditions for DRFrame models, there is some world x such that $x \not\models^{\mathsf{h}} C$. Contraposing the argument yields

$$(\text{for all } x \in X,\ x \models^{\mathsf{h}} C) \text{ implies } \vdash^{\mathsf{h}} C.$$

Since every formula can be in only one local logic (at, say, h), this statement is true for all h. □

The work in this section shows that from any DRAlg, a *canonical* frame can be constructed. In the process, a 1-1 representation homomorphism is constructed into the DRAlg of sets derived from the canonical frame. The following theorem is a recap of similar theorems in Jónsson and Tarski [13] and Dunn [11].

Definition 3.4.2. Let $\mathcal{DA}_{\mathsf{h}}$ be a LCAlg at h. The *local canonical frame* at h is $\mathcal{DF}_{\mathsf{h}*} = (\mathsf{H}, \mathbb{H}, \mathring{\mathsf{H}}, {}^{\smile})$ where H is the collection of all proper, maximal filters (as points), $\mathring{\mathsf{H}}$ is defined with

$$z \in_{\mathsf{h}} \mathring{\mathsf{H}} \text{ iff } 1 \in_{\mathsf{h}} z,$$

and

$$x^{\smile} = \{a^{\smile} \mid a \in x\}.$$

$\mathbb{H}$ is the set of all sets of the form

$$\beta_{\mathsf{h}} a = \{x \mid a \in x \text{ and } x \text{ is a maximal filter}\}.$$

THEOREM 3.4.3. *Let $\mathcal{DA}_{\mathsf{h}}$ be a LCAlg at h, the local canonical frame $\mathcal{DF}_{\mathsf{h}*}$ is a local frame.*

PROOF. The Frame Conditions FA hold. From Stone's Representation Theorem [20], $\mathbb{H}$ is known to be a Boolean algebra, i.e., $\mathbb{H}$ is closed under the set operations $\cap, \vee$, and $\neg$. From Axioms M5 and M6 and Lemma 3 of the Appendix, it is clear that $x^{\smile}$ is well-defined and that $x^{\smile\smile} = x$. By unwinding definitions, it is clear that $\beta_{\mathsf{h}} a^{\smile} = (\beta_{\mathsf{h}} a)^{\smile}$, hence $\mathbb{H}$ is closed under $^{\smile}$:

$$\begin{aligned}
\beta_{\mathsf{h}} a^{\smile} &\overset{\mathsf{h}}{=} \{x \mid a^{\smile} \in x\} \\
&\overset{\mathsf{h}}{=} \{x^{\smile} \mid a \in x^{\smile}\} \\
&\overset{\mathsf{h}}{=} \{x^{\smile} \mid x^{\smile} \in \beta a\} \\
&\overset{\mathsf{h}}{=} (\beta_{\mathsf{h}} a)^{\smile}.
\end{aligned}$$

□

DEFINITION 3.4.4. Let $\mathcal{DA}$ be a DRAlg, then the canonical frame $\mathcal{DA}_*$ has a local canonical frame for every $\mathsf{h} \in \mathfrak{G}$. The relations and hence multicategory morphisms are defined for $h, l, k \mathrel{\underline{\in}} \mathsf{hlk} \in \mathfrak{C}$ via

$$\mathcal{I}^{\mathsf{hh}} xy \text{ iff } [\iota]\, a \in x \text{ implies } a \in y,$$

$$\mathcal{R}^{hlk} xyz \text{ iff } a \in x \text{ and } b \in y \text{ implies } a \overset{hlk}{\circ} b \in z.$$

THEOREM 3.4.5. *Let $\mathcal{DA}$ be a DRAlg, then the canonical frame $\mathcal{DA}_*$ is a DRFrame.*

PROOF. The Frame Conditions FG hold. The distribution structure $\mathfrak{C}$ is provided by the DRel for this canonical frame. Hence Frame Conditions FG1 and FG2 are satisfied. The definition 3.4.4 causes FG3 to hold. From Theorem 3.4.3, the Frame Conditions FA hold.

The Frame Conditions FB hold. We show FB1 as an example. Assume left-associativity,

$$(a \overset{\mathsf{hlm}}{\circ} b) \overset{\mathsf{mno}}{\circ} c \leq_{\circ} a \overset{\mathsf{hno}}{\circ} (b \overset{\mathsf{lkn}}{\circ} c)$$

is present in the algebra. The notation $\lambda p.q \circ p$ refers to a function of p with q previously fixed to some value. $(\lambda p.q \circ p)^{-1}_{q \in u}$ refers to the inverse image of this function with values for q taken from maximal filter u. $(v \overset{\mathsf{lkn}}{\circ} y) \uparrow$ is the upper set defined by applying $\overset{\mathsf{lkn}}{\circ}$ componentwise to the elements of v and y.

Assume there is some x such that

$$\mathcal{R}^{\mathsf{hlm}}uvx \text{ and } \mathcal{R}^{\mathsf{mno}}xyz.$$

Then we create an w such that

$$\mathcal{R}^{\mathsf{hno}}uwz \text{ and } \mathcal{R}^{\mathsf{lkn}}vyw.$$

Define the filter U and the ideal V by

$$U = (v \overset{\mathsf{lkn}}{\circ} y) \uparrow \qquad V = (\lambda p.q \overset{\mathsf{hno}}{\circ} p)^{-1}_{q \in u} \overline{z}$$

where $\overline{z}$ is the set complement of the maximal filter z and thus a maximal ideal. Then $U \cap V = \emptyset$: assume the opposite and let $\hat{d} \in U \cap V$. Then there exists some $d \in v$, $d' \in y$, and $q \in u$ such that $d \overset{\mathsf{lkn}}{\circ} d' \leq^{n} \hat{d}$ and $q \overset{\mathsf{hno}}{\circ} \hat{d} \in \overline{z}$. Using the lattice properties, $(d \overset{\mathsf{lkn}}{\circ} d') \vee \hat{d} \overset{\mathsf{n}}{=} \hat{d}$. Distribution of $\overset{\mathsf{hno}}{\circ}$ over $\vee$ yields

$$q \overset{\mathsf{hno}}{\circ} ((d \overset{\mathsf{lkn}}{\circ} d') \vee \hat{d}) \overset{\mathsf{o}}{=} (q \overset{\mathsf{hno}}{\circ} (d \overset{\mathsf{lkn}}{\circ} d')) \vee (q \overset{\mathsf{hno}}{\circ} \hat{d}).$$

Since $q \overset{hno}{\circ} ((d \overset{\mathsf{lkn}}{\circ} d') \vee \hat{d}) \overset{\mathsf{o}}{=} q \overset{\mathsf{hno}}{\circ} \hat{d}$, $q \overset{\mathsf{hno}}{\circ} \hat{d} \in \overline{z}$, and $\overline{z}$ is an ideal, then $q \overset{\mathsf{hno}}{\circ} (d \overset{\mathsf{lkn}}{\circ} d') \in \overline{z}$.

From associativity, $(q \overset{\mathsf{hlm}}{\circ} d) \overset{\mathsf{mno}}{\circ} d' \in \overline{z}$. $\mathcal{R}^{\mathsf{hlm}}uvx$ tells us $q \overset{\mathsf{hlm}}{\circ} d \in x$ but $\mathcal{R}^{\mathsf{mno}}xyz$ tells us that $(q \overset{\mathsf{hlm}}{\circ} d) \overset{\mathsf{mno}}{\circ} d' \in z$ which is a contradiction. So $U \cap V = \emptyset$. This gives us a disjoint filter-ideal pair (U, V) which we extend to a maximal pair (w_1, w_2). By definition, $\mathcal{R}^{\mathsf{hno}}uwz$ and $\mathcal{R}^{\mathsf{lkn}}vyw$. hold.

The Frame Conditions FC hold. Relational composition is associative. $\mathcal{I}^{\mathsf{hh}}xx$ holds iff $[\iota]\, a \in x$ iff $a \in x$ since $[\iota]\, a = a$. Hence the diagonal relations supply the identity morphisms. □

The following theorem (or at least one close to the following) in conjunction with Theorem 3.4.3 is similar to the one found in Jónsson and Tarski [13], Dunn [10], and Routley-Meyer [17]. We follow Dunn [10].

THEOREM 3.4.6. *The function* $\beta_{\mathsf{h}} \colon \mathcal{DA}_{\mathsf{h}} \to (\mathcal{DA}_{\mathsf{h}_*})^*$ *defined by*

$$\beta_{\mathsf{h}} a = \{x \mid a \in x \text{ and } x \textit{ is a maximal filter.}\}$$

is a 1–1 LCAlg homomorphism.

PROOF. The proof that β_{h} is 1–1 stems from the Stone's representation theorem for Boolean algebras. That β_{h} is a homomorphism is a result of Stone's theorem, and Theorem 3.4.3. □

THEOREM 3.4.7. *The function β, which is β_{h} extended to all of $\mathcal{DA}$, is a faithful functor.*

PROOF. β_{h} is already known to be a 1-1 LCAlg lattice homomorphism from Theorem 3.4.6. The distribution graph $\mathfrak{C}$ is shared by the domain of β and the codomain of β because β_{h} and β_{k} for $\mathsf{h} \neq \mathsf{k}$ cannot mix maximal filters from the algebras at h and k.

What is left is to show β preserves the $[\iota]$ modalities and is a distributed semigroup homomorphism. Combined with β being 1-1 a function, this shows that β is a faithful functor. Note that $\mathcal{I}^{\mathsf{hh}}xx$ always holds as a consequence of $[\iota]\,a = a$ for all $a \in \mathsf{h}$. We then have the following chain of iffs: $x \in \beta([\iota]\,a)$ iff $[\iota]\,a \in x$ iff $a \in x$ iff $x \in \beta a$ iff ($\mathcal{I}^{\mathsf{hh}}xx$ implies $x \in \beta a$) iff $x \in [\iota]\,\beta a$. Now we show β is a distributed semigroup homomorphism. We must show

$$\beta(a \overset{\mathsf{hlk}}{\circ} b) = \beta a \overset{\mathsf{hlk}}{\circ} \beta b.$$

The set containment $\beta a \overset{\mathsf{hlk}}{\circ} \beta b \subseteq \beta(a \overset{\mathsf{hlk}}{\circ} b)$ is a direct result of the definitions.

We show $\beta(a \overset{\mathsf{hlk}}{\circ} b) \subseteq \beta a \overset{\mathsf{hlk}}{\circ} \beta b$. Assume $z \in \beta(a \overset{\mathsf{hlk}}{\circ} b)$, an x and y must be generated such that $\mathcal{R}^{\mathsf{hlk}}xyz$ and $a \in x$ and $b \in y$. Consider the principle filters $a{\uparrow} \subseteq \mathsf{H}$ and $b{\uparrow} \subseteq \mathsf{L}$ where H is the set of points at h and L is the set of points at l. It is clear that $a{\uparrow} \overset{\mathsf{hlk}}{\circ} b{\uparrow} \subseteq z$ from the order properties of $\overset{\mathsf{hlk}}{\circ}$ and assuming the $\overset{\mathsf{hlk}}{\circ}$ is applied pointwise to the elements of $a{\uparrow}$ and $b{\uparrow}$.

The filters $a{\uparrow}$ and $b{\uparrow}$ will be expanded to become prime filters, say, x and y, and will be done so that the relation $x \overset{\mathsf{hlk}}{\circ} y \subseteq z$ is preserved. The following set is nonempty:

$$F = \{(u, v) \mid u, v \text{ are filters and } a \in u, b \in v, u \overset{\mathsf{hlk}}{\circ} v \subseteq z\},$$

since it contains $(a{\uparrow}, b{\uparrow})$. The $\overset{\mathsf{hlk}}{\circ}$ operator is normal (Lemma 3.1.6) so that $c \overset{\mathsf{h}}{=} \bot$ or $d \overset{\mathsf{l}}{=} \bot$ implies $c \overset{\mathsf{hlk}}{\circ} d \overset{\mathsf{k}}{=} \bot$ where $\bot$ is the bottom of the lattice at k. This allows that only proper filters need to be considered.

Define a partial order on F by

$$(u, v) \le_{\mathsf{h}\times\mathsf{l}} (\hat{u}, \hat{v}) \text{ iff } u \subseteq_{\mathsf{h}} \hat{u} \text{ and } v \subseteq_{\mathsf{l}} \hat{v}.$$

Each chain in the $\le_{\mathsf{h}\times\mathsf{l}}$ order has an upper bound: let E be a chain in F with E_{h} be the chain of filters in the first position of a tuple, and E_{l} the chain of filters in the second position. Now define

$$\bigvee E \overset{\mathsf{h}\times\mathsf{l}}{=} \left(\bigcup E_{\mathsf{h}}, \bigcup E_{\mathsf{l}}\right).$$

It must be shown that $\bigvee E \in F$. The union of any chain of filters is clearly a filter, and $a \in \bigcup E_{\mathsf{h}}$ and $b \in \bigcup E_{\mathsf{l}}$ by definition for membership in F. So the item that needs to be checked is that

$$\bigcup E_{\mathsf{h}} \overset{\mathsf{hlk}}{\circ} \bigcup E_{\mathsf{l}} \subseteq z.$$

Suppose that $c \in \bigcup E_{\mathsf{h}}$ and $d \in \bigcup E_{\mathsf{l}}$, then there is some i, j such that $c \in u_i \in E_{\mathsf{h}}$ and $d \in v_j \in E_{\mathsf{l}}$. Without loss of generality, assume $j \leq i$, then $(u_i, v_i) \in F$. Since $(u_i, v_i) \in F$, $u_i \overset{\mathsf{hlk}}{\circ} v_i \subseteq z$, each chain has an upper bound. From Zorn's Lemma, F has a maximal element, call it (x, y).

Now x and y must be shown to be prime. Suppose $c \vee \hat{c} \in x$ (the argument for y is the same). For reductio, suppose $c \notin x$ and $\hat{c} \notin x$. Let $f(x, c)$ and $f(x, \hat{c})$ be the filters generated by x and the elements c and $\hat{c}$. That is, $f(x, c) = \{d \wedge c \mid d \in x\}$. Since $x \subseteq f(x, c), f(x, \hat{c})$, then $(x, y) \subseteq (f(x, c), y), (f(x, \hat{c}), y)$. Each of $(f(x, c), y)$ and $(f(x, \hat{c}), y)$ must fail to be in F since they each contain an element, c or $\hat{c}$, which is not in the maximal element x. This can only occur if there are elements c_1, d_1 and c_2, d_2 such that

$$c_1 \in f(x, c) \text{ and } d_1 \in y \text{ and } c_1 \overset{\mathsf{hlk}}{\circ} d_1 \notin z,$$

and

$$c_2 \in f(x, \hat{c}) \text{ and } d_2 \in y \text{ and } c_2 \overset{\mathsf{hlk}}{\circ} d_2 \notin z.$$

Now let $p = c_1 \vee c_2$ and $q = d_1 \wedge d_2$. It is clear that $p \in f(x, c), f(x, \hat{c})$ since $c_1, c_2 \leq p$ and $f(x, c), f(x, \hat{c})$ are both filters. Also note that $q = d_1 \wedge d_2 \in y$.

Since $(x, y) \in F$, $x \circ y \subseteq z$. This implies that $p \overset{\mathsf{hlk}}{\circ} q \in z$. Using the definition of p,

$$(c_1 \vee c_2) \overset{\mathsf{hlk}}{\circ} q \in z \text{ iff } (c_1 \overset{\mathsf{hlk}}{\circ} q) \vee (c_2 \overset{\mathsf{hlk}}{\circ} q) \in z,$$

and so either $c_1 \overset{\mathsf{hlk}}{\circ} q \in z$ or $c_2 \overset{\mathsf{hlk}}{\circ} q \in z$ since z is a prime filter. Assume the former. Since $q \leq d_1$ and the fact that $\overset{\mathsf{hlk}}{\circ}$ is monotone in each position, then $c_1 \overset{\mathsf{hlk}}{\circ} d_1 \in z$ contradicting $c_1 \overset{\mathsf{hlk}}{\circ} d_1 \notin z$ above. Similarly, $c_2 \overset{\mathsf{hlk}}{\circ} q \in z$ yields a contradiction. The reductio is complete and x is prime. A similar argument shows y is also prime. Prime filters in a Boolean lattice are maximal since for any prime filter x, $c \vee \neg c = \top \in x$ and hence either $c \in x$ or $\neg c \in x$. $\square$

4. Conclusions

The notion of *distribution* has wide applicability as shown in [4] and the current research. Distribution is orthogonal to many constructions in logic in that it does not prevent them. The distribution structure generally is not some abstract, unmotivated structure but rather comes about because the logic is to be applied to some specific domain of discourse. It is for this reason that we designed the distribution structure to be parametric to the logic. For the authors, one of the primary domains is System-on-a-Chip (SoC) architectures. The subcomponents on a chip form a distribution structure. The relations used in a distributed logic over a domain of this kind come from operational behavior and interaction among the subcomponents. Paper length prevents us from going into this here. A relatively complicated SoC will require both modal and two-place intensional connectives in the logic.

Category theory is a good theory of typing. It also is a convenient model of logical morphisms when the logical morphisms are deemed to be intensional connectives in a logic. Modal distributed logic uses a category as opposed to a multicategory. The jump in arity from the modal case seems to indicate that more complicated category theory might be required for more sophisticated logical connectives. One can go the other way around and attempt to discover new logical connections from higher category theory.

Domains of discourse which are naturally distributed should have their distribution structure lifted directly into a logic over those domains. As logicians, we should be interested in making our logics more expressive to increase the utility of those logics. The typing structure also tends to bring about a certain discipline to one's reasoning in the same way that category theory brings about a discipline in reasoning about mathematics, or that type systems in programming languages enforces discipline in reasoning about programs. One cannot merely apply connectives without taking the typing into account.

In the semantics of logic, morphisms are frequently used. As logicians, we should be interested in abstracting these morphisms into the logic and not leave them as meta-logical furniture of the semantics. In modal logic, one can abstract similarity relations into a distributed modal logic [3]. Lifting the morphisms into the logic allows them to be assigned properties with logical axioms and rules. This represents an extension to logic.

Distributed logic is also not restricted to normal connectives; consequently, neighborhoods are employed in the semantics of modal distributed logic. Neighborhood morphisms between logics can be lifted into the logic. The result is very close to Markov transition systems save for the probabilistic aspect. This gives us the possibility that one can analyze a system logically and then by changing the interpretation, treat the morphisms as measurable relations and assigning measurements to logical formulas. This is a very seductive notion for applied logics where the real world is never black and white but admits shades of grey. We have not done so in this paper owing to its length but it opens up new possibilities for future research. We would like to allow the 2-place distributed connectives to be non-normal and thus requiring something like neighborhood maps, as opposed to relations, in the semantics. We would also like a measurement interpretation to widen the utility of distributed logics.

Another area we are beginning to explore is distributed epistemic logics where one can associate a local logic with an agent. The relations between logic are then used to interpret distributed epistemic connectives. This has a ready application in security for System-on-a-Chip architectures. There are several logics for security that are epistemic. When an SoC is under security attack, it is important to know which subcomponents might be compromised. In that situation, it is important to know what a subcomponent can "know" about another. In a more human realm, people can be taken to constitute a distributed system. The distributed connectives describe what one person can know about another, or using higher-arity relations, what information is available to an individual about groupings of individuals.

Appendix

Logic Toolbox

The statements of Lemma 1 easy to prove in DRel.

Lemma 1.

$\vdash T^{\smile} \equiv T$

$\vdash (A \vee B)^{\smile} \equiv A^{\smile} \vee B^{\smile}$

$\vdash F^{\smile} \equiv F$

$\vdash T \supset A^{\smile} \vee (\neg A)^{\smile}$

$\vdash A^{\smile} \wedge (\neg A)^{\smile} \supset F$

Logic Proofs

Most uses of classical logic and theorems 2.1.2 and 2.2.1 in the proofs of this section are omitted, generally only the axioms and rules of the current paper are cited.

Lemma 2.1.1. *The rule*

$$\frac{A \overset{h}{\supset} \hat{A}, \quad B \in l, \quad h, l, k \Subset \mathsf{hlk} \in \mathfrak{C}}{A \overset{hlk}{\circ} B \supset \hat{A} \overset{hlk}{\circ} B} \text{ Right } \circ \text{ Monotonicity}$$

is derivable.

Proof.

1 $A^{\smile} \overset{h}{\supset} \hat{A}^{\smile}$ . . . . Instance of the premise

2 $B^{\smile} \overset{lhk}{\circ} A^{\smile} \overset{k}{\supset} B^{\smile} \overset{lhk}{\circ} \hat{A}^{\smile}$ . . . Rule Left $\circ$ Monotonicity, line 1

3 $(B^{\smile} \overset{lhk}{\circ} A^{\smile})^{\smile} \overset{k}{\supset} (B^{\smile} \overset{lhk}{\circ} \hat{A}^{\smile})^{\smile}$ . . Rule $^{\smile}$ Monotonicity, line 2

4 $A^{\smile\smile} \overset{hlk}{\circ} B^{\smile\smile} \overset{k}{\supset} \hat{A}^{\smile\smile} \overset{hlk}{\circ} B^{\smile\smile}$ . . . . Axiom B4, line 3

5 $A \overset{hlk}{\circ} B \overset{k}{\supset} \hat{A} \overset{hlk}{\circ} B$ . . . . Axiom A3, line 4

□

Theorem 2.2.1. $(\neg A)^{\smile} \equiv \neg(A^{\smile})$.

Proof. It is a theorem of classical logic that if $\vdash T \supset C \vee D$ and $\vdash C \wedge D \supset F$, then $\vdash C \equiv \neg D$. Using this fact and Lemmas 1 and 1, it follows that $\vdash (\neg A)^{\smile} \equiv \neg(A^{\smile})$. □

Theorem 2.2.2. *The Axiom B5 and Axiom B5$^{\smile}$ are inter-derivable:*

Proof.

1 $A^{\smile\smile} \overset{hlk}{\circ} \neg(A^{\smile} \overset{hkl}{\circ} C^{\smile}) \overset{k}{\supset} \neg C^{\smile}$ . . . . . . Instance of Axiom B5

2 $A \overset{hlk}{\circ} \neg(A^{\smile} \overset{hkl}{\circ} C^{\smile}) \overset{k}{\supset} \neg C^{\smile}$ . . . . Axiom A3, line 1

3 $(A \overset{hlk}{\circ} \neg(A^{\smile} \overset{hkl}{\circ} C^{\smile}))^{\smile} \overset{k}{\supset} \neg C^{\smile\smile}$ . . Rule $^{\smile}$ Monotonicity, line 2

4 $(A \overset{hlk}{\circ} \neg(A^{\smile} \overset{hkl}{\circ} C^{\smile}))^{\smile} \overset{k}{\supset} \neg C$ . . . . Axiom A3, line 3

5 $\neg(A^{\smile} \overset{hkl}{\circ} C^{\smile})^{\smile} \overset{lhk}{\circ} A^{\smile} \overset{k}{\supset} \neg C$ . . . . Axiom B4, line 4

6 $\neg(C^{\smile\smile} \overset{khl}{\circ} A^{\smile\smile}) \overset{lhk}{\circ} A^{\smile} \overset{k}{\supset} \neg C$ . . . . Axiom B4, line 5

7 $\neg(C \overset{khl}{\circ} A) \overset{lhk}{\circ} A^{\smile} \overset{k}{\supset} \neg C$ . . . . Axiom A3, line 6

The other direction is similar. □

Theorem 2.2.4. *Rule* $\twoheadrightarrow$ Residuation *is derivable from Axiom* B5, *and Rule* $\twoheadleftarrow$ Residuation *is derivable from Axiom* B5$^{\smile}$.

Proof.

1	$A \overset{hlk}{\circ} B \overset{k}{\supset} C$	Assume
2	$\neg C \overset{k}{\supset} \neg(A \overset{hlk}{\circ} B)$	$\overset{k}{\supset}$ contraposition, line 1
3	$A^{\smile} \overset{hkl}{\circ} \neg C \overset{l}{\supset} A^{\smile} \overset{hkl}{\circ} \neg(A \overset{hlk}{\circ} B)$	Rule Left $\circ$ Monotonicity, line 2
4	$A^{\smile} \overset{hkl}{\circ} \neg(A \overset{hlk}{\circ} B) \overset{l}{\supset} \neg B$	Axiom B5
5	$A^{\smile} \overset{hkl}{\circ} \neg C \overset{l}{\supset} \neg B$	$\overset{l}{\supset}$ transitivity, lines 3, 4
6	$\neg\neg B \overset{l}{\supset} \neg(A^{\smile} \overset{hkl}{\circ} \neg C)$	$\neg$ contraposition, line 5
7	$B \overset{l}{\supset} \neg(A^{\smile} \overset{hkl}{\circ} \neg C)$	Classical negation, line 6
8	$B \overset{l}{\supset} A \xrightarrow{hlk} C$	Encoding, line 7

It is easy to go back the other way. There are similar proofs showing that Axiom B5$^{\smile}$ and Rule $\twoheadleftarrow$ Residuation are inter-derivable. ☐

Theorem 2.3.2. *Letting*

$$\vec{X} \overset{\text{def}}{=} \langle\langle A, \hat{A}\rangle, \langle B, \hat{B}\rangle\rangle, \quad \vec{Y} \overset{\text{def}}{=} \langle\langle C, \hat{C}\rangle, \langle D, \hat{D}\rangle\rangle,$$

the following equivalence is provable:

$$\Big[\overset{\mathsf{n_1n_2o}}{\circ} \cdot \big[(\overset{\mathsf{h_3l_3n_1}}{\circ} \times \overset{\mathsf{k_3m_3n_2}}{\circ}) \cdot ((\overset{\mathsf{h_1h_2h_3}}{\circ} \times \overset{\mathsf{l_1l_2l_3}}{\circ}) \times (\overset{\mathsf{k_1k_2k_3}}{\circ} \times \overset{\mathsf{m_1m_2m_3}}{\circ}))\big]\Big]\langle\vec{X}, \vec{Y}\rangle \overset{\mathsf{o}}{\equiv}$$
$$\Big[\big[\overset{\mathsf{n_1n_2o}}{\circ} \cdot (\overset{\mathsf{h_3l_3n_1}}{\circ} \times \overset{\mathsf{k_3m_3n_2}}{\circ})\big] \cdot ((\overset{\mathsf{h_1h_2h_3}}{\circ} \times \overset{\mathsf{l_1l_2l_3}}{\circ}) \times (\overset{\mathsf{k_1k_2k_3}}{\circ} \times \overset{\mathsf{m_1m_2m_3}}{\circ}))\Big]\langle\vec{X}, \vec{Y}\rangle$$

for $\mathsf{h_1h_2h_3}$, $\mathsf{l_1l_2l_3}$, $\mathsf{k_1k_2k_3}$, $\mathsf{m_1m_2m_3}$, $\mathsf{h_3l_3n_1}$, $\mathsf{k_3m_3n_2}$, $\mathsf{n_1n_2o}, \in \mathfrak{C}$.

Proof. Let $\circ_1 = \overset{\mathsf{n_1n_2o}}{\circ}$, $\circ_2 = \overset{\mathsf{h_3l_3n_1}}{\circ}$, $\circ_3 = \overset{\mathsf{k_3m_3n_2}}{\circ}$, $\circ_4 = \overset{\mathsf{h_1h_2h_3}}{\circ}$, $\circ_6 = \overset{\mathsf{l_1l_2l_3}}{\circ}$, $\circ_5 = \overset{\mathsf{k_1k_2k_3}}{\circ}$, $\circ_7 = \overset{\mathsf{m_1m_2m_3}}{\circ}$.

1	$\circ_3(\circ_5 \times \circ_7)\vec{Y} \overset{\mathsf{n_2}}{\equiv} [\circ_3 \cdot (\circ_5 \times \circ_7)]\vec{Y}$	Axiom C1
2	$(\circ_2(\circ_4 \times \circ_6)\vec{X}) \circ_1 (\circ_3(\circ_5 \times \circ_7)\vec{Y}) \overset{\mathsf{o}}{\equiv}$ $(\circ_2(\circ_4 \times \circ_6)\vec{X}) \circ_1 ([\circ_3 \cdot (\circ_5 \times \circ_7)]\vec{Y})$	Rule Left $\circ$ Monotonicity, line 1
3	$\circ_2(\circ_4 \times \circ_6)\vec{X} \overset{\mathsf{o}}{\equiv} [\circ_2 \cdot (\circ_4 \times \circ_6)]\vec{X}$	Axiom C1
4	$\big(\circ_2(\circ_4 \times \circ_6)\vec{X}\big) \circ_1 \big([\circ_3 \cdot (\circ_5 \times \circ_7)]\vec{Y}\big) \overset{\mathsf{o}}{\equiv}$ $\big([\circ_2 \cdot (\circ_4 \times \circ_6)]\vec{X}\big) \circ_1 \big([\circ_3 \cdot (\circ_5 \times \circ_7)]\vec{Y}\big)$	Lemma 2.1.1, line 3
5	$\big(\circ_2(\circ_4 \times \circ_6)\vec{X}\big) \circ_1 \big(\circ_3(\circ_5 \times \circ_7)\vec{Y}\big) \overset{\mathsf{o}}{\equiv}$ $\big([\circ_2 \cdot (\circ_4 \times \circ_6)]\vec{X}\big) \circ_1 \big([\circ_3 \cdot (\circ_5 \times \circ_7)]\vec{Y}\big)$	Transitivity of $\overset{\mathsf{o}}{\equiv}$, lines 2,4

6 $\circ_1\langle \circ_2(\circ_4 \times \circ_6)\vec{X}, \circ_3(\circ_5 \times \circ_7)\vec{Y}\rangle \circeq \Big([\circ_2 \cdot (\circ_4 \times \circ_6)]\vec{X}\Big) \circ_1 \Big([\circ_3 \cdot (\circ_5 \times \circ_7)]\vec{Y}\Big)$ Def. 2.3.1: Base Case, line 5

7 $\circ_1(\circ_2 \times \circ_3)\langle(\circ_4 \times \circ_6)\vec{X}, (\circ_5 \times \circ_7)\vec{Y}\rangle \circeq \Big([\circ_2 \cdot (\circ_4 \times \circ_6)]\vec{X}\Big) \circ_1 \Big([\circ_3 \cdot (\circ_5 \times \circ_7)]\vec{Y}\Big)$ . . . . . . Def. 2.3.1: App Reorder, line 6

8 $\circ_1(\circ_2 \times \circ_3)((\circ_4 \times \circ_6) \times (\circ_5 \times \circ_7))\langle\vec{X}, \vec{Y}\rangle \circeq \Big([\circ_2 \cdot (\circ_4 \times \circ_6)]\vec{X}\Big) \circ_1 \Big([\circ_3 \cdot (\circ_5 \times \circ_7)]\vec{Y}\Big)$ . . . . . Def. 2.3.1: App Reorder, line 7

9 $\circ_1(\circ_2 \times \circ_3)((\circ_4 \times \circ_6) \times (\circ_5 \times \circ_7))\langle\vec{X}, \vec{Y}\rangle \circeq \circ_1\langle[\circ_2 \cdot (\circ_4 \times \circ_6)]\vec{X}, [\circ_3 \cdot (\circ_5 \times \circ_7)]\vec{Y}\rangle$ . . . . . . Def. 2.3.1: Base Case, line 8

10 $\circ_1(\circ_2 \times \circ_3)((\circ_4 \times \circ_6) \times (\circ_5 \times \circ_7))\langle\vec{X}, \vec{Y}\rangle \circeq \circ_1([\circ_2 \cdot (\circ_4 \times \circ_6)] \times [\circ_3 \cdot (\circ_5 \times \circ_7)])\langle\vec{X}, \vec{Y}\rangle$ . . . . Def. 2.3.1: App Reorder, line 9

11 $\circ_1([\circ_2 \cdot (\circ_4 \times \circ_6)] \times [\circ_3 \cdot (\circ_5 \times \circ_7)])\langle\vec{X}, \vec{Y}\rangle \circeq \circ_1(\circ_2 \times \circ_3) \cdot ((\circ_4 \times \circ_6) \times (\circ_5 \times \circ_7))\langle\vec{X}, \vec{Y}\rangle$ . . . Def. 2.3.1: Arr Reorder, line 10

12 $\circ_1(\circ_2 \times \circ_3)((\circ_4 \times \circ_6) \times (\circ_5 \times \circ_7))\langle\vec{X}, \vec{Y}\rangle \circeq \circ_1(\circ_2 \times \circ_3) \cdot ((\circ_4 \times \circ_6) \times (\circ_5 \times \circ_7))\langle\vec{X}, \vec{Y}\rangle$ . . . . . $\circeq$ Transitivity, lines 10, 11

13 $\circ_1(\circ_2 \times \circ_3) \cdot ((\circ_4 \times \circ_6) \times (\circ_5 \times \circ_7))\langle\vec{X}, \vec{Y}\rangle \circeq [\circ_1 \cdot [(\circ_2 \times \circ_3) \cdot ((\circ_4 \times \circ_6) \times (\circ_5 \times \circ_7))]]\langle\vec{X}, \vec{Y}\rangle$ . . . Axiom C1

14 $\circ_1(\circ_2 \times \circ_3)((\circ_4 \times \circ_6) \times (\circ_5 \times \circ_7))\langle\vec{X}, \vec{Y}\rangle \circeq [\circ_1 \cdot [(\circ_2 \times \circ_3) \cdot ((\circ_4 \times \circ_6) \times (\circ_5 \times \circ_7))]]\langle\vec{X}, \vec{Y}\rangle$ . . Transitivity of $\circeq$, lines 12, 13

We also have

1 $\circ_1(\circ_2 \times \circ_3)(\langle\circ_4, \circ_6\rangle \times (\circ_5 \times \circ_7))\langle\vec{X}, \vec{Y}\rangle \circeq [\circ_1 \cdot (\circ_2 \times \circ_3)]((\circ_4 \times \circ_6) \times (\circ_5 \times \circ_7))\langle\vec{X}, \vec{Y}\rangle$ . . . . . . Axiom C1

2 $[\circ_1 \cdot (\circ_2 \times \circ_3)]((\circ_4 \times \circ_6) \times (\circ_5 \times \circ_7))\langle\vec{X}, \vec{Y}\rangle \circeq \Big[[\circ_1 \cdot (\circ_2 \times \circ_3)] \cdot ((\circ_4 \times \circ_6) \times (\circ_5 \times \circ_7))\Big]\langle\vec{X}, \vec{Y}\rangle$ . . . Axiom C1

3 $\circ_1(\circ_2 \times \circ_3)((\circ_4 \times \circ_6) \times (\circ_5 \times \circ_7))\langle\vec{X}, \vec{Y}\rangle \circeq \Big[[\circ_1 \cdot (\circ_2 \times \circ_3)] \cdot ((\circ_4 \times \circ_6) \times (\circ_5 \times \circ_7))\Big]\langle\vec{X}, \vec{Y}\rangle$ . . . . . $\circeq$ transitivity, lines 2, 3

Tying the two proofs together:

$$\Big[\circ_1 \cdot [(\circ_2 \times \circ_3) \cdot ((\circ_4 \times \circ_6) \times (\circ_5 \times \circ_7))]\Big]\langle\vec{X}, \vec{Y}\rangle \circeq \circ_1\,(\circ_2 \times \circ_3)((\circ_4 \times \circ_6) \times (\circ_5 \times \circ_7))\langle\vec{X}, \vec{Y}\rangle \circeq \Big[[\circ_1 \cdot (\circ_2 \times \circ_3)] \cdot ((\circ_4 \times \circ_6) \times (\circ_5 \times \circ_7))\Big]\langle\vec{X}, \vec{Y}\rangle \quad \square$$

Theorem 2.4.1. $\overset{\mathsf{h}}{\vDash} A \xrightarrow{\mathsf{hhh}} \hat{A}$ *implies* $\overset{\mathsf{h}}{\vDash} A \overset{\mathsf{h}}{\supset} \hat{A}$.

Proof.

1	$\overset{\mathsf{h}}{\vDash} A \xrightarrow{\mathsf{hhh}} \hat{A}$	Assume
2	$\overset{\mathsf{h}}{\vDash} T \overset{\mathsf{h}}{\supset} (A \xrightarrow{\mathsf{hhh}} \hat{A})$	Classical logic, line 1
3	$\overset{\mathsf{h}}{\vDash} t \overset{\mathsf{h}}{\supset} T$	For any A, $A \overset{\mathsf{h}}{\supset} T$
4	$\overset{\mathsf{h}}{\vDash} t \overset{\mathsf{h}}{\supset} (A \xrightarrow{\mathsf{hhh}} \hat{A})$	Transitivity of $\overset{\mathsf{h}}{\supset}$, lines 2, 3
5	$\overset{\mathsf{h}}{\vDash} A \overset{\mathsf{hhh}}{\circ} t \overset{\mathsf{h}}{\supset} \hat{A}$	Residuation, line 4
6	$\overset{\mathsf{h}}{\vDash} A \overset{\mathsf{hhh}}{\circ} t \overset{\mathsf{h}}{\equiv} A$	Axiom B2
7	$\overset{\mathsf{h}}{\vDash} A \overset{\mathsf{h}}{\supset} \hat{A}$	Replacement, lines 5, 6

□

Algebraic Proofs

Algebraic Toolbox

The items from Lemma 2 through Lemma 6 are used in the succeeding Section 4.

Lemma 2. *From Boolean algebras it is known that* $a \vee b = \top$ *and* $a \wedge b = \bot$ *implies* $a = \neg b$.

Lemma 3. *From relation algebras it is known that*

$$\top^{\smile} = \top$$
$$\bot^{\smile} = \bot$$
$$a \leq b \text{ implies } a^{\smile} \leq b^{\smile}$$
$$(a \wedge b)^{\smile} = a^{\smile} \wedge b^{\smile}$$
$$(\neg a)^{\smile} = \neg(a^{\smile})$$

Corollary 4. $\sim$ *is a period 2 operator.*

The proof is easy applications of classical negation and the lemma.

Lemma 5. *If* $\mathsf{hhh} \in \mathfrak{C}$, *then* $1_{\mathsf{h}}^{\smile} \overset{\mathsf{h}}{\equiv} 1_{\mathsf{h}}$.

Proof.

$1_{\mathsf{h}}^{\smile} \overset{\mathsf{hhh}}{\circ} 1_{\mathsf{h}} \overset{\mathsf{h}}{\equiv} 1_{\mathsf{h}}^{\smile}$	Axiom M4
$(1_{\mathsf{h}}^{\smile} \overset{\mathsf{hhh}}{\circ} 1_{\mathsf{h}})^{\smile} \overset{\mathsf{h}}{\equiv} 1_{\mathsf{h}}^{\smile\smile}$	Apply $^{\smile}$
$1_{\mathsf{h}}^{\smile} \overset{\mathsf{hhh}}{\circ} 1_{\mathsf{h}}^{\smile\smile} \overset{\mathsf{h}}{\equiv} 1_{\mathsf{h}}$	Axiom M7

$$
\begin{array}{rcl}
1_{\mathsf{h}}^{\smile} \overset{\mathsf{hhh}}{\circ} 1_{\mathsf{h}} \overset{\mathsf{h}}{=} 1_{\mathsf{h}} & & \text{Axiom M5} \\
1_{\mathsf{h}}^{\smile} \overset{\mathsf{h}}{=} 1_{\mathsf{h}} & & \text{Axiom M4}
\end{array}
$$

□

Lemma 6. *If* $\mathsf{hhh} \in \mathfrak{C}$, *then* 1_{h} *is a two-sided identity.*

Proof.

$$
\begin{array}{rcl}
(1_{\mathsf{h}} \overset{\mathsf{hhh}}{\circ} a)^{\smile} \overset{\mathsf{h}}{=} (1_{\mathsf{h}} \overset{\mathsf{hhh}}{\circ} a)^{\smile} & & \text{Equality} \\
(1_{\mathsf{h}} \overset{\mathsf{hhh}}{\circ} a)^{\smile} \overset{\mathsf{h}}{=} a^{\smile} \overset{\mathsf{hhh}}{\circ} 1_{\mathsf{h}}^{\smile} & & \text{Axiom M7} \\
(1_{\mathsf{h}} \overset{\mathsf{hhh}}{\circ} a)^{\smile} \overset{\mathsf{h}}{=} a^{\smile} \overset{\mathsf{hhh}}{\circ} 1_{\mathsf{h}} & & \text{Lemma 5} \\
(1_{\mathsf{h}} \overset{\mathsf{hhh}}{\circ} a)^{\smile} \overset{\mathsf{h}}{=} a^{\smile} & & \text{Axiom M4} \\
(1_{\mathsf{h}} \overset{\mathsf{hhh}}{\circ} a)^{\smile\smile} \overset{\mathsf{h}}{=} a^{\smile\smile} & & \text{Apply } {}^{\smile} \\
1_{\mathsf{h}} \overset{\mathsf{hhh}}{\circ} a \overset{\mathsf{h}}{=} a & & \text{Axiom M5}
\end{array}
$$

The other side is similar but uses this proof in place of Axiom M4. □

Proofs for the Algebras and Frames (Section 3)

Note that the numbers on the Lemmas and Theorems match the numbers from the Algebras and Frames Section 3.

Lemma 3.1.4.

$$\frac{b \leq_l \hat{b}, \quad a \in h}{a \overset{hlk}{\circ} b \leq_k a \overset{hlk}{\circ} \hat{b}} \circ \text{ Left Monotone}$$

$$\frac{a \leq_h \hat{a}, \quad b \in l}{a \overset{hlk}{\circ} b \leq_k \hat{a} \overset{hlk}{\circ} b} \circ \text{ Right Monotone}$$

The tonicity of $\xrightarrow{hlk}$ *and* $\xleftarrow{hlk}$ *is contained in the following derived rules:*

$$\frac{a \leq_h \hat{a}, \quad c \in k}{\hat{a} \xrightarrow{hlk} c \leq_l a \xrightarrow{hlk} c} \rightarrow \text{ Left Antitone}$$

$$\frac{c \leq_k \hat{c}, \quad a \in h}{a \xrightarrow{hlk} c \leq_l a \xrightarrow{hlk} \hat{c}} \rightarrow \text{ Right Monotone}$$

$$\frac{c \leq_k \hat{c}, \quad b \in l}{c \xleftarrow{hlk} b \leq_h \hat{c} \xleftarrow{hlk} b} \leftarrow \text{ Left Monotone}$$

$$\frac{b \leq_l \hat{b}, \quad c \in k}{c \xleftarrow{hlk} \hat{b} \leq_h c \xleftarrow{hlk} b} \leftarrow \text{ Right Antitone}$$

Proof.

1 $b \le_l \hat{b}$ and $a \in h$ . . . Assume
2 $b \vee \hat{b} \overset{l}{=} \hat{b}$ . . . Lattice properties
3 $a \overset{hlk}{\circ} (b \vee \hat{b}) \overset{k}{=} a \overset{hlk}{\circ} \hat{b}$ . . . apply $\overset{hlk}{\circ}$, line 2
4 $(a \overset{hlk}{\circ} (b \vee \hat{b}))^{\smile} \overset{k}{=} (a \overset{hlk}{\circ} \hat{b})^{\smile}$ . . . Apply $^{\smile}$, line 3
5 $(b \vee \hat{b})^{\smile} \overset{lhk}{\circ} a^{\smile} \overset{k}{=} (a \overset{hlk}{\circ} \hat{b})^{\smile}$ . . . Axiom M7, line 4
6 $(b^{\smile} \vee \hat{b}^{\smile}) \overset{lhk}{\circ} a^{\smile} \overset{k}{=} (a \overset{hlk}{\circ} \hat{b})^{\smile}$ . . . Axiom M6, line 5
7 $(b^{\smile} \overset{lhk}{\circ} a^{\smile}) \vee (\hat{b}^{\smile} \overset{lhk}{\circ} a^{\smile}) \overset{k}{=} (a \overset{hlk}{\circ} \hat{b})^{\smile}$ . . . Axiom M3, line 6
8 $((b^{\smile} \overset{lhk}{\circ} a^{\smile}) \vee (\hat{b}^{\smile} \overset{lhk}{\circ} a^{\smile}))^{\smile} \overset{k}{=} (a \overset{hlk}{\circ} \hat{b})^{\smile\smile}$ . . . Apply $^{\smile}$, line 7
9 $(b^{\smile} \overset{lhk}{\circ} a^{\smile})^{\smile} \vee (\hat{b}^{\smile} \overset{lhk}{\circ} a^{\smile})^{\smile} \overset{k}{=} (a \overset{hlk}{\circ} \hat{b})^{\smile\smile}$ . . . Axiom M6, line 8
10 $(b^{\smile} \overset{lhk}{\circ} a^{\smile})^{\smile} \vee (\hat{b}^{\smile} \overset{lhk}{\circ} a^{\smile})^{\smile} \overset{k}{=} a \overset{hlk}{\circ} \hat{b}$ . . . Axiom M5, line 9
11 $(a^{\smile\smile} \overset{hlk}{\circ} b^{\smile\smile}) \vee (a^{\smile\smile} \overset{hlk}{\circ} \hat{b}^{\smile\smile}) \overset{k}{=} a \overset{hlk}{\circ} \hat{b}$ . . . Axiom M7, line 10
12 $(a \overset{hlk}{\circ} b) \vee (a \overset{hlk}{\circ} \hat{b}) \overset{k}{=} a \overset{hlk}{\circ} \hat{b}$ . . . Axiom M5, line 11
13 $a \overset{hlk}{\circ} b \le_k a \overset{hlk}{\circ} \hat{b}$ . . . Lattice properties, line 12

and

1 $a \le_h \hat{a}$ and $b \in l$ . . . Assume
2 $a \vee \hat{a} \overset{h}{=} \hat{a}$ . . . Lattice properties
3 $(a \vee \hat{a}) \overset{hlk}{\circ} b \overset{k}{=} \hat{a} \overset{hlk}{\circ} b$ . . . Apply $\overset{hlk}{\circ}$, line 2
4 $(a \overset{hlk}{\circ} b) \vee (\hat{a} \overset{hlk}{\circ} b) \overset{k}{=} \hat{a} \overset{hlk}{\circ} b$ . . . Axiom M6, line 3
5 $a \overset{hlk}{\circ} b \le_k \hat{a} \overset{hlk}{\circ} b$ . . . Lattice properties, line 4

and

1 $a^{\smile} \le_h \hat{a}^{\smile}$ and $c \in k$ . . . Assume
2 $a^{\smile} \overset{hkl}{\circ} \neg c \le_l \hat{a}^{\smile} \overset{hkl}{\circ} \neg c$ . . . proof $\circ$ Right Monotone (above), line 1
3 $\neg(\hat{a}^{\smile} \overset{hkl}{\circ} \neg c) \le_l \neg(a^{\smile} \overset{hkl}{\circ} \neg c)$ . . . Boolean negation, line 2
4 $\hat{a} \xrightarrow{hlk} c \le_l a \xrightarrow{hlk} c$ . . . Encoding, line 3

and

1 $c \le_k \hat{c}$ and $a \in h$ . . . Assume
2 $\neg\hat{c} \le_k \neg c$ . . . Boolean negation, line 1
3 $a^{\smile} \overset{hkl}{\circ} \neg\hat{c} \le_l a^{\smile} \overset{hkl}{\circ} \neg c$ . . . proof $\circ$ Left Monotone (above), , line 2
4 $\neg(a^{\smile} \overset{hkl}{\circ} \neg c) \le_l \neg(a^{\smile} \overset{hkl}{\circ} \neg\hat{c})$ . . . Boolean negation, line 3
5 $a \xrightarrow{hlk} c \overset{l}{=} a \xrightarrow{hlk} \hat{c}$ . . . Encoding, line 4

The proofs involving $\leftarrow$ are similar. ☐

LEMMA 3.1.3. *Axioms M8 and M8$^{\smile}$ are equivalent.*

PROOF.

1 $a^{\smile\smile} \overset{hlk}{\circ} \neg(a^{\smile} \overset{hkl}{\circ} c^{\smile}) \leq_k \neg c^{\smile}$ Instance of Axiom M8
2 $a \overset{hlk}{\circ} \neg(a^{\smile} \overset{hkl}{\circ} c^{\smile}) \leq_k \neg c^{\smile}$ Axiom M5, line 1
3 $(a \overset{hlk}{\circ} \neg(a^{\smile} \overset{hkl}{\circ} c^{\smile}))^{\smile} \leq_k \neg c^{\smile\smile}$ Lemma 3, line 2
4 $(a \overset{hlk}{\circ} \neg(a^{\smile} \overset{hkl}{\circ} c^{\smile}))^{\smile} \leq_k \neg c$ Axiom M5, line 3
5 $\neg(a^{\smile} \overset{hkl}{\circ} c^{\smile})^{\smile} \overset{lhk}{\circ} a^{\smile} \leq_k \neg c$ Axiom M7, line 4
6 $\neg(c^{\smile\smile} \overset{khl}{\circ} a^{\smile\smile}) \overset{lhk}{\circ} a^{\smile} \leq_k \neg c$ Axiom M7, line 5
7 $\neg(c \overset{khl}{\circ} a) \overset{lhk}{\circ} a^{\smile} \leq_k \neg c$ Axiom M5, line 6

That Axiom M8$^{\smile}$ implies Axiom M8 is similar. □

THEOREM 3.1.5. *Axioms M8, M8$^{\smile}$, N8, N8$^{\neg}$, N8$^{\smile}$, and N8$^{\smile\neg}$ are all equivalent.*

PROOF. Axioms M8 and N8 are equivalent.

1 $a^{\smile\smile} \overset{hlk}{\circ} \neg(a^{\smile} \overset{hkl}{\circ} \neg c) \leq_k \neg\neg c$ Instance of Axiom M8
2 $a \overset{hlk}{\circ} \neg(a^{\smile} \overset{hkl}{\circ} \neg c) \leq_k \neg\neg c$ Axiom M5, line 1
3 $a \overset{hlk}{\circ} \neg(a^{\smile} \overset{hkl}{\circ} \neg c) \leq_k c$ Boolean negation, line 2
4 $a \overset{hlk}{\circ} (a \xrightarrow{hlk} c) \leq_k c$ Encoding, line 3

and

1 $a^{\smile} \overset{hlk}{\circ} (a^{\smile} \xrightarrow{hlk} \neg c) \leq_k \neg c$ Instance of Axiom N8
2 $a^{\smile} \overset{hlk}{\circ} \neg(a^{\smile\smile} \overset{hkl}{\circ} \neg\neg c) \leq_k \neg c$ Encoding, line 1
3 $a^{\smile} \overset{hlk}{\circ} \neg(a^{\smile\smile} \overset{hkl}{\circ} c) \leq_k \neg c$ Boolean negation, line 2
4 $a^{\smile} \overset{hlk}{\circ} \neg(a \overset{hkl}{\circ} c) \leq_k \neg c$ Axiom M5, line 3

The other proofs are similar. □

THEOREM 3.1.6. *The distributed semigroup operators are normal: for $h, l, k \Subset \mathsf{hlk} \in \mathfrak{C}$, $\overset{hlk}{\circ}$ distributes over $\vee$ from both sides and*

$$a \overset{hlk}{\circ} \bot_l \overset{k}{=} \bot_k = \bot_l \overset{lhk}{\circ} a.$$

PROOF. Distribution of $\circ$ over $\vee$ from the right is Axiom M3, from the left is easily proven given the rest of the axioms.

Since $\bot_l \leq_l a \xrightarrow{hlk} \bot_k$ ($\bot_l \leq_l b$ for all $b \in l$), $a \overset{hlk}{\circ} \bot_l \leq_k \bot_k$. So $a \overset{hlk}{\circ} \bot_l \overset{k}{=} \bot_k$. Similarly, $\bot_l \leq_l \bot_k \xleftarrow{lhk} a$, so $\bot_l \overset{lhk}{\circ} a \leq_k \bot_k$ and $\bot_l \overset{lhk}{\circ} a \overset{k}{=} \bot_k$. □

References

[1] Adámek, J., and J. Rosický, *Locally Presentable and Accessible Categories*, Lecture Note Series 189, London Mathematical Society, 1994. DOI: 10.1017/CBO9780511600579

[2] Allwein, G., H. Demir, and L. Pike, "Logics for classes of Boolean monoids", *Journal of Logic, Language and Information*, 13, 3 (2004): 241–266. DOI: 10.1023/B:JLLI.0000028336.64373.f6

[3] Allwein, G., W. Harrison, and D. Andrews, "Simulation logic", *Logic and Logical Philosophy*, 23, 3 (2014): 277–299. DOI: 10.12775/LLP.2013.027

[4] Allwein, G., and W. L. Harrison, "Distributed modal logic", pages 331–362 in Katalin Bimbó (ed.), *J. Michael Dunn on Information Based Logic*, Outstanding Contributions to Logic, Springer-Verlag, 2016. DOI: 10.1007/978-3-319-29300-4_16

[5] Anderson, A. R., and N. D. Belnap, Jr., *Entailment: The Logic of Relevance and Necessity*, volume I, Princeton University Press, 1975.

[6] Birkhoff, G., *Lattice Theory*, Third Edition, American Mathematical Society, 1967.

[7] Birkhoff, G., and J. D. Lipson, "Heterogeneous algebras", *Journal of Computational Theory*, 8 (1968): 115–133.

[8] Blackburn, P., M. de Rijke, and Y. Venema, *Modal Logic*, Cambridge Tracts in Theoretical Computer Science, No. 53, Cambridge University Press, 2001. DOI: 10.1017/CBO9781107050884

[9] Chin, L. H., and A. Tarski, "Distributive and modular laws in the arithmetic of relation algebras", *University of California Publications in Mathematics*, 1951.

[10] Dunn, J. M., "Gaggle theory: An abstraction of Galois connections and residuation with applications to negation and various logical operations", pages 31–51 in *Logics in AI, Proceedings European Workshop JELIA, LNCS 478*, Springer-Verlag, 1990. DOI: 10.1007/BFb001843

[11] Dunn, J. M., and G. Hardegree, *Algebraic Methods in Philosophical Logic*, Oxford Logic Guides 41, Oxford University Press, 2001.

[12] Hoare, C. A. R., and J. He, "A weakest pre-specification", *Inform. Process. Lett.*, 24 (1987): 127–132.

[13] Jónsson, B., and A. Tarski, "Boolean algebras with operators. Parts 1 and 2", *American Journal of Mathematics*, 73–74 (1951–1952): 891–939, 127–162. DOI: 10.2307/2372074

[14] Jónsson, B., and C. Tsinakis, "Relation algebras as residuated Boolean algebras", *Algebra Universalis*, 30 (1993): 469–478.

[15] Ng, K. C., "Relation algebras with transitive closure", PhD thesis, University of California, Berkeley, 1984.

[16] Pratt, V. R., "Dynamic algebras as a well behaved fragment of relation algebras", Chapter 5 in *Proceedings, Algebra and Computer Science*, Lecture Notes in Computer Science, Springer-Verlag, 1990. DOI: 10.1007/BFb0043079
[17] Routley, R., and R. K. Meyer, "The semantics of entailment", pages 194–243 in H. Leblanc (ed.), *Truth, Syntax, and Modality*, Studies in Logic and the Foundations of Mathematics, North Holland, 1973. DOI: 10.1016/S0049-237X(08)71541-6
[18] Routley, R., R. K. Meyer, V. Plumwood, and R. T. Brady, *Relevant Logics and Their Rivals*, Ridgeview Publishing Company, 1982.
[19] Sahlqvist, H., "Completeness and correspondence in the first and second order semantics for modal logic", pages 110–143 in *Proceedings of the Third Scandanavian Logic Symposium, Uppsala, 1973*, Studies in Logic and the Foundations of Mathematics, North-Holland Publishing Company, 1975. DOI: 10.1016/S0049-237X(08)70728-6
[20] Stone, M. H., "The theory of representation for Boolean algebras", *Transactions of the American Mathematical Society*, 40 (1936): 37–111. DOI: 10.2307/1989664
[21] van Benthem, J., *Modal Logic and Classical Logic*, Bibliopolis, 1983.

Gerard Allwein
Naval Research Laboratory
Code 5543
Washington, DC 20375, USA
gerard.allwein@nrl.navy.mil

William L. Harrison and Thomas Reynolds
Dept. of Computer Science
University of Missouri
Columbia, Missouri, USA
harrisonwl@missouri.edu