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NEIGHBORHOOD SEMANTICS FOR BASIC AND INTUITIONISTIC LOGIC

Abstract. In this paper we present a neighborhood semantics for Intuitionistic Propositional Logic (IPL). We show that for each Kripke model of the logic there is a pointwise equivalent neighborhood model and vice versa. In this way, we establish soundness and completeness of IPL with respect to the neighborhood semantics. The relation between neighborhood and topological semantics are also investigated. Moreover, the notions of *bisimulation* and *n-bisimulation* between neighborhood models of IPL are defined naturally and some of their basic properties are proved. We also consider Basic Propositional Logic (BPL), a logic weaker than IPL introduced by Albert Visser, and introduce and study its neighborhood models in the same manner.

Keywords: Intuitionistic Logic; Basic Logic; Kripke models; neighborhood models; bisimulation; modal logic; topological semantics.

Mathematics Subject Classification: 03B20, 03B45

1. Introduction

There are several semantics for intuitionistic logic that are sound and complete. Kripke semantics for intuitionistic logic and modal logic developed in the late 1950s and early 1960s. Saul Kripke proved completeness of intuitionistic logic w.r.t. Kripke models in 1963. Neighborhood model for modal logic was first studied by Dana Scott and Richard Montague (independently in [6, 8]). In a neighborhood model for modal logic, each state is associated with a collection of subsets of the universe (called its

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neighborhood), and a modal formula $\Box\varphi$ is true at a world w, if the set of all states in which φ is true is a neighborhood of w. See [2] for more details on neighborhood semantics for modal logic.

An interesting question is whether one can define similar neighborhood semantics for Intuitionistic Propositional Logic (IPL). In this paper, we give an answer to this question. The definition enables us to naturally define some notions familiar in the context of modal logic for intuitionistic logic. Among these notions are *disjoint union*, *bounded morphism* and *bisimulation*.

The mentioned question can also be asked about subsystems of IPL. In this regard, we consider Basic Propositional Logic (BPL) and introduce neighborhood models of it by just omitting one condition in the definition of the corresponding models of IPL. BPL was introduced by Visser in 1981 [10]. His motivation was to interpret implication as formal provability. To protect his system from the liar paradox, modus ponens is weakened in the system. He gave an axiomatization of BPL in the natural deduction form and proved completeness of BPL with respect to finite irreflexive Kripke models. Afterwards, Wim Ruitenburg [7], considering philosophical criticisms of the intuitionistic interpretation of the logical connectives, reintroduced BPL and a first order extension (BQC).

The structure of this paper is as follows. In Section 2, we review some basic results concerning Kripke models of IPL and neighborhood models of propositional modal logic. We also bring Visser's axiomatization of BPL in the natural deduction style. In Section 3, we introduce neighborhood models for IPL, show that for every Kripke model there is a pointwise equivalent neighborhood model and prove soundness and completeness. In Section 4, we discuss the relation between topological and neighborhood semantics of intuitionistic logic. In Section 5, we introduce and study the notions of *disjoint union*, *bounded morphism*, *bisimulation* and *n-bisimulation* for neighborhood models and prove some of their elementary properties. In Section 6, we present neighborhood semantics for BPL.

2. Preliminaries

In this section we recall definitions of Kripke models for Intuitionistic Propositional Logic (IPL) and for Basic Propositional Logic (BPL). We also give an axiomatization of IPL and Visser's axiomatization of BPL in the natural deduction form.

We work in a fixed propositional language throughout this paper, for which P is the set of propositional letters.

The rules of IPL in the natural deduction form are the following. *Group I*:

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi}\,(\wedge\mathrm{I}) \qquad \frac{\varphi \wedge \psi}{\varphi} \quad \frac{\varphi \wedge \psi}{\psi}\,(\wedge\mathrm{E})$$

$$\frac{\varphi}{\varphi \vee \psi} \quad \frac{\psi}{\varphi \vee \psi}\,(\vee\mathrm{I}) \qquad \frac{\bot}{\varphi}\,(\bot\mathrm{E}) \qquad \frac{\varphi \quad \varphi \to \psi}{\psi}\,(\to\mathrm{E})$$

Group II:

$$\frac{\begin{array}{c}[\varphi]\\ \vdots\\ \psi\end{array}}{\varphi \to \psi}\,(\to\mathrm{I}) \qquad \frac{\varphi \vee \psi \quad \begin{array}{c}[\varphi]\\ \vdots\\ \chi\end{array} \quad \begin{array}{c}[\psi]\\ \vdots\\ \chi\end{array}}{\chi}\,(\vee\mathrm{E})$$

Recall that the rules in Group II cancel their hypotheses. For a set Γ and a formula φ, we say φ is deducible from Γ in IPL and write $\Gamma \vdash \varphi$ if there is a deduction for φ using the rules of IPL with the property that all of its uncancelled hypotheses are in Γ.

A *Kripke model for* IPL is any triple $\langle K, \preceq, V\rangle$, where K is a non-empty set (its elements are called *worlds* or *nodes*), $\preceq$ is a preorder on K (i.e., reflexive and transitive relation on K), and V is a valuation function from P into 2^K such that for all $k, k' \in K$ and $p \in \mathrm{P}$ we have:

- if $k \in V(p)$ and $k \preceq k'$, then $k' \in V(p)$.

For any Kripke model for IPL we introduce a binary relation $\Vdash$ on $K \times$ the set of formulas, which in a node $k \in K$ is defined inductively as follows:

1. $k \Vdash p$ iff $k \in V(p)$, for any $p \in \mathrm{P}$;
2. $k \Vdash \varphi \wedge \psi$ iff $k \Vdash \varphi$ and $k \Vdash \psi$;
3. $k \Vdash \varphi \vee \psi$ iff $k \Vdash \varphi$ or $k \Vdash \psi$;
4. $k \Vdash \varphi \to \psi$ iff $\forall_{k' \in W}(k \Vdash \varphi$ and $k \preceq k' \Rightarrow k' \Vdash \psi)$;
5. $k \nVdash \bot$ (i.e., no elements of K forces $\bot$).

By 4 and 5, for the intuitionistic negation, $\neg\varphi := \varphi \to \bot$, we obtain:

- $k \Vdash \neg\varphi$ iff $\forall_{k' \in W}(k \preceq k' \;\Rightarrow\; k' \nVdash \varphi)$.

We shall use the expression "k forces φ" (or "φ is true at k") for $k \Vdash \varphi$.

The following two lemmas are standard and can be found in [9].

LEMMA 2.1. *For any formula φ of* IPL *we have monotonicity*:

$$\forall_{k,k' \in K}(k \Vdash \varphi \text{ and } k \preceq k' \Longrightarrow k' \Vdash \varphi).$$

Lemma 2.2 (Model Existence Lemma). *If $\Gamma \nvdash \varphi$ (in* IPL*), then there is a Kripke model with a bottom node k_0 such that $k_0 \Vdash \Gamma$ and $k_0 \nVdash \varphi$.*

BPL is a subsystem of IPL with the following rules. *Group I*: the rules ($\wedge$I), ($\wedge$E), ($\vee$I), ($\bot$E), and

$$\frac{\varphi \to \psi \quad \psi \to \chi}{\varphi \to \chi}\,(\mathrm{Tr}) \qquad \frac{\varphi \to \psi \quad \varphi \to \chi}{\varphi \to (\psi \wedge \chi)}\,(\wedge\mathrm{If})$$

$$\frac{\varphi \to \chi \quad \psi \to \chi}{(\varphi \vee \psi) \to \chi}\,(\vee\mathrm{Ef})$$

Group II: the rules ($\to$I) and ($\vee$E).

A *Kripke model for* BPL is any triple $\langle K, R, V\rangle$, where K is a non-empty set, R is a transitive relation on K, and V is a valuation function from P into 2^K such that for all $k, k' \in K$ and $p \in \mathrm{P}$ we have:

- if $k \in V(p)$ and kRk', then $k' \in V(p)$.

Thus, Kripke models for BPL obtained from Kripke models for IPL by removing the reflexivity condition for relations.

For any Kripke model for BPL we introduce a binary relation $\Vdash$, as for Kripke models for PPL.

We write $w \Vdash \Gamma$, if w forces each formula in Γ. The semantical consequence relation $\Gamma \Vdash \varphi$ in IPL (resp. BPL) holds, if for every Kripke model $\langle K, \preceq, \Vdash\rangle$ (resp. $\langle K, R, \Vdash\rangle$) for any $k \in K$: if $k \Vdash \Gamma$, then $k \Vdash \varphi$.

The following theorem belongs to Visser [10].

Theorem 2.3. BPL *is sound and complete with respect to the class of Kripke models for* BPL.

3. Neighborhood semantics for intuitionistic logic

In this section we introduce neighborhood semantics for intuitionistic propositional logic and show that IPL is sound and complete with respect to this semantics.

Definition 3.1 (Neighborhood frames). A pair $\langle W, N\rangle$ is called a *neighborhood frame of* IPL, if W is a non-empty set and N is a (neighborhood) function from W into 2^{2^W} such that for each $w \in W$ we have:

1. $w \in \bigcap N(w)$;
2. $\bigcap N(w) \in N(w)$;

3. if $X \in N(w)$ and $X \subseteq Y$, then $Y \in N(w)$; (*superset*)
4. if $X \in N(w)$, then $\{u \in W \mid X \in N(u)\} \in N(w)$;

Lemma 3.2 ([2, p. 220]). *From clauses 2 and 3 for any $w \in W$ we have:*
1. $W \in N(w)$.
2. *If $\mathscr{X} \subseteq N(w)$ then $\bigcap \mathscr{X} \in N(w)$. So also: if $X, Y \in N(w)$, then $X \cap Y \in N(w)$.*
3. *If $\bigcap N(w) \subseteq X$ then $X \in N(w)$.*

Definition 3.3 (Neighborhood models). A *neighborhood model of* IPL is a tuple $\langle W, N, V\rangle$, where $\langle W, N\rangle$ is a neighborhood frame of IPL and V is a valuation function from P into 2^W such that for all $w \in W$ and $p \in \mathrm{P}$ we have:
- if $w \in V(p)$ then $V(p) \in N(w)$.

Definition 3.4 (Truth in neighborhood models). Let $\langle W, N, V\rangle$ be any neighborhood model of IPL. Truth of a propositional formula in a world $w \in W$ is defined inductively as follows:

1. $w \Vdash p$ iff $w \in V(p)$, for any $p \in \mathrm{P}$;
2. $w \nVdash \bot$;
3. $w \Vdash \varphi \wedge \psi$ iff $w \Vdash \varphi$ and $w \Vdash \psi$;
4. $w \Vdash \varphi \vee \psi$ iff $w \Vdash \varphi$ or $w \Vdash \psi$;
5. $w \Vdash \varphi \to \psi$ iff $\{u \in W \mid u \nVdash \varphi \text{ or } u \Vdash \psi\} \in N(w)$.

Remark 3.5. For the intuitionistic negation, $\neg\varphi := \varphi \to \bot$, we obtain: $w \Vdash \neg\varphi$ iff $\{u \in W \mid u \nVdash \varphi\} \in N(w)$.

We put $V(\varphi) := \{w \in W \mid w \Vdash \varphi\}$. Then we obtain:

Lemma 3.6. *If $w \Vdash \varphi$ then $V(\varphi) \in N(w)$.*

Proof. By induction on the complexity of formulas. The case where φ is a proposition letter follows from the clause 1 of Definition 3.4 and Definition 3.3. Moreover, by the clause 2 of Definition 3.4 we have $w \nVdash \bot$.

Let $w \Vdash \varphi \wedge \psi$; so $w \Vdash \varphi$ and $w \Vdash \psi$, by the clause 3 of Definition 3.4. Then, by induction hypothesis, $V(\varphi) \in N(w)$ and $V(\psi) \in N(w)$. So $V(\varphi \wedge \psi) = V(\varphi) \cap V(\psi) \in N(w)$, by the clause 2 of Lemma 3.2.

Let $w \Vdash \varphi \vee \psi$; so $w \Vdash \varphi$ or $w \Vdash \psi$, by the clause 4 of Definition 3.4. Then $V(\varphi)$ or $V(\psi)$ belongs to $N(w)$, by induction hypothesis. Thus, $V(\varphi \vee \psi) = V(\varphi) \cup V(\psi) \in N(w)$, by the clause 3 of Definition 3.1.

Now let $w \Vdash \varphi \to \psi$; so $X := \{v \in W \mid v \nVdash \varphi \text{ or } v \Vdash \psi\} \in N(w)$. Then $V(\varphi \to \psi) = \{u \in W \mid X \in N(u)\} \in N(w)$, by the clause 4 of Definition 3.1, and the clause 5 of Definition 3.4. $\dashv$

Example 3.7. Let $W := \{w_1, w_2\}$, with $w_1 \neq w_2$. We define the following neighborhood model $\langle W, N, V\rangle$ where: $N(w_1) := \{W\}$, $N(w_2) := \{W, \{w_2\}\}$, and $V(p) := \{w_2\}$. We have $w_1 \nVdash p \vee \neg p$, since $w_1 \notin V(p)$ and $\{u \mid u \nVdash p\} = \{w_1\} \notin N(w_1)$.

We write $w \Vdash \Gamma$, if all formulas in Γ are true in w. The *semantical consequence relation* $\Gamma \models \varphi$ holds, if for every neighborhood model $\langle W, N, V\rangle$ of IPL, for any $w \in W$: if $w \Vdash \Gamma$, then $w \Vdash \varphi$.

Theorem 3.8 (Soundness). *If $\Gamma \vdash \varphi$, then $\Gamma \models \varphi$.*

Proof. Suppose that $\Gamma \vdash \varphi$. We use induction on the complexity of derivations from Γ. Let $\langle W, N, V\rangle$ be any neighborhood model.

If a proof D consists of just φ, then for each $w \in W$: $w \Vdash \Gamma$ implies $w \Vdash \varphi$.

Now suppose that D ends with an application of a derivation rule. We only check the case of implication.

($\to$I) By the induction hypothesis, for any $u \in W$: if $u \Vdash \Gamma \cup \{\varphi\}$ then $u \Vdash \psi$. Let $w \Vdash \Gamma$. We show that $w \Vdash \varphi \to \psi$, that is $X := \{u \mid u \nVdash \varphi$ or $u \Vdash \psi\} \in N(w)$. By the clause 3 of Lemma 3.2, it is sufficient to show that $\bigcap N(w) \subseteq X$. To do this, let $u \in \bigcap N(w)$ and $u \Vdash \varphi$. Since $w \Vdash \Gamma$, so — by Lemma 3.6 — for any $\gamma \in \Gamma$ we have $V(\gamma) \in N(w)$. Hence $u \in \bigcap N(w) \subseteq V(\gamma)$, for any $\gamma \in \Gamma$. So $u \Vdash \Gamma$. Therefore, by the induction hypothesis, $u \Vdash \psi$.

($\to$E) By the induction hypothesis, for any $u \in W$: if $u \Vdash \Gamma$ than $u \Vdash \varphi$ and $u \Vdash \varphi \to \psi$. Let $w \Vdash \Gamma$. By assumption and the induction hypothesis, $w \Vdash \varphi$ and $X := \{u \mid u \nVdash \varphi$ or $u \Vdash \psi\} \in N(w)$. Hence, by the clause 1 of Definition 3.1, we have $w \in \bigcap N(w) \subseteq X$. So $w \Vdash \psi$. $\dashv$

We say that a Kripke model and a neighborhood model with the same universe are *pointwise equivalent*, if in every world in this models are true the same formulas.

Theorem 3.9. *For every Kripke model $\langle W, \preceq, V\rangle$ there is a pointwise equivalent neighborhood model $\langle W, N, V\rangle$.*

Proof. Let $M_{\mathrm{K}} = \langle W, \preceq, V\rangle$ be a Kripke model and $w \in W$. For each $w \in W$, we define $R(w) := \{u \in W \mid w \preceq u\}$ and $N(w) := \{X \mid R(w) \subseteq X\}$. We show that $M_{\mathrm{n}} := \langle W, N, V\rangle$ is a neighborhood model.

By definition of $N(w)$, clauses 1–3 of Definition 3.1, are immediate. For the clause 4, let $X \in N(w)$, i.e., $R(w) \subseteq X$. We show that $\{u \mid X \in N(u)\} \in N(w)$. That is we must show $R(w) \subseteq \{u \mid X \in N(u)\}$.

Let $x \in R(w)$; i.e., $w \preceq x$. To complete the proof we need only to show $X \in N(x)$, i.e., $R(x) \subseteq X$. Since $\preceq$ is transitive, so $R(x) \subseteq R(w)$. Hence $R(x) \subseteq X$, since $R(w) \subseteq X$.

Let $w \in V(p)$. Then, by definition of the Kripke model $\langle W, \preceq, V\rangle$, for any $x \in W$: if $w \preceq x$ then $x \in V(p)$. Hence $R(w) \subseteq V(p)$, i.e., $V(p) \in N(w)$.

Now we proof that M_{K} and M_{n} are pointwise equivalent. The proof is by induction on the complexity of formulas. We only consider the case $\varphi \to \psi$, where the argument goes like this. Let $w \Vdash_{M_{\mathrm{K}}} \varphi \to \psi$. We want to show that $w \Vdash_{M_{\mathrm{n}}} \varphi \to \psi$, i.e., $Y := \{u \mid u \nVdash_{M_{\mathrm{n}}} \varphi$ or $u \Vdash_{M_{\mathrm{n}}} \psi\} \in N(w)$. By the clause 3 of Lemma 3.2, it is sufficient to show that $\bigcap N(w) \subseteq Y$. Let $u \in \bigcap N(w)$ and $u \Vdash_{M_{\mathrm{n}}} \varphi$. Then $w \preceq u$ and $u \Vdash_{M_{\mathrm{K}}} \varphi$, by the induction hypothesis. Hence $u \Vdash_{M_{\mathrm{K}}} \psi$, by the assumption. So $u \Vdash_{M_{\mathrm{n}}} \psi$, by the induction hypothesis.

Now let $w \Vdash_{M_{\mathrm{n}}} \varphi \to \psi$. We show that $w \Vdash_{M_{\mathrm{K}}} \varphi \to \psi$. So let $w \preceq u$ and $u \Vdash_{M_{\mathrm{K}}} \varphi$. Then $u \Vdash_{M_{\mathrm{n}}} \varphi$, by the induction hypothesis. Since $u \in \bigcap N(w) \subseteq \{u \mid u \nVdash_{M_{\mathrm{n}}} \varphi$ or $u \Vdash_{M_{\mathrm{n}}} \psi\}$, so $u \Vdash_{M_{\mathrm{n}}} \psi$. Hence $u \Vdash_{M_{\mathrm{K}}} \psi$, by the induction hypothesis. That is, $w \Vdash_{M_{\mathrm{K}}} \varphi \to \psi$. $\dashv$

By Lemma 2.1 and Theorem 3.9 we obtain.

Lemma 3.10. *If $\Gamma \nvdash \varphi$, then there is a neighborhood model $\langle W, N, V\rangle$ with a world w such that $w \Vdash \Gamma$ and $w \nVdash \varphi$.*

Finally, by the above lemma we obtain:

Theorem 3.11 (Completeness). *If $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$.*

Thus, in theorems 3.8 and 3.11 we have soundness and completeness of IPL with respect to the neighborhood semantics for IPL. Notice that we obtain (see p. 342): $\Gamma \models \varphi$ iff $\Gamma \Vdash \varphi$.

Moreover, below we obtain a theorem that is inverse to Theorem 3.9. We use the following lemma.

Lemma 3.12. *Let $\langle W, N, V\rangle$ be a neighborhood model. Then for all $w \in W$ and $u \in \bigcap N(w)$ we have $N(w) \subseteq N(u)$.*

Proof. Let $u \in \bigcap N(w)$ and $X \in N(w)$. Then, by the clause 4 of Definition 3.1, $\{v \in W \mid X \in N(v)\} \in N(w)$. Hence $u \in \bigcap N(w) \subseteq \{v \mid X \in N(v)\}$. So $X \in N(u)$. Thus, $N(w) \subseteq N(u)$. $\dashv$

Theorem 3.13. *For every neighborhood model $\langle W, N, V\rangle$ there is a pointwise equivalent Kripke model $\langle W, \preceq, V\rangle$.*

Proof. Let $M_{\mathrm{n}} = \langle W, N, V\rangle$ be a neighborhood model for IPL. We define the following binary relation $\preceq$ on $W \times W$: $w \preceq u$ iff $N(w) \subseteq N(u)$.

We first show that $M_{\mathrm{K}} := \langle W, \preceq, V\rangle$ is a Kripke model. By definition, reflexivity and transitivity of $\preceq$ are immediate.

Let $w \in V(p)$ and $w \preceq u$. Hence, by definition, $N(w) \subseteq N(u)$. Moreover, by assumption, $V(p) \in N(w)$. So $V(p) \in N(u)$. Hence $\bigcap N(u) \subseteq V(p)$. So $u \in V(p)$, since $u \in \bigcap N(u)$, by the clause 1 of Definition 3.1.

Next, by induction on the complexity of formulas, we show that M_{K} and M_{n} are pointwise equivalent. The only case of interest is implication where the argument goes like this.

Let $w \Vdash_{M_{\mathrm{n}}} \varphi \to \psi$, i.e., $\{v \mid v \nVdash_{M_{\mathrm{n}}} \varphi \text{ or } v \Vdash_{M_{\mathrm{n}}} \psi\} \in N(w)$. We must show that $w \Vdash_{M_{\mathrm{K}}} \varphi \to \psi$. So let $w \preceq u$ and $u \Vdash_{M_{\mathrm{K}}} \varphi$. Then $u \Vdash_{M_{\mathrm{n}}} \varphi$, by the induction hypothesis. Hence $u \Vdash_{M_{\mathrm{n}}} \psi$, by the assumption. So $\nu \Vdash_{M_{\mathrm{K}}} \psi$, again by the induction hypothesis.

Now let $w \Vdash_{M_{\mathrm{K}}} \varphi \to \psi$. We must show that $w \Vdash_{M_{\mathrm{n}}} \varphi \to \psi$, i.e., $X := \{v \mid v \nVdash_{M_{\mathrm{n}}} \varphi \text{ or } v \Vdash_{M_{\mathrm{n}}} \psi\} \in N(w)$. By the clause 3 of Lemma 3.2, it is enough to show that $\bigcap N(w) \subseteq X$. To do this, let $u \in \bigcap N(w)$ and $u \Vdash_{M_{\mathrm{n}}} \varphi$. By Lemma 3.12, $N(w) \subseteq N(u)$ and so $w \preceq u$. Therefore $u \Vdash_{M_{\mathrm{K}}} \psi$ and so $u \Vdash_{M_{\mathrm{n}}} \psi$. That is $w \Vdash_{M_{\mathrm{n}}} \varphi \to \psi$. ⊣

4. Neighborhood semantics vs. topological semantics

In this section we discuss the relation of neighborhood semantics to the topological semantics. This semantics for some families of modal logic and its relation to the neighborhood semantics have been studied extensively, see e.g. [1]. The book [3] contains much materials on the connection between algebraic and topological semantics for IPL. At the end of this section we consider topological models of IPL.

Definition 4.1. Let $\langle X, \mathscr{O}\rangle$ be a topological space, where X is a nonempty set and $\mathscr{O}$ is a family of open subsets of X, i.e., $\mathscr{O} \subseteq 2^X$ and $\mathscr{O}$ satisfies the following conditions:

1. $\varnothing, X \in \mathscr{O}$;
2. if $A, B \in \mathscr{O}$, then $A \cap B \in \mathscr{O}$;
3. if $\mathscr{A} \subseteq \mathscr{O}$, then $\bigcup \mathscr{A} \in \mathscr{O}$.

Let V be any valuation function from P into $\mathscr{O}$. This function we inductively extended to the set of all formulas as follows:

1. $V(\varphi \vee \psi) = V(\varphi) \cup V(\psi)$;
2. $V(\varphi \wedge \psi) = V(\varphi) \cap V(\psi)$;
3. $V(\varphi \to \psi) = \mathrm{Int}((X \setminus V(\varphi)) \cup V(\psi))$;
4. $V(\neg\varphi) = \mathrm{Int}(X \setminus V(\varphi))$,

where for any $Y \in 2^X$, $\mathrm{Int}\, Y = \bigcap\{A \in \mathscr{O} \mid Y \subseteq A\}$.

A topological space $\langle X, \mathscr{O}\rangle$ is called an *Alexandroff space*, if the intersection of any family of open sets is open, i.e.:

- if $\mathscr{A} \subseteq \mathscr{O}$, then $\bigcap \mathscr{A} \in \mathscr{O}$.

It is easy to see that a topological space is an Alexandroff space iff every point has a least open neighborhood (which is the intersection of all open neighborhoods).

Theorem 4.2. *For any Alexandroff space $\langle X, \mathscr{O}\rangle$ and $V\colon \mathrm{P} \to \mathscr{O}$ there is a pointwise equivalent neighborhood model $\langle X, N_{\mathscr{O}}, V\rangle$.*

Proof. Let $\langle X, \mathscr{O}\rangle$ be an Alexandroff space. Then for any $x \in X$ we put $N_{\mathscr{O}}(x) := \{Y \in 2^X \mid \text{for some } A \in \mathscr{O} \text{ we have } x \in A \subseteq Y\}$, i.e., $N_{\mathscr{O}}(x)$ is the family of neighborhoods of x. We must show that $\langle X, N_{\mathscr{O}}, V\rangle$ satisfied all conditions of Definition 3.1.

For 1. $x \in \bigcap\{Y \in 2^X \mid \exists_{A\in\mathscr{O}}\, x \in A \subseteq Y\}$.

For 2. For every $Y \in N_{\mathscr{O}}(x)$ we choose an open set A_Y such that $x \in A_Y \subseteq Y$. The set $A := \bigcap\{A_Y \mid Y \in N_{\mathscr{O}}(x)\}$ is open, since $\langle X, \mathscr{O}\rangle$ is an Alexandroff space. Since $x \in A \subseteq \bigcap N_{\mathscr{O}}(x)$, so $\bigcap N_{\mathscr{O}}(x) \in N_{\mathscr{O}}(x)$.

For 3. Of course, if $Y \in N_{\mathscr{O}}(x)$ and $Y \subseteq Z$, then $Z \in N_{\mathscr{O}}(x)$.

For 4. Let $Y \in N_{\mathscr{O}}(x)$, i.e., for some $A \in \mathscr{O}$ we have $x \in A \subseteq Y$. Of course, $A \in N_{\mathscr{O}}(x)$ and $A = \{z \in X \mid A \in N_{\mathscr{O}}(z)\}$. Hence $A \subseteq \{z \mid Y \in N_{\mathscr{O}}(z)\}$, since $A \subseteq Y$. So $\{z \mid Y \in N_{\mathscr{O}}(z)\} \in N_{\mathscr{O}}(x)$, by 3.

The proof that $\langle X, \mathscr{O}\rangle$ with valuation V and $\langle X, N_{\mathscr{O}}, V\rangle$ are pointwise equivalent is by an easy induction on the complexity of formulas. ⊣

Theorem 4.3. *For every neighborhood model $M = \langle X, N, V\rangle$ there is an Alexandroff space $\tau = \langle X, \mathscr{O}\rangle$ such that τ with valuation V is pointwise equivalent to M, i.e., for all formula φ and $x \in X$: $x \in V(\varphi)$ iff $x \Vdash_M \varphi$.*

Proof. Let $M = \langle X, N, V\rangle$ be a neighborhood model, we define a family $\mathscr{O}_N$ of open sets as follow: $A \in \mathscr{O}_N$ iff $\forall_{x\in A}\, A \in N(x)$. Then $\tau := \langle X, \mathscr{O}_N\rangle$ is an Alexandroff space.

The proof that τ with valuation V and M are pointwise equivalent is by induction on the complexity of formulas. We only consider the case $\varphi \to \psi$.

Let $x \in V(\varphi \to \psi) = \mathrm{Int}((X \setminus V(\varphi)) \cup V(\psi))$. Then there is $A \in \mathscr{O}_N$ such that $x \in A \subseteq (X \setminus V(\varphi)) \cup V(\psi)$ and $A \in N(x)$. Hence, by *superset*, also $(X \setminus V(\varphi)) \cup V(\psi) \in N(x)$. But, by the induction hypothesis, $(X \setminus V(\varphi)) \cup V(\psi) = \{z \in X \mid z \nVdash_M \varphi \text{ or } z \Vdash_M \psi\}$. That is $x \Vdash_M \varphi \to \psi$.

Now let $x \Vdash_M \varphi \to \psi$, i.e., $\{z \in X \mid z \nVdash_M \varphi \text{ or } \Vdash_M \psi\} \in N(x)$. Then, by the induction hypothesis, $(X \setminus V(\varphi)) \cup V(\psi) \in N(x)$. Hence $\bigcap N(x) \subseteq (X \setminus V(\varphi)) \cup V(\psi)$. We show that $x \in V(\varphi \to \psi) = \mathrm{Int}((X \setminus V(\varphi)) \cup V(\psi))$. Since $x \in \bigcap N(x)$, by 1 of Definition 3.1, so it is sufficient to show that $\bigcap N(x) \in \mathscr{O}_N$, i.e., we must show that for any $y \in \bigcap N(x)$ we have $\bigcap N(x) \in N(y)$. But, by Lemma 3.12, if $y \in \bigcap N(x)$, then $N(x) \subseteq N(y)$. But $\bigcap N(x) \in N(x)$, by the condition 2 of Definition 3.1. Thus, $\bigcap N(x) \in N(y)$. That is $x \in \mathrm{Int}((X \setminus V(\varphi)) \cup V(\psi))$. ⊣

5. Invariance results for neighborhood models

In this section, we introduce and study the notions of *disjoint union*, *bounded morphism*, *bisimulation* and *n-bisimulation* for neighborhood models of IPL and prove some of their basic properties. Some ideas came from [4, 5].

Definition 5.1 (Disjoint unions). Let $M_i = \langle W_i, N_i, V_i\rangle$, $i \in I$, be a collection of neighborhood models of IPL with disjoint universes. The *disjoint union* of this collection is the structure $M_I = \langle W, N, V\rangle$, where $W := \bigcup_{i\in I} W_i$, $V(p) := \bigcup_{i\in I} V_i(p)$, and for any $i \in I$ and $w \in W_i$: $N(w) := \{X \in 2^W \mid X \cap W_i \in N_i(w)\}$.

Proposition 5.2. *Let $M_I = \langle W, N, V\rangle$ be the disjoint union of a collection $M_i = \langle W_i, N_i, V_i\rangle$, $i \in I$, of neighborhood models of* IPL *with disjoint universes. Then for each formula φ and for all $i \in I$ and $w \in W_i$:*

$$w \Vdash_{M_I} \phi \quad \text{iff} \quad \exists_{i\in M}\, M_i, w \Vdash \phi.$$

Proof. The proof is by induction on the complexity of formulas. Let $i \in I$ and $w \in W_i$. The atomic and Boolean cases are straightforward.

For $\varphi \to \psi$ we put $X_i := \{u \in W_i \mid u \nVdash_{M_i} \varphi \text{ or } u \Vdash_{M_i} \psi\}$ and $X := \{u \in W \mid u \nVdash_{M_I} \varphi \text{ or } u \Vdash_{M_I} \psi\}$. We need to show that $X_i \in N_i(w)$ iff $X \in N(w)$.

Suppose that $X \in N(w)$. Then $X \cap W_i \in N_i(w)$. Now the induction hypothesis tells us that $X \cap W_i = X_i$, and hence $X_i \in N_i(w)$. The other direction is immediate. ⊣

Definition 5.3 (Bounded morphism). Let $M_1 = \langle W_1, N_1, V_1\rangle$ and $M_2 = \langle W_2, N_2, V_2\rangle$ be two neighborhood models of IPL. A function $f\colon W_1 \to W_2$ is a *bounded morphism* from M_1 to M_2, if for any $w \in W$ the following conditions are satisfied:

1. for any $p \in \mathrm{P}$: $w \Vdash_{M_1} p$ iff $f(w) \Vdash_{M_2} p$, i.e., w and $f(w)$ satisfy the same proposition letters;
2. $f[\bigcap N_1(w)] = \bigcap N_2(f(w))$.

Proposition 5.4. *Let $M_1 = \langle W_1, N_1, V_1\rangle$ and $M_2 = \langle W_2, N_2, V_2\rangle$ be two neighborhood models. If $f\colon W_1 \to W_2$ is a bounded morphism from M_1 to M_2, then for each formula φ and each $w \in W_1$:*

$$w \Vdash_{M_1} \varphi \quad \textit{iff} \quad f(w) \Vdash_{M_2} \varphi.$$

Proof. The proof is by induction on the complexity of formulas. Let $w \Vdash_{M_1} \varphi \to \psi$. We show that $f(w) \Vdash_{M_2} \varphi \to \psi$. By *superset* it is sufficient to show that $\bigcap N_2(f(w)) \subseteq \{\nu \mid M_2, \nu \Vdash \varphi \Rightarrow M_2, \nu \Vdash \psi\}$. Let $y \in \bigcap N_2(f(w))$ and $M_2, y \Vdash \varphi$. Since f is bounded morphism, hence there is $\nu \in \bigcap N_1(w)$ such that $f(\nu) = y$. Now the induction hypothesis tells us that $M_1, \nu \Vdash \varphi$ and so $M_1, \nu \Vdash \psi$. Again by the induction hypothesis $M_2, y \Vdash \psi$. The other direction is similar. ⊣

Definition 5.5 (Bisimulation). Let $M_1 = \langle W_1, N_1, V_1\rangle$ and $M_2 = \langle W_2, N_2, V_2\rangle$ be two neighborhood models. A non-empty binary relation $R \subseteq W_1 \times W_2$ is called a *bisimulation between* M_1 and M_2 (write: $R\colon M_1 \rightleftarrows M_2$), if for all $w_1 \in W_1$, $w_2 \in W_2$ such that $w_1 R w_2$ we have:

1. for any $p \in \mathrm{P}$: $w_1 \Vdash_{M_1} p$ iff $w_2 \Vdash_{M_2} p$, i.e., w_1 and w_2 satisfy the same proposition letters;
2. for any $y \in \bigcap N_2(w_2)$ there is $x \in \bigcap N_1(w_1)$ such that xRy;
3. for any $x \in \bigcap N_1(w_1)$ there is $y \in \bigcap N_2(w_2)$ such that xRy.

When R is a bisimulation linking two states $w_1 \in W_1$ and $w_2 \in W_2$ we say that w_1 and w_2 are *bisimilar*, and we write $R\colon w_1 \rightleftarrows w_2$.

Lemma 5.6. *Let $M_1 = \langle W_1, N_1, V_1\rangle$ and $M_2 = \langle W_2, N_2, V_2\rangle$ be two neighborhood models and R be a bisimulation. Then $w_1 R w_2$ implies that w_1 and w_2 force the same formulas.*

Proof. By induction on the complexity of formulas. We only consider the case of implication.

Let $w_1 \Vdash_{M_1} \varphi \to \psi$. We show that $w_2 \Vdash_{M_2} \varphi \to \psi$. By *superset* it is enough to show that $\bigcap N_2(w_2) \subseteq \{u \mid u \nVdash_{M_2} \varphi \text{ or } u \Vdash_{M_2} \psi\}$.

Let $y \in \bigcap N_2(w_2)$ and $y \Vdash_{M_2} \varphi$. By definition of bisimulation there is $x \in \bigcap N_1(w_1)$ such that xRy. So $x \Vdash_{M_1} \varphi$ and $x \Vdash_{M_1} \psi$. Again by the induction hypothesis, $y \Vdash_{M_2} \psi$. The converse direction can be similarly verified using the clause 3 of Definition 5.5. $\dashv$

Now we introduce the notion of *n-bisimulation*. To do this we need to define a degree for complexity of formulas.

Definition 5.7 (Degree). We define the *degree* of formulas as follows:

- $\deg(p) = 0$, for any $p \in \mathrm{P}$;
- $\deg(\perp) = 0$;
- $\deg(\varphi \vee \psi) = \deg(\varphi \wedge \psi) = \max\{\deg(\varphi), \deg(\psi)\}$;
- $\deg(\varphi \to \psi) = 1 + \max\{\deg(\varphi), \deg(\psi)\}$.

The following proposition is obviously true.

Proposition 5.8. *Assume that the collection of all proposition letters be finite. Then we have:*

1. *For any n, up to logical equivalence there are only finitely many formulas of degree at most n.*
2. *For all n, neighborhood model M, and world w of M, the set of all formulas of degree at most n that are satisfied by w is equivalent to a single formula.*

Definition 5.9 (n-bisimulation). Let M_1 and M_2 be two neighborhood models, and let w_1 and w_2 be worlds of M_1 and M_2, respectively. We say that w_1 and w_2 are *n-bisimilar* (denoted by: $w_1 \rightleftarrows_n w_2$), if there exists a sequence of binary relations $R_n \subseteq \ldots \subseteq R_0$ with the following properties (for $i + 1 \leqslant n$):

1. $w_1 R_n w_2$;
2. if $u_1 R_0 u_2$, then u_1 and u_2 agree on all proposition letters;
3. if $w_1 R_{i+1} w_2$, then for any $y \in \bigcap N_2(w_2)$ there is $x \in \bigcap N_1(w_1)$ such that xR_iy;
4. if $w_1 R_{i+1} w_2$, then for any $x \in \bigcap N_1(w_1)$ there is $y \in \bigcap N_2(w_2)$ such that xR_iy.

Proposition 5.10. *Let Π be a finite set of proposition letters. Let M_1 and M_2 be any neighborhood models for language of Π. Then for all w_1 in M_1 and w_2 in M_2 the following are equivalent:*

(i) $w_1 \rightleftarrows_n w_2$,
(ii) *w_1 and w_2 agree on all proposition formulas of degree at most n.*

PROOF. "(i) $\Rightarrow$ (ii)" By induction on n. The case $n = 0$ is obvious. Let $w_1 \rightleftarrows_{n+1} w_2$. Then, by Proposition 5.8, the set of all formulas of degree at most $n + 1$ that are satisfied by w_1 is equivalent to a single formula such as φ_1. Also, the set of all formulas of degree at most $n+1$ that are satisfied by w_2 is equivalent to a single formula such as φ_2. We show that φ_2 is satisfied by w_1 and φ_1 is satisfied by w_2. Now, let $\deg(\varphi_1) = n+1$. Then φ_1 is a Boolean combination of proposition letters and formulas of the form $\psi = \psi_1 \to \psi_2$ such that $\max\{\deg(\psi_1), \deg(\psi_2)\} = n$. Clearly, it is enough to show that if $w_1 \Vdash_{M_1} \psi_1 \to \psi_2$, then $w_2 \Vdash_{M_2} \psi_1 \to \psi_2$. By *superset* it is enough to show that $\bigcap N_2(w_2) \subseteq \{u \mid u \nVdash_{M_2} \psi_1$ or $u \Vdash_{M_2} \psi_2\}$. Let $y \in \bigcap N_2(w_2)$ and $y \Vdash_{M_2} \psi_1$. By definition of n-bisimulation there is $x \in \bigcap N_1(w_1)$ such that $xR_n y$. We have $\deg(\psi_1) \leqslant n$. Hence $x \Vdash_{M_1} \psi_1$ and $x \Vdash_{M_1} \psi_2$. Again, by the induction hypothesis, $y \Vdash_{M_2} \psi$. The converse direction can be similarly verified using the clause 4 of Definition 5.9.

"(ii) $\Rightarrow$ (i)" Let w_1 and w_2 agree on all proposition formulas of degree at most n. We define

$$
\begin{aligned}
&R_n = \{\langle x, y\rangle \mid x \in W_1, y \in W_2, \deg(\varphi) \leqslant n, x \Vdash \varphi \Leftrightarrow y \Vdash \varphi\};\\
&R_{n-1} = \{\langle x, y\rangle \mid x \in W_1, y \in W_2, \deg(\varphi) \leqslant n-1, x \Vdash \varphi \Leftrightarrow y \Vdash \varphi\};\\
&\vdots\\
&R_0 = \{\langle x, y\rangle \mid x \in W_1, y \in W_2, \deg(\varphi) = 0, x \Vdash \varphi \Leftrightarrow y \Vdash \varphi\}.
\end{aligned}
$$

Therefore $R_n \subseteq R_{n-1} \subseteq \ldots \subseteq R_0$. We must show that this sequence is an n-bisimulation. Clauses 1 and 2 of Definition 5.9 are immediate. Now suppose that $u_1 R_{i+1} u_2$ namely ν_1 and ν_2 agree on all proposition formulas of degree at most i+1. We must show that

$$\forall_{y \in \cap N_2(u_2)} \exists_{x \in \cap N_1(u_1)} \ s.t.\ \ x \rightleftarrows_i y.$$

Assume on the contrary that there is $y \in \cap N_2(\nu_2)$ such that for any $x \in \bigcap N_1(\nu_1)$ there is φ_x sauch that $\deg(\varphi_x) \leqslant i$ and:

$$
\begin{array}{ccc}
y \Vdash_{M_2} \varphi_x & & x \nVdash_{M_1} \varphi_x\\
& \text{or} & \\
y \nVdash_{M_2} \varphi_x & & x \Vdash_{M_1} \varphi_x.
\end{array}
$$

Let Γ be the set of all the formulas φ_x with the above property. By Proposition 5.8, Γ is (up to equivalence) finite. Consider the following two subsets of Γ:

$$
\begin{aligned}
\Gamma_0 &:= \{\varphi_x \in \Gamma \mid y \Vdash_{M_2} \varphi_x \text{ and } x \nVdash_{M_1} \varphi_x\},\\
\Gamma_1 &:= \{\varphi_x \in \Gamma \mid y \nVdash_{M_2} \varphi_x \text{ and } x \Vdash_{M_1} \varphi_x\}.
\end{aligned}
$$

Obviously, Γ_0 and Γ_1 cannot be both empty. Let us define the following formula:

$$\theta := \begin{cases} \bigwedge \Gamma_0 & \text{if } \Gamma_1 = \varnothing \\ \bigwedge \Gamma_0 \to \bigvee \Gamma_1 & \text{if } \Gamma_0 \neq \varnothing,\ \Gamma_1 \neq \varnothing \\ \bigvee \Gamma_1 & \text{if } \Gamma_0 = \varnothing \end{cases}$$

Let us note that when $\Gamma_1 = \varnothing$, $y \Vdash_{M_2} \theta$, and hence $u_2 \nVdash_{M_2} \neg\theta$. Also since for any $x \in \bigcap N_1(u_1)$, $x \nVdash_{M_1} \theta$, so $\{u \mid u \nVdash_{M_1} \theta\} \in N_1(u_1)$. Therefore by the Remark 3.5, $\nu_1 \Vdash_{M_1} \theta \to \bot$ and this is a contradiction, since $\deg(\neg\theta) \leqslant i+1$.

Similarly, when $\Gamma_0 = \varnothing$ we have $y \nVdash_{M_2} \theta$. Thus, $u_2 \nVdash_{M_2} \theta$. Moreover, for any $x \in X$, $x \Vdash_{M_1} \theta$. So in particular, $u_1 \Vdash_{M_1} \theta$. Then since $\deg(\theta) \leqslant i+1$, we get a contradiction.

So, let us now consider the case when $\Gamma_0 \neq \varnothing$ and $\Gamma_1 \neq \varnothing$. Let

$$K := \{u \mid u \Vdash \bigwedge \Gamma_0 \Rightarrow u \Vdash \bigvee \Gamma_1\}.$$

Since for any $x \in \bigcap N_1(u_1)$, $x \nVdash_{M_1} \bigwedge \Gamma_0$, so $\bigcap N_1(u_1) \subseteq K$ and $K \in N_1(u_1)$. That is $\nu_1 \Vdash_{M_1} \theta$. But $y \Vdash_{M_2} \wedge\Gamma_0$ and since $y \nVdash_{M_2} \bigvee \Gamma_1$, so $K \notin N_2(\nu_2)$, i.e., $u_2 \nVdash_{M_2} \theta$. This is contradiction, since $\deg(\theta) \leqslant i+1$.

The clause 4 of Definition 5.9 can be similarly verified. $\dashv$

6. Neighborhood semantics for BPL

In this section we introduce neighborhood semantics for BPL and show that BPL is sound and complete with respect to this semantics.

Neighborhood models of BPL are defined similarly to such models of IPL. The only difference is that the condition 1 of Definition 3.1 in the case of BPL is dropped. So for BPL we also obtain Lemma 3.2.

The *semantical consequence relation* $\Gamma \models \varphi$ holds, if for every neighborhood model $\langle W, N, V\rangle$ of BPL, for any $w \in W$: if $w \Vdash \Gamma$, then $w \Vdash \varphi$.

THEOREM 6.1 (Soundness). *If $\Gamma \vdash \varphi$, then $\Gamma \models \varphi$.*

PROOF. By induction on the complexity of derivations from Γ. We only check the cases of (Tr) and ($\vee$Ef). Let $\langle W, N, V\rangle$ be any neighborhood model.

(Tr) By the induction hypothesis, for any $u \in W$: if $u \Vdash \Gamma$, then $u \Vdash \varphi \to \psi$ and $u \Vdash \psi \to \chi$. Let $w \Vdash \Gamma$. So, by the induction hypothesis, (1) $\{u \mid u \nVdash \varphi$ or $u \Vdash \psi\} \in N(w)$ and (2) $\{u \mid u \nVdash \psi$ or $u \Vdash \chi\} \in N(w)$.

We want to show that $w \Vdash \varphi \to \chi$, that is $X := \{u \mid u \nVdash \varphi$ or $u \Vdash \chi\} \in N(w)$. By the clause 3 of Lemma 3.2, it is sufficient to show that $\bigcap N(w) \subseteq X$. Now let $x \in \bigcap N(w)$ and $u \Vdash \varphi$. Then $u \Vdash \psi$, by (1). Hence $u \Vdash \psi$, by (2). So we conclude that $u \Vdash \chi$.

($\vee$Ef) By the induction hypothesis, for any $u \in W$: if $u \Vdash \Gamma$, then $u \Vdash \varphi \to \chi$ and $u \Vdash \psi \to \chi$. Let $w \Vdash \Gamma$. So, by the induction hypothesis, (1) $\{u \mid u \nVdash \varphi$ or $u \Vdash \chi\} \in N(w)$ and (2) $\{u \mid u \nVdash \psi$ or $u \Vdash \chi\} \in N(w)$.

We want to show that $w \Vdash \varphi \vee \psi \to \chi$, that is $X := \{u \mid u \nVdash \varphi \vee \psi$ or $u \Vdash \chi\} \in N(w)$. By the clause 3 of Lemma 3.2, it is sufficient to show that $\bigcap N(w) \subseteq X$. Now let $u \in \bigcap N(w)$ and $u \Vdash \varphi \vee \psi$. If $u \Vdash \varphi$ then $u \Vdash \chi$, by (1). If $u \Vdash \psi$ then $u \Vdash \chi$, by (2). So we conclude that $u \Vdash \chi$. ⊣

The proof of the following theorem is completely similar to the proof of Theorem 3.9.

Theorem 6.2. *For each Kripke model $\langle W, R, V\rangle$ of* BPL, *there is a pointwise equivalent basic neighborhood model $\langle W, N, V\rangle$ of* BPL.

Visser proved the following fact for BPL in [10].

Lemma 6.3. *If $\Gamma \nvdash \varphi$* (*in* BPL) *then there is a Basic Kripke model with a node k such that $k \Vdash \Gamma$ and $k \nVdash \varphi$.*

Use Theorem 6.2 and Lemma 6.3 we obtain:

Lemma 6.4. *If $\Gamma \nvdash \varphi$* (*in* BPL), *then there is a basic neighborhood model with a world w such that $w \Vdash \Gamma$ and $w \nVdash \varphi$.*

Finally, by the above lemma we obtain:

Theorem 6.5 (Completeness). *If $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$.*

Thus, in theorems 6.1 and 6.5 we have soundness and completeness of BPL with respect to the neighborhood semantics for BPL. Notice that we obtain (see p. 342): $\Gamma \models \varphi$ iff $\Gamma \Vdash \varphi$.

Moreover, below we obtain a theorem that is inverse to Theorem 6.2.

Theorem 6.6. *For every neighborhood model $\langle W, N, V\rangle$ of* BPL *there is a pointwise equivalent Kripke model $\langle W, R, V\rangle$ of* BPL.

PROOF. Let $M_{\mathrm{n}} = \langle W, N, V\rangle$ be a neighborhood model. For all $w \in W$ and $\nu \in W$ we define R as follows:

$$wRu \quad \text{iff} \quad \forall_{X \in N(w)}\, u \in X\,.$$

For transitivity. Let wRu and uRv; that is (1) for any $X \in N(w)$ we have $u \in X$ and (2) for any $X \in N(u)$ we have $v \in X$. We must show that for any $X \in N(w)$, $v \in X$. Let $X \in N(w)$. Then, by the condition 4 of Definition 3.1, $\{z \mid X \in N(z)\} \in N(w)$. Hence, $X \in N(u)$, by (1). So $v \in X$, by (2). Therefore wRv.

For models. Let $w \in V(p)$ and wRu. Then $V(p) \in N(w)$ and for any $X \in N(w)$ we have $u \in X$. So $u \in V(p)$.

The proof that $M_{\mathrm{n}} = \langle W, N, V\rangle$ and $M_{\mathrm{K}} = \langle W, R, V\rangle$ are pointwise equivalent is by the induction on the complexity of formulas. The only case of interest is implication.

Let $w \Vdash_{M_{\mathrm{n}}} \varphi \to \psi$, that is $X := \{v \mid v \nVdash_{M_{\mathrm{n}}} \varphi \text{ or } v \Vdash_{M_{\mathrm{n}}} \psi\} \in N(w)$. We show that $w \Vdash_{M_{\mathrm{K}}} \varphi \to \psi$. So let wRu and $u \Vdash_{M_{\mathrm{K}}} \varphi$. Then, $u \Vdash_{M_{\mathrm{n}}} \varphi$, by the induction hypothesis. Hence, by assumptions, $u \in X$; so $u \Vdash_{M_{\mathrm{n}}} \psi$. So, again by the induction hypothesis, $\nu \Vdash_{M_{\mathrm{K}}} \psi$.

Now let $w \Vdash_{M_{\mathrm{K}}} \varphi \to \psi$. We must show that $w \Vdash_{M_{\mathrm{n}}} \varphi \to \psi$; that is $X := \{v \mid v \nVdash_{M_{\mathrm{n}}} \varphi \text{ or } v \Vdash_{M_{\mathrm{n}}} \psi\} \in N(w)$. Then, by the clause 3 of Lemma 3.2, it is sufficient to show that $\bigcap N(w) \subseteq X$. Indeed, let $u \in \bigcap N(w)$ and $u \Vdash_{M_{\mathrm{n}}} \varphi$. Then, by the induction hypothesis, $u \Vdash_{M_{\mathrm{K}}} \varphi$. Now, by definition of R, we have wRu. (Indeed, if $Z \in N(w)$, then $\bigcap N(w) \subseteq Z$. So $u \in \bigcap N(w) \subseteq Z$.) Hence $u \Vdash_{M_{\mathrm{K}}} \psi$, by assumption. So, again by the induction hypothesis, $u \Vdash_{M_{\mathrm{n}}} \psi$. ⊣

Remark 6.7. 1. In the proofs of theorems 3.13 and 6.6 we use, respectively, the following relations:

$$wR_1u \text{ iff } N(w) \subseteq N(u)\,,$$
$$wR_2u \text{ iff } \forall_{X \in N(w)}\, u \in X\,.$$

Notice that for all $w, u \in W$:

$$wR_2u \text{ iff } u \in \bigcap N(w)\,.$$

Indeed, since $N(w) = \{X \mid X \in N(w)\}$: $u \in \bigcap N(w)$ iff $\forall_{X \in N(w)}\, u \in X$.

2. In all BPL-neighborhood models we have $R_2 \subseteq R_1$. Indeed, suppose that wR_2u. Then $u \in \bigcap N(w)$. So $N(w) \subseteq N(u)$, by Lemma 3.2. That is wR_1u.

3. In all IPL-neighborhood models we have $R_1 = R_2$. Indeed, let wR_1u, i.e., $N(w) \subseteq N(u)$. Then $u \in \bigcap N(u) \subseteq \bigcap N(w)$, by the clause 1 of Definition 3.1. So wR_2u.

4. In the proof of Theorem 6.6 we can not use the relation R_1. Indeed, without the clause 1 of Definition 3.1, we can not obtain the claim *for models* in the proof of Theorem 6.6 with R_1.

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