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## A PROPOSITIONAL LOGIC OF TEMPORAL CONNECTIVES

**Abstract.** We investigate how to formalize reasoning that takes account of time by using connectives like “before” and “after.” We develop semantics for a formal logic, which we axiomatize. In proving that the axiomatization is strongly complete we show how a temporal ordering of propositions can yield a linear timeline. We formalize examples of ordinary language sentences to illustrate the scope and limitations of this method. We then discuss ways to deal with some of those limitations.

**Keywords:** temporal logic; propositional logic; completeness theorem; linear orderings; subjective experience of time

### Introduction

In this paper we present a propositional logic that takes account of time using connectives similar to “before” and “after,” as in “Suzy called Zoe before Dick came home.” Each atomic sentence is meant as a description of the world at some particular time and — viewed as either a proposition or as representing a proposition — is true or false. This distinguishes our approach from systems built using temporal operators on sentences in the tradition of Arthur Prior, in which a sentence such as “John loves Mary” is not treated as a proposition but as a scheme, true at some times and false at others. In a companion paper (“Reflections on temporal and modal logic”, [4]) Epstein shows that the methodology of that approach is incoherent.

## 1. Propositions

### 1.1. Propositions and future tense propositions

In logic we're concerned with what is true and with what follows from what. Those things that are true or that follow are what we call "propositions."<sup>1</sup>

DEFINITION. A *proposition* is a part of speech which we agree to view as being true or false but not both.

Before we can invoke the methods and analyses of propositional logics as set out in Epstein's *Propositional Logics* [2], we must settle the issue of whether sentences about the future can be viewed as being true or false. Can we reason with "Dick will marry Zoe" as a proposition?

There are some future-tense sentences we have good reason to believe. We, the authors, have good reason to believe "No dog lives more than 30 years" and also "Birta is a dog," so we can conclude "Birta will be dead 31 years from now." Every scientific law is meant as true of all times, including the future. For example, "Electrons have spin" is meant not as a proposition that is true of just the present or the past, summarizing our previous experiences, but true for all times, including the future where it serves as a prediction: any electron in the future will have spin.

On the other hand, as we write this we have no reason to believe that "Richard L. Epstein will be alive three years from now" is true; nor do we have reason to believe it is false. When we do not have good reason to believe that a particular sentence about the future is true or to believe that it is false, we can treat it as we treat any proposition whose truth-value we do not know. We consider ways in which it could be true and ways in which it could be false in order to investigate its consequences.<sup>2</sup> We can do so whether we believe that there are contingent propositions about the future or only that we are factoring into our reasoning our ignorance of the future. This is how we treat many

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<sup>1</sup> Linguistic propositions as we define them can be understood to be representatives or expressions of thoughts or abstract propositions. See Epstein's [2] for motivation and a survey of how such different views of propositions enter into analyses of propositional logics. See Epstein's [3] for a discussion of this definition and these issues in the general context of both formal and informal logic.

<sup>2</sup> Or in some cases, perhaps, to find that the sentence cannot be true or false, as with the liar paradox "This sentence is false."

propositions about the past: we agree, for whatever reason, to view them as true or false, perhaps in an attempt to decide which they are. The definition of “proposition” serves well for this purpose, and with it we can reason with future-tense sentences as propositions.

### 1.2. The time of a proposition

We assume that each proposition is a description that is true or false of some part of the “world,” however we may construe that. Taking account of time with the use of temporal connectives we need to assume further that each such description is about some time. We might take that “Spot is barking” to be about now, while “Spot barked” to be about yesterday, and “Spot chased Puff” to be about three days ago.<sup>3</sup> The time we associate with a proposition need not be indicated in it. All we require is that we can agree that there is a time — whether imprecise or quite specific — of which the proposition is meant to be a description. If we consider numbers to be abstract, outside of time and space, then “ $2 + 2 = 4$ ” will be about no time or place, and hence that proposition will be outside the scope of our methods here.

A proposition such as “Spot barked” is meant as a description of the world of some time, a description that may be true or may be false. If true, “Spot barked” is true of the world at that time; if false, it is false of the world at that time. This is not to say that “Spot barked” becomes true at some specific time or that it stops being true at a later time. If “Spot barked” is true of three days ago, it is true then, it is true now, and it was true whenever it is or was uttered with the intention of talking about that time. *A proposition is true or false of or about a time; it is not true or false at a time.*

### 1.3. Truth-value ascriptions and time ascriptions

We might not know whether a proposition such as “Socrates was over 1.6 m tall” is true or false. We consider models in which it is true and models in which it is false even if we believe that given a semi-formal language there is just one model that corresponds to “reality” where all the propositions in it are given their “correct” truth-values. We can say that our use of such models is just a way to consider consequences of propositions given our ignorance of their actual truth-values.

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<sup>3</sup> “Spot” refers to a dog.

Each model has only one collection of truth values: true and false. But there seems to be no single, canonical collection of times to use for all models. Even if there were just one correct set of times to ascribe to propositions, and even supposing that we could know how to use those in our models, we still might want to use simpler ones. If time to the closest day is good enough for our purposes, we might want the time ascriptions in our model to be much smaller than the full “correct” set.

Moreover, with a proposition that carries only a very general indication of time, such as “Birta barked,” it’s not clear what the “correct” time assignment would be. The proposition is about some time in the past, but even given a collection of times that we believe correspond to our conception of time in the world, we would still have great leeway in assigning a time to that proposition. We could assign yesterday, or last year, or all of the decade 2000–2009, or just the interval between 8 and 9 minutes ago. It seems, then, that the ascription of a time to a proposition is a subjective choice.

But if subjective, that subjectivity is also part of the ascription of a truth-value to the proposition. If we take “Spot barked” as a proposition, we must agree that it is about some time whether we are taking time into account in our reasoning or not. Implicit in our agreement to take that sentence to be a proposition, to have a truth-value, is an agreement that it is about some time. The sentence “Spot barked” is not a proposition independent of our understanding well enough both the words and the time of which it is meant to be true in order to agree that it has a truth-value. Now we are making that agreement about time into an explicit semantic assignment.

## **2. The basic temporal connectives**

### **2.1. Four basic temporal connectives**

We use in our speech and reasoning a number of temporal connectives such as “before,” “after,” “at the same time as,” “and then,” “while.” As we commonly use them, however, each has various readings.

Consider, for example, “Spot barked before Dick yelled.” We understand this to mean that Spot started barking before Dick yelled. But do we also mean that Spot finished barking before Dick started to yell? Or do we mean only that he finished barking before Dick stopped yelling?

When we say “Zoe was talking on the phone while Dick washed the dishes,” do we mean that she was talking the whole time Dick washed the dishes? Or do we mean only that there was some overlap in the time when she was talking and the time that Dick was washing dishes?

Just as we regiment our analyses of “and,” “or,” “not,” and “if ... then” in classical propositional logic, we need to specify exactly how we will interpret the temporal connectives we will formalize.

To begin our analysis, we will look at just “before” and try to regiment uses of that. But consider:

- (a) Spot began barking before Dick began yelling.
- (b) Spot ended barking before Dick began yelling.
- (c) Spot began barking before Dick ended yelling.
- (d) Spot ended barking before Dick ended yelling.

Where in (a)–(d) are the propositions “Spot barked” and “Dick yelled”?

In English we modify the verb with auxiliaries such as “began” and “ended,” or “started” and “stopped,” so that we take (a) rather than “Spot barked beginning before the ending of Dick yelled” or some other clumsy locution. Moreover, because all sentences in English are tensed, we modify those auxiliaries to reflect the tense of the propositions that are meant to be connected. Thus, joining “Spot will bark” and “Dick will yell” we might say, “Spot will begin barking before Dick will end yelling.”

We will regiment these kinds of constructions by taking as basic the following four connectives.

$A \wedge_{bb} B$  meant to formalize “A beginning before the beginning of B”.

This is true iff both A and B are true and A is about a time that begins before the beginning of the time of which B is about.

$A \wedge_{eb} B$  meant to formalize “A ending before the beginning of B”.

This is true iff both A and B are true and A is about a time that ends before the beginning of the time of which B is about.

$A \wedge_{be} B$  meant to formalize “A beginning before the ending of B”.

This is true iff both A and B are true and A is about a time that begins before the ending of the time of which B is about.

$A \wedge_{ee} B$  meant to formalize “A ending before the ending of B”.

This is true iff both A and B are true and A is about a time that ends before the ending of the time of which B is about.

These are all temporal versions of “and,” which is why we have chosen to use the usual symbol for conjunction in them.

Note that we are talking about the time a proposition is meant to describe. We are not taking about “events,” or “situations,” or “actions.” You may, if you wish, add to the metaphysics of what we are doing and say, for example, that “Spot barked” is a description of an action and the time we ascribe to it is from the moment that Spot began to bark until the moment that Spot ended barking. The development here is compatible with that, but does not depend on it. It’s enough that for whatever reason we can ascribe a truth-value and a time to each proposition we are considering.<sup>4</sup>

## 2.2. Compound propositions

The only structure of propositions we will be concerned with here is how they are formed from other propositions using certain connectives. Those include the four we just looked at as well as formalizations of the usual ones studied in propositional logics: “and,” “or,” “not,” and “if . . . then . . .”. We’ll use the symbols  $\wedge$ ,  $\vee$ ,  $\neg$ ,  $\supset$  for the formalizations of those, calling them the *standard connectives*.

The simplest propositions we’ll study are *atomic*: they have no structure in terms of any these connectives. All others involve the use of one or more of the connectives and are called compound. We use the standard connectives to join not only atomic propositions but also compound ones, formalizing “If Spot barked and Dick yelled, then Zoe got upset” as “(Spot barked  $\wedge$  Dick yelled)  $\supset$  Zoe got upset,” where the parentheses make clear the grouping for the connectives. Do we want or need to use our temporal connectives to join compound propositions?

Consider:

Spot barked before Dick yelled and Zoe got upset.

If we understand “before” as meaning that the time of when Spot barked began is before the beginning of when Dick yelled and Zoe got upset, then we should formalize this as:

Spot barked  $\wedge_{bb}$  (Dick yelled  $\wedge$  Zoe got upset)

To do so we need to use our temporal connectives to join propositions compounded using the standard connectives. So we’ll have to assign

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<sup>4</sup> In [5] Epstein discusses issues of metaphysics, beginnings and endings, and more about how to reason taking account of time, especially with regard to the analyses of Jean Buridan.

times to propositions compounded using the standard propositional connectives.

It's not clear, though, how we could do that for conditionals. "If Spot barked, then Dick chased Spot" is hypothetical, not clearly meant as a description of the time of either the antecedent or the consequent or perhaps any other time. However, if we take classical propositional logic as our base logic, interpreting our formalizations of "and," "or," "not," and "if ... then ..." according to the classical tables in which the only semantic values that are considered are truth and falsity, then a formalization of a conditional has no hypothetical character: it is equivalent to  $\neg(A \wedge \neg B)$ . So let's use classical logic for our interpretation of the standard connectives. That's been studied extensively, and what we need about it can be found in [2]. In order to simplify our work more, let's take just  $\neg$  and  $\wedge$  as primitive; from those we can define the classical interpretations of disjunction and the conditional.

Do we also want to allow temporal connectives to join propositions which themselves are temporal compounds? Consider:

- (1) Tom laughed and Dick chortled and Maria guffawed after Zoe stumbled.

If we use  $\wedge$  to formalize "and" in (1), then the formalization could be true if Tom laughed on Wednesday, Dick on Thursday, and Maria on Friday. More naturally we read (1) as asserting that Tom laughed, Dick chortled, and Maria guffawed at (roughly) the same time. Using the connectives we have, we'll see how to formalize "at the same time as" with a connective  $\wedge_{\approx}$ , and then we can formalize (1) as:

Zoe stumbled  $\wedge_{\text{eb}}$  ((Tom laughed  $\wedge_{\approx}$  Dick chortled)  $\wedge_{\approx}$  Maria guffawed)

To do so we need to allow for temporal connectives to join propositions that also involve temporal connectives. That means we assume that a temporal compound such as "Tom laughed  $\wedge_{\approx}$  Dick chortled" is meant as a description of the world at some time. So we'll need to assign times to temporal compounds, too.

### 2.3. The semantic values of atomic propositions and of compound propositions

We take each atomic proposition to be true or to be false. We also take each atomic proposition to be about some particular time. What

is meant by this will depend on the way we construe the nature of time. Many choices are available: time as linear, circular, or branching; time as continuous or discrete; time as infinite or finite; . . . . To develop a logic we must make a choice. Let's assume that time is linear, composed of instants, without assuming anything about the divisibility or indivisibility of those instants. This is how we will start, hoping that others will use these methods to analyze reasoning based on other conceptions of time.

We could assign to each atomic proposition a single instant of time. But what counts as an instant might vary from one model to another, so that 9:22 pm Mountain Standard Time, March 24, 2012 could be an instant in one model and an interval one minute long in another. For generality, let's assign to each atomic proposition an interval of time, though the interval might be just a single instant.

How should the time of a compound proposition be related to the times of its parts? We understand "Spot didn't bark" as the contradictory of "Spot barked," so those two propositions can't be meant as descriptions of different times. If they were, they could have the same truth-value: Spot didn't bark yesterday but did bark the day before yesterday. If we take  $\neg A$  to be the contradictory of  $A$ , the time assigned to  $\neg A$  must be the same as the time assigned to  $A$ .

When we use both "Spot barked" and "Dick was chasing Spot" together as a description of the world, the description is meant to cover the world at both those times. The time assigned to  $A \wedge B$  should be the sum of the times assigned to  $A$  and to  $B$ , which needn't be an interval. This doesn't mean that in the compound "Spot barked  $\wedge$  Dick was chasing Spot," the part "Spot barked" is now meant to be true or false of an expanded time which includes the time of which "Dick was chasing Spot" is meant to be true. The time fixed for "Spot barked" does not change. The time of the compound is all the times which it is meant, at least in part, to describe.

Similarly, "Spot began barking before Dick stopped opening the door," which we'll formalize as "Spot barked  $\wedge_{eb}$  Dick opened the door," is meant as a description of the world at the times when Spot barked and when Dick opened the door. So the time assigned to  $A \wedge_{eb} B$  should be the sum of the times assigned to  $A$  and the times assigned to  $B$ . Likewise, the time assigned to  $A \wedge_{be} B$ , to  $A \wedge_{bb} B$ , and to  $A \wedge_{ee} B$  should be the sum of the times assigned to  $A$  and to  $B$ . Thus we'll take the time of any compound proposition to be the sum of the times of its parts.

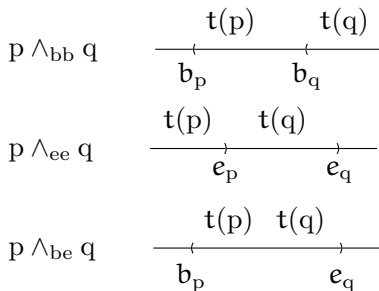


To extend the truth-value assignment to all propositions of the semi-formal language we need to agree on how to evaluate the propositional connectives. We've agreed to evaluate  $\neg$  and  $\wedge$  by the classical truth-tables.

To evaluate compounds whose main connective is a temporal connective, we first note that each of those connectives is meant as a temporal version of "and." Hence, for the compound to be true both parts must be true. Further, the times associated with the parts must be correctly related. It is the beginning and ending of the time assigned to its parts that determine that relation. It is essential, then, that we describe the time assigned to a proposition in terms of a beginning point and an ending point, whether those are included in the interval or not. This is not to say that we can point to a particular instant at which Spot's barking began. It is only to say that we will describe the time which we consider that proposition to be about to be bounded at the beginning and at the end.

For a proposition such as "Electrons have spin," though, we'll want to assign all times. We can accommodate such an assignment by requiring that whatever collection of times we are considering for a model has a beginning point and an ending point. Doing so need not commit us to the view that there is a beginning and ending of all time, no more than talking of points at infinity need commit us to any reality of a beginning and an ending of the real numbers. We can treat the beginning and ending points of the entire collection as just labels that simplify our discussions. They are not "instants" and will not appear in any interval.

Letting  $p$  and  $q$  stand for atomic propositions, and  $t(p)$ ,  $t(q)$  the intervals of time assigned to those, and  $b_p$ ,  $b_q$ ,  $e_p$ ,  $e_q$  as the beginning and ending points of those intervals, respectively, we can picture the relations that govern the evaluation of a temporal compound of  $p$  and  $q$ :



$$p \wedge_{eb} q \quad \frac{t(p) \quad t(q)}{e_p \quad b_q}$$

(We've drawn the intervals as containing neither the beginning nor ending points, though we also allow for the possibility that one or both are included in the interval.)

There will also be a beginning and ending point for the time assigned to a compound proposition, since that time is just the union of the times assigned to its parts, each of which has a beginning point and an ending point. So we can use the same relations to govern the evaluation of the temporal connectives when those join compound propositions.<sup>5</sup>

$$A \wedge_{bb} B \quad \frac{t(A) \quad t(B)}{b_A \quad b_B}$$

$$A \wedge_{ee} B \quad \frac{t(A) \quad t(B)}{e_A \quad e_B}$$

$$A \wedge_{be} B \quad \frac{t(A) \quad t(B)}{b_A \quad e_B}$$

$$A \wedge_{eb} B \quad \frac{t(A) \quad t(B)}{e_A \quad b_B}$$

Now we can present a formal logic.

### 3. Classical Propositional Logic with Temporal Connectives

#### 3.1. The formal language and realizations

**The formal language.** Using the usual methods as in [2], we can define the formal language:

$$\underline{L(\neg, \wedge, \wedge_{bb}, \wedge_{ee}, \wedge_{be}, \wedge_{eb}, p_0, p_1, \dots)}$$

<sup>5</sup> It is crucial that we can extend the relations governing the truth-tables from ones for atomic propositions to ones for compound ones. That's the point where using simply "before" and "after" as primitives becomes intractable. The relation governing the evaluation of "before" would be that the interval assigned to the one part comes entirely before the interval assigned to the other, or perhaps entirely before or overlapping. But then there doesn't seem to be any way to extend the relation to compound propositions in accord with how we would want to evaluate that connective.

Call this language L. In it we define:

$$\begin{aligned} A \supset B &\equiv_{\text{df}} \neg(A \wedge \neg B) \\ A \vee B &\equiv_{\text{df}} \neg(\neg A \wedge \neg B) \end{aligned}$$

We write  $\bigvee_{p \text{ in } A} B(p)$  for the disjunction of the formulas  $B(p)$  such that  $p$  is a propositional variable appearing in  $A$  associating to the left. We write  $\bigwedge_{p \text{ in } A} B(p)$  for the conjunction of those formulas associating to the left.

**Realizations and semi-formal languages.** A realization is an assignment of propositions to the propositional variables. These are the *atomic propositions*, ones whose internal structure we will not consider in our reasoning. Each is meant to be a description of the world of some time. No word or phrase we could formalize with one or a combination of the formal connectives of L should appear in any atomic proposition. The collection of all wffs of L whose variables are replaced by their realizations is called the *semi-formal language* L.

We'll use:

- $q, r, s$ , and indexed versions of these as well as the letter  $p$  as metavariables ranging over propositional variables or atomic propositions;
- $A, B, C, D$ , and indexed versions of these to range over wffs of L or of the semi-formal language;
- $\Gamma, \Delta, \Sigma$  and indexed versions of these to stand for collection of wffs of the formal language or of the semi-formal language.

We'll trust to context to make clear whether we intend a formal or semi-formal item. We use  $]$  and  $[$  to stand informally for parentheses.

Consider now:

(2) Spot barked. Then Dick yelled. And then Spot barked.

We all recognize that the use of “and then” indicates that the first and second occurrence of “Spot barked” are meant to refer to different times even though the words “Spot” and “barked” do not change in meaning. In taking sentences to be propositions, we normally treat both words and propositions to be types (see [2, Chapter 1] or [3]). If we wish to reason with (1), even informally, we must distinguish the two occurrences of “Spot barked.” We can do so by indexing them, writing (2) as:

(Spot barked)<sub>1</sub>. Then Dick yelled at Spot. Then (Spot barked)<sub>2</sub>.

This is not to give the propositions time indices, for neither “1” nor “2” indicates any time. The indices only show that the propositions are distinct, and hence may differ in their semantic values.

We continue to treat words as types, unchanging in their meaning in any particular discussion or model. So given two equiform sentences that we take to be propositions meant to describe the world at the same time, they have the same truth-value. There’s no point in distinguishing them. We use indexed versions of a single sentence to stand for distinct atomic propositions only if their time assignments differ, in which case one could be true and the other false. Because they can have distinct semantic values, they are distinct propositions. We codify this with the following condition on realizations and models.

*Assumption* (The same words used for different propositions). If the same words in the same order are meant as two distinct atomic propositions, we index them with distinct subscripts and require them to realize distinct propositional variables. In that case they must be assigned different times in a model.

If we interpret “then” to mean that the first begins before the second begins, we can formalize (2) as:

$$((\text{Spot barked})_1 \wedge_{bb} \text{Dick yelled}) \wedge_{bb} (\text{Spot barked})_2.$$

### 3.2. Linear orderings

Before we define the models of our logic we need to examine linear orderings.

DEFINITION (Strict orderings). A pair  $\langle T, < \rangle$  is a (*strictly ordered set*), if the relation  $<$  on the non-empty set  $T$  is an ordering, that is,  $<$  satisfies for all  $a, b, c$  in  $T$ :

$$\begin{array}{ll} \text{not } a < a & < \text{ is } \textit{anti-reflexive} \\ \text{if } a < b \text{ and } b < c, \text{ then } a < c & < \text{ is } \textit{transitive} \end{array}$$

Note that from these two conditions for all  $a, b$  in  $T$  we have:

$$\text{if } a < b, \text{ then not } b < a \quad < \text{ is } \textit{asymmetric}$$

It is a *linear ordering*, if satisfies in addition for all  $a$  and  $b$  in  $T$ :

$$a < b \text{ or } b < a \text{ or } b = a \quad \textit{trichotomy}$$



We define

$$a \leq b \equiv_{\text{df}} a < b \text{ or } a = b.$$

The ordering  $\mathcal{T} = \langle T, < \rangle$  has *endpoints*, if there are  $b, e \in T$  such that  $b \neq e$  and for all  $c \in T$ ,  $b \leq c \leq e$ . In that case  $b$  is a *left* or *beginning point*, and  $e$  is a *right* or *ending point*.

LEMMA 1. *In a linearly ordered set  $\mathcal{T}$ : if  $c$  and  $d$  are beginning (ending) points of  $\mathcal{T}$ , then  $c = d$ .*

Hence, in a linearly ordered set  $\mathcal{T}$  with endpoints, we denote the beginning point of  $\mathcal{T}$  as  $b_{\mathcal{T}}$  and the ending point as  $e_{\mathcal{T}}$ .

In some linear orderings a point may have an immediate successor or an immediate predecessor as defined here.

DEFINITION (Immediate successors and predecessors). Given  $a$ , an *immediate successor* of  $a$  is some  $b$  such that  $a < b$  and there is no  $c$  such that  $a < c < b$ . An *immediate predecessor* of  $a$  is some  $b$  such that  $b < a$  and there is no  $c$  such that  $b < c < a$ .

LEMMA 2. *In a linearly ordered set: if  $b$  and  $c$  are immediate successors (predecessors) of  $a$ , then  $b = c$ .*

DEFINITION (Intervals). Given a linearly ordered set  $\mathcal{T} = \langle T, < \rangle$  with endpoints, an *interval* (with beginning and ending points) is a non-empty set  $I \subsetneq T$  such that there are  $c, d$  in  $T$  and at least one of the following holds:

- (a)  $I = \{x : c < x < d\}$ ,
- (b)  $I = \{x : c \leq x < d \text{ and } x \neq b_{\mathcal{T}}\}$ ,
- (c)  $I = \{x : c < x \leq d \text{ and } x \neq e_{\mathcal{T}}\}$ ,
- (d)  $I = \{x : c \leq x \leq d \text{ and } b_{\mathcal{T}} \neq x \neq e_{\mathcal{T}}\}$ .

We call  $c$  a *left* or *beginning point* and  $d$  a *right* or *ending point* of the interval. An interval of type (a) is *open*. An interval of type (b) is *closed on the left* and *open on the right*. An interval of type (c) is *open on the left* and *closed on the right*. An interval of type (d) is *closed*.

*Remark.* A singleton set  $I = \{a\}$  can be an interval. Every closed interval with  $c = d$  is a singleton set. Note also that if  $I$  is an interval, then  $I \neq T$  since neither  $b_{\mathcal{T}}$  nor  $e_{\mathcal{T}}$  can be in  $I$ .

It is possible for an interval to be of more than one type. For example, if  $T = \{c_1, c_2, c_3, c_4\}$  and  $c_1 < c_2 < c_3 < c_4$ , then the following are the

same:  $\{x : c_1 < x < c_4\}$ ,  $\{x : c_2 \leq x < c_4\}$ ,  $\{x : c_1 < x \leq c_3\}$ , and  $\{x : c_2 \leq x \leq c_3\}$ . However, the only case when this occurs is when a beginning point of an interval closed on the left has an immediate predecessor or an ending point of an interval closed on the right has an immediate successor. In order to be able to say that each interval has a unique beginning point and ending point, we need to choose a standard for how we will describe intervals.

DEFINITION (Canonical descriptions of intervals). If  $I$  is an interval, a *canonical description* of  $I$  has the form:

$$I = \{x : c \leq x \leq d\} \text{ if that is possible.}$$

If that is not possible, then a canonical description has the form:

$$I = \{x : c < x \leq d\} \text{ if that is possible.}$$

If that is not possible, then a canonical description has the form:

$$I = \{x : c \leq x < d\} \text{ if that is possible.}$$

Otherwise, a canonical description has the form:

$$I = \{x : c < x < d\}.$$

*Remark.* Every interval has one and only one canonical description.

DEFINITION (Beginning points and ending points of an interval). If  $I$  is an interval, the *beginning point* of  $I$ , denoted by  $b_I$ , is the beginning point of its canonical description. The *ending point* of  $I$ , denoted by  $e_I$ , is the ending point of its canonical description.

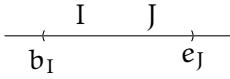
We now define some relations on intervals. Our pictures are of open intervals, though the definitions apply to all kinds of intervals.

DEFINITION (The four basic relations on intervals).

$$\begin{array}{c} \text{I} \qquad \qquad \text{J} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{b}_I \qquad \qquad \text{b}_J \end{array} \qquad I <_{bb} J \text{ means } b_I < b_J$$

$$\begin{array}{c} \text{I} \qquad \qquad \text{J} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{e}_I \qquad \qquad \text{e}_J \end{array} \qquad I <_{ee} J \text{ means } e_I < e_J$$

$$\begin{array}{c} \text{I} \qquad \qquad \text{J} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{e}_I \qquad \qquad \text{b}_J \end{array} \qquad I <_{eb} J \text{ means } e_I < b_J$$



$I <_{be} J$  means  $b_I < e_J$

- $I =_{bb} J \equiv_{df} \text{not } (I <_{bb} J) \text{ and not } (J <_{bb} I),$
- $I =_{ee} J \equiv_{df} \text{not } (I <_{ee} J) \text{ and not } (J <_{ee} I),$
- $I =_{be} J \equiv_{df} \text{not } (I <_{be} J) \text{ and not } (J <_{eb} I).$

DEFINITION (Beginning points and ending points of finite unions of intervals). A subset  $A$  of a linearly ordered set is a *finite union of intervals*, if there are  $I_1, \dots, I_n$  (for some  $n > 0$ ) such that  $A = I_1 \cup \dots \cup I_n$ . In that case we define:

- $b_A \equiv_{df} \min\{b_{I_i} : 1 \leq i \leq n\}$ , the *beginning point* of  $A$ ,
- $e_A \equiv_{df} \max\{e_{I_i} : 1 \leq i \leq n\}$ , the *ending point* of  $A$ .

We use  $A, B, C$ , and subscripted versions of these as variables ranging over intervals or finite unions of intervals.

LEMMA 3. *If  $A = I_1 \cup \dots \cup I_n$ , then:*

- (a) *If there is some  $k$  such that  $b_A = b_{I_k}$ , where  $I_k$  is closed on the left, then (i) for all  $a$  in  $A$ ,  $b_A \leq a$ , and (ii) for any  $d$ , if for all  $a$  in  $A$ ,  $d \leq a$ , then  $d \leq b_A$ .*
- (b) *If there is no  $k$  such that  $b_A = b_{I_k}$ , where  $I_k$  is closed on the left, then (i) for all  $a$  in  $A$ ,  $b_A < a$ , and (ii) for any  $d$ , if for all  $a$  in  $A$ ,  $d < a$ , then  $d \leq b_A$ .*
- (c) *If there is some  $k$  such that  $e_A = e_{I_k}$  and  $I_k$  is closed on the right, then (i) for all  $a$  in  $A$ ,  $a \leq e_A$ , and (ii) for any  $d$ , if for all  $a$  in  $A$ ,  $a \leq d$ , then  $e_A \leq d$ .*
- (d) *If there is no  $k$  such  $e_A = e_{I_k}$ , where  $I_k$  is closed on the right, then (i) for all  $a$  in  $A$ ,  $a < e_A$ , and (ii) for any  $d$ , if for all  $a$  in  $A$ ,  $a < d$  then  $e_A \leq d$ .*

We now extend the definitions of the relations on intervals to finite unions of intervals.

DEFINITION (Relations on finite unions of intervals).  $A <_{bb} B$  means  $b_A < b_B$

$A <_{ee} B$  means  $e_A < e_B$

$A <_{eb} B$  means  $e_A < b_B$

$A <_{be} B$  means  $b_A < e_B$

$$A =_{bb} B \equiv_{df} \text{not}(A <_{bb} B) \text{ and not}(B <_{bb} A)$$

$$A =_{ee} B \equiv_{df} \text{not}(A <_{ee} B) \text{ and not}(B <_{ee} A)$$

$$A =_{be} B \equiv_{df} \text{not}(A <_{be} B) \text{ and not}(B <_{eb} A).$$

LEMMA 4. *If each of  $A$  and  $B$  is an interval or a finite union of intervals, then:*

- (a)  $A =_{bb} B$  iff  $b_A = b_B$ ,
- (b)  $A =_{ee} B$  iff  $e_A = e_B$ ,
- (c)  $=_{ee}$  and  $=_{bb}$  are equivalence relations,
- (d)  $A =_{be} B$  iff  $b_A = e_B$  iff  $B =_{eb} A$ ,
- (e)  $=_{be}$  is not an equivalence relation,
- (f)  $A =_{be} A$  iff  $b_A = e_A$ ,
- (g)  $A =_{be} B$  and  $B =_{be} A$  iff for some  $c$ ,  $A = B = \{c\}$ .

LEMMA 5 (Trichotomies for relations on intervals and unions of intervals). *If each of  $A$  and  $B$  is an interval or a finite union of intervals, then:*

- (a) *One and only one of the following holds:*

$$A <_{bb} B \text{ or } B <_{bb} A \text{ or } A =_{bb} B.$$

- (b) *One and only one of the following hold:*

$$A <_{ee} B \text{ or } B <_{ee} A \text{ or } A =_{ee} B.$$

- (c) *One and only one of the following hold:*

$$A <_{be} B \text{ or } B <_{eb} A \text{ or } A =_{be} B.$$

LEMMA 6 (Characterizing relations on unions of intervals in terms of relations on their parts).

- (a)  $A <_{bb} B$  iff for some  $I$  in  $A$ , for all  $J$  in  $B$ ,  $I <_{bb} J$ .
- (b)  $A <_{ee} B$  iff for some  $J$  in  $B$ , for all  $I$  in  $A$ ,  $I <_{ee} J$ .
- (c)  $A <_{eb} B$  iff for some  $I$  in  $A$ , for all  $I'$  in  $A$ ,  $I' <_{ee} I$  or  $I =_{ee} I'$ , and for some  $J$  in  $B$ , for all  $J'$  in  $B$ ,  $J <_{bb} J'$  or  $J =_{bb} J'$ , and  $I <_{eb} J$ .
- (d)  $A <_{be} B$  iff for some  $I$  in  $A$ , for all  $I'$  in  $A$ ,  $I <_{bb} I'$  or  $I =_{ee} I'$ , and for some  $J$  in  $B$ , for all  $J'$  in  $B$ ,  $J' <_{ee} J$  or  $J =_{bb} J'$ , and  $I <_{be} J$ .



### 3.3. Models

DEFINITION (Models). A *model*  $\mathfrak{M}$  for the language  $L$  is a realization of  $L$  along with:

- a valuation  $\mathbf{v}$  that assigns to each atomic proposition either the value true or the value false, notated respectively as  $\mathbf{v}(p) = \mathbf{T}$  or  $\mathbf{v}(p) = \mathbf{F}$ ;
- a linearly ordered set  $\mathcal{T} = \langle \mathbf{T}, < \rangle$  with endpoints;  $\mathbf{T}$  is called the *collection of instants* with  $\mathbf{b}_{\mathcal{T}}$  the beginning point and  $\mathbf{e}_{\mathcal{T}}$  the ending point;
- a time assignment  $\mathbf{t}$  that assigns to each atomic proposition an interval of  $\langle \mathbf{T}, < \rangle$ , notated as  $\mathbf{t}(p)$ ; the beginning point of  $\mathbf{t}(p)$  is denoted  $\mathbf{b}_p$  and the ending point as  $\mathbf{e}_p$ .

The time assignment  $\mathbf{t}$  is extended to all wffs of the realization by:

$$\mathbf{t}(A) = \bigcup \{ \mathbf{t}(p) : p \text{ is an atomic proposition that appears in } A \}.$$

The beginning and ending points of  $\mathbf{t}(A)$  are notated respectively as  $\mathbf{b}_A$  and  $\mathbf{e}_A$ .

The valuation is extended to all wffs of the realization by

$$\begin{aligned} \mathbf{v}(\neg A) &= \mathbf{T} \text{ iff } \mathbf{v}(A) = \mathbf{F}, \\ \mathbf{v}(A \wedge B) &= \mathbf{T} \text{ iff } \mathbf{v}(A) = \mathbf{T} \text{ and } \mathbf{v}(B) = \mathbf{T}, \\ \mathbf{v}(A \wedge_{\mathbf{bb}} B) &= \mathbf{T} \text{ iff } \mathbf{v}(A) = \mathbf{T} \text{ and } \mathbf{v}(B) = \mathbf{T} \text{ and } \mathbf{t}(A) <_{\mathbf{bb}} \mathbf{t}(B), \\ \mathbf{v}(A \wedge_{\mathbf{ee}} B) &= \mathbf{T} \text{ iff } \mathbf{v}(A) = \mathbf{T} \text{ and } \mathbf{v}(B) = \mathbf{T} \text{ and } \mathbf{t}(A) <_{\mathbf{ee}} \mathbf{t}(B), \\ \mathbf{v}(A \wedge_{\mathbf{be}} B) &= \mathbf{T} \text{ iff } \mathbf{v}(A) = \mathbf{T} \text{ and } \mathbf{v}(B) = \mathbf{T} \text{ and } \mathbf{t}(A) <_{\mathbf{be}} \mathbf{t}(B), \\ \mathbf{v}(A \wedge_{\mathbf{eb}} B) &= \mathbf{T} \text{ iff } \mathbf{v}(A) = \mathbf{T} \text{ and } \mathbf{v}(B) = \mathbf{T} \text{ and } \mathbf{t}(A) <_{\mathbf{eb}} \mathbf{t}(B). \end{aligned}$$

A model is designated  $\mathfrak{M} = \langle \mathbf{v}, \langle \mathbf{T}, < \rangle, \mathbf{t} \rangle$ .<sup>6</sup>

A wff  $A$  is *true* in  $\mathfrak{M}$  means that  $\mathbf{v}(A) = \mathbf{T}$ , notated as  $\mathfrak{M} \models A$ . A wff  $A$  is *false* in  $\mathfrak{M}$  means that  $\mathbf{v}(A) = \mathbf{F}$ , notated as  $\mathfrak{M} \not\models A$ . A wff  $A$  is *valid* or a *tautology* means that for every model  $\mathfrak{M}$ ,  $\mathfrak{M} \models A$ . A wff  $B$  is a *semantic consequence* of a collection of wffs  $\Gamma$ , written  $\Gamma \models B$ , means that for every model  $\mathfrak{M}$ , if for every  $A$  in  $\Gamma$ ,  $\mathfrak{M} \models A$ , then  $\mathfrak{M} \models B$ .

Two models  $\mathfrak{M}$  and  $\mathfrak{M}'$  are *elementarily equivalent* iff for every wff  $A$ ,  $\mathfrak{M} \models A$  iff  $\mathfrak{M}' \models A$ .

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<sup>6</sup> The only formal systems that we have seen where times are taken to be intervals either use time-makers (names of times, variables for times) in predicate logic or are temporal operators in the tradition of Arthur Prior. Such work is not directly applicable here.

DEFINITION. The *propositional logic of linear time based on classical logic*,  $\mathbf{TL}_{PC}$ , is the collection of these models and the semantic consequence relation.

DEFINITION (Fully General Abstraction). Any function  $\mathbf{v}$  from the set of propositional variables to  $\{\mathbf{T}, \mathbf{F}\}$  plus any linearly ordered set  $\langle \mathbf{T}, < \rangle$  and any function  $\mathbf{t}$  from the set of propositional variables to intervals of  $\mathbf{T}$  comprises a model.

#### 4. Finite, Infinite, Dense, and Discrete Models

In a model the only instants that enter into evaluations of wffs are the beginning and ending points of the intervals assigned to the atomic propositions. To show this we'll construct reduced models that include only those points. However, since the beginning and ending points of the entire time line can't be in an interval, we have to add two new points.

DEFINITION (Reduced models). Given any model  $\mathfrak{M} = \langle \mathbf{v}, \langle \mathbf{T}, < \rangle, \mathbf{t} \rangle$ , the associated *reduced* model is  $\mathfrak{M}_r = \langle \mathbf{v}_r, \langle \mathbf{T}_r, <_r \rangle, \mathbf{t}_r \rangle$ , where:

$$\mathbf{T}_r = \bigcup \{ \{ \mathbf{b}_p, \mathbf{e}_p \} : p \text{ is a propositional variable} \} \cup \{ \mathbf{x}, \mathbf{y} \},$$

where  $\mathbf{x}$  and  $\mathbf{y}$  are letters that do not appear in  $\mathbf{T}$ ,

$$\mathbf{v}_r(p) = \mathbf{v}(p),$$

$$z <_r w \text{ iff } z, w \notin \{ \mathbf{x}, \mathbf{y} \} \text{ and } z < w, \text{ or } z = \mathbf{x}, \text{ or } w = \mathbf{y},$$

$$\mathbf{t}_r(p) = \{ \mathbf{b}_q : \mathbf{b}_p \leq \mathbf{b}_q \leq \mathbf{e}_p \} \cup \{ \mathbf{e}_q : \mathbf{b}_p \leq \mathbf{e}_q \leq \mathbf{e}_p \}.$$

Given a wff  $A$ , the associated *reduced model* for  $A$  is

$$\mathfrak{M}_A = \langle \mathbf{v}_A, \langle \mathbf{T}_A, <_A \rangle, \mathbf{t}_A \rangle,$$

where:

$$\mathbf{T}_A = \bigcup \{ \{ \mathbf{b}_p, \mathbf{e}_p \} : p \text{ is a propositional variable appearing in } A \} \cup \{ \mathbf{x}, \mathbf{y} \},$$

$$\mathbf{v}_A(p) = \mathbf{v}(p),$$

$$z <_A w \text{ iff } z, w \notin \{ \mathbf{x}, \mathbf{y} \} \text{ and } z < w, \text{ or } z = \mathbf{x}, \text{ or } w = \mathbf{y},$$

$$\mathbf{t}_A(p) = \begin{cases} \{ \mathbf{b}_q : \mathbf{b}_p \leq \mathbf{b}_q \leq \mathbf{e}_p \} \cup \{ \mathbf{e}_q : \mathbf{b}_p \leq \mathbf{e}_q \leq \mathbf{e}_p \} & \text{if } p \text{ appears in } A \\ \mathbf{T}_r \setminus \{ \mathbf{x}, \mathbf{y} \} & \text{otherwise} \end{cases}$$

THEOREM 7 (Models and reduced models).

- (a) Given any model  $\mathfrak{M}$ , for all  $A$ ,  $\mathfrak{M}_r \models A$  iff  $\mathfrak{M} \models A$ .
- (b) Given any model  $\mathfrak{M}$  and wff  $A$ ,  $\mathfrak{M}_A \models A$  iff  $\mathfrak{M} \models A$ .

PROOF. We'll let you show that for every  $p$ ,  $t_r(p)$  is an interval and in  $\mathfrak{M}_r$  the beginning point of  $t_r(p)$  is  $b_p$  and the ending point is  $e_p$ .

The proof is by induction. We'll do just one case as an example and leave the rest, including part (b), to you.

$$\begin{aligned}
 v(p \wedge_{bb} q) = T & \text{ iff } v(p) = v(q) = T \text{ and } t(p) <_{bb} t(q) \\
 & \text{ iff } v(p) = v(q) = T \text{ and } e_p < e_q \\
 & \text{ iff } v_r(p) = v_r(q) = T \text{ and } e_p <_r e_q \\
 & \text{ iff } v_r(p) = v_r(q) = T \text{ and } t(p) <_{rbb} t(q) \\
 & \text{ iff } v_r(p \wedge_{bb} q) = T. \qquad \dashv
 \end{aligned}$$

We say that a model is *finite* if  $T$  is finite; otherwise it is *infinite*. Note that for every wff  $A$  and for every  $\mathfrak{M}$ ,  $\mathfrak{M}_A$  is finite.

COROLLARY 8. For every  $A$ , if for some model  $\mathfrak{M}$ ,  $\mathfrak{M} \not\models A$ , then there is a finite model  $\mathfrak{M}'$  such that  $\mathfrak{M}' \not\models A$ .

COROLLARY 9.  $\models A$  iff for every finite model  $\mathfrak{M}$ ,  $\mathfrak{M} \models A$

COROLLARY 10.  $\mathbf{TL}_{PC}$  is decidable.

PROOF. Suppose  $A$  has exactly  $n$  propositional variables. If  $\models A$ , then  $A$  is true in every finite model. If  $\not\models A$ , then  $A$  fails in some model in which there are at most  $2n + 2$  instants. There are only a finite number such models, and in each the truth-value of  $A$  can be calculated.<sup>7</sup> So  $A$  is valid iff it is true in each one of those.  $\dashv$

The tautologies of our logic are the wffs true in every finite model. But equally, as we'll show, they are the wffs true in every infinite model. Actually, we'll show something stronger after we make two definitions.

DEFINITION (Dense models). A model  $\mathfrak{M} = \langle T, < \rangle$  is *dense* iff the ordering  $<$  on  $T$  is dense, i.e., for every  $a, b$  in  $T$  such that  $a < b$  there is a  $c$  in  $T$  such that  $a < c < b$ .

<sup>7</sup> If someone could provide a formula for the number of different ordered sets of  $2n + 2$  instants that can serve for a model for  $n$  propositional variables we could say exactly many distinct models must be checked.

Note that every dense linear ordering is infinite.

DEFINITION (Discrete models). A model  $\mathfrak{M} = \langle T, < \rangle$  is *discrete* iff the ordering  $<$  on  $T$  is discrete, i.e., for every  $a$ , if there is a  $b$  such that  $a < b$ , then there is an immediate successor of  $a$  in  $T$  and if there is a  $b$  such that  $b < a$ , then there is an immediate predecessor of  $a$  in  $T$ .

THEOREM 11. For every wff  $A$ ,

- (a)  $\models A$  iff for every discrete model  $\mathfrak{M}$ ,  $\mathfrak{M} \models A$ .
- (b)  $\models A$  iff for every dense model  $\mathfrak{M}$ ,  $\mathfrak{M} \models A$ .

PROOF. Part (a) comes from Corollary 9, since every finite model is discrete.

For part (b), if  $\models A$ , then  $A$  is true in every dense model. So suppose that  $\not\models A$ . Then by Corollary 8, there is a finite model  $\mathfrak{M}$  with  $\mathcal{T} = \langle T, < \rangle$  such that  $\mathfrak{M} \not\models A$ . Let the linear ordering for that model be  $b_{\mathcal{T}} < c_1 < \dots < c_n < e_{\mathcal{T}}$ . Since every time-assignment is not empty and since  $b_{\mathcal{T}} \neq e_{\mathcal{T}}$  and neither  $b_{\mathcal{T}}$  nor  $e_{\mathcal{T}}$  is in any time assignment, we have that  $n \geq 1$ . Define a model  $\mathfrak{M}'$  with the same valuation on propositional variables and  $T' = \{q : q \text{ is a rational number and } 0 \leq q \leq 1\}$ , and:

$$t'(p) = \{q : \frac{i}{n} \leq q \leq \frac{j}{n} \text{ where } b_p = c_i \text{ and } e_p = c_j\}$$

We'll let you show that for any wff  $B$ ,  $\mathfrak{M}' \models B$  iff  $\mathfrak{M} \models B$ . Hence,  $\mathfrak{M}' \not\models A$ . ⊥

COROLLARY 12.  $\models A$  iff for every infinite model  $\mathfrak{M}$ ,  $\mathfrak{M} \models A$ .

Valid wffs cannot distinguish between finite and infinite models, nor between dense and discrete models. However, we do have the following.

THEOREM 13. There is a computable collection of wffs  $\Gamma$  such that if  $\mathfrak{M} \models \Gamma$  then  $\mathfrak{M}_r$  is dense and hence infinite.

PROOF. Let  $f$  be a computable function that enumerates the rational numbers between 0 and 1 without repetitions (see [6]). That is, for each  $n$ ,  $f(n) = s$ , where  $s$  is a rational number such that  $0 \leq s \leq 1$ , and for every rational number  $s$  there is one and only one  $n$  such that  $f(n) = s$ . Set:

$$\begin{aligned} \Gamma = \{ & \neg(p_n \wedge \neg p_n) \wedge_{bb} \neg(p_m \wedge \neg p_m) : f(n) < f(m) \} \\ & \cup \{ \neg(p_n <_{be} p_n) \wedge \neg(p_n <_{eb} p_n) : n \geq 0 \}. \end{aligned}$$

Note that  $\neg(p_n \wedge \neg p_n) \wedge_{bb} \neg(p_m \wedge \neg p_m)$  is true in a model iff  $t(p_n) <_{bb} t(p_m)$ , and  $\neg(p_n <_{be} p_n) \wedge \neg(p_n <_{eb} p_n)$  is true in a model iff  $b_p = e_p$ .



We'll let you show that if  $\mathfrak{M} \models \Gamma$  then  $\langle T_r, <_r \rangle$  is order isomorphic to the rational numbers between 0 and 1, and hence is dense.  $\dashv$

We can't claim that every model of  $\Gamma$  is dense because we can always add "irrelevant" points to the ordered set of a model to get an elementarily equivalent model that is not dense.

## 5. An Axiom System for the Logic of Linear Time Based on Classical Logic

Our goal is to define a notion of theorem and syntactic consequence, which we'll denote as  $\vdash$ , that is strongly complete for this logic: For any wff  $A$  and collection  $\Gamma$  of wffs of the language  $L$ ,  $\Gamma \vdash A$  iff  $\Gamma \models A$ .

We first make the following abbreviations:

$$\begin{aligned}
 A <_{bb} B &\equiv_{df} \neg(A \wedge \neg A) \wedge_{bb} \neg(B \wedge \neg B) \\
 A <_{ee} B &\equiv_{df} \neg(A \wedge \neg A) \wedge_{ee} \neg(B \wedge \neg B) \\
 A <_{be} B &\equiv_{df} \neg(A \wedge \neg A) \wedge_{be} \neg(B \wedge \neg B) \\
 A <_{eb} B &\equiv_{df} \neg(A \wedge \neg A) \wedge_{eb} \neg(B \wedge \neg B) \\
 A \approx_{bb} B &\equiv_{df} \neg(A <_{bb} B) \wedge \neg(B <_{bb} A) \\
 A \approx_{ee} B &\equiv_{df} \neg(A <_{ee} B) \wedge \neg(B <_{ee} A) \\
 A \approx_{be} B &\equiv_{df} \neg(A <_{be} B) \wedge \neg(B <_{be} A)
 \end{aligned}$$

LEMMA 14. *In any model:*

$$\begin{aligned}
 \mathbf{v}(A <_{bb} B) = \mathbf{T} &\text{ iff } \mathbf{t}(A) <_{bb} \mathbf{t}(B), \\
 \mathbf{v}(A <_{ee} B) = \mathbf{T} &\text{ iff } \mathbf{t}(A) <_{ee} \mathbf{t}(B), \\
 \mathbf{v}(A <_{be} B) = \mathbf{T} &\text{ iff } \mathbf{t}(A) <_{be} \mathbf{t}(B), \\
 \mathbf{v}(A <_{eb} B) = \mathbf{T} &\text{ iff } \mathbf{t}(A) <_{eb} \mathbf{t}(B), \\
 \mathbf{v}(A \approx_{bb} B) = \mathbf{T} &\text{ iff } \mathbf{b}_A = \mathbf{b}_B, \\
 \mathbf{v}(A \approx_{ee} B) = \mathbf{T} &\text{ iff } \mathbf{e}_A = \mathbf{e}_B, \\
 \mathbf{v}(A \approx_{be} B) = \mathbf{T} &\text{ iff } \mathbf{b}_A = \mathbf{e}_B.
 \end{aligned}$$

Our base logic will be classical propositional logic, **PC**, using the primitives  $\neg$  and  $\wedge$ . We take the strongly complete axiom system for that from [2, Chapter II.7], where the schema of that axiom system now range over wffs of  $L(\neg, \wedge, \wedge_{bb}, \wedge_{ee}, \wedge_{be}, \wedge_{eb}, p_0, p_1, \dots)$ :

$$\begin{aligned}
& B \supset (A \supset B) \\
& (A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C)) \\
& (A \wedge B) \supset A \\
& (A \wedge B) \supset B \\
& A \supset (B \supset (A \wedge B)) \\
& \neg A \supset (A \supset B) \\
& (A \supset B) \supset ((\neg A \supset B) \supset B)
\end{aligned}$$

rule  $\frac{A, A \supset B}{B}$  *material detachment*

Now we present an axiom system for **TL<sub>PC</sub>** using the defined connectives. It is long and likely to be unclear until you read the proof of strong completeness below, after which we hope it will seem natural. In that proof we mark in boldface the first use of each axiom so you can see why the axiom is adopted.

**TL<sub>PC</sub>** in  $L(\neg, \wedge, \wedge_{bb}, \wedge_{ee}, \wedge_{be}, \wedge_{eb}, p_0, p_1, \dots)$ . The axiom schema of **PC** in this language plus the following schema:

1.  $\neg(A <_{bb} A)$
2.  $\neg(A <_{ee} A)$
3.  $(A \approx_{bb} B) \wedge (B \approx_{bb} C) \supset (A \approx_{bb} C)$
4.  $(A \approx_{ee} B) \wedge (B \approx_{ee} C) \supset (A \approx_{ee} C)$
5.  $(A \approx_{bb} B) \wedge (B \approx_{be} C) \supset (A \approx_{be} C)$
6.  $(A \approx_{be} B) \wedge (C \approx_{be} B) \supset (A \approx_{bb} C)$
7.  $(A \approx_{be} B) \wedge (B \approx_{ee} C) \supset (A \approx_{be} C)$
8.  $(A \approx_{ee} B) \wedge (C \approx_{be} B) \supset (C \approx_{be} A)$
9.  $(B \approx_{be} A) \wedge (B \approx_{be} C) \supset (A \approx_{ee} C)$
10.  $(B \approx_{be} A) \wedge (B \approx_{bb} C) \supset (C \approx_{be} A)$
11.  $[(A <_{bb} B) \wedge (A \approx_{bb} C) \wedge (B \approx_{bb} D)] \supset (C <_{bb} D)$
12.  $[(A <_{bb} B) \wedge (A \approx_{bb} C) \wedge (B \approx_{be} D)] \supset (C <_{be} D)$
13.  $[(A <_{bb} B) \wedge (A \approx_{be} C) \wedge (B \approx_{bb} D)] \supset (C <_{bb} D)$
14.  $[(A <_{bb} B) \wedge (A \approx_{be} C) \wedge (B \approx_{be} D)] \supset (C <_{ee} D)$
15.  $[(A <_{be} B) \wedge (A \approx_{bb} C) \wedge (B \approx_{be} D)] \supset (C <_{be} D)$
16.  $[(A <_{be} B) \wedge (A \approx_{be} C) \wedge (D \approx_{be} C)] \supset (C <_{eb} D)$
17.  $[(A <_{be} B) \wedge (A \approx_{bb} C) \wedge (B \approx_{ee} D)] \supset (C <_{be} D)$

18.  $[(A <_{be} B) \wedge (A \approx_{be} C) \wedge (B \approx_{bb} D)] \supset (C <_{bb} D)$
19.  $[(A <_{eb} B) \wedge (A \approx_{be} C) \wedge (B \approx_{bb} D)] \supset (C <_{bb} D)$
20.  $[(A <_{eb} B) \wedge (A \approx_{ee} C) \wedge (B \approx_{bb} D)] \supset (C <_{eb} D)$
21.  $[(A <_{eb} B) \wedge (C \approx_{be} C) \wedge (B \approx_{be} D)] \supset (C <_{be} D)$
22.  $[(A <_{eb} B) \wedge (A \approx_{ee} C) \wedge (B \approx_{be} D)] \supset (C <_{ee} D)$
23.  $[(A <_{ee} B) \wedge (C \approx_{be} A) \wedge (D \approx_{be} B)] \supset (C <_{bb} D)$
24.  $[(A <_{ee} B) \wedge (A \approx_{ee} C) \wedge (D \approx_{be} C)] \supset (C <_{eb} D)$
25.  $[(A <_{ee} B) \wedge (C \approx_{be} A) \wedge (B \approx_{ee} D)] \supset (C <_{be} D)$
26.  $[(A <_{ee} B) \wedge (A \approx_{ee} C) \wedge (B \approx_{ee} D)] \supset (C <_{ee} D)$
27.  $(A <_{bb} B) \supset \neg(B <_{bb} A)$
28.  $(A <_{ee} B) \supset \neg(B <_{ee} A)$
29.  $(A <_{be} B) \supset \neg(B <_{eb} A)$
30.  $(A <_{eb} B) \supset \neg(B <_{be} A)$
31.  $(A <_{bb} B) \wedge (B <_{bb} C) \supset (A <_{bb} C)$
32.  $(A <_{bb} B) \wedge (B <_{be} C) \supset (A <_{be} C)$
33.  $(A <_{be} B) \wedge (B <_{eb} C) \supset (A <_{bb} C)$
34.  $(A <_{be} B) \wedge (B <_{ee} C) \supset (A <_{be} C)$
35.  $(A <_{ee} B) \wedge (B <_{ee} C) \supset (A <_{ee} C)$
36.  $(A <_{ee} B) \wedge (B <_{eb} C) \supset (A <_{eb} C)$
37.  $(A <_{eb} B) \wedge (B <_{be} C) \supset (A <_{ee} C)$
38.  $(A <_{eb} B) \wedge (B <_{bb} C) \supset (A <_{eb} C)$
39.  $A \wedge_{bb} B \equiv (A \wedge B) \wedge (A <_{bb} B)$
40.  $A \wedge_{ee} B \equiv (A \wedge B) \wedge (A <_{ee} B)$
41.  $A \wedge_{be} B \equiv (A \wedge B) \wedge (A <_{be} B)$
42.  $A \wedge_{eb} B \equiv (A \wedge B) \wedge (A <_{eb} B)$
43.  $(A <_{bb} B) \equiv \bigvee_{p \text{ in } A} (\bigwedge_{q \text{ in } B} (p <_{bb} q))$
44.  $(A <_{ee} B) \equiv \bigvee_{q \text{ in } B} (\bigwedge_{p \text{ in } A} (p <_{ee} q))$
45.  $(A <_{eb} B) \equiv \bigvee_{p \text{ in } A} \bigvee_{q \text{ in } B} [\bigwedge_{p' \text{ in } A} (p' <_{ee} p \vee [\neg(p <_{ee} p') \wedge \neg(p' <_{ee} p)]) \wedge \bigwedge_{q' \text{ in } B} (q <_{bb} q' \vee [\neg(q <_{bb} q') \wedge \neg(q' <_{bb} q)]) \wedge p <_{eb} q]$
46.  $(A <_{be} B) \equiv \bigvee_{p \text{ in } A} \bigvee_{q \text{ in } B} [\bigwedge_{p' \text{ in } A} (p <_{bb} p' \vee [\neg(p <_{bb} p') \wedge \neg(p' <_{bb} p)]) \wedge \bigwedge_{q' \text{ in } B} (q' <_{ee} q \vee [\neg(q <_{ee} q') \wedge \neg(q' <_{ee} q)]) \wedge p <_{be} q]$

rule  $\frac{A, A \supset B}{B}$

We adopt the usual definitions of completeness, consistency, and theory as for **PC**:

- $\Gamma$  is *consistent* iff for no  $A$  does  $\Gamma \vdash A$  and  $\Gamma \vdash \neg A$ .
- $\Gamma$  is *complete* iff for every  $A$ ,  $\Gamma \vdash A$  or  $\Gamma \vdash \neg A$ .
- $\Gamma$  is a *theory* iff for every  $A$ , if  $\Gamma \vdash A$  then  $A$  is in  $\Gamma$ .

Using the methods of [2, Chapter II] it is routine to prove the following.

- LEMMA 15. (a) (The Deduction Theorem)  $\Gamma \cup \{A\} \vdash B$  iff  $\Gamma \vdash A \supset B$ .  
 (b) If  $\Gamma$  is complete and consistent, then  $\Gamma$  is a theory.  
 (c) If  $\Gamma \not\vdash A$ , then there is some complete and consistent  $\Sigma \supseteq \Gamma$  such that  $A \notin \Sigma$ .  
 (d) If  $A$  has the form of a valid wff in **PC**, then  $\vdash A$ .

In the proofs in this section we write “by **PC**” to mean that the wff under consideration is **PC**-valid and so by Lemma 15 is a theorem of this system.

The proof of strong completeness now reduces to proving the following theorem. We include all the details in the proof because, though after reading the proof many steps may seem routine and easy, we have found that it is difficult to formulate them correctly.

THEOREM 16. If  $\Gamma$  is a complete and consistent collection of wffs of  $L$ , then  $\Gamma$  has a model.

PROOF. Given  $\Gamma$  we will construct a model  $\langle \mathbf{v}, \langle \mathbf{T}, \langle \rangle, \mathbf{t} \rangle \rangle$  such that  $\mathbf{v}(A) = \mathbf{T}$  iff  $A \in \Gamma$ .

First note that by Lemma 15,  $\Gamma$  is a theory, and  $\Gamma \supseteq \mathbf{PC}$ .

In order to construct a linearly ordered set, we first define:

$$S = \{x_p, y_p : p \text{ is a propositional variable}\} \cup \{\mathbf{b}, \mathbf{e}\}.$$

The letters  $x_{p_0}, y_{p_0}, x_{p_1}, y_{p_1}, \dots, \mathbf{b}, \mathbf{e}$  do not stand for anything; they are not symbols but simply letters.

We define a relation  $<$  on  $S$ . For all propositional variables  $p, q$ ,

$$\begin{array}{ll} \mathbf{b} < x_p & \\ \mathbf{b} < y_p & x_p < x_q \text{ iff } (p <_{bb} q) \in \Gamma \\ x_p < \mathbf{e} & y_p < y_q \text{ iff } (p <_{ee} q) \in \Gamma \\ y_p < \mathbf{e} & x_p < y_q \text{ iff } (p <_{be} q) \in \Gamma \\ \mathbf{b} < \mathbf{e} & y_p < x_q \text{ iff } (p <_{eb} q) \in \Gamma \end{array}$$





The points  $x_p$ ,  $y_p$  are meant to be the beginning and ending points of the interval assigned to the propositional variable  $p$ . But for distinct  $p$ ,  $q$  we may have that  $(p \approx_{bb} q) \in \Gamma$ , yet  $x_p \neq y_p$  since those are different letters. So we define an equivalence relation on  $S$  and take the equivalence classes as the instants of our models.

$$\begin{aligned}
 & \mathbf{b} \sim \mathbf{b} \\
 & \mathbf{e} \sim \mathbf{e} \\
 & x_p \sim x_q \text{ iff } (p \approx_{bb} q) \in \Gamma \\
 & y_p \sim y_q \text{ iff } (p \approx_{ee} q) \in \Gamma \\
 & x_p \sim y_q \text{ iff } (p \approx_{be} q) \in \Gamma \\
 & y_q \sim x_p \text{ iff } (p \approx_{be} q) \in \Gamma
 \end{aligned}$$

Note that the defining condition for the last two equivalences are the same. Below we will note the use of an axiom that justifies a step.

LEMMA A. *The relation  $\sim$  is an equivalence relation.*

PROOF. We must show that  $\sim$  is reflexive, symmetric, and transitive.

*Reflexive* For all  $z$ ,  $z \sim z$ .

$$\begin{aligned}
 x_p \sim x_p & \qquad \qquad \qquad \neg(A <_{bb} A) \\
 y_p \sim y_p & \qquad \qquad \qquad \neg(A <_{ee} A) \\
 \mathbf{b} \sim \mathbf{b} \text{ and } \mathbf{e} \sim \mathbf{e} & \text{ by definition}
 \end{aligned}$$

*Symmetric* For all  $z$ ,  $w$ , if  $z \sim w$ , then  $w \sim z$ .

If  $z = w$ , this follows by reflexivity.

If  $z$  is  $\mathbf{b}$ , then  $w = \mathbf{b}$ , and we are done by reflexivity.

If  $z$  is  $\mathbf{e}$ , then  $w = \mathbf{e}$ , and we are done by reflexivity.

$$\begin{aligned}
 x_p \sim x_q \text{ iff } (p \approx_{bb} q) \in \Gamma & \qquad \qquad \text{by definition} \\
 \text{iff } \neg(p <_{bb} q) \wedge \neg(q <_{bb} p) \in \Gamma & \qquad \qquad \text{by PC} \\
 \text{iff } \neg(q <_{bb} p) \wedge \neg(p <_{bb} q) \in \Gamma & \qquad \qquad \text{by PC} \\
 \text{iff } (q \approx_{bb} p) \in \Gamma & \\
 \text{iff } x_q \sim x_p &
 \end{aligned}$$

If  $y_p \sim y_q$ , then  $y_p \sim y_q$  follows similarly.

$$\begin{aligned}
 y_p \sim x_q \text{ iff } (p \approx_{be} q) \in \Gamma \\
 \text{iff } x_p \sim y_q
 \end{aligned}$$

*Transitive* For all  $z, w, u$ , if  $z \sim w$  and  $w \sim u$ , then  $z \sim u$ .

If  $z = w$  or  $w = u$ , we are done.

If  $z, w$ , or  $u$  is  $b$ , then all three are, and we are done.

If  $z, w$ , or  $u$  is  $e$ , then all three are, and we are done.

1. If  $x_p \sim x_q$  and  $x_q \sim x_r$  then  $x_p \sim x_r$   
 $(A \approx_{bb} B) \wedge (B \approx_{bb} C) \supset (A \approx_{bb} C)$ .
2. If  $x_p \sim x_q$  and  $x_q \sim y_r$  then  $x_p \sim y_r$   
 $(A \approx_{bb} B) \wedge (B \approx_{be} C) \supset (A \approx_{be} C)$ .
3. If  $x_p \sim y_q$  and  $y_q \sim x_r$  then  $x_p \sim x_r$   
 $(A \approx_{be} B) \wedge (C \approx_{be} B) \supset (A \approx_{bb} C)$ .
4. If  $x_p \sim y_q$  and  $y_q \sim y_r$  then  $x_p \sim y_r$   
 $(A \approx_{be} B) \wedge (B \approx_{ee} C) \supset (A \approx_{be} C)$ .
5. If  $y_p \sim y_q$  and  $y_q \sim y_r$  then  $y_p \sim y_r$   
 $(A \approx_{ee} B) \wedge (B \approx_{ee} C) \supset (A \approx_{ee} C)$ .
6. If  $y_p \sim y_q$  and  $y_q \sim x_r$  then  $y_p \sim x_r$   
 $(A \approx_{ee} B) \wedge (C \approx_{be} B) \supset (C \approx_{be} A)$ .
7. If  $y_p \sim x_q$  and  $x_q \sim y_r$  then  $y_p \sim y_r$   
 $(B \approx_{be} A) \wedge (B \approx_{be} C) \supset (A \approx_{ee} C)$ .
8. If  $y_p \sim x_q$  and  $x_q \sim x_r$  then  $y_p \sim x_r$   
 $(B \approx_{be} A) \wedge (B \approx_{bb} C) \supset (C \approx_{be} A)$ .

This completes the proof of Lemma A. ⊢

LEMMA B. *The relation  $\sim$  respects  $<$ . That is,*

*if  $z < w$  and  $z \sim u$  and  $w \sim v$ , then  $u < v$ .*

PROOF. If  $b < w$  and  $b \sim u$  and  $w \sim v$ , then  $u$  is  $b$ , and  $w \neq b$ . Hence  $v \neq b$ , so  $b \sim v$ . We'll leave to you the other cases when any of  $z, w, u$ , or  $v$  is either  $b$  or  $e$ .

1. If  $x_p < x_q$  and  $x_p \sim x_r$  and  $x_q \sim x_s$ , then  $x_r < x_s$   
 $[(A <_{bb} B) \wedge (A \approx_{bb} C) \wedge (B \approx_{bb} D)] \supset (C <_{bb} D)$ .
2. If  $x_p < x_q$  and  $x_p \sim x_r$  and  $x_q \sim y_s$ , then  $x_r < y_s$   
 $[(A <_{bb} B) \wedge (A \approx_{bb} C) \wedge (B \approx_{be} D)] \supset (C <_{be} D)$ .
3. If  $x_p < x_q$  and  $x_p \sim y_r$  and  $x_q \sim x_s$ , then  $y_r < x_s$   
 $[(A <_{bb} B) \wedge (A \approx_{be} C) \wedge (B \approx_{bb} D)] \supset (C <_{eb} D)$ .
4. If  $x_p < x_q$  and  $x_p \sim y_r$  and  $x_q \sim y_s$ , then  $y_r < y_s$   
 $[(A <_{bb} B) \wedge (A \approx_{be} C) \wedge (B \approx_{be} D)] \supset (C <_{ee} D)$ .

5. If  $x_p < y_q$  and  $x_p \sim x_r$  and  $y_q \sim x_s$ , then  $x_r < x_s$   

$$[(A <_{be} B) \wedge (A \approx_{bb} C) \wedge (D \approx_{be} B)] \supset (C <_{bb} D).$$
6. If  $x_p < y_q$  and  $x_p \sim y_r$  and  $y_q \sim x_s$ , then  $y_r < x_s$   

$$[(A <_{be} B) \wedge (A \approx_{be} C) \wedge (D \approx_{be} B)] \supset (C <_{eb} D).$$
7. If  $x_p < y_q$  and  $x_p \sim y_r$  and  $y_q \sim y_s$ , then  $x_r < y_s$   

$$[(A <_{be} B) \wedge (A \approx_{bb} C) \wedge (B \approx_{ee} D)] \supset (C <_{be} D).$$
8. If  $x_p < y_q$  and  $x_p \sim x_r$  and  $y_q \sim y_s$ , then  $x_r < y_s$   

$$[(A <_{be} B) \wedge (A \approx_{bb} C) \wedge (B \approx_{ee} D)] \supset (C <_{be} D).$$
9. If  $y_p < x_q$  and  $y_p \sim x_r$  and  $x_q \sim x_s$ , then  $x_r < x_s$   

$$[(A <_{eb} B) \wedge (C \approx_{be} A) \wedge (B \approx_{bb} D)] \supset (C <_{bb} D).$$
10. If  $y_p < x_q$  and  $y_p \sim y_r$  and  $x_q \sim x_s$ , then  $y_r < x_s$   

$$[(A <_{eb} B) \wedge (A \approx_{ee} C) \wedge (B \approx_{bb} D)] \supset (C <_{eb} D).$$
11. If  $y_p < x_q$  and  $y_p \sim x_r$  and  $x_q \sim y_s$ , then  $x_r < y_s$   

$$[(A <_{eb} B) \wedge (C \approx_{be} C) \wedge (D \approx_{be} B)] \supset (C <_{be} D).$$
12. If  $y_p < x_q$  and  $y_p \sim y_r$  and  $x_q \sim y_s$ , then  $y_r < y_s$   

$$[(A <_{eb} B) \wedge (A \approx_{ee} C) \wedge (B \approx_{be} D)] \supset (C <_{ee} D).$$
13. If  $y_p < y_q$  and  $y_p \sim x_r$  and  $y_q \sim x_s$ , then  $x_r < x_s$   

$$[(A <_{ee} B) \wedge (C \approx_{be} A) \wedge (D \approx_{be} B)] \supset (C <_{bb} D).$$
14. If  $y_p < y_q$  and  $y_p \sim x_r$  and  $y_q \sim y_s$ , then  $x_r < y_s$   

$$[(A <_{ee} B) \wedge (C \approx_{be} A) \wedge (B \approx_{ee} D)] \supset (C <_{be} D).$$
15. If  $y_p < y_q$  and  $y_p \sim y_r$  and  $y_q \sim x_s$ , then  $y_r < x_s$   

$$[(A <_{ee} B) \wedge (A \approx_{ee} C) \wedge (D \approx_{be} C)] \supset (C <_{eb} D).$$
16. If  $y_p < y_q$  and  $y_p \sim y_r$  and  $y_q \sim y_s$ , then  $y_r < y_s$   

$$[(A <_{ee} B) \wedge (A \approx_{ee} C) \wedge (B \approx_{ee} D)] \supset (C <_{ee} D).$$

This completes the proof of Lemma B. ⊣

For  $z \in S$ , denote the equivalence class of  $z$  by  $[z]$ . Note that the  $[b] = \{b\}$  and  $[e] = \{e\}$ .

Define  $T = \{[z] : z \in S\}$  and  $[z] < [w]$  iff  $z < w$ . By Lemma B this is well-defined.

LEMMA C.  $\langle T, \langle \rangle$  is a linearly ordered set with endpoints.

PROOF. We first show that it has endpoints.

For every  $z \neq b$ ,  $b < z$  and not  $b < b$ . Hence, for every  $[z] \neq [b]$ ,  $[b] < [z]$ , and also not  $[b] < [b]$ . And similarly, for every  $[z] \neq [e]$ ,  $[z] < [e]$ , while not  $[e] < [e]$ .

We now show that  $<$  is anti-reflexive and transitive.

*Anti-reflexive* For any  $[z]$ , not  $[z] < [z]$ .

If  $[z] = [b]$  or  $[z] = [e]$  we are done by what we have just noted above. Otherwise, for some  $p$ ,  $[z] = [x_p]$  or  $[z] = [y_p]$ . We have not  $x_p < x_p$  by Axiom (schema) 1, and we have not  $y_p < y_p$  by Axiom 2.

*Transitive* If  $[z] < [w]$  and  $[w] < [v]$ , then  $[z] < [v]$ .

We'll leave to you the cases when any one of  $[z]$ ,  $[w]$ , or  $[v]$  is  $[b]$  or  $[e]$ . We'll abbreviate  $[z] < [w]$  and  $[w] < [v]$  as  $[z] < [w] < [v]$ .

1. If  $[x_p] < [x_q] < [x_r]$  then  $[x_p] < [x_r]$   
 $(A <_{bb} B) \wedge (B <_{bb} C) \supset (A <_{bb} C)$ .
2. If  $[x_p] < [x_q] < [y_r]$  then  $[x_p] < [y_r]$   
 $(A <_{bb} B) \wedge (B <_{be} C) \supset (A <_{be} C)$ .
3. If  $[x_p] < [y_q] < [x_r]$  then  $[x_p] < [x_r]$   
 $(A <_{be} B) \wedge (B <_{eb} C) \supset (A <_{bb} C)$ .
4. If  $[x_p] < [y_q] < [y_r]$  then  $[x_p] < [y_r]$   
 $(A <_{be} B) \wedge (B <_{ee} C) \supset (A <_{be} C)$ .
5. If  $[y_p] < [y_q] < [y_r]$  then  $[y_p] < [y_r]$   
 $(A <_{ee} B) \wedge (B <_{ee} C) \supset (A <_{ee} C)$ .
6. If  $[y_p] < [y_q] < [x_r]$  then  $[y_p] < [x_r]$   
 $(A <_{ee} B) \wedge (B <_{eb} C) \supset (A <_{eb} C)$ .
7. If  $[y_p] < [x_q] < [y_r]$  then  $[y_p] < [y_r]$   
 $(A <_{eb} B) \wedge (B <_{be} C) \supset (A <_{ee} C)$ .
8. If  $[y_p] < [x_q] < [x_r]$  then  $[y_p] < [x_r]$   
 $(A <_{eb} B) \wedge (B <_{bb} C) \supset (A <_{eb} C)$ .

This completes the proof of Lemma C. ⊢

Now we define the valuation  $\mathbf{v}$  and time-assignment  $\mathbf{t}$  for our model.

$$\begin{aligned} \mathbf{v}(p) &= \mathbf{T} \text{ iff } p \in \Gamma, \\ \mathbf{t}(p) &= \{[z] : [x_p] \leq [z] \leq [y_p]\}. \end{aligned}$$

Then  $\mathbf{t}$  is extended to all wffs by the standard condition for models:

$$\mathbf{t}(A) = \bigcup \{ \mathbf{t}(p) : p \text{ appears in } A \}.$$

And  $\mathbf{v}$  is extended to all wffs by the tables for the connectives.

In this ordering,  $\mathbf{b}_p$  is  $[x_p]$ ,  $\mathbf{e}_p$  is  $[y_p]$ , and for all  $p$ ,  $\mathbf{b} \notin \mathbf{t}(p)$  and  $\mathbf{e} \notin \mathbf{t}(p)$ .

LEMMA D.  $\mathbf{v}(A) = \mathbf{T}$  iff  $A \in \Gamma$ .

PROOF. If  $A$  has length 1,  $A$  is  $p$  and the lemma follows by definition.

If  $A$  has length 2 and  $A$  is  $p \wedge q$  or  $\neg p$ , then the proof follows as for **PC** (see [2]). Otherwise, we have:

$$\begin{aligned}
 \mathbf{v}(p \wedge_{bb} q) = \mathbf{T} & \text{ iff } \mathbf{v}(p) = \mathbf{v}(q) = \mathbf{T} \text{ and } \mathbf{t}(p) <_{bb} \mathbf{t}(q) \\
 & \text{ iff } p \in \Gamma \text{ and } q \in \Gamma \text{ and } [x_p] < [x_q] \\
 & \text{ iff } (p \wedge q) \in \Gamma \text{ and } [x_p] < [x_q] & \text{ by } \mathbf{PC} \\
 & \text{ iff } (p \wedge q) \in \Gamma \text{ and } x_p < x_q \\
 & \text{ iff } (p \wedge q) \in \Gamma \text{ and } (p <_{bb} q) \in \Gamma & \text{ by construction} \\
 & \text{ iff } (p \wedge_{bb} q) & A \wedge_{bb} B \equiv (A \wedge B) \wedge (A <_{bb} B)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{v}(p \wedge_{ee} q) = \mathbf{T} & \text{ iff } \mathbf{v}(p) = \mathbf{v}(q) = \mathbf{T} \text{ and } \mathbf{t}(p) <_{ee} \mathbf{t}(q) \\
 & \text{ iff } (p \wedge q) \in \Gamma \text{ and } [y_p] < [y_q] \\
 & \text{ iff } (p \wedge q) \in \Gamma \text{ and } y_p < y_q \\
 & \text{ iff } (p \wedge q) \in \Gamma \text{ and } (p <_{ee} q) \in \Gamma & \text{ by construction} \\
 & \text{ iff } (p \wedge_{bb} q) & A \wedge_{ee} B \equiv (A \wedge B) \wedge (A <_{ee} B)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{v}(p \wedge_{be} q) = \mathbf{T} & \text{ iff } \mathbf{v}(p) = \mathbf{v}(q) = \mathbf{T} \text{ and } \mathbf{t}(p) <_{be} \mathbf{t}(q) \\
 & \text{ iff } (p \wedge q) \in \Gamma \text{ and } [x_p] < [y_q] \\
 & \text{ iff } (p \wedge q) \in \Gamma \text{ and } x_p < y_q \\
 & \text{ iff } (p \wedge q) \in \Gamma \text{ and } (p <_{be} q) \in \Gamma & \text{ by construction} \\
 & \text{ iff } (p \wedge_{bb} q) & A \wedge_{be} B \equiv (A \wedge B) \wedge (A <_{be} B)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{v}(p \wedge_{eb} q) = \mathbf{T} & \text{ iff } \mathbf{v}(p) = \mathbf{v}(q) = \mathbf{T} \text{ and } \mathbf{t}(p) <_{eb} \mathbf{t}(q) \\
 & \text{ iff } (p \wedge q) \in \Gamma \text{ and } [y_p] < [x_q] \\
 & \text{ iff } (p \wedge q) \in \Gamma \text{ and } y_p < x_q \\
 & \text{ iff } (p \wedge q) \in \Gamma \text{ and } (p <_{eb} q) \in \Gamma & \text{ by construction} \\
 & \text{ iff } (p \wedge_{bb} q) & A \wedge_{eb} B \equiv (A \wedge B) \wedge (A <_{eb} B)
 \end{aligned}$$

Now suppose that  $A$  has length 3 or greater and the lemma is true for all shorter wffs. If  $A$  is of the form  $A \wedge B$  or  $\neg A$ , then the proof follows as for **PC**. Otherwise, we have:

$$\begin{aligned}
 \mathbf{v}(A \wedge_{bb} B) = \mathbf{T} & \text{ iff } \mathbf{v}(A) = \mathbf{v}(B) = \mathbf{T} \text{ and } \mathbf{t}(A) <_{bb} \mathbf{t}(B) \\
 & \text{ iff } A \in \Gamma \text{ and } B \in \Gamma \text{ and } \mathbf{t}(A) <_{bb} \mathbf{t}(B) \\
 & \text{ iff } (A \wedge B) \in \Gamma \text{ and for some } p \text{ in } A, \text{ all } q \text{ in } B, \\
 & \quad \mathbf{t}(p) <_{bb} \mathbf{t}(q) & \text{ by Lemma 7}
 \end{aligned}$$

$$\begin{aligned}
 & \text{iff } (A \wedge B) \in \Gamma \text{ and for some } p \text{ in } A, \text{ all } q \text{ in } B, \\
 & \quad (p <_{bb} q) \in \Gamma \quad \text{by construction} \\
 & \text{iff } (A \wedge B) \in \Gamma \text{ and } (A <_{bb} B) \in \Gamma \\
 & \quad (A <_{bb} B) \equiv \mathbb{W}_{p \text{ in } A} (\mathbb{M}_{q \text{ in } B} (p <_{bb} q)) \\
 & \quad \text{and Axiom 39 as in the evaluation of } \mathfrak{v}(p \wedge_{bb} q)
 \end{aligned}$$

The other cases are done similarly using Lemma 5 and the following axioms:

$$\begin{aligned}
 (A <_{ee} B) & \equiv \mathbb{W}_{q \text{ in } B} (\mathbb{M}_{p \text{ in } A} (p <_{ee} q)) \\
 (A <_{eb} B) & \equiv \mathbb{W}_{p \text{ in } A} \mathbb{W}_{q \text{ in } B} [\mathbb{M}_{p' \text{ in } A} (p' <_{ee} p \vee \\
 & \quad [\neg(p <_{ee} p') \wedge \neg(p' <_{ee} p)]) \\
 & \quad \wedge \mathbb{M}_{q' \text{ in } B} (q <_{bb} q' \vee [\neg(q <_{bb} q') \wedge \neg(q' <_{bb} q)]) \wedge p <_{eb} q] \\
 (A <_{be} B) & \equiv \mathbb{W}_{p \text{ in } A} \mathbb{W}_{q \text{ in } B} [\mathbb{M}_{p' \text{ in } A} (p <_{bb} p' \vee \\
 & \quad [\neg(p <_{bb} p') \wedge \neg(p' <_{bb} p)]) \\
 & \quad \wedge \mathbb{M}_{q' \text{ in } B} (q' <_{ee} q \vee [\neg(q <_{ee} q') \wedge \neg(q' <_{ee} q)]) \wedge p <_{be} q]
 \end{aligned}$$

This completes the proof of Lemma D. ⊢

Lemma D concludes the proof of Theorem 16. ⊢

**THEOREM 17 (Strong Completeness).** *For any set  $\Gamma \cup \{A\}$  of wffs:  $\Gamma \vdash A$  iff  $\Gamma \models A$ .*

**PROOF.** We'll leave to you to check that each of the axioms is true in every model and that the rule preserves truth in a model. Hence, if  $\Gamma \vdash A$  then  $\Gamma \models A$ .

In the other direction, suppose  $\Gamma \not\models A$ . Then by Lemma 15(c) there is a complete and consistent  $\Sigma$  such that  $\Sigma \supseteq \Gamma$  and  $A \notin \Sigma$ . By Theorem 16 there is a model  $\mathfrak{M}$  such that for all wffs  $B$ ,  $\mathfrak{M} \models B$  iff  $B \in \Sigma$ . Hence,  $\mathfrak{M} \not\models A$ , and so  $\Sigma \not\models A$ , and hence  $\Gamma \not\models A$ . ⊢

**COROLLARY 18.** (a) *If  $\Gamma \not\models A$ , then there is a model  $\mathfrak{M}$  in which every time assignment is a closed interval and  $\mathfrak{M} \not\models A$ .*

(b) *If  $\Gamma \not\models A$ , then there is a model  $\mathfrak{M}$  in which every time assignment is an open interval and  $\mathfrak{M} \not\models A$ .*

PROOF. (a) Every time assignment in the model constructed in Theorem 16 is closed.

(b) We can modify the construction of the model in Theorem 16 to have only open intervals as time assignments. We just have to ensure that the time assignments are not empty since we may have that for no  $z$  is  $x_p < z < y_p$ . For the construction of  $S$  we add the following step:

For any  $p$  such that for no  $z$ ,  $x_p < z < y_p$  add a point  $m_p$  to  $S$  with the conditions:

$$\begin{aligned} z < m_p &\text{ iff } z \leq x_p \\ m_p < z &\text{ iff } y_p \leq z \end{aligned}$$

Then in the construction of the model set  $t(p) = \{[z] : [x_p] < [z] < [y_p]\}$ .

COROLLARY 19. (a)  $\Gamma \vdash A$  iff for every model  $\mathfrak{M}$  that validates  $\Gamma$  and in which every time-assignment is a closed interval,  $\mathfrak{M} \models A$ .

(b)  $\Gamma \vdash A$  iff for every model  $\mathfrak{M}$  that validates  $\Gamma$  and in which every time-assignment is an open interval,  $\mathfrak{M} \models A$ .

It might be possible to simplify the axiom system for  $\mathbf{TL}_{PC}$  by using the definitions of the defined connectives and noting that the following are valid:

$$\begin{array}{ll} A <_{eb} B \supset A <_{ee} B & A <_{eb} B \supset A <_{be} B \\ A <_{eb} B \supset A <_{bb} B & A <_{bb} B \supset A <_{be} B \end{array}$$

## 6. Time as a Subjective Ordering of Experience

We set out to formalize how to take time into account in our reasoning. That sounds as if we're assuming that time has a reality in the world distinct from us. Viewing time as a linear ordering seemed to confirm that we were viewing time as objective.

But nothing we've done depends on our taking time to be real beyond our experience of before and after. Our experiences, at least those we can communicate and investigate together, are expressed in linguistic propositions. The import of the proof of Theorem 16 is that we can construct "time" from a complete and consistent ordering of propositions in terms of when we say their descriptions are meant to begin and to end. In reasoning about time in this logic we need not assume that time is anything more than our ordering of experiences, and to use this logic together we need only agree on that ordering.

The intersubjectivity of our ordering of experiences may be due to the reality of time external to us which we perceive or are forced to acknowledge. But equally this conception allows for that intersubjective ordering to be based solely on our sharing subjective evaluations. For example, Zoe may “know” that “Dick yelled at Spot” is a true description of a time after “Spot was barking” because that’s how it seems to her. If Dick agrees that “Dick yelled at Spot” is a true description of a time after “Spot was barking,” then their orderings are the same for that part of their experience. If lots of people agree that “Dick yelled at Spot” is a true description of a time after “Spot was barking,” then they have an intersubjective ordering of time for those experiences. On the other hand, that Zoe knows that “Zoe forgot her keys” came after “Zoe decided to go to Suzy’s house” is a subjective ordering that others can only infer, though, again, it could be that her memory is grounded in a reality of time external to her that is shared by us all.<sup>8</sup>

## 7. Examples of Formalization

### 1. *Spot barked and then Dick yelled.*

Spot barked  $\wedge_{\text{eb}}$  Dick yelled

ANALYSIS Unless context suggests otherwise, let’s formalize “and then” to mean that the first proposition describes a time completely before the second.

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<sup>8</sup> A subjective conception of time could also allow for a metric. An experience may be the recording of an oscillation of an atom. Then the recording of a successive oscillation is another experience, which could give us a standard minimal difference in times. Accumulating those, projecting those backwards and forwards, we have not only a linear ordering but a metric. One trillion trillion of those “experiences” would be equal — if we were to have them — to the time between when Spot barked and Dick yelled. Time, when we put a metric on it, is then not only our actual experiences but what we take to be our possible experiences. To say that time is dense in this subjective conception of time is to say that given any two experiences, it is possible to have another experience that comes between them. Compare J. F. Allen in [1]:

There seems to be a strong intuition that, given an event, we can always “turn up the magnification” and look at its structure. This has certainly been the experience so far in physics. Since the only times we consider will be times of events, it appears that we can always decompose times into subparts. Thus the formal notion of a time point, which would not be decomposable, is not useful. [1, p. 834]



2. *Spot barked. Then Dick yelled.*

Same as for Example 1

ANALYSIS The word “then” is used to relate two propositions as with “and then.”

3. *Dick yelled after Spot barked.*

Same as for Example 1.

ANALYSIS The order of the propositions is reversed in the formalization, understanding “after” to be the reverse of “and then.”

4. *Suzy took off her clothes and went to bed.*

Suzy took off her clothes  $\wedge_{eb}$  Suzy went to bed

ANALYSIS We understand “and” here to mean “and then.”

5. *If Suzy went to bed, then she off her clothes.*

$\neg$ (Suzy went to bed  $\wedge$   $\neg$ (Suzy took off her clothes))

ANALYSIS The “then” here is not meant temporally but only as part of the conditional. Indeed, in normal circumstances if this is true, then “Suzy took off her clothes” would be about a time before that of “Suzy went to bed.” But that understanding is not part of what is asserted here.

6. *Spot barked before Dick yelled.*

ANALYSIS As noted in Section 2.1, the use of “before” as a connective in English is ambiguous. We have at least two choices:

Spot started barking before Dick began to yell.

Spot barked  $\wedge_{bb}$  Dick yelled.

Spot finished barking before Dick began to yell.

Spot barked  $\wedge_{eb}$  Dick yelled.

In the latter case, we do not need to add that Spot started barking before Dick began to yell, since  $(A \wedge_{eb} B) \supset (A \wedge_{bb} B)$  is valid. Nor do we need to add that Spot started barking before Dick ended yelling, since  $(A \wedge_{eb} B) \supset (A \wedge_{be} B)$  is valid.

7. *Spot barked at 7:12 am May 5th, 2005.*

*Spot barked around 7:12 am May 5th, 2005.*

*Spot barked at exactly 7:12 am May 5th, 2005.*

ANALYSIS We can distinguish these only by what time-assignment we give to “Spot barked.” To distinguish them syntactically we would have to be able to refer to specific times in the formal language.

8. *Sometime after Dick finished eating, Spot began to bark.*

Dick ate  $\wedge_{eb}$  Spot barked

ANALYSIS Despite the apparent quantification over times in the example, we can formalize it without quantifying. Note that we modify the sentences in the original to eliminate “finished” and “began.”

9. *Dick yelled at the same time as Spot barked.*

(Spot barked  $\wedge$  Dick yelled)  $\wedge$  (Spot barked  $\approx_{bb}$  Dick yelled)  
 $\wedge$  (Spot barked  $\approx_{ee}$  Dick yelled)

ANALYSIS We need no quantification or reference to a time in the formalization.

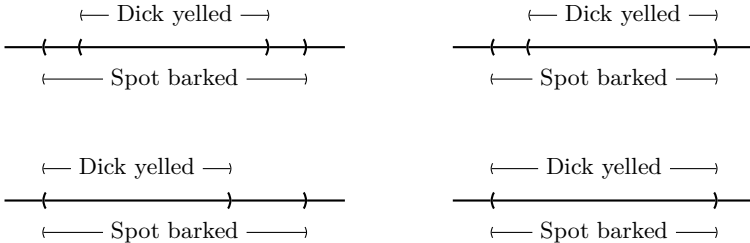
We can define a connective to formalize at the same time as:

$$A \wedge_{\approx} B \equiv_{df} (A \wedge B) \wedge (A \approx_{bb} B) \wedge (A \approx_{ee} B)$$

10. *Dick yelled within the time that Spot was barking.*

(Spot barked  $\wedge$  Dick yelled)  $\wedge$   $\neg$ (Dick yelled  $\wedge_{bb}$  Spot barked)  
 $\wedge$   $\neg$ (Spot barked  $\wedge_{ee}$  Dick yelled)

ANALYSIS We have four possibilities for the times assigned to “Spot barked” and “Dick yelled” that can make this true:



All are covered by the formalization here. We can define a connective to formalize within the time that:

$$A \wedge_{in} B \equiv_{df} (A \wedge B) \wedge \neg(A \wedge_{bb} B) \wedge \neg(B \wedge_{ee} A)$$

11. *Dick yelled during the time that Spot barked.*

ANALYSIS This is ambiguous. We have (at least) two choices:

Dick yelled within the time that Spot barked.

Formalization as for Example 10.

Dick yelled through all the time that Spot yelled, beginning before and ending after.

$$(\text{Spot barked} \wedge \text{Dick yelled}) \wedge (\text{Dick yelled} \wedge_{\text{bb}} \text{Spot barked}) \\ \wedge (\text{Spot barked} \wedge_{\text{ee}} \text{Dick yelled})$$

12. *There was a time when both Dick yelled and Spot barked.*

$$(\text{Spot barked} \wedge \text{Dick yelled}) \wedge [(\text{Dick yelled} <_{\text{bb}} \text{Spot barked}) \\ \wedge (\text{Spot barked} <_{\text{be}} \text{Dick yelled})] \vee [(\text{Spot barked} <_{\text{bb}} \text{Dick} \\ \text{yelled}) \wedge (\text{Dick yelled} <_{\text{be}} \text{Spot barked})] \vee (\text{Spot barked} \wedge_{\text{in}} \\ \text{Dick yelled}) \vee (\text{Dick yelled} \wedge_{\text{in}} \text{Spot barked})$$

ANALYSIS Though the ordinary English involves a quantification over time, we need no quantification in the formalization to formalize overlapping times.

13. *Dick yelled, then Spot began barking, and then Spot finished barking before Dick stopped yelling.*

$$(\text{Dick yelled} \wedge_{\text{bb}} \text{Spot barked}) \wedge (\text{Spot barked} \wedge_{\text{ee}} \text{Dick yelled})$$

ANALYSIS It seems that we have four propositions here:

Dick yelled.

Spot began barking.

Spot finished barking.

Dick stopped yelling.

But we're treating "began," "finished," "stopped," and other words like those as modifiers of verbs, incorporated into the connectives rather than within the propositions since their role is to relate times of propositions. Moreover, we understand the second and third to be modifications, in this context, of the same proposition, "Spot barked."

14. *Dick yelled while Spot was barking.*

ANALYSIS The use of "while" as a connective in English is ambiguous. We have (at least) the following choices:

There was a time when both Dick yelled and Spot barked.

Formalized as in Example 12.

Dick yelled at the same time that Spot barked.

Formalized as in Example 9.

Dick yelled within the time that Spot barked.

Formalized as in Example 10.

Dick yelled through all the time that Spot barked.

$$(\text{Dick yelled} \wedge \text{Spot barked}) \wedge \neg(\text{Spot barked} \wedge_{\text{bb}} \text{Dick yelled}) \\ \wedge \neg(\text{Spot} \wedge_{\text{ee}} \text{Dick yelled})$$

15. *Dick will cook dinner when Zoe arrives.*

Zoe will arrive  $\wedge_{\text{eb}}$  Dick will cook dinner

ANALYSIS We use the present tense in “Zoe arrives” to talk about some future time, formalizing the example to mean that Dick will start cooking only after Zoe arrives.

16. *Dick will start cooking dinner the moment when Zoe arrives.*

$$(\text{Dick will cook dinner} \wedge \text{Zoe will arrive}) \wedge \\ (\text{Zoe will arrive} \approx_{\text{eb}} \text{Dick will cook dinner})$$

ANALYSIS This works regardless of whether the times assigned to the propositions are open or closed intervals.

17. *Dick will begin cooking as soon as Zoe arrives home.*

Formalization the same as for Example 16.

ANALYSIS Even in ordinary speech we need not talk of quantification over times.

18. *Sometime between when Zoe arrived and Dick started cooking, Spot ran away.*

$$((\text{Zoe arrived}) \wedge_{\text{eb}} (\text{Spot ran away})) \wedge (\text{Spot ran away} \wedge_{\text{eb}} \\ \text{Dick cooked})$$

ANALYSIS We can formalize the time of a proposition being strictly between the times of two other propositions by defining a new connective:

$$\text{SB}(A, B, C) \equiv_{\text{df}} (A <_{\text{eb}} B) \wedge (B <_{\text{eb}} C)$$

Alternatively, we can formalize that the time of a proposition comes between the times of two other propositions though not strictly by defining a new connective:

$$B(A, B, C) \equiv_{\text{df}} \neg(B <_{\text{eb}} A) \wedge \neg(C <_{\text{eb}} B)$$

19. *There was some time between when Zoe arrived and Dick started cooking.*

Not formalizable?

ANALYSIS Unless we know a proposition that is true whose time is between that of the time assigned to “Zoe arrived” and the time assigned to “Dick cooked” we can’t point to any time that comes between those. We can only say that the first is true of a time ending before than the start of the second with “(Zoe arrived)  $\wedge_{\text{eb}}$  (Dick cooked)”. Whether there are any “times” between those — not just in our model but in “reality” — is connected to the question of whether we can point to such a time with a proposition that describes a (possible) experience, as discussed in Section 6.

20. *Spot barked and then Dick yelled. But before that, Zoe shouted. This was all after Suzy arrived and before Tom arrived.*

$$((\text{Suzy arrived}) \wedge_{\text{eb}} ((\text{Zoe shouted}) \wedge_{\text{eb}} (\text{Spot barked} \wedge_{\text{eb}} \text{Dick yelled}))) \wedge_{\text{eb}} (\text{Tom arrived})$$

ANALYSIS We can take account of the relative times of any number of propositions.

21. *If Dick studied, then Zoe will cook.*

ANALYSIS We can read “if ... then ...” classically to formalize the example as:

$$\text{Dick studied} \supset \text{Zoe will cook}$$

Alternatively, we can read “then” in “if ... then ...” as indicating later in time. In that case we can formalize the example:

$$(\text{Dick studies} \supset \text{Zoe cooks}) \wedge (\text{Dick studies} <_{\text{eb}} \text{Zoe cooks})$$

We can formalize “if ... then (later) ...” by introducing a new connective:

$$A \supset_{\text{eb}} B \equiv_{\text{df}} (A \supset B) \wedge (A <_{\text{eb}} B)$$

22. *The temperatures in New Mexico in 2012 were the highest they have been since the last ice age.*

Not formalizable.

ANALYSIS To formalize comparisons of times (“the last”) as well as temperatures we need a more ample language, such as that of predicate logic.

23. *When Zoe cooks, Dick washes up.*

Not formalizable.

ANALYSIS The example is meant to be true of all times. That is, “Zoe cooks  $\supset$  Dick washes up” is always true. But we don’t want all of time to be assigned to the union of the times assigned to “Zoe cooks” and “Dick washes up.” It seems that to formalize this example we need to quantify over times, which we can do in predicate logic. We can make the implicit quantification explicit by rewriting the example as “Whenever Zoe cooks, Dick washes up.”

24. *Tom shot, skinned, and butchered a deer.*

Not formalizable.

ANALYSIS The first comma and the word “and” here are meant as “and then.” But we can’t formalize this as “(Tom shot a deer  $\wedge_{\text{eb}}$  Tom skinned a deer)  $\wedge_{\text{eb}}$  Tom butchered a deer” because the example clearly means that it’s the same deer that Tom shot, skinned, and butchered. We need a way to formalize cross-referencing for this example, which is possible in predicate logic even though no quantification is involved.

25. *Dick talked only when Spot wasn’t barking.*

Not formalizable.

ANALYSIS We can take “Spot barked” as an atomic proposition and assign a time to it. Then “ $\neg$ (Spot barked)” is meant as its contradictory and hence is about the same time. We can talk of times when Spot wasn’t barking by considering propositions that are true of times before and times after the time when Spot barked. But “Dick talked” is an atomic proposition, and hence we can only assert either “Dick talked  $\wedge_{\text{eb}}$  Spot barked,” in which case we’re asserting that Dick talked before Spot barked, or “Spot barked  $\wedge_{\text{be}}$  Dick talked,” in which case we’re

asserting that Dick talked after Spot barked. Or we could take “(Dick talked)<sub>1</sub>” and “(Dick talked)<sub>2</sub>” to be distinct propositions and assert:

$$((\text{Dick talked})_1 \wedge_{\text{eb}} \text{Spot barked}) \vee \text{Spot barked} \wedge_{\text{be}} (\text{Dick talked})_2$$

But that doesn’t formalize that we intend “Dick talked” to be about lots of times, perhaps some before Spot barked and some after Spot barked.

The problem is that the complement of the time assigned to “Spot barked” or of “Spot barked  $\wedge$  Dick talked” need not be an interval or a union of intervals. In Section 9 we’ll look at how we might extend our logic to allow for a temporal negation.

26. *Spot will bark and then Dick yelled.*

$$\text{Spot will bark} \wedge_{\text{eb}} \text{Dick yelled}$$

ANALYSIS This can be true in a model since we can assign an interval of time to “Spot will bark” that comes before the time assigned to “Dick yelled.” Yet the example can’t be true, since the tenses indicate that “Dick yelled” is about a time in the past and “Spot will bark” is about a time in the future. To ensure that this is formalized as an anti-tautology we should take account of tenses directly in the language.<sup>9</sup>

Alternatively, we can stipulate an informal convention that if A is in the past or present tense and B is in the future tense, then we use  $A \wedge_{\text{eb}} B$  as part of a formalization involving A and B, with similar conditions for other tenses. Let’s look at how we might do that.

## 8. Past, Present, and Future

Part of our conception of time is of the past, present, and future. The present is *right now*. As you are reading this perhaps you’ll understand that to mean the present is exactly as we were writing this. Or perhaps you’ll understand it to mean that the present is exactly as you are reading this, which would have been in the future as we were writing. In any case, both those “times” are past now.

We have a subjective sense of the present or *right now*. But in our reasoning we talk of the present as any time that’s used to split propositions into those meant to be about a time before and those meant to be about a time after. The present when we are writing this, the present

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<sup>9</sup> A way to do that is sketched in [4] and is developed in [5].

of an essay written in 1774, the present of a science-fiction novel set in 2095, the present since the last ice age, all can be understood by us as “the present” in a conversation.

Given some collection of atomic propositions with which we want to reason and the semi-formal language of those, there are various ways we can talk about the world at a particular time, whether that’s what we want to call the present or simply some other time we want to distinguish. We do so by giving a description that is meant to be about that time, say “Spot barked  $\wedge$  Dick yelled.” Then the fullest description we can give of that time would be the collection of atomic propositions that are true of that time:  $\{p : p \text{ is atomic, and } \mathbf{v}(p) = \mathbf{T}, \text{ and } \mathbf{t}(p) \subseteq \mathbf{t}(\text{Spot barked } \wedge \text{ Dick yelled})\}$ . If we can choose one particular proposition, whether compound or atomic, as a description of the world at “the present,” then we can divide all propositions into those that are meant to be about times before that and those that are meant to be about times after that.

Explicitly, given a realization let  $N$  be a wff of the semi-formal language that we choose to be a description of the “the whole” present. Then we can define:

$$\text{Past}(A) \equiv_{\text{df}} A <_{\text{eb}} N$$

$$\text{In a model, } \mathbf{v}(\text{Past}(A)) = \mathbf{T} \text{ iff } \mathbf{v}(A) = \mathbf{T} \text{ and } \mathbf{t}(A) <_{\text{eb}} \mathbf{t}(N)$$

$$\text{Present}(A) \equiv_{\text{df}} \neg(A <_{\text{bb}} N) \wedge \neg(N <_{\text{ee}} A)$$

$$\text{In a model, } \mathbf{v}(\text{Pres}(A)) = \mathbf{T} \text{ iff } \mathbf{t}(A) \subseteq \mathbf{t}(N)$$

$$\text{Future}(A) \equiv_{\text{df}} N <_{\text{eb}} A$$

$$\text{In a model, } \mathbf{v}(\text{Future}(A)) = \mathbf{T} \text{ iff } \mathbf{t}(N) <_{\text{eb}} \mathbf{t}(A)$$

Note that  $\text{Past}(A)$  means  $A$  is about some time in the past.  $\text{Past}(A)$  does not mean  $A$  is about all of the past. When we say “the past” or “the future” we needn’t be construed as talking about all of the past or all of the future as if those were some things. Similarly, when we say “the present” we needn’t be talking of all of the time we consider to be the present.

27. *Tom had a dog.*

$$\text{Tom had a dog} \wedge \text{Past}(\text{Tom had a dog})$$

ANALYSIS Because of the tensed form of “to have” we know that this is meant to be about some time in the past. In a narrative we can usually identify one or several propositions as being about the “now” of





it. Suppose, in this case, we know that “Spot is Dick and Zoe’s dog,” and “Suzy loves Tom,” and “Ralph is a dog” are all about the present and are ample enough together to cover the entire present of the narrative. Then we can take:

$N \equiv_{df} \text{Spot is Dick and Zoe's dog} \wedge \text{Suzy loves Tom} \wedge \text{Ralph is a dog}$

We needn’t agree that  $N$  is true. Then we can formalize the example as an assertion not only that the proposition is true but is also about the past.

28. *Tom will have a dog.*

$\text{Tom will have a dog} \wedge \text{Future}(\text{Tom will have a dog})$

ANALYSIS Taking  $N$  to be as in the previous example, we have the formalization.

29. *Tom has a dog.*

$\text{Tom has a dog} \wedge \text{Present}(\text{Tom has a dog})$

ANALYSIS We formalize with the same assumption about the time of the present as in the previous two examples.

30. *Spot will bark and then Dick yelled.*

$(\text{Spot will bark} \wedge_{eb} \text{Dick yelled}) \wedge \text{Future}(\text{Spot will bark})$   
 $\wedge \text{Past}(\text{Dick yelled})$

ANALYSIS Using the same conventions about the time of the present, the formalization here is an anti-tautology. In a model, we can’t have  $t(N) < t(\text{Spot will bark})$ , and  $t(\text{Dick yelled}) < t(N)$ , and  $t(\text{Spot will bark}) < t(\text{Dick yelled})$ , since  $<$  is transitive and linear.

We can take account of tenses in our formalizations by first agreeing that the times of a particular (small) collection of atomic propositions cover the present. Then relative to the conjunction of those we can define  $\text{Past}(A)$ ,  $\text{Present}(A)$ , and  $\text{Future}(A)$ .<sup>10</sup> We could, perhaps, be more subtle in devising ways to distinguish, say, the past perfect from the past. But all these conventions are about how to formalize certain kinds of ordinary language propositions. They are not directly formalizations of

<sup>10</sup> These bear a superficial resemblance to operators used in the tradition of Arthur Prior, but they are quite different, as explained in [4].

tenses. In English tenses are modifiers of verbs, and to deal directly with them we need to look at the internal structure of atomic propositions.<sup>11</sup>

## 9. Timeless Propositions

Some say that there are propositions about no time at all. They say that numbers, for example, are abstract objects existing outside space and time. Hence, “ $2 + 2 = 4$ ” is not true of any time at all. It is not true of all times. It is timelessly true.

We can extend our temporal propositional logic to accomodate reasoning based on the assumption that there are such propositions. We add a unary connective  $\emptyset$  to the language of our logic, where  $\emptyset A$  is meant to be an assertion that  $A$  is true and  $A$  is about no time. We modify our models to allow that the empty set is also an interval, so that  $t(A) = \emptyset$  is allowed. We take the valuation:

$$v(\emptyset A) = T \text{ iff } v(A) = T \text{ and } t(A) = \emptyset$$

We can define a further connective:

$$\begin{aligned} \underline{\emptyset} A &\equiv_{\text{df}} \emptyset(\neg(A \wedge \neg A)) \\ v(\underline{\emptyset} A) &= T \text{ iff } t(A) = \emptyset \end{aligned}$$

Call this system  $\mathbf{TL}_{\mathbf{PC}} + \emptyset$ . With it we can formalize some talk of abstract things, such as numbers and beauty, along with talk about dogs and cats. We’ll leave to those who have developed a greater intuition than we have for the nature of abstract things to provide examples and analyses.

It might seem simple to axiomatize this logic. We can modify each scheme  $A$  of the axiomatization of  $\mathbf{TL}_{\mathbf{PC}}$  other than those for  $\mathbf{PC}$  itself to read:

$$(\bigwedge_{p \text{ in } A} \neg \underline{\emptyset} p) \supset A$$

We can also add an axiom scheme:

$$\begin{aligned} \underline{\emptyset} A \supset [ &\neg(A <_{\text{bb}} B) \wedge \neg(B <_{\text{bb}} A) \wedge \neg(A <_{\text{ee}} B) \wedge \neg(B <_{\text{ee}} A) \\ &\wedge \neg(A <_{\text{be}} B) \wedge \neg(B <_{\text{be}} A) \wedge \neg(A <_{\text{eb}} B) \wedge \neg(B <_{\text{eb}} A) \\ &\wedge \neg(A \approx_{\text{bb}} B) \wedge \neg(A \approx_{\text{ee}} B) \wedge \neg(A \approx_{\text{be}} B) \wedge \neg(B \approx_{\text{be}} A) ] \end{aligned}$$

However, the axiomatization of  $\mathbf{TL}_{\mathbf{PC}}$  uses in an essential way that  $\approx_{\text{bb}}$ ,  $\approx_{\text{be}}$ ,  $\approx_{\text{ee}}$  are defined connectives.

<sup>11</sup> See the discussion in [4].



## 10. Omnitemporal Propositions and Temporal Negation

### 10.1. Omnitemporal propositions

A proposition such as “Electrons have spin” is meant to be a true description of all times. At any time whatsoever, every electron has spin. We call propositions meant to be true of all times *omnitemporal*.

Suppose we have a realization where  $A$  is meant to be a true description of every time. Then we can assign  $t(A) = \{z : b_{\mathcal{T}} < z < e_{\mathcal{T}}\}$ , which is the collection of all times in our model, since the endpoints of the entire time line are meant as markers only. We then have for every  $p$  in the model:

$$\text{not: } t(p) <_{bb} t(A)$$

$$\text{not: } t(A) <_{ee} t(p)$$

We would like to pick out such propositions in our reasoning. We could do so if we introduce the following connective:

*omnitemporal om*

$$v(\text{om } A) = \text{T iff } v(A) = \text{T and } t(A) = \{z : b_{\mathcal{T}} < z < e_{\mathcal{T}}\}$$

This cannot be defined in  $\mathbf{TL}_{PC}$  because given any scheme  $S(A)$ , even with parameters,  $t(S(A))$  will be the union of only a finite number of intervals, those assigned to the propositional variables in  $S(A)$ , yet another propositional variable could be assigned an interval that lies outside those.

### 10.2. Temporal negation

We take  $\neg A$  to be the contradictory of  $A$  and hence to be about the same time as  $A$ . So “Spot barked” and “ $\neg$ (Spot barked)” are about the same time. But then how can we formalize the following?

Dick talked only during the time that Spot didn’t bark.

We need a way to talk about all times other than those in the interval of the time assigned to “Spot barked.” We could do so if we introduce the following connective, where “ $-$ ” indicates the set-theoretic complement of the given set:

*temporal negation n*

$$t(n(A)) = -t(A)$$

$$v(n(A)) = \text{T iff } v(A) = \text{F}$$

This can't be defined in  $\mathbf{TL}_{PC}$  because given any scheme  $S(A)$  with parameters  $q_1, \dots, q_n$ ,

$$t(S(A)) = \bigcup \{t(p) : p \text{ appears in } A\} \cup \{t(q_i) : i \geq 1\} \supseteq t(A).$$

### 10.3. Temporal intersection

Consider:

Dick talked only during the time that both Zoe was talking and Spot was sleeping.

For this to be true we need that each of “Dick talked,” “Zoe was talking,” and “Spot was sleeping” are true and the time assigned to “Dick talked” is contained in both the time assigned to “Zoe was talking” and “Spot was sleeping.” We could formalize this if we introduce in addition to the temporal negation connective the following connective:

$$\begin{aligned} & \textit{temporal intersection } X \\ t(A \times B) &= t(A) \cap t(B) \\ v(A \times B) &= T \text{ iff } v(A) = v(B) = T \end{aligned}$$

Just as for the connective  $\mathbf{n}$ , this cannot be defined in  $\mathbf{TL}_{PC}$ .

### 10.4. Extending $\mathbf{TL}_{PC}$ ?

We have seen limitations on what we can formalize with  $\mathbf{TL}_{PC}$ . It might seem that we could simply add the connectives  $\mathbf{om}$ ,  $\mathbf{on}$ , and  $\mathbf{X}$  to the language and then use the evaluations given above. Then we could define any connective whose evaluation or time assignment is given in terms of the truth-values of the atomic propositions in it and any combination using union, intersection, and complement of the times assigned to the atomic propositions in it. But if we do, then the time assigned to a compound wff need no longer be a union of intervals, so that comparing the times of compound propositions will not be straightforward.

## Summary

We set out to formalize how to reason taking account of time when the only structure of propositions we would pay attention to is how they are built from other propositions using sentential connectives. We take a proposition to be a sentence that is a description of the world at

some particular time. Thus, a proposition has two semantic values: a truth-value and a time it is meant to be about. We saw how to extend those semantic values to compound propositions that use the standard propositional connectives as well as connectives that formalize the “before” relation in terms of the beginning point and ending point of the intervals of times assigned to their compounds. This allowed us to give an axiomatization for which any complete and consistent collection of propositions has a model. From that we have that our logic can be understood as formalizing a subjective conception of time in which the timeline can be derived from an ordering of propositions describing our experiences. A series of examples of formalizing ordinary language sentences showed how the logic can be applied. Those also showed limitations on what we can formalize using this logic. We saw that we could formalize some reasoning with tenses by taking a particular proposition as being about the present and comparing others to it. We saw how we could extend our logic to allow for reasoning about timeless propositions. We considered also connectives that formalize omnitemporal propositions, temporal negation, and temporal intersection, but it was not clear how to modify our formal system to incorporate those.

This is a work in progress, a beginning only.

**Acknowledgements.** We are grateful to Walter Carnielli and two anonymous referees for suggestions that have improved this paper.

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