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QUANTIFIED TEMPORAL ALETHIC-DEONTIC LOGIC

Abstract. The purpose of this paper is to describe a set of quantified temporal alethic-deontic systems, i.e., systems that combine temporal alethic-deontic logic with predicate logic. We consider three basic kinds of systems: constant, variable and constant and variable domain systems. These systems can be augmented by either necessary or contingent identity, and every system that includes identity can be combined with descriptors. All logics are described both semantically and proof theoretically. We use a kind of possible world semantics, inspired by the so-called T × W semantics, to characterize them semantically and semantic tableaux to characterize them proof theoretically. We also show that all systems are sound and complete with respect to their semantics.

Keywords: Quantified modal logic; T × W logics; temporal logic; deontic logic; semantic tableaux; Barcan formulas, possibilism; eternalism; actualism; presentism; Graham Priest.

1. Introduction

The purpose of this paper is to describe a set of quantified temporal alethic-deontic systems, i.e., systems that combine temporal alethic-deontic logic with predicate logic. We consider three basic kinds of systems: constant, variable and constant and variable domain systems. These systems can be augmented by either necessary or contingent identity, and every system that includes identity can be combined with descriptors. All logics are described both semantically and proof theoretically. We use a kind of possible world semantics, inspired by the so-called T × W semantics, to characterize them semantically and semantic

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tableaux to characterize them proof theoretically. We also show that all systems are sound and complete with respect to their semantics.

Many philosophers and logicians have developed temporal alethic-deontic logics (e.g. Chellas [13], Bailhache [1, 2], van Eck [15], Thomason [39, 40], Åqvist and Hoepelman [46], Åqvist [45], Bartha [5], Horty [25], Belnap, Perloff and Xu [6], Brown [9, 10]). The systems we describe in this essay are extensions of the systems introduced in Rönnedal [37], which includes information about non-quantified temporal alethic-deontic logic and some further relevant references to the literature (see also [38], [41] and [44]).

Some brief remarks about quantified modal logic can be found in Lewis [32] and Lewis and Langford [33]. The first systematic discussions were presented by Barcan (Barcan-Marcus) [3, 4], and Carnap [11, 12]. Since the 50s, several philosophers and logicians have been interested in modal predicate logic, e.g. Kanger [27], Kripke [28, 29, 30, 31] and Hintikka [21, 22, 23, 24]. Other early contributions are [8, 17, 26, 34, 35, 42].[1] However, the literature contains surprisingly few attempts to combine predicate logic with systems including both temporal and alethic concepts and even fewer attempts to combine predicate logic with systems that contain temporal, alethic and deontic concepts (but see e.g. [15, 43]). As far as I know, all systems of this kind are axiomatic. No one seems to have developed tableau systems that combine predicate logic with temporal alethic-deontic logic. The systems discussed in this paper are therefore entirely new, as far as I know. So, I think that the present study is technically well motivated.

I also think that the paper is philosophically well motivated. Unfortunately we cannot consider all the philosophical reasons why. But I will briefly mention three.

(i) The systems can be used to analyze many interesting sentences or principles in natural languages, for instance some normative propositions. And we can use them to make important distinctions. E.g. consider the following claims: (1) "*Everyone ought always* to be honest" ("*Everyone ought always* to act rationally", "*Everyone ought always* to do the action that will bring about the most good"), (2) "*Everyone always ought* to be honest" ("*Everyone always ought* to act rationally", "*Everyone always ought* to do the action that will bring about the most good"). What does "everyone", "always" and "ought" mean in these

[1] Introductions to quantified modal logic can be found in e.g. [16, 18, 19, 26, 36].

principles? Does "everyone" mean everyone that exists now, everyone that exists or has existed, everyone that exists, has existed or will exist, or every possible person? Are there any semantic differences between the expressions "everyone ought always", "everyone always ought", "always everyone ought" etc.? Is "*Everyone ought always* to be honest" logically equivalent with "*Everyone always ought* to be honest". In our systems we can investigate such questions and give very precise answers.

If "everyone" is interpreted as every possible person (and we suppose our quantifiers are restricted to persons), we can symbolize (1) as (1′) $\Pi x \mathbf{O} \mathsf{A} Hx$ (or as (1″) $\Pi x \mathbf{O} \mathsf{G} Hx$ if "always" means always in the future), and (2) as (2′) $\Pi x \mathsf{A} \mathbf{O} Hx$ (or as (2″) $\Pi x \mathsf{G} \mathbf{O} Hx$ if "always" means always in the future). In fact, (1′) is not equivalent with (2′) (and (1″) is not equivalent with (2″)) (see Theorem 7). So, it turns out to be important exactly how we formulate our normative principles.

(ii) The systems can be used to shed some light on some philosophical problems and debates, for instance the discussion between possibilists and actualists, and between presentists and eternalists, even though the systems in themselves don't solve the problems. According to actualists (at least according to one interpretation) everything exists. Possibilists deny this, according to them there are things that do not exist. Is actualism true or not? This seems to depend on what "everything" and "exists" mean? Given some interpretations this thesis is true; given some it is false. Presentism is the view that only the present is real, while eternalism is the view that past and future times are just as real as the present time. More precisely, presentism (at least according to one interpretation) is the doctrine that everything (presently) exists; while according to eternalism past or future individuals are just as real as present individuals: they just happen to exist prior to the present, or after the present (non-present objects are like spatially distant objects; they exist, but not where we are). According to growing block theorists the spatio-temporal world is a growing four-dimensional block, where past and present (but no future) objects exist. As time goes by, new entities come into existence and the universe grows by accretion. Is presentism true or not? Is eternalism or the growing block theory true? That seems to depend on what we mean by "everything" and "(presently) exists". Given some interpretations presentism is true, given some it is false.

We will, in fact, distinguish between 27 different interpretations of the concept "everything" and between 27 different interpretations of the concept "exists" in this paper (see below definitions). Some of these seem more plausible than others, but no one is obviously "the correct one". If "everything" means everything that exists now, has existed or will exist and "exists" means exists now, then presentism isn't true. But if "everything" means everything that exists now and "exists" means everything that exists now, or if "everything" means everything that exists now, has existed or will exist, and "exists" means everything that exists now, has existed or will exist, then presentism is true.

(iii) The systems can be used to analyze and evaluate many arguments that cannot be adequately analyzed in other systems. Consider, for instance, the following argument. Everyone ought always (in the future) to be honest. Hence, no one is permitted ever (in the future) to be dishonest. This argument seems clearly valid, given some natural interpretations of the concepts in the premise and conclusion. But to prove this, it seems that we must have a system that includes deontic and temporal operators, as well as quantifiers that can be used to symbolize the expression "everyone". And we can in fact prove that it is valid in our systems. $\neg\Sigma x \mathsf{P}\mathsf{F}\neg Hx$ is derivable from $\Pi x \mathsf{O}\mathsf{G}Hx$ in all our constant and constant and variable domain systems.

The paper is divided into 6 main sections. Section 2 deals with the syntax and Section 3 with the semantics of our systems. In Section 4 I describe the proof theory of our logics and Section 5 includes some examples of theorems. Finally, Section 6 contains soundness and completeness proofs for every system.

2. Syntax

Our languages will be constructed from the following *alphabet*:

(i) a set of variables x_0, x_1, x_2, x_3, $\ldots$;
(ii) a set of (non-temporal, rigid) constants c_0, c_1, c_2, c_3, $\ldots$;
(iii) a set of (non-temporal, non-rigid) constants (descriptors) α_0, α_1, α_2, α_3, $\ldots$;
(iv) a set NT of names of times (temporal constants) t_0, t_1, t_2, t_3 $\ldots$;
(v) for every natural number $n > 0$, n-place predicate symbols P^0_n, P^1_n, P^2_n, P^3_n, $\ldots$;
(vi) the monadic existence predicate E;
(vii) the dyadic identity predicate $=$;
(viii) the primitive truth-functional connectives $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (material implication), and $\leftrightarrow$ (material equivalence);

(ix) the alethic operators $\mathbf{U}$, $\mathbf{M}$, $\Box$, and $\Diamond$;
(x) the temporal operators R (followed by a name in NT), A, S, G, H, F, and P;
(xi) the deontic operators $\mathbf{O}$ and $\mathbf{P}$;
(xii) the "possibilist" quantifiers Π and Σ;
(xiii) the "actualist" quantifiers $\forall$ and $\exists$;
(xiv) the brackets (and).

I will use x, y and z for arbitrary variables, a, b, c for arbitrary (non-temporal) rigid constants, α, β, γ for arbitrary descriptors, and t for an arbitrary temporal constant (name in NT) (possibly with primes or subscripts). Note that I also use s and t (with or without primes or subscripts) for arbitrary terms. I will use F_n, G_n, H_n for arbitrary n-place predicates and I will omit the subscript if it can be read off from the context.

The symbols Π and Σ (resp. $\forall$ and $\exists$) are called *"possibilist"* (resp. *"actualist"*) *quantifiers*, because it is most natural to take them to vary over the class of every possible (resp. existing) object.[2]

Languages. We will consider several languages in this essay. They are all constructed from the following clauses.

Terms:

(i) Any (non-temporal, rigid) constant or variable is a term.
(ii) Any (non-temporal, non-rigid) constant (descriptor) is a term.
(iii) Nothing else is a term.

Formulas:

(i) If $t_1, \ldots, t_n$ are any terms and P is any n-place predicate, $Pt_1 \ldots t_n$ is an atomic formula.
(ii) If t is a term, $\mathrm{E}t$ ("t exists") is an atomic formula.
(iii) If s and t are terms, then $s = t$ ("s is identical with t") is an atomic formula.
(iv) If A and B are formulas, so are $\neg A$, $(A \wedge B)$, $(A \vee B)$, $(A \to B)$ and $(A \leftrightarrow B)$.
(v) If A is a formula, then $\mathbf{U}A$ ("it is universally (or absolutely) necessary that A"), $\mathbf{M}A$ ("it is universally (or absolutely) possible that

[2] Although most natural, I do believe that there are interpretations of the quantifiers on which Π and Σ do vary over existing things, and $\forall$ and $\exists$ over a subset of the existing things. However, I will not say anything more about this in the present paper.

A"), $\Box A$ ("it is (historically) necessary (or settled) that A"), and $\Diamond A$ ("it is (historically) possible (or open) that A") are formulas.

(vi) If A is a formula, so are $\mathsf{A}A$ ("It is always the case that A"), $\mathsf{S}A$ ("It is sometimes the case that A"), $\mathsf{G}A$ ("it is always going to be the case that A"), $\mathsf{H}A$ ("it has always been the case that A"), $\mathsf{F}A$ ("it will some time in the future be the case that A"), and $\mathsf{P}A$ ("it was some time in the past the case that A").

(vii) If A is a formula, then $\mathbf{O}A$ ("it ought to be the case that A") and $\mathbf{P}A$ ("it is permitted that A") are formulas.

(viii) if A is a formula and t is in NT, then $\mathsf{R}tA$ ("it is realized at time t that A") is a formula.

(ix) If A is any formula and x is any variable, then ΠxA ("For every (possible) x: A") and ΣxA ("For some (possible) x: A") are formulas.

(x) If A is any formula and x is any variable, then $\forall xA$ ("For every (existing) x: A") and $\exists xA$ ("For some (existing) x: A") are formulas.

(xi) Nothing else is a formula.

The letters A, B, C stand for arbitrary formulas, and Γ, Φ for sets of formulas. The concepts of bound and free variable, open and closed formula, are defined in the usual way. $(A)[t/x]$ is the formula obtained by substituting t for every free occurrence of x in A. The definition is standard. Brackets around formulas are usually dropped if the result is not ambiguous. In some languages it is possible to define E, or $\forall xA$ and $\exists xA$ (see below). In [37] A and S were introduced by definitions. For our purposes in this essay it is more natural to take them to be primitive.

All constant domain systems include the possibilist quantifiers as primitive, but not the actualist quantifiers. All variable domain systems include the actualist quantifiers as primitive, but not the possibilist quantifiers. All systems that combine constant and variable domains include both the possibilist and actualist quantifiers.

Definitions. (i) Alethic operators $\Diamond\!\!\!\!\!-\,A$ ("it is (historically) impossible that A") $:= \neg\Diamond A$, $\boxminus A$ ("It is (historically) unnecessary (non-necessary) that A") $:= \neg\Box A$, ∇A ("it is (historically) contingent that A") $:= \Diamond A \wedge \Diamond\neg A$, $\triangle A$ ("it is (historically) non-contingent that A") $:= \neg\nabla A$ (or $\Box A \vee \Box\neg A$), $A \Rightarrow B := \Box(A \to B)$, $A \Leftrightarrow B := \Box(A \leftrightarrow B)$.

(ii) The deontic operator $\mathbf{F}A$ ("it is forbidden that A") $:= \neg\mathbf{P}A$.

(iii) The temporal operators $[\mathsf{G}]A := A \wedge \mathsf{G}A$, $\langle\mathsf{F}\rangle A := \neg[\mathsf{G}]\neg A$ (or $A \vee \mathsf{F}A$), $[\mathsf{H}]A := A \wedge \mathsf{H}A$, $\langle\mathsf{P}\rangle A := \neg[\mathsf{H}]\neg A$ (or $A \vee \mathsf{P}A$). Let $\odot$ be $[\mathsf{H}]$, H, P, $\langle\mathsf{P}\rangle$, $[\mathsf{G}]$, G, F, or $\langle\mathsf{F}\rangle$. Then $t_{\odot}A := \mathsf{R}t \odot A$.

We shall say that the following operators are *positive modal operators*: $\mathbf{U}$, $\mathbf{M}$, A, S, $[\mathsf{H}]$, $\langle\mathsf{P}\rangle$, H, P, $[\mathsf{G}]$, $\langle\mathsf{F}\rangle$, G, F, $\Box$, $\Diamond$, $\mathbf{O}$, $\mathbf{P}$, $\mathsf{R}t$, $t_{[\mathsf{H}]}$, $t_{\langle\mathsf{P}\rangle}$, t_{H}, t_{P}, $t_{[\mathsf{G}]}$, $t_{\langle\mathsf{F}\rangle}$, t_{G}, t_{F}. So, there are 25 different positive modal operators. $\mathbf{U}$, A, $[\mathsf{H}]$, H, $[\mathsf{G}]$, G, $\Box$, $\mathbf{O}$, $\mathsf{R}t$, $t_{[\mathsf{H}]}$, t_{H}, $t_{[\mathsf{G}]}$, and t_{G} are *necessity-like operators*. $\mathsf{R}t$ and all other positive modal operators are *possibility-like*. So, $\mathsf{R}t$ is both necessity- and possibility-like.

(iv) Let $\odot$ be a positive modal operator (or empty). Then $\mathrm{E}_{\odot}t := \odot \mathrm{E}t$ for every t, $\forall_{\odot}xA := \Pi x(\mathrm{E}_{\odot}x \rightarrow A) = \Pi x(\odot \mathrm{E}x \rightarrow A)$, and $\exists_{\odot}xA := \Sigma x(\mathrm{E}_{\odot}x \wedge A) = \Sigma x(\odot \mathrm{E}x \wedge A)$. Note that when $\odot$ is empty, $\forall_{\odot}xA = \forall xA$ and $\exists_{\odot}xA = \exists xA$. So, $\forall xA = \Pi x(\mathrm{E}x \rightarrow A)$ and $\exists xA = \Sigma x(\mathrm{E}x \wedge A)$. Consequently, $\forall$ is definable in terms of Π and E, and $\exists$ is definable in terms of Σ and E.

(v) $\mathrm{E}x := \exists yy = x$, for all x. So, E is definable in terms of $\exists$ and $=$.

(vi) $\mathbf{E}x := \Sigma yy = x$, for all x.

The symbols E, $\mathbf{E}$ and all of the form $\mathrm{E}_{\odot}$ are called *existence predicates*; Π, $\forall$ and all symbols of the form $\forall_{\odot}$ are called (*universal*) *quantifiers*; Σ, $\exists$ and all symbols of the form $\exists_{\odot}$ — (*particular*) *quantifiers*. Accordingly, since there are 25 positive modal operators, there are 27 existence predicates, 27 universal quantifiers, and 27 particular quantifiers. The existence predicates can be used to obtain 27 possible interpretations of what we mean by "exists", the universal quantifiers 27 possible interpretations of what we mean by "everything", and the particular quantifiers 27 possible interpretations of what we mean by "something". Σ is the dual of Π, $\exists$ is the dual of $\forall$, and $\exists_{\odot}$ is the dual of $\forall_{\odot}$ (i.e., $\exists_{\mathbf{M}}$ is the dual of $\forall_{\mathbf{M}}$, $\exists_{[\mathsf{H}]}$ is the dual of $\forall_{[\mathsf{H}]}$ etc.). If Π and Σ are taken to vary over all possible objects and $\forall$ and $\exists$ over all existing objects, $\mathrm{E}x$ is most naturally read "x exists" and $\mathbf{E}x$ as "x is a possible object" (not as "x exists"). A presentist might want to use E to symbolize the English word "exists", an eternalist the symbol E_{S}, a growing block theorist $\mathrm{E}_{\langle\mathsf{P}\rangle}$ etc.

Let X and Y be two positive modal operators. Then if X is the dual of Y, then Y is the dual of X. $\mathbf{M}$ is the dual of $\mathbf{U}$, S is the dual of A, etc. Hence, $\mathsf{R}t$ is the dual of $\mathsf{R}t$.

3. Semantics

We will consider several different kinds of quantified temporal alethic-deontic systems in this paper (see Section 4). Every system is either constant, variable or constant and variable. Every system can be combined with either necessary or contingent identity; and all systems that include identity can also be augmented by descriptors. In this section we describe the semantics of these various systems.

3.1. Constant domain semantics

All our constant domain systems include the "possibilist" quantifiers as the only quantifiers. A (*quantified temporal alethic-deontic*) *constant domain model* $\mathcal{M}$ is a relational structure $\langle D, W, T, <, R, S, v\rangle$, where D is a non-empty set of objects, W is a non-empty set of possible worlds, T is a non-empty set of times, $<$ is a binary relation on T ($< \subseteq T \times T$), R and S are two ternary accessibility relations ($R \subseteq W \times W \times T$ and $S \subseteq W \times W \times T$) and v is an interpretation function.

The relation R "corresponds" to the alethic operators $\Box$ and $\Diamond$, the relation $<$ to the temporal operators G, F, H and P, and the relation S to the deontic operators $\mathbf{O}$ and $\mathbf{P}$. Informally, $\tau < \tau'$ says that the time τ is before the time τ' (or that τ' is later than τ), $R\omega\omega'\tau$ says that the possible world ω' is alethically accessible from the possible world ω at time τ, and $S\omega\omega'\tau$ says that ω' is deontically accessible from ω at τ.

The function v assigns each temporal name, t, in NT a time, $v(t)$, in T, each (non-temporal, rigid) constant, c, an element, $v(c)$, of D, and each pair comprising a world-moment pair, $\langle\omega, \tau\rangle$, and an n-place predicate, P, a subset, $v_{\omega\tau}(P)$ (the extension of P in ω at τ), of D^n. In other words, $v_{\omega\tau}(P)$ is the set of n-tuples that satisfy P in the world ω at time τ (in the world-moment pair $\langle\omega, \tau\rangle$). So, the extension of a predicate may change from world-moment pair to world-moment pair and it may be empty at a world-moment pair.

The language of a model $\mathcal{M}$, $\mathcal{L}(\mathcal{M})$, is obtained by adding a constant k_d, such that $v(k_d) = d$, to the language for every member $d \in D$.

Every closed formula, A, is assigned exactly one truth value ($1 =$ True or $0 =$ False), $v_{\omega\tau}(A)$, in each world ω at every time τ (in each world-moment pair $\langle\omega, \tau\rangle$). For closed atomic formulas, for A and S, and for the "possibilist" quantifiers the truth conditions are as follows:

$$v_{\omega\tau}(Pa_1 \ldots a_n) = 1 \quad \text{iff} \quad \langle v(a_1), \ldots, v(a_n)\rangle \in v_{\omega\tau}(P),$$

$$\begin{aligned}
v_{\omega\tau}(AA) = 1 &\text{ iff } \text{for all } \tau' \in T,\ v_{\omega\tau'}(A) = 1,\\
v_{\omega\tau}(SA) = 1 &\text{ iff } \text{for some } \tau' \in T,\ v_{\omega\tau'}(A) = 1,\\
v_{\omega\tau}(\Pi x A) = 1 &\text{ iff } \text{for all } d \in D,\ v_{\omega\tau}(A[k_d/x]) = 1,\\
v_{\omega\tau}(\Sigma x A) = 1 &\text{ iff } \text{for some } d \in D,\ v_{\omega\tau}(A[k_d/x]) = 1.
\end{aligned}$$

The truth conditions for all other sentences are as in [37].

3.2. Variable domain semantics

All our variable domain systems include the existence predicate E and the "actualist" quantifiers as the only quantifiers. A (*quantified temporal alethic-deontic*) *variable domain model* $\mathcal{M}$ is a relational structure $\langle D, W, T, <, R, S, v\rangle$, where D, W, $<$, R, S and v are the same as in the constant domain case, except that for every world-moment pair $\langle \omega, \tau\rangle$, where $\omega \in W$ and $\tau \in T$, v maps $\langle \omega, \tau\rangle$ to a subset, $v(\omega\tau)$, of D. The domain of a world-moment pair, $v(\omega\tau)$ or $D_{\omega\tau}$, is the set of all things we quantify over in this world at this time. It is natural to take this to be the class of all things that exist in this world at this time. For any n-place predicate, P, $v_{\omega\tau}(P) \subseteq D^n$ (not $D^n_{\omega\tau}$), and $v_{\omega\tau}(\mathrm{E})$ is $D_{\omega\tau}$. So, the extension of a predicate at a world-moment pair may change from world-moment pair to world-moment pair, it may include things that are not in the domain of this world-moment pair, and it may be empty at some world-moment pair. Also note that D still is non-empty, but that $D_{\omega\tau}$ may be empty, and that the constants in our language may denote something in a world-moment pair that is not in the domain of this world-moment pair.

The truth conditions for the "actualist" quantifiers are as follows:

$$\begin{aligned}
v_{\omega\tau}(\exists x A) = 1 &\text{ iff } \text{for some } d \in D_{\omega\tau},\ v_{\omega\tau}(A[k_d/x]) = 1,\\
v_{\omega\tau}(\forall x A) = 1 &\text{ iff } \text{for all } d \in D_{\omega\tau},\ v_{\omega\tau}(A[k_d/x]) = 1.
\end{aligned}$$

The truth conditions for other sentences in our language are as in the constant domain case (Section 3.1).

3.3. Constant and variable domain semantics

The constant and variable domain semantics is the same as the variable domain semantics, with the exception that the truth conditions for the possibilist quantifiers are added. In particular, a constant and variable domain model is exactly the same as a variable domain model. The difference between variable domain and constant and variable domain systems lies at the syntactic level: our constant and variable domain systems include both the possibilist and the actualist quantifiers. In a variable domain system we cannot define the possibilist quantifiers. But if we add the existence predicate, E, to a constant domain system, we can define the actualist quantifiers in this system (see definitions).

3.4. Necessary identity semantics

Up until now we have assumed that the identity predicate is not part of our language. We will now see what happens when we add this. We will consider two kinds of semantics for the predicate: necessary and contingent. We can combine constant, variable and constant and variable systems with necessary identity or with contingent identity. According to the necessary identity semantics the denotation of the identity predicate is the same at every world-moment pair in a model, i.e., $v_{\omega\tau}(=) := \{\langle d, d\rangle : d \in D\}$.

We will only add descriptors to systems that contain the identity predicate. Rigid constants have a denotation that is world-moment-invariant. That is why we call them "rigid" and write $v(c)$ and not $v_{\omega\tau}(c)$. Non-rigid constants or descriptors, however, may refer to different things at different world-moment pairs. In a necessary identity system with descriptors, v assigns each descriptor α an object $d \in D$ at each world-moment pair: $v_{\omega\tau}(\alpha)$ is the denotation of α in the world ω at the time τ. For all rigid constants, a, let $v_{\omega\tau}(a)$ be $v(a)$. Then the truth conditions for closed atomic sentences are as follows:

$$v_{\omega\tau}(Pt_1 \dots t_n) \ \text{ iff } \ \langle v_{\omega\tau}(t_1), \dots, v_{\omega\tau}(t_n)\rangle \in v_{\omega\tau}(P).$$

For all other sentences, the truth conditions are the usual.

3.5. Contingent identity semantics

We now turn to the semantics for our contingent identity systems. A (*quantified temporal alethic-deontic, constant, variable or constant and variable domain*) *model* $\mathcal{M}$ is now a relational structure $\langle D, H, W, T, <, R, S, v\rangle$, where W, $<$, R, and S are the same as in the constant, variable or constant and variable domain cases. The elements of D are now functions from $W \times T$ to H. Note that D is still non-empty, but does not have to comprise all such functions. H is a set of objects, which we will call substrata or manifestations. If $d \in D$, $\omega \in W$ and $\tau \in T$, we shall say that $d(\langle \omega, \tau \rangle)$, or $|d|_{\omega\tau}$ as we shall also write, is the manifestation or substratum of d at the world-moment pair $\langle \omega, \tau \rangle$. For every (non-temporal, rigid) constant, c, $v(c) \in D$, and for every world-moment pair, $\langle \omega, \tau \rangle$, and n-place predicate, P, $v_{\omega\tau}(P)$ is a subset of H^n, not D^n. The interpretation of the identity predicate, $v_{\omega\tau}(=)$, is the world-moment-invariant set $\{\langle h, h \rangle : h \in H\}$. Let $\mathcal{M}$ be a variable domain model. Then $v(\omega\tau) := D_{\omega\tau} := \{d \in D : |d|_{\omega\tau} \in v_{\omega\tau}(\mathrm{E})\}$.

The truth conditions for closed atomic formulas are as follows:

$$v_{\omega\tau}(Pa_1 \dots a_n) = 1 \ \text{ iff } \ \langle |v(a_1)|_{\omega\tau}, \dots, |v(a_n)|_{\omega\tau} \rangle \in v_{\omega\tau}(P).$$

For all the other sentences, the truth conditions remain the same.

We can add descriptors to a contingent identity system. Every descriptor then denotes a member of D at each world-moment pair, possibly different members at different pairs. Let $v_{\omega\tau}(t)$ be $v(t)$, if t is a (non-temporal) rigid constant Then the truth conditions for closed atomic sentences are as follows:

$$v_{\omega\tau}(Pt_1 \dots t_n) = 1 \ \text{ iff } \ \langle |v_{\omega\tau}(t_1)|_{\omega\tau}, \dots, |v_{\omega\tau}(t_n)|_{\omega\tau} \rangle \in v_{\omega\tau}(P).$$

The concepts of validity, satisfiability, logical consequence etc. are essentially defined as in [37]. The definitions are the same for all our semantics.

3.6. Conditions on models

In [37] various frame- and modelconditions were mentioned. All of these conditions may also be imposed on our quantified temporal alethic-deontic models, with the exception that the conditions on the valuation function have to be modified slightly (see Table 2).

In this section we will also consider some further conditions. We mention three new conditions on $<$: C-C, as in comparability; C-UB, as in upper bounds; and C-LB, as in lower bounds (see Table 1); and some domain-inclusion (or Barcan) conditions, which say something about the relations between the different domains of different world-moment pairs (see Table 3). C-ACBF, for instance, says that if the world ω_j is alethically accessible from ω_i at τ, then the domain of ω_i at τ is a subset of the domain of ω_j at τ. The other rules are interpreted similarly.

Condition	Formalization of condition
C-C	For all $\tau_i, \tau_j : \tau_i < \tau_j$ or $\tau_i = \tau_j$ or $\tau_j < \tau_i$.
C-UB	For all τ_i, τ_j, τ_k : if $\tau_i < \tau_j$ and $\tau_i < \tau_k$, then for some $\tau_l : \tau_j < \tau_l$ and $\tau_k < \tau_l$.
C-LB	For all τ_i, τ_j, τ_k : if $\tau_j < \tau_i$ and $\tau_k < \tau_i$, then for some $\tau_l : \tau_l < \tau_j$ and $\tau_l < \tau_k$.

Table 1. Conditions on the relation $<$

Condition	Formalization of condition
C-FT	If $R\omega_1\omega_2\tau$ and A is an atomic sentence true in ω_1 at τ, then A is true in ω_2 at τ.
C-BT	If $R\omega_1\omega_2\tau$ and A is an atomic sentence true in ω_2 at τ, then A is true in ω_1 at τ.

Table 2. Conditions on the valuation function v in a model

Condition	Formalization of condition
C-TBF	If $\tau_i < \tau_j$, then $D_{\omega\tau_j} \subseteq D_{\omega\tau_i}$.
C-TCBF	If $\tau_i < \tau_j$, then $D_{\omega\tau_i} \subseteq D_{\omega\tau_j}$.
C-ABF	If $R\omega_i\omega_j\tau$, then $D_{\omega_j\tau} \subseteq D_{\omega_i\tau}$.
C-ACBF	If $R\omega_i\omega_j\tau$, then $D_{\omega_i\tau} \subseteq D_{\omega_j\tau}$.
C-DBF	If $S\omega_i\omega_j\tau$, then $D_{\omega_j\tau} \subseteq D_{\omega_i\tau}$.
C-DCBF	If $S\omega_i\omega_j\tau$, then $D_{\omega_i\tau} \subseteq D_{\omega_j\tau}$.

Table 3. Domain-inclusion (Barcan) conditions

3.7. Model classes and the logic of a class of models

In [37] 31 modelconditions were mentioned, and in this paper we have introduced 9 new (2 conditions are modifications of the conditions on the valuation function). So, all in all we have 40 different conditions. These can be used to obtain a categorization of the set of all models into various kinds. There are 2^{40} different combinations of these conditions. In general, we shall say that $\mathbf{CM}(C_1, \ldots, C_n)$ is the class of all constant domain models that satisfy the conditions $C_1, \ldots, C_n$. $\mathbf{VM}(C_1, \ldots, C_n)$ is the class of all variable domain models that satisfy the conditions C_1, $\ldots$, C_n. Moreover, $\mathbf{CVM}(C_1, \ldots, C_n)$ is the class of all constant and variable domain models that satisfy the conditions $C_1, \ldots, C_n$. If we are considering necessary or contingent identity semantics, we add **NI** or **CI**, respectively. And if the underlying language includes descriptors, we add a **D**. For example, $\mathbf{CVM}$(C-FT, C-BT)**NID** is the class of all constant

Π	Σ	$\neg\Pi$	$\neg\Sigma$
$\Pi x A, w_i t_j$ $\downarrow$ $A[a/x], w_i t_j$ for every constant a on the branch, a new if there are no constants on the branch	$\Sigma x A, w_i t_j$ $\downarrow$ $A[c/x], w_i t_j$ where c is new to the branch	$\neg\Pi x A, w_i t_j$ $\downarrow$ $\Sigma x \neg A, w_i t_j$	$\neg\Sigma x A, w_i t_j$ $\downarrow$ $\Pi x \neg A, w_i t_j$

Table 4. Possibilist quantifiers

$\forall$	$\exists$	$\neg\forall$	$\neg\exists$
$\forall x A, w_i t_j$ $\swarrow \searrow$ $\neg Ea, w_i t_j \quad A[a/x], w_i t_j$ for every constant a on the branch, a new if there are no constants on the branch	$\exists x A, w_i t_j$ $\downarrow$ $Ec, w_i t_j$ $A[c/x], w_i t_j$ where c is new to the branch	$\neg\forall x A, w_i t_j$ $\downarrow$ $\exists x \neg A, w_i t_j$	$\neg\exists x A, w_i t_j$ $\downarrow$ $\forall x \neg A, w_i t_j$

Table 5. Actualist quantifiers

and variable domain models with necessary identity and descriptors that satisfy the conditions C-FT and C-BT. The concept of the logical system of a class of models is defined as in [37].

4. Proof theory

4.1. Semantic tableaux. Tableau rules

The kind of semantic tableau system we use is mainly inspired by Graham Priest [36]. For more information about tableau methods, see e.g. Fitting and Mendelsohn [16], and D'Agostino *et al.* [14]. The concepts of semantic tableau, branch, open and closed branch etc. are essentially defined as in [37]. Also in [37] a large set of tableau rules were introduced: propositional rules, basic temporal, alethic and deontic rules, temporal, alethic, deontic accessibility rules, alethic-deontic accessibility rules etc. We use these and some new rules discussed in this section to construct our tableau systems (see Tables 4 and 5).

The derived rules in Table 6 hold in every constant and variable domain system that includes the definitions.

$\forall_{\odot}$	$\exists_{\odot}$	$\neg\forall_{\odot}$	$\neg\exists_{\odot}$
$\forall_{\odot}xA, w_it_j$ $\mathrm{E}_{\odot}a, w_it_j$ $\downarrow$ $A[a/x], w_it_j$	$\exists_{\odot}xA, w_it_j$ $\downarrow$ $\mathrm{E}_{\odot}c, w_it_j$ $A[c/x], w_it_j$ where c is new to the branch	$\neg\forall_{\odot}xA, w_it_j$ $\downarrow$ $\exists_{\odot}x\neg A, w_it_j$	$\neg\exists_{\odot}xA, w_it_j$ $\downarrow$ $\forall_{\odot}x\neg A, w_it_j$

Table 6. Derived rules

Note that in Table 6 the symbols a and c in the quantifier rules are rigid constants; we never instantiate with descriptors. a is any constant on the branch and c is a constant new to the branch.

A	$\neg\mathsf{A}$	S	$\neg\mathsf{S}$
$\mathsf{A}A, w_it_j$ $\downarrow$ A, w_it_k for every t_k on the branch	$\neg\mathsf{A}A, w_it_j$ $\downarrow$ $\mathsf{S}\neg A, w_it_j$	$\mathsf{S}A, w_it_j$ $\downarrow$ A, w_it_k where t_k is new to the branch	$\neg\mathsf{S}A, w_it_j$ $\downarrow$ $\mathsf{A}\neg A, w_it_j$

Table 7. Temporal rules

T-C	T-UB	T-LB
t_i, t_j $\swarrow \quad \downarrow \quad \searrow$ $t_i < t_j \quad t_i = t_j \quad t_j < t_i$	$t_i < t_j$ $t_i < t_k$ $\downarrow$ $t_j < t_l$ $t_k < t_l$ where t_l is new to the branch	$t_j < t_i$ $t_k < t_i$ $\downarrow$ $t_l < t_j$ $t_l < t_k$ where t_l is new to the branch

Table 8.

Note that in Table 9 T-ACBF is derivable from T-ABF and T-aB, and T-ABF from T-ACBF and T-aB. Furthermore, T-ABF is derivable from T-BT and T-ACBF from T-FT. T-DBF follows from T-ABF and T-MO and T-DCBF from T-ACBF and T-MO.[3]

[3] In the domain-inclusion rules, the second "T" stands for "temporal", "A" for "alethic", "D" for "deontic", "C" for "converse", and "BF" for "Barcan formula". T-aB, T-FT, T-BT and T-MO are introduced in [37].

T-ABF	T-DBF	T-TBF
$\mathrm{E}a, w_j t_k$ $rw_i w_j t_k$ $\downarrow$ $\mathrm{E}a, w_i t_k$	$\mathrm{E}a, w_j t_k$ $sw_i w_j t_k$ $\downarrow$ $\mathrm{E}a, w_i t_k$	$\mathrm{E}a, w_i t_k$ $t_j < t_k$ $\downarrow$ $\mathrm{E}a, w_i t_j$
T-ACBF	**T-DCBF**	**T-TCBF**
$\mathrm{E}a, w_i t_k$ $rw_i w_j t_k$ $\downarrow$ $\mathrm{E}a, w_j t_k$	$\mathrm{E}a, w_i t_k$ $sw_i w_j t_k$ $\downarrow$ $\mathrm{E}a, w_j t_k$	$\mathrm{E}a, w_i t_j$ $t_j < t_k$ $\downarrow$ $\mathrm{E}a, w_i t_k$

Table 9. Domain-inclusion (Barcan) rules

T-R=	T-S=	T-N=	T-D=
$*$ $\downarrow$ $t = t, w_i t_j$ for every t on the branch	$s = t, w_i t_j$ $A[s/x], w_i t_j$ $\downarrow$ $A[t/x], w_i t_j$ where A is atomic	$a = b, w_i t_j$ $\downarrow$ $a = b, w_k t_l$ for any w_k and t_l	$*$ $\downarrow$ $c = \alpha, w_i t_j$ where c is new to the branch for every α, w_i and t_j on the branch

Table 10. Identity rules and the descriptor rule

Moreover, in Table 10 (T-R=) and (T-S=) can be applied to both rigid constants and descriptors. (T-N=) is applied only if both terms, a and b, are rigid constants. In (T-D=) c is a rigid constant and α is a descriptor. (T-S=) is applied only within "world-moment pairs", and we only apply the rule when A is atomic. However, in systems with necessary identity, it is in general true that $A[b/x]$ follows from $a = b$ and $A[a/x]$. And in systems with contingent identity this is true when x is not in the scope of a modal operator.[4]

4.2. Tableau systems

The concepts of tableau system, temporal alethic-deontic tableau system etc. are essentially defined as in [37], with the exception that every temporal alethic-deontic tableau system now also includes the rules A, $\neg\mathsf{A}$, S, and $\neg\mathsf{S}$.

[4] In the identity rules and the descriptor rule "R" stands for "reflexive", "S" for "substitution (of identities)", "N" for "necessary identity" and "D" for "descriptor".

Let S be a temporal alethic-deontic tableau system. Then *CS*, a *constant domain quantified temporal alethic-deontic tableau system*, is S augmented by the rules for the possibilist quantifiers; *VS*, a *variable domain quantified temporal alethic-deontic tableau system*, is S augmented by the rules for the actualist quantifiers; and *CVS*, a *constant and variable domain quantified temporal alethic-deontic tableau system*, is S augmented by the rules for the possibilist and actualist quantifiers.

Any tableau system that includes T-C also includes Id(I) and Id(II) (see [37]). Any subset of the domain-inclusion (Barcan) rules may also be added to our systems.

Let S be a quantified temporal alethic-deontic tableau system without identity. Then *SNI* is S augmented by the rules for necessary identity, i.e., (T-R=), (T-S=) and (T-N=); *SCI* is S augmented by the rules for contingent identity, i.e., (T-R=) and (T-S=); *SNID* is *SNI* augmented by the rule for descriptors, i.e., (T-D=); and *SCID* is *SCI* augmented by (T-D=).

For example, *CVaTdDadMOOCt4CadtSPABFACBFNID* is the constant and variable domain quantified temporal alethic-deontic tableau system with necessary identity and descriptors that includes (T-aT), (T-dD), (T-MO), (T-OC), (T-t4), (T-SP), Id(I), Id(II) (introduced in [37]), (T-C), (T-ABF), (T-ACBF), the rules for the possibilist and actualist quantifiers, (T-R=), (T-S=), (T-N=) and (T-D=).

4.3. Some proof-theoretical concepts

The concepts of proof, theorem, derivation, consistency, inconsistency in a system, the logic of a tableau system etc. are defined as in [37]. An arbitrary formula A (a schema) is a theorem in the system S just in case every (closed) instance of A is a theorem in S. We also speak about theorems, derivations etc. with assumptions. These concepts were used in [37] but never explicitly defined. Here are the definitions.

Let $\triangleleft$ be $<$ or $=$. Then A is a theorem in S with the assumptions $v(t_1) \triangleleft v(t_2)$, $v(t_3) \triangleleft v(t_4)$, ... iff there is a closed S-tableau whose initial list comprises $t_1 \triangleleft t_2$, $t_3 \triangleleft t_4$, ... and $\neg A, w_0 t_0$.

A derivation in the system S of B from the (finite) set of formulas Γ with the assumptions $v(t_1) \triangleleft v(t_2)$, $v(t_3) \triangleleft v(t_4)$, ..., is a closed S-tableau whose initial list comprises $t_1 \triangleleft t_2$, $t_3 \triangleleft t_4$, ..., $A, w_0 t_0$ for every $A \in \Gamma$ and $\neg B, w_0 t_0$. Etc.

Obviously, we can also speak about validity of sentences and arguments with assumptions in models in a similar sense. Our soundness and completeness theorems can then be extended so that they include theorems and derivations with assumptions in a straightforward way.

Some theorems with assumptions were mentioned in [37]. For example, $\mathsf{R}t'\Box\mathsf{R}tA \to \mathsf{R}t\Box\mathsf{R}tA$ is a theorem in the system *adtSP* given that $v(t') < v(t)$ (for a proof, see [38, p. 299]). Since this system is sound with respect to the class of all models that satisfy C-SP, $\mathsf{R}t'\Box\mathsf{R}tA \to \mathsf{R}t\Box\mathsf{R}tA$ is valid on this class given that $v(t') < v(t)$. We can in fact prove something slightly stronger, namely that $\mathsf{R}t'\Box\mathsf{R}t''A \to \mathsf{R}t\Box\mathsf{R}t''A$ is a theorem in the system *adtSP* given that $v(t') < v(t)$. According to this theorem, if it is necessary that at time $v(t'')$ it is the case that A, then at every later time it is also necessary that at $v(t'')$ A. E.g. if it is now (today) (historically) necessary that I was in Stockholm yesterday, then at any time after today, it will be (historically) necessary that I was in Stockholm on that day (see [15]).

5. Examples of theorems

In this section we will consider some theorems in some systems. If not otherwise stated S will denote a constant and variable domain system that includes all definitions of all non-primitive concepts. All proofs are omitted; in most cases they are straightforward.

$$
\begin{array}{ccccccc}
 & \mathbf{U}A & & & & & \\
\swarrow & & \searrow & & & & \\
\mathsf{A}A & \downarrow & \mathsf{A}A & & & \mathsf{A}A & \\
\downarrow & & \downarrow & & \swarrow & & \searrow \\
[\mathsf{H}]A & \Box A & [\mathsf{G}]A & \Box A & t_{[\mathsf{H}]}A & \downarrow & t_{[\mathsf{G}]}A \\
\downarrow\ \searrow & \downarrow & \swarrow\ \downarrow & \downarrow & \downarrow\ \searrow & & \swarrow\ \downarrow \\
\mathsf{H}A & & \mathsf{G}A & \mathbf{O}A & t_{\mathsf{H}}A & & t_{\mathsf{G}}A \\
\downarrow & A & \downarrow & \downarrow & \downarrow & \mathsf{R}tA & \downarrow \\
\mathsf{P}A & \swarrow\ \downarrow\ \searrow & \mathsf{F}A & \mathbf{P}A & t_{\mathsf{P}}A & \swarrow\ \downarrow\ \searrow & t_{\mathsf{F}}A \\
\downarrow & & \downarrow & \downarrow & \downarrow & & \downarrow \\
\langle\mathsf{P}\rangle A & \Diamond A & \langle\mathsf{F}\rangle A & \Diamond A & t_{\langle\mathsf{P}\rangle}A & & t_{\langle\mathsf{P}\rangle}A \\
\downarrow & & \downarrow & & \searrow & & \swarrow \\
\mathsf{S}A & \downarrow & \mathsf{S}A & & & \mathsf{S}A & \\
\searrow & & \swarrow & & & & \\
 & \mathbf{M}A & & & & &
\end{array}
$$

Figure 1.

$$
\begin{array}{ccccccc}
 & \mathrm{E}_U a & & & & & \\
\swarrow & & \searrow & & & & \\
\mathrm{E}_{\mathsf{A}} a & \downarrow & \mathrm{E}_{\mathsf{A}} a & \mathrm{E}_{\Box} a & & \mathrm{E}_{\mathsf{A}} a & \\
\downarrow & & \downarrow & \downarrow & \swarrow & & \searrow \\
\mathrm{E}_{[\mathsf{H}]} a & \mathrm{E}_{\Box} a & \mathrm{E}_{[\mathsf{G}]} a & \mathrm{E}_{O} a & \mathrm{E}_{t_{[\mathsf{H}]}} a & \downarrow & \mathrm{E}_{t_{[\mathsf{G}]}} a \\
\downarrow \searrow & \downarrow & \swarrow \downarrow & \downarrow & \downarrow \searrow & & \swarrow \downarrow \\
\mathrm{E}_{\mathsf{H}} a & & \mathrm{E}_{\mathsf{G}} a & \mathrm{E}_{P} a & \mathrm{E}_{t_{\mathsf{H}}} a & & \mathrm{E}_{t_{\mathsf{G}}} a \\
\downarrow & \mathrm{E} a & \downarrow & \downarrow & \downarrow & \mathrm{E}_{\mathsf{R}t} a & \downarrow \\
\mathrm{E}_{\mathsf{P}} a & \swarrow \downarrow \searrow & \mathrm{E}_{\mathsf{F}} a & \mathrm{E}_{\Diamond} a & \mathrm{E}_{t_{\mathsf{P}}} a & \swarrow \downarrow \searrow & \mathrm{E}_{t_{\mathsf{F}}} a \\
\downarrow & & \downarrow & \downarrow & \downarrow & & \downarrow \\
\mathrm{E}_{\langle \mathsf{P} \rangle} a & \mathrm{E}_{\Diamond} a & \mathrm{E}_{\langle \mathsf{F} \rangle} a & \mathrm{E}_{M} a & \mathrm{E}_{t_{\langle \mathsf{P} \rangle}} a & & \mathrm{E}_{t_{\langle \mathsf{F} \rangle}} a \\
\downarrow & & \downarrow & \downarrow & \searrow & & \swarrow \\
\mathrm{E}_{\mathsf{S}} a & \downarrow & \mathrm{E}_{\mathsf{S}} a & \mathbf{E} a & & \mathrm{E}_{\mathsf{S}} a & \\
\searrow & & \swarrow & & & & \\
 & \mathrm{E}_M a & & & & &
\end{array}
$$

Figure 2.

THEOREM 1. (i) $\mathsf{H}A$ *implies* $\mathsf{P}A$ *and* $t_{\mathsf{H}}A$ *implies* $t_{\mathsf{P}}A$ *in every* S *that includes* T-PD. $\mathsf{G}A$ *implies* $\mathsf{F}A$ *and* $t_{\mathsf{G}}A$ *implies* $t_{\mathsf{F}}A$ *in every* S *that includes* T-FD. $\Box A$ *implies* A *and* A *implies* $\Diamond A$ *in every* S *that includes* T-aT. $\Box A$ *implies* $\mathbf{O}A$ *and* $\mathbf{P}A$ *implies* $\Diamond A$ *in every* S *that includes* T-MO. $\mathbf{O}A$ *implies* $\mathbf{P}A$ *in every* S *that includes* T-dD. *All other implications in Figure* 1 *hold in every* S.

(ii) *All implications in Figure* 2 *hold in* S *with similar provisos as in the part* (i).

(iii) $\forall xA$ *implies* $\forall_{\Box}xA$, $\forall_{\Diamond}xA$ *implies* $\forall xA$, $\exists_{\Box}xA$ *implies* $\exists xA$, *and* $\exists xA$ *implies* $\exists_{\Diamond}xA$ *in every* S *that includes* T-aT. *If* S *includes* T-FD, *then all of the following implications hold in* S: $\forall_{\mathsf{F}}xA \to \forall_{\mathsf{G}}xA$, $\exists_{\mathsf{G}}xA \to \exists_{\mathsf{F}}xA$, $\forall_{t_{\mathsf{F}}}xA \to \forall_{t_{\mathsf{G}}}xA$, $\exists_{t_{\mathsf{G}}}xA \to \exists_{t_{\mathsf{F}}}xA$. *If* S *includes* T-PD, *then all of the following implications hold in* S: $\forall_{\mathsf{P}}xA \to \forall_{\mathsf{H}}xA$, $\exists_{\mathsf{H}}xA \to \exists_{\mathsf{P}}xA$, $\forall_{t_{\mathsf{P}}}xA \to \forall_{t_{\mathsf{H}}}xA$, $\exists_{t_{\mathsf{H}}}xA \to \exists_{t_{\mathsf{P}}}xA$. *If* S *includes* T-MO, *then all of the following implications hold in* S: $\exists_{\Box}xA \to \exists_{O}xA$, $\exists_{P}xA \to \exists_{\Diamond}xA$, $\forall_{\Diamond}xA \to \forall_{P}xA$ *and* $\forall_{O}xA \to \forall_{\Box}xA$. *If* S *includes* T-dD, *then the following implications hold in* S: $\exists_{O}xA \to \exists_{P}xA$ *and* $\forall_{P}xA \to \forall_{O}xA$. *All other implications in Figures* 3 *and* 4 *hold in every* S.

THEOREM 2. (i) *Let* $\odot$ *be a positive modal operator or empty.* (*When* $\odot$ *is empty,* $\forall_{\odot}xFx = \forall xFx$ *and* $\exists_{\odot}xFx = \exists xFx$.) *Then all formulas in Table* 11 *are theorems in* S.

$$\begin{tikzcd}
 & \exists_U xA \arrow[dl] \arrow[dd] \arrow[dr] & & \\
\exists_{\mathsf{A}} xA \arrow[d] & & \exists_{\mathsf{A}} xA \arrow[d] & \\
\exists_{[\mathsf{H}]} xA \arrow[d] \arrow[ddr] & \exists_{\Box} xA \arrow[dd] & \exists_{[\mathsf{G}]} xA \arrow[ddl] \arrow[d] & \exists_{\Box} xA \arrow[d] \\
\exists_{\mathsf{H}} xA \arrow[dd] & & \exists_{\mathsf{G}} xA \arrow[dd] & \exists_O xA \arrow[dd] \\
 & \exists xA \arrow[ddl] \arrow[dd] \arrow[ddr] & & \\
\exists_{\mathsf{P}} xA \arrow[d] & & \exists_{\mathsf{F}} xA \arrow[d] & \exists_P xA \arrow[d] \\
\exists_{\langle \mathsf{P} \rangle} xA \arrow[d] & \exists_{\Diamond} xA \arrow[dd] & \exists_{\langle \mathsf{F} \rangle} xA \arrow[d] & \exists_{\Diamond} xA \\
\exists_{\mathsf{S}} A \arrow[dr] & & \exists_{\mathsf{S}} xA \arrow[dl] & \\
 & \exists_M xA \arrow[d] & & \\
 & \Sigma xA & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & \Pi xA \arrow[d] & \\
 & & \forall_M xA \arrow[dl] \arrow[dd] \arrow[dr] & \\
 & \forall_{\mathsf{S}} xA \arrow[d] & & \forall_{\mathsf{S}} xA \arrow[d] \\
\forall_{\Diamond} xA \arrow[d] & \forall_{\langle \mathsf{P} \rangle} xA \arrow[d] \arrow[ddr] & \forall_{\Diamond} xA \arrow[dd] & \forall_{\langle \mathsf{F} \rangle} xA \arrow[ddl] \arrow[d] \\
\forall_P xA \arrow[dd] & \forall_{\mathsf{P}} xA \arrow[dd] & & \forall_{\mathsf{F}} xA \arrow[dd] \\
 & & \forall xA \arrow[ddl] \arrow[dd] \arrow[ddr] & \\
\forall_O xA \arrow[d] & \forall_{\mathsf{H}} xA \arrow[d] & & \forall_{\mathsf{G}} xA \arrow[d] \\
\forall_{\Box} xA & \forall_{[\mathsf{H}]} xA \arrow[d] & \forall_{\Box} xA \arrow[dd] & \forall_{[\mathsf{G}]} xA \arrow[d] \\
 & \forall_{\mathsf{A}} A \arrow[dr] & & \forall_{\mathsf{A}} xA \arrow[dl] \\
 & & \forall_U xA &
\end{tikzcd}$$

Figure 3.

$$\begin{tikzcd}
 & \exists_{\mathsf{A}} A \arrow[dl] \arrow[ddd] \arrow[dr] & \\
\exists_{t_{[\mathsf{H}]}} A \arrow[d] \arrow[ddr] & & \exists_{t_{[\mathsf{G}]}} A \arrow[ddl] \arrow[d] \\
\exists_{t_{\mathsf{H}}} A \arrow[dd] & & \exists_{t_{\mathsf{G}}} A \arrow[dd] \\
 & \exists_{\mathsf{R}t} A \arrow[ddl] \arrow[ddd] \arrow[ddr] & \\
\exists_{t_{\mathsf{P}}} A \arrow[d] & & \exists_{t_{\mathsf{F}}} A \arrow[d] \\
\exists_{t_{\langle \mathsf{P} \rangle}} A \arrow[dr] & & \exists_{t_{\langle \mathsf{F} \rangle}} A \arrow[dl] \\
 & \exists_{\mathsf{S}} A &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \forall_{\mathsf{S}} A \arrow[dl] \arrow[ddd] \arrow[dr] & \\
\forall_{t_{\langle \mathsf{P} \rangle}} A \arrow[d] \arrow[ddr] & & \forall_{t_{\langle \mathsf{F} \rangle}} A \arrow[ddl] \arrow[d] \\
\forall_{t_{\mathsf{P}}} A \arrow[dd] & & \forall_{t_{\mathsf{F}}} A \arrow[dd] \\
 & \forall_{\mathsf{R}t} A \arrow[ddl] \arrow[ddd] \arrow[ddr] & \\
\forall_{t_{\mathsf{H}}} A \arrow[d] & & \forall_{t_{\mathsf{G}}} A \arrow[d] \\
\forall_{t_{[\mathsf{H}]}} A \arrow[dr] & & \forall_{t_{[\mathsf{G}]}} A \arrow[dl] \\
 & \forall_{\mathsf{A}} A &
\end{tikzcd}$$

Figure 4.

(ii) *Not all instances of the schemas in Table 12 are theorems in every S.*

(iii) *Replace every occurrence of $\forall_{\odot}$ with Π and every occurrence of $\exists_{\odot}$ with Σ in the schemas in Tables 11 and 12. Then the resulting formulas are theorems in every S. (Π and Σ behave as classical quantifiers, and $\forall_{\odot}$ and $\exists_{\odot}$ as quantifiers in positive free logic.)*

THEOREM 3. *Let $\forall_S$ and $\forall_W$ be two universal quantifiers such that $\forall_S$ is stronger than $\forall_W$ in S (i.e., $\forall_S A$ implies $\forall_W A$ in S), and let $\exists_S$ and $\exists_W$*

$\forall_{\odot}xFx \leftrightarrow \neg\exists_{\odot}x\neg Fx$	$\exists_{\odot}xFx \leftrightarrow \neg\forall_{\odot}x\neg Fx$
$\forall_{\odot}x\neg Fx \leftrightarrow \neg\exists_{\odot}xFx$	$\exists_{\odot}x\neg Fx \leftrightarrow \neg\forall_{\odot}xFx$
$(\forall_{\odot}xFx \wedge E_{\odot}a) \rightarrow Fa$	$(Fa \wedge E_{\odot}a) \rightarrow \exists_{\odot}xFx$
$\forall_{\odot}x(Fx \rightarrow Gx) \leftrightarrow \neg\exists_{\odot}x(Fx \wedge \neg Gx)$	$\forall_{\odot}x(Fx \rightarrow \neg Gx) \leftrightarrow \neg\exists_{\odot}x(Fx \wedge Gx)$
$\exists_{\odot}x(Fx \wedge Gx) \leftrightarrow \neg\forall_{\odot}x(Fx \rightarrow \neg Gx)$	$\exists_{\odot}x(Fx \wedge \neg Gx) \leftrightarrow \neg\forall_{\odot}x(Fx \rightarrow Gx)$
$\forall_{\odot}x(Fx \leftrightarrow Fx)$	$\forall_{\odot}x(Fx \vee \neg Fx)$
$\neg\exists_{\odot}x(Fx \wedge \neg Fx)$	$\forall_{\odot}x\neg(Fx \wedge \neg Fx)$
$\neg\exists_{\odot}xFx \rightarrow \forall_{\odot}x(Fx \rightarrow Gx)$	$\exists_{\odot}x(Fx \rightarrow Gx) \rightarrow (\forall_{\odot}xFx \rightarrow \exists_{\odot}xGx)$
$\forall_{\odot}xFx \leftrightarrow \forall_{\odot}yFy$	$\exists_{\odot}xFx \leftrightarrow \exists_{\odot}yFy$
$\forall_{\odot}x\forall_{\odot}yFxy \leftrightarrow \forall_{\odot}y\forall_{\odot}xFxy$	$\exists_{\odot}x\exists_{\odot}yFxy \leftrightarrow \exists_{\odot}y\exists_{\odot}xFxy$
$\forall_{\odot}x(Fx \wedge Gx) \leftrightarrow (\forall_{\odot}xFx \wedge \forall_{\odot}xGx)$	$\exists_{\odot}x(Fx \wedge Gx) \rightarrow (\exists_{\odot}xFx \wedge \exists_{\odot}xGx)$
$\exists_{\odot}x(Fx \vee Gx) \leftrightarrow (\exists_{\odot}xFx \vee \exists_{\odot}xGx)$	$(\forall_{\odot}xFx \vee \forall_{\odot}xGx) \rightarrow \forall_{\odot}x(Fx \vee Gx)$
$\forall_{\odot}x(Fx \rightarrow Gx) \rightarrow (\forall_{\odot}xFx \rightarrow \forall_{\odot}xGx)$	$\forall_{\odot}x(Fx \rightarrow Gx) \rightarrow (\exists_{\odot}xFx \rightarrow \exists_{\odot}xGx)$
$\forall_{\odot}x(Fx \leftrightarrow Gx) \rightarrow (\forall_{\odot}xFx \leftrightarrow \forall_{\odot}xGx)$	$\forall_{\odot}x(Fx \leftrightarrow Gx) \rightarrow (\exists_{\odot}xFx \leftrightarrow \exists_{\odot}xGx)$

Table 11.

$\forall_{\odot}xFx \rightarrow Fa$	$Fa \rightarrow \exists_{\odot}xFx$
$\forall_{\odot}xFx \rightarrow \exists_{\odot}xFx$	$\exists_{\odot}xFx \vee \exists_{\odot}x\neg Fx$
$\neg(\forall_{\odot}xFx \wedge \forall_{\odot}x\neg Fx)$	$\forall_{\odot}x\neg Fx \rightarrow \neg\forall_{\odot}xFx$
$(\forall_{\odot}xFx \wedge \forall_{\odot}xGx) \rightarrow \exists_{\odot}x(Fx \wedge Gx)$	$\forall_{\odot}x(Fx \wedge Gx) \rightarrow (\exists_{\odot}xFx \wedge \exists_{\odot}xGx)$
$(\forall_{\odot}xFx \vee \forall_{\odot}xGx) \rightarrow \exists_{\odot}x(Fx \vee Gx)$	$\forall_{\odot}x(Fx \vee Gx) \rightarrow (\exists_{\odot}xFx \vee \exists_{\odot}xGx)$
$(\exists_{\odot}xFx \rightarrow \exists_{\odot}xGx) \rightarrow \exists_{\odot}x(Fx \rightarrow Gx)$	$(\forall_{\odot}xFx \rightarrow \forall_{\odot}xGx) \rightarrow \exists_{\odot}x(Fx \rightarrow Gx)$

Table 12.

$\forall_S x(A \wedge B) \rightarrow (\forall_W A \wedge \forall_W B)$	$\forall_S x(A \leftrightarrow B) \rightarrow (\forall_W A \leftrightarrow \forall_W B)$
$(\forall_S xA \vee \forall_S xB) \rightarrow \forall_W (A \vee B)$	$\forall_S x(A \leftrightarrow B) \rightarrow (\exists_W xA \leftrightarrow \exists_W xB)$
$(\forall_S xA \wedge \forall_S xB) \rightarrow \forall_W (A \wedge B)$	$\forall_S x(A \leftrightarrow B) \rightarrow (\neg\exists_W xA \leftrightarrow \neg\exists_W xB)$
$\exists_W x(A \wedge B) \rightarrow (\exists_S xA \wedge \exists_S xB)$	$(\forall_W xA \wedge \forall_S x(A \rightarrow B)) \rightarrow \forall_W xB$
$\exists_W x(A \vee B) \rightarrow (\exists_S xA \vee \exists_S xB)$	$\forall_S x(A \rightarrow B) \rightarrow (\forall_W xA \rightarrow \forall_W xB)$
$(\exists_W xA \vee \exists_W xB) \rightarrow \exists_S x(A \vee B)$	$(\exists_W xA \wedge \forall_S x(A \rightarrow B)) \rightarrow \exists_W xB$
$\neg\exists_S x(A \vee B) \rightarrow (\neg\exists_W xA \wedge \neg\exists_W xB)$	$\forall_S x(A \rightarrow B) \rightarrow (\exists_W xA \rightarrow \exists_W xB)$
$(\neg\exists_S xA \vee \neg\exists_S xB) \rightarrow \neg\exists_W x(A \wedge B)$	$(\neg\exists_W xB \wedge \forall_S x(A \rightarrow B)) \rightarrow \neg\exists_W xA$
$(\neg\exists_S xA \wedge \neg\exists_S xB) \rightarrow \neg\exists_W x(A \vee B)$	$\forall_S x(A \rightarrow B) \rightarrow (\neg\exists_W xB \rightarrow \neg\exists_W xA)$
$\forall_S x(A \rightarrow B) \rightarrow (\forall_S xA \rightarrow \forall_W xB)$	$\forall_S x(A \rightarrow B) \rightarrow (\exists_W xA \rightarrow \exists_S xB)$
$\forall_S x(A \rightarrow B) \rightarrow (\neg\exists_S xB \rightarrow \neg\exists_W xA)$	$(\forall_W x(A \vee B) \wedge \neg\exists_S xB) \rightarrow \forall_W xA$

Table 13.

be the duals of $\forall_S$ and $\forall_W$, respectively. Then all formulas in Tables 13 and 14 are theorems in S.

THEOREM 4. (i) *All formulas in Tables 15, 16 and 17 are theorems in S.*

(ii) *Let A be a formula in Table 15, 16 or 17. Let B be the result of replacing every occurrence of $\forall$ in A by any universal quantifier, Q, every occurrence of $\exists$ with the dual of Q, every occurrence of $\Box$ by any*

$\forall_S x((A \vee B) \to C) \to ((\forall_W xA \vee \forall_W xB) \to \forall_W xC)$
$\forall_S x((A \vee B) \to C) \to ((\exists_W xA \vee \exists_W xB) \to \exists_W xC)$
$\forall_S x((A \vee B) \to C) \to (\neg\exists_W xC \to (\neg\exists_W xA \wedge \neg\exists_W xB))$
$\forall_S x(A \to (B \vee C)) \to (\exists_W xA \to (\exists_W xB \vee \exists_W xC))$
$\forall_S x(A \to (B \vee C)) \to ((\neg\exists_W xB \wedge \neg\exists_W xC) \to \neg\exists_W xA)$
$\forall_S x((A \wedge B) \to C) \to ((\forall_W xA \wedge \forall_W xB) \to \forall_W xC)$
$\forall_S x(A \to (B \wedge C)) \to (\forall_W xA \to (\forall_W xB \wedge \forall_W xC))$
$\forall_S x(A \to (B \wedge C)) \to (\exists_W xA \to (\exists_W xB \wedge \exists_W xC))$
$\forall_S x(A \to (B \wedge C)) \to ((\neg\exists_W xB \vee \neg\exists_W xC) \to \neg\exists_W xA)$
$(\forall_W x(A \vee B) \wedge (\forall_S x(A \to C) \wedge \forall_S x(B \to C))) \to \forall_W xC$
$(\forall_W x(A \vee B) \wedge (\forall_S x(A \to C) \wedge \forall_S x(B \to D))) \to \forall_W x(C \vee D)$
$(\forall_W xA \wedge (\forall_S x(A \to B) \wedge \forall_S x(A \to C))) \to (\forall_W xB \wedge \forall_W xC)$
$(\forall_W x(A \wedge B) \wedge (\forall_S x(A \to C) \vee \forall_S x(B \to D))) \to \forall_W x(C \vee D)$
$(\forall_W xA \wedge (\forall_S x(A \to B) \vee \forall_S x(A \to C))) \to \forall_W x(B \vee C)$
$(\forall_W x(A \wedge B) \wedge (\forall_S x(A \to C) \wedge \forall_S x(B \to D))) \to (\forall_W xC \wedge \forall_W xD)$

Table 14.

$\Box\forall x(A \to B) \leftrightarrow \diamondminus\exists x(A \wedge \neg B)$	$\Box\forall x(A \to \neg B) \leftrightarrow \diamondminus\exists x(A \wedge B)$
$\Box\exists x(A \wedge B) \leftrightarrow \diamondminus\forall x(A \to \neg B)$	$\Box\exists x(A \wedge \neg B) \leftrightarrow \diamondminus\forall x(A \to B)$
$\forall x\Box(A \wedge B) \leftrightarrow \forall x(\Box A \wedge \Box B)$	$\exists x\Diamond(A \vee B) \leftrightarrow \exists x(\Diamond A \vee \Diamond B)$
$\forall x\Diamond(A \vee B) \leftrightarrow \forall x(\Diamond A \vee \Diamond B)$	$\exists x\Box(A \wedge B) \leftrightarrow \exists x(\Box A \wedge \Box B)$
$\forall x\Diamond(A \wedge B) \to \forall x(\Diamond A \wedge \Diamond B)$	$\exists x(\Box A \vee \Box B) \to \exists x\Box(A \vee B)$
$\exists x\Diamond(A \wedge B) \to \exists x(\Diamond A \wedge \Diamond B)$	$\forall x(\Box A \vee \Box B) \to \forall x\Box(A \vee B)$
$\forall x\diamondminus(A \vee B) \leftrightarrow \forall x(\diamondminus A \wedge \diamondminus B)$	$\exists x\diamondminus(A \vee B) \leftrightarrow \exists x(\diamondminus A \wedge \diamondminus B)$
$\forall x(\diamondminus A \vee \diamondminus B) \to \forall x\diamondminus(A \wedge B)$	$\exists x(\diamondminus A \vee \diamondminus B) \to \exists x\diamondminus(A \wedge B)$
$\forall x\Box(A \to B) \to \forall x(\Box A \to \Box B)$	$\forall x\Box(A \to B) \to \forall x(\Diamond A \to \Diamond B)$
$\forall x\Box(A \leftrightarrow B) \to \forall x(\Box A \leftrightarrow \Box B)$	$\forall x\Box(A \leftrightarrow B) \to \forall x(\Diamond A \leftrightarrow \Diamond B)$
$\forall x\Box(A \to B) \to \forall x(\diamondminus B \to \diamondminus A)$	$\forall x\Box(A \leftrightarrow B) \to \forall x(\diamondminus A \leftrightarrow \diamondminus B)$

Table 15.

positive necessity-like modal operator, L, every occurrence of $\Diamond$ by the dual, M, of L, $\boxminus$ by $\neg L$ and $\diamondminus$ by $\neg M$. Then B is a theorem in S.

THEOREM 5. *Let $\Box$ and $\mathbf{O}$ be two (positive) necessity-like modal operators such that $\Box$ is stronger than $\mathbf{O}$ in S (i.e., $\Box A \to \mathbf{O}A$ is a theorem in S), let $\Diamond$ and $\mathbf{P}$ be the duals of $\Box$ and $\mathbf{O}$, respectively, and let $\forall$ be any universal quantifier, and $\exists$ the dual of $\forall$. Then all of the formulas in Tables 18 and 19 hold in S.*

THEOREM 6. (i) *If S includes* T-ABF, *then $\forall x\Box A \to \Box\forall xA$ and $\Diamond\exists xA \to \exists x\Diamond A$ are theorems in S. If it includes* T-ACBF, *then $\Box\forall xA \to \forall x\Box A$, $\exists x\Diamond A \to \Diamond\exists xA$, $\Diamond\forall xA \to \forall x\Diamond A$, and $\exists x\Box A \to \Box\exists xA$ are theorems in S.*

$\Box\forall x(A \wedge B) \leftrightarrow (\Box\forall xA \wedge \Box\forall xB)$	$\Diamond\exists x(A \vee B) \leftrightarrow (\Diamond\exists xA \vee \Diamond\exists xB)$
$\Diamond\forall x(A \wedge B) \rightarrow (\Diamond\forall xA \wedge \Diamond\forall xB)$	$(\Box\exists xA \vee \Box\exists xB) \rightarrow \Box\exists x(A \vee B)$
$\forall x\Box(A \wedge B) \leftrightarrow (\forall x\Box A \wedge \forall x\Box B)$	$\exists x\Diamond(A \vee B) \leftrightarrow (\exists x\Diamond A \vee \exists x\Diamond B)$
$\forall x\Diamond(A \wedge B) \rightarrow (\forall x\Diamond A \wedge \forall x\Diamond B)$	$(\exists x\Box A \vee \exists x\Box B) \rightarrow \exists x\Box(A \vee B)$
$(\Box\forall xA \vee \Box\forall xB) \rightarrow \Box\forall x(A \vee B)$	$\Diamond\exists x(A \wedge B) \rightarrow (\Diamond\exists xA \wedge \Diamond\exists xB)$
$(\Diamond\forall xA \vee \Diamond\forall xB) \rightarrow \Diamond\forall x(A \vee B)$	$\Box\exists x(A \wedge B) \rightarrow (\Box\exists xA \wedge \Box\exists xB)$
$(\forall x\Box A \vee \forall x\Box B) \rightarrow \forall x\Box(A \vee B)$	$\exists x\Diamond(A \wedge B) \rightarrow (\exists x\Diamond A \wedge \exists x\Diamond B)$
$(\forall x\Diamond A \vee \forall x\Diamond B) \rightarrow \forall x\Diamond(A \vee B)$	$\exists x\Box(A \wedge B) \rightarrow (\exists x\Box A \wedge \exists x\Box B)$
$\forall x\diamondminus(A \vee B) \leftrightarrow (\forall x\diamondminus A \wedge \forall x\diamondminus B)$	$(\exists x \boxminus A \vee \exists x \boxminus B) \leftrightarrow \exists x \boxminus (A \wedge B)$
$\exists x\diamondminus(A \vee B) \rightarrow (\exists x\diamondminus A \wedge \exists x\diamondminus B)$	$(\forall x \boxminus A \vee \forall x \boxminus B) \rightarrow \forall x \boxminus (A \wedge B)$
$(\exists x\diamondminus A \vee \exists x\diamondminus B) \rightarrow \exists x\diamondminus(A \wedge B)$	$\forall x \boxminus (A \vee B) \rightarrow (\forall x \boxminus A \wedge \forall x \boxminus B)$
$(\forall x\diamondminus A \vee \forall x\diamondminus B) \rightarrow \forall x\diamondminus(A \wedge B)$	$\exists x \boxminus (A \vee B) \rightarrow (\exists x \boxminus A \wedge \exists x \boxminus B)$

Table 16.

$\Box\forall x(A \rightarrow B) \rightarrow (\Box\forall xA \rightarrow \Box\forall xB)$	$\Box\forall x(A \rightarrow B) \rightarrow (\Diamond\forall xA \rightarrow \Diamond\forall xB)$
$\Box\forall x(A \rightarrow B) \rightarrow (\Box\exists xA \rightarrow \Box\exists xB)$	$\Box\forall x(A \rightarrow B) \rightarrow (\Diamond\exists xA \rightarrow \Diamond\exists xB)$
$\forall x\Box(A \rightarrow B) \rightarrow (\forall x\Box A \rightarrow \forall x\Box B)$	$\forall x\Box(A \rightarrow B) \rightarrow (\forall x\Diamond A \rightarrow \forall x\Diamond B)$
$\forall x\Box(A \rightarrow B) \rightarrow (\exists x\Box A \rightarrow \exists x\Box B)$	$\forall x\Box(A \rightarrow B) \rightarrow (\exists x\Diamond A \rightarrow \exists x\Diamond B)$
$\Box\forall x(A \rightarrow B) \rightarrow (\diamondminus\forall xB \rightarrow \diamondminus\forall xA)$	$\Box\forall x(A \rightarrow B) \rightarrow (\diamondminus\exists xB \rightarrow \diamondminus\exists xA)$
$\forall x\Box(A \rightarrow B) \rightarrow (\forall x\diamondminus B \rightarrow \forall x\diamondminus A)$	$\forall x\Box(A \rightarrow B) \rightarrow (\exists x\diamondminus B \rightarrow \exists x\diamondminus A)$
$\Box\forall x(A \leftrightarrow B) \rightarrow (\Box\forall xA \leftrightarrow \Box\forall xB)$	$\Box\forall x(A \leftrightarrow B) \rightarrow (\Diamond\forall xA \leftrightarrow \Diamond\forall xB)$
$\Box\forall x(A \leftrightarrow B) \rightarrow (\Box\exists xA \leftrightarrow \Box\exists xB)$	$\Box\forall x(A \leftrightarrow B) \rightarrow (\Diamond\exists xA \leftrightarrow \Diamond\exists xB)$
$\forall x\Box(A \leftrightarrow B) \rightarrow (\forall x\Box A \leftrightarrow \forall x\Box B)$	$\forall x\Box(A \leftrightarrow B) \rightarrow (\forall x\Diamond A \leftrightarrow \forall x\Diamond B)$
$\forall x\Box(A \leftrightarrow B) \rightarrow (\exists x\Box A \leftrightarrow \exists x\Box B)$	$\forall x\Box(A \leftrightarrow B) \rightarrow (\exists x\Diamond A \leftrightarrow \exists x\Diamond B)$
$\Box\forall x(A \leftrightarrow B) \rightarrow (\diamondminus\forall xA \leftrightarrow \diamondminus\forall xB)$	$\Box\forall x(A \leftrightarrow B) \rightarrow (\diamondminus\exists xA \leftrightarrow \diamondminus\exists xB)$
$\forall x\Box(A \leftrightarrow B) \rightarrow (\forall x\diamondminus A \leftrightarrow \forall x\diamondminus B)$	$\forall x\Box(A \leftrightarrow B) \rightarrow (\exists x\diamondminus A \leftrightarrow \exists x\diamondminus B)$

Table 17.

$\forall x\Box(A \wedge B) \rightarrow (\forall x\mathbf{O}A \wedge \forall x\mathbf{O}B)$	$\forall x\Box(A \leftrightarrow B) \rightarrow (\forall x\mathbf{O}A \leftrightarrow \forall x\mathbf{O}B)$
$(\forall x\Box A \vee \forall x\Box B) \rightarrow \forall x\mathbf{O}(A \vee B)$	$\forall x\Box(A \leftrightarrow B) \rightarrow (\forall x\mathbf{P}A \leftrightarrow \forall x\mathbf{P}B)$
$(\forall x\Box A \wedge \forall x\Box B) \rightarrow \forall x\mathbf{O}(A \wedge B)$	$\forall x\Box(A \leftrightarrow B) \rightarrow (\forall x\neg\mathbf{P}A \leftrightarrow \forall x\neg\mathbf{P}B)$
$\forall x\mathbf{P}(A \wedge B) \rightarrow (\forall x\Diamond A \wedge \forall x\Diamond B)$	$(\forall x\mathbf{O}A \wedge \forall x\Box(A \rightarrow B)) \rightarrow \forall x\mathbf{O}B$
$\exists x\mathbf{P}(A \vee B) \rightarrow (\exists x\Diamond A \vee \exists x\Diamond B)$	$\forall x\Box(A \rightarrow B) \rightarrow (\forall x\mathbf{O}A \rightarrow \forall x\mathbf{O}B)$
$(\exists x\mathbf{P}A \vee \exists x\mathbf{P}B) \rightarrow \exists x\Diamond(A \vee B)$	$(\forall x\mathbf{P}A \wedge \forall x\Box(A \rightarrow B)) \rightarrow \forall x\mathbf{P}B$
$\forall x\neg\Diamond(A \vee B) \rightarrow (\forall x\neg\mathbf{P}A \wedge \forall x\neg\mathbf{P}B)$	$\forall x\Box(A \rightarrow B) \rightarrow (\forall x\mathbf{P}A \rightarrow \forall x\mathbf{P}B)$
$(\exists x\neg\Diamond A \vee \exists x\neg\Diamond B) \rightarrow \exists x\neg\mathbf{P}(A \wedge B)$	$(\forall x\neg\mathbf{P}B \wedge \forall x\Box(A \rightarrow B)) \rightarrow \forall x\neg\mathbf{P}A$
$(\forall x\neg\Diamond A \wedge \forall x\neg\Diamond B) \rightarrow \forall x\neg\mathbf{P}(A \vee B)$	$\forall x\Box(A \rightarrow B) \rightarrow (\forall x\neg\mathbf{P}B \rightarrow \forall x\neg\mathbf{P}A)$
$\forall x\Box(A \rightarrow B) \rightarrow (\forall x\Box A \rightarrow \forall x\mathbf{O}B)$	$\forall x\Box(A \rightarrow B) \rightarrow (\forall x\mathbf{P}A \rightarrow \forall x\Diamond B)$
$\forall x\Box(A \rightarrow B) \rightarrow (\forall x\neg\Diamond B \rightarrow \forall x\neg\mathbf{P}A)$	$(\forall x\mathbf{O}(A \vee B) \wedge \forall x\neg\Diamond B) \rightarrow \forall x\mathbf{O}A$

Table 18.

(ii) *If* S *includes* T-DBF, *then* $\forall x\mathbf{O}A \rightarrow \mathbf{O}\forall xA$ *and* $\mathbf{P}\exists xA \rightarrow \exists x\mathbf{P}A$ *are theorems in* S. *If it includes* T-DCBF, *then* $\mathbf{O}\forall xA \rightarrow \forall x\mathbf{O}A$,

$\forall x\Box((A \vee B) \to C) \to ((\exists x\mathbf{O}A \vee \exists x\mathbf{O}B) \to \exists x\mathbf{O}C)$
$\forall x\Box((A \vee B) \to C) \to ((\exists x\mathbf{P}A \vee \exists x\mathbf{P}B) \to \exists x\mathbf{P}C)$
$\forall x\Box((A \vee B) \to C) \to (\forall x\neg\mathbf{P}C \to (\forall x\neg\mathbf{P}A \wedge \forall x\neg\mathbf{P}B))$
$\forall x\Box(A \to (B \vee C)) \to (\exists x\mathbf{P}A \to (\exists x\mathbf{P}B \vee \exists x\mathbf{P}C))$
$\forall x\Box(A \to (B \vee C)) \to ((\forall x\neg\mathbf{P}B \wedge \forall x\neg\mathbf{P}C) \to \forall x\neg\mathbf{P}A)$
$\forall x\Box((A \wedge B) \to C) \to ((\forall x\mathbf{O}A \wedge \forall x\mathbf{O}B) \to \forall x\mathbf{O}C)$
$\forall x\Box(A \to (B \wedge C)) \to (\forall x\mathbf{O}A \to (\forall x\mathbf{O}B \wedge \forall x\mathbf{O}C))$
$\forall x\Box(A \to (B \wedge C)) \to (\forall x\mathbf{P}A \to (\forall x\mathbf{P}B \wedge \forall x\mathbf{P}C))$
$\forall x\Box(A \to (B \wedge C)) \to ((\exists x\neg\mathbf{P}B \vee \exists x\neg\mathbf{P}C) \to \exists x\neg\mathbf{P}A)$
$(\forall x\mathbf{O}(A \vee B) \wedge (\forall x\Box(A \to C) \wedge \forall x\Box(B \to C))) \to \forall x\mathbf{O}C$
$(\forall x\mathbf{O}(A \vee B) \wedge (\forall x\Box(A \to C) \wedge \forall x\Box(B \to D))) \to \forall x\mathbf{O}(C \vee D)$
$(\forall x\mathbf{O}A \wedge (\forall x\Box(A \to B) \wedge \forall x\Box(A \to C))) \to (\forall x\mathbf{O}B \wedge \forall x\mathbf{O}C)$
$(\forall x\mathbf{O}(A \wedge B) \wedge \forall x(\Box(A \to C) \vee \Box(B \to D))) \to \forall x\mathbf{O}(C \vee D)$
$(\forall x\mathbf{O}A \wedge \forall x(\Box(A \to B) \vee \Box(A \to C))) \to \forall x\mathbf{O}(B \vee C)$
$(\forall x\mathbf{O}(A \wedge B) \wedge (\forall x\Box(A \to C) \wedge \forall x\Box(B \to D))) \to (\forall x\mathbf{O}C \wedge \forall x\mathbf{O}D)$

Table 19.

$\exists x\mathbf{P}A \to \mathbf{P}\exists xA$, $\mathbf{P}\forall xA \to \forall x\mathbf{P}A$, *and* $\exists x\mathbf{O}A \to \mathbf{O}\exists xA$ *are theorems in* S.

(iii) *If* S *includes* T-TBF, *then the following sentences are theorems in* S: $\forall x\mathsf{G}A \to \mathsf{G}\forall xA$, $\mathsf{F}\exists xA \to \exists x\mathsf{F}A$, $\mathsf{H}\forall xA \to \forall x\mathsf{H}A$, $\exists x\mathsf{P}A \to \mathsf{P}\exists xA$, $\forall x[\mathsf{G}]A \to [\mathsf{G}]\forall xA$, $\langle\mathsf{F}\rangle\exists xA \to \exists x\langle\mathsf{F}\rangle A$, $[\mathsf{H}]\forall xA \to \forall x[\mathsf{H}]A$, $\exists x\langle\mathsf{P}\rangle A \to \langle\mathsf{P}\rangle\exists xA$, $\mathsf{P}\forall xA \to \forall x\mathsf{P}A$, $\exists x\mathsf{H}A \to \mathsf{H}\exists xA$, $\langle\mathsf{P}\rangle\forall xA \to \forall x\langle\mathsf{P}\rangle A$, $\exists x[\mathsf{H}]A \to [\mathsf{H}]\exists xA$.

(iv) *If* S *includes* T-TCBF, *then the following sentences are theorems in* S: $\mathsf{G}\forall xA \to \forall x\mathsf{G}A$, $\exists x\mathsf{F}A \to \mathsf{F}\exists xA$, $\forall x\mathsf{H}A \to \mathsf{H}\forall xA$, $\mathsf{P}\exists xA \to \exists x\mathsf{P}A$, $[\mathsf{G}]\forall xA \to \forall x[\mathsf{G}]A$, $\exists x\langle\mathsf{F}\rangle A \to \langle\mathsf{F}\rangle\exists xA$, $\forall x[\mathsf{H}]A \to [\mathsf{H}]\forall xA$, $\langle\mathsf{P}\rangle\exists xA \to \exists x\langle\mathsf{P}\rangle A$, $\mathsf{F}\forall xA \to \forall x\mathsf{F}A$, $\exists x\mathsf{G}A \to \mathsf{G}\exists xA$, $\langle\mathsf{F}\rangle\forall xA \to \forall x\langle\mathsf{F}\rangle A$, $\exists x[\mathsf{G}]A \to [\mathsf{G}]\exists xA$.

(v) *If* S *doesn't contain any domain-inclusion rules, then the actualist quantifiers do not "commute" with any modal operators.*

Theorem 7. *The following equivalences hold in every* S.

(i) $\Pi x\mathbf{O}\mathsf{A}B \leftrightarrow \mathbf{O}\Pi x\mathsf{A}B \leftrightarrow \mathbf{O}\mathsf{A}\Pi xB \leftrightarrow \neg\Sigma x\mathbf{P}\mathsf{S}\neg B \leftrightarrow \neg\mathbf{P}\Sigma x\mathsf{S}\neg B \leftrightarrow \neg\mathbf{P}\mathsf{S}\Sigma x\neg B$.

(ii) $\Pi x\mathsf{A}\mathbf{O}B \leftrightarrow \mathsf{A}\Pi x\mathbf{O}B \leftrightarrow \mathsf{A}\mathbf{O}\Pi xB \leftrightarrow \neg\Sigma x\mathsf{S}\mathbf{P}\neg B \leftrightarrow \neg\mathsf{S}\Sigma x\mathbf{P}\neg B \leftrightarrow \neg\mathsf{S}\mathbf{P}\Sigma x\neg B$.

(iii) $\Pi x\mathbf{O}\mathsf{G}B \leftrightarrow \mathbf{O}\Pi x\mathsf{G}B \leftrightarrow \mathbf{O}\mathsf{G}\Pi xB \leftrightarrow \neg\Sigma x\mathbf{P}\mathsf{F}\neg B \leftrightarrow \neg\mathbf{P}\Sigma x\mathsf{F}\neg B \leftrightarrow \neg\mathbf{P}\mathsf{F}\Sigma x\neg B$.

(iv) $\Pi x\mathsf{G}\mathbf{O}B \leftrightarrow \mathsf{G}\Pi x\mathbf{O}B \leftrightarrow \mathsf{G}\mathbf{O}\Pi xB \leftrightarrow \neg\Sigma x\mathsf{F}\mathbf{P}\neg B \leftrightarrow \neg\mathsf{F}\Sigma x\mathbf{P}\neg B \leftrightarrow \neg\mathsf{F}\mathbf{P}\Sigma x\neg B$.

(v) *The formulas in* (i) *are not logically equivalent with the formulas in* (ii), *and the formulas in* (iii) *are not logically equivalent with the formulas in* (iv).

Let A be atomic. Then $\mathbf{O}\mathsf{A}A \leftrightarrow (\mathsf{H}A \wedge A \wedge \mathbf{O}\mathsf{G}A)$, $\mathbf{O}[\mathsf{G}]A \leftrightarrow (A \wedge \mathbf{O}\mathsf{G}A)$, $\mathbf{O}\langle\mathsf{F}\rangle A \leftrightarrow (A \vee \mathbf{O}\mathsf{F}A)$, $\mathbf{O}\mathsf{S}A \leftrightarrow (\mathsf{P}A \vee A \vee \mathbf{O}\mathsf{F}A)$ hold in $aTB4dD5adMOtCadtSPBTFT$.

So, it is problematic to symbolize a sentence such as "Everyone ought always to be honest" as $\Pi x \mathbf{O}\mathsf{A}Hx$ in some logics, since this is false if someone in the past wasn't honest or someone now isn't honest. In these systems, which are plausible if we assume that the past and present are settled, a norm is "reasonable" only when it is future oriented (see [37]). But we can get a similar result by using G instead of A. "Everyone ought always (in the future) to be honest" can then be symbolized as $\Pi x \mathbf{O}\mathsf{G}Hx$. The interesting norms in these systems often have one of the following forms: $\mathbf{O}\mathsf{G}A$, $\mathbf{O}\mathsf{F}A$, $\mathbf{O}\mathsf{R}tA$, $\mathbf{O}t_{\mathsf{P}}A$, $\mathbf{O}t_{\mathsf{F}}A$, $\mathbf{O}t_{\langle\mathsf{P}\rangle}A$, $\mathbf{O}t_{\langle\mathsf{F}\rangle}A$, where $v(t)$ lies in the future.

THEOREM 8. (i) *Let* $\blacksquare = \mathbf{U}$, A, $[\mathsf{G}]$, G, $\Box$, O, $[\mathsf{H}]$, H, $\mathsf{R}t$, $t_{[\mathsf{H}]}$, t_{H}, $t_{[\mathsf{G}]}$ *or* t_{H}. *Let* $\blacklozenge = \mathbf{M}$, S, $\langle\mathsf{F}\rangle$, F, $\Diamond$, P, $\langle\mathsf{P}\rangle$, P, $\mathsf{R}t$, $t_{\langle\mathsf{P}\rangle}$, t_{P}, $t_{\langle\mathsf{F}\rangle}$ *or* t_{F}. *Then every formula in Table 20 is a theorem in* S.

(ii) *Let* $\blacksquare = \mathsf{A}$, $[\mathsf{G}]$, G, $[\mathsf{H}]$, H, $\mathsf{R}t$, $t_{[\mathsf{H}]}$, t_{H}, $t_{[\mathsf{G}]}$ *or* t_{H}. *Let* $\blacklozenge = \mathsf{S}$, $\langle\mathsf{F}\rangle$, F, $\langle\mathsf{P}\rangle$, P, $\mathsf{R}t$, $t_{\langle\mathsf{P}\rangle}$, t_{P}, $t_{\langle\mathsf{F}\rangle}$ *or* t_{F}. *Then every formula in Table 21 is a theorem in* S.

$\Pi x \blacksquare A \leftrightarrow \blacksquare \Pi x A$	$\Sigma x \blacklozenge A \leftrightarrow \blacklozenge \Sigma x A$
$\Sigma x \blacksquare A \rightarrow \blacksquare \Sigma x A$	$\blacklozenge \Pi x A \rightarrow \Pi x \blacklozenge A$
$\forall_M x \blacksquare A \leftrightarrow \blacksquare \forall_M x A$	$\exists_M x \blacklozenge A \leftrightarrow \blacklozenge \exists_M x A$
$\exists_M x \blacksquare A \rightarrow \blacksquare \exists_M x A$	$\blacklozenge \forall_M x A \rightarrow \forall_M x \blacklozenge A$
$\forall_U x \blacksquare A \leftrightarrow \blacksquare \forall_U x A$	$\exists_U x \blacklozenge A \leftrightarrow \blacklozenge \exists_U x A$
$\exists_U x \blacksquare A \rightarrow \blacksquare \exists_U x A$	$\blacklozenge \forall_U x A \rightarrow \forall_U x \blacklozenge A$

Table 20.

$\forall_{\mathsf{S}} x \blacksquare A \leftrightarrow \blacksquare \forall_{\mathsf{S}} x A$	$\exists_{\mathsf{S}} x \blacklozenge A \leftrightarrow \blacklozenge \exists_{\mathsf{S}} x A$
$\exists_{\mathsf{S}} x \blacksquare A \rightarrow \blacksquare \exists_{\mathsf{S}} x A$	$\blacklozenge \forall_{\mathsf{S}} x A \rightarrow \forall_{\mathsf{S}} x \blacklozenge A$
$\forall_{\mathsf{A}} x \blacksquare A \leftrightarrow \blacksquare \forall_{\mathsf{A}} x A$	$\exists_{\mathsf{A}} x \blacklozenge A \leftrightarrow \blacklozenge \exists_{\mathsf{A}} x A$
$\exists_{\mathsf{A}} x \blacksquare A \rightarrow \blacksquare \exists_{\mathsf{A}} x A$	$\blacklozenge \forall_{\mathsf{A}} x A \rightarrow \forall_{\mathsf{A}} x \blacklozenge A$
$\forall_{\mathsf{R}t} x \blacksquare A \leftrightarrow \blacksquare \forall_{\mathsf{R}t} x A$	$\exists_{\mathsf{R}t} x \blacklozenge A \leftrightarrow \blacklozenge \exists_{\mathsf{R}t} x A$
$\exists_{\mathsf{R}t} x \blacksquare A \rightarrow \blacksquare \exists_{\mathsf{R}t} x A$	$\blacklozenge \forall_{\mathsf{R}t} x A \rightarrow \forall_{\mathsf{R}t} x \blacklozenge A$

Table 21.

THEOREM 9. *Let S be any system with necessary identity and let $\odot$ be any necessity-like positive modal operator. Then $\forall x \forall y(x = y \to \odot(x = y))$ and $\forall x \forall y(\neg x = y \to \odot\neg(x = y))$ are theorems in S.*

THEOREM 10. *Let $\blacksquare = \mathsf{R}t$, A, H, G, $[\mathsf{H}]$ or $[\mathsf{G}]$, let $\blacklozenge$ be the dual of $\blacksquare$, and let $\odot = t_{\mathsf{P}}, t_{\mathsf{H}}, t_{\langle \mathsf{P} \rangle}, t_{[\mathsf{H}]}, t_{\mathsf{F}}, t_{\mathsf{G}}, t_{\langle \mathsf{F} \rangle}$ or $t_{[\mathsf{G}]}$. Then the following hold in every S: $\forall_{\odot} x \blacksquare A \leftrightarrow \blacksquare \forall_{\odot} x A$, $\exists_{\odot} x \blacklozenge A \leftrightarrow \blacklozenge \exists_{\odot} x A$, $\exists_{\odot} x \blacksquare A \to \blacksquare \exists_{\odot} x A$, $\blacklozenge \forall_{\odot} x A \to \forall_{\odot} x \blacklozenge A$.*

6. Soundness and completeness theorems

We are now in a position to prove that every system in this essay is sound and complete with respect to its semantics. The concepts of soundness and completeness are defined as usual (see e.g. [37]).[5]

6.1. Constant domain logics

Let us first consider all systems with a constant domain. We start with the weakest logic, *CS*, and then take a look at modifications required for stronger systems. At this stage we assume that our language doesn't contain the identity predicate or any descriptors.

LEMMA 11 (Locality). *Let $\mathcal{M}_1 = \langle D, W, T, <, R, S, v_1 \rangle$, $\mathcal{M}_2 = \langle D, W, T, <, R, S, v_2 \rangle$ be two (constant domain) models. The language of the two, which we call $\mathcal{L}$, is the same, for they have the same domain. Let A be any closed formula of $\mathcal{L}$ such that v_1 and v_2 agree on the denotations of all the predicates and constants in it. Then for all $\omega \in W$ and $\tau \in T$: $v_{1\omega\tau}(A) = v_{2\omega\tau}(A)$.*

PROOF. The proof is by recursion on the sentences in our language. "IH" refers to the induction hypothesis.

Atomic formulas. $v_{1\omega\tau}(Pa_1 \dots a_n) = 1$ iff $\langle v_1(a_1), \dots, v_1(a_n) \rangle \in v_{1\omega\tau}(P)$ iff $\langle v_2(a_1), \dots, v_2(a_n) \rangle \in v_{2\omega\tau}(P)$ iff $v_{2\omega\tau}(Pa_1 \dots a_n) = 1$.

Truth-functional connectives. Straightforward.

($\Box$) $v_{1\omega\tau}(\Box B) = 1$ iff for all ω' such that $R\omega\omega'\tau$, $v_{1\omega'\tau}(B) = 1$ iff for all ω' such that $R\omega\omega'\tau$, $v_{2\omega'\tau}(B) = 1$ (IH) iff $v_{2\omega\tau}(\Box B) = 1$.

[5] The proofs in this section combine techniques from [36] and [37]. In all the soundness and completeness theorems, the new steps for our new temporal rules are straightforward and omitted.

(G) $v_{1\omega\tau}(\mathsf{G}B) = 1$ iff for all τ' such that $\tau < \tau'$, $v_{1\omega\tau'}(B) = 1$ iff for all τ' such that $\tau < \tau'$, $v_{2\omega\tau'}(B) = 1$ (IH) iff $v_{2\omega\tau}(\mathsf{G}B) = 1$.

The cases for the other primitive positive modal operators are similar.

(Π) $v_{1\omega\tau}(\Pi x B) = 1$ iff for all $d \in D$, $v_{1\omega\tau}(B[k_d/x]) = 1$ iff for all $d \in D$, $v_{2\omega\tau}(B[k_d/x]) = 1$ ((IH), and the fact that $v_{1\omega\tau}(k_d) = v_{2\omega\tau}(k_d) = d$) iff $v_{2\omega\tau}(\Pi x B) = 1$.

The case for the particular quantifier is similar. ⊣

LEMMA 12 (Denotation). *Let $\mathcal{M} = \langle D, W, T, <, R, S, v\rangle$ be any (constant domain) model. Let A be any formula of $\mathcal{L}(\mathcal{M})$ with at most one free variable, x, and a and b be any two (non-temporal) constants such that $v(a) = v(b)$. Then for any $\omega \in W$ and $\tau \in T$: $v_{\omega\tau}(A[a/x]) = v_{\omega\tau}(A[b/x])$.*

PROOF. The proof is by induction on the complexity of A.

Atomic formulas. (To illustrate, we assume that the formula has one occurrence of "a", distinct from each a_i.) $v_{\omega\tau}(Pa_1...a...a_n) = 1$ iff $\langle v(a_1), ..., v(a), ..., v(a_n)\rangle \in v_{\omega\tau}(P)$ iff $\langle v(a_1), ..., v(b), ..., v(a_n)\rangle \in v_{\omega\tau}(P)$ iff $v_{\omega\tau}(Pa_1...b...a_n) = 1$.

Truth-functional connectives. Straightforward.

($\Box$) $v_{\omega\tau}(\Box B[a/x]) = 1$ iff for all ω' such that $R\omega\omega'\tau$, $v_{\omega'\tau}(B[a/x]) = 1$ iff for all ω' such that $R\omega\omega'\tau$, $v_{\omega'\tau}(B[b/x]) = 1$ (IH) iff $v_{\omega\tau}(\Box B[b/x]) = 1$.

(G) $v_{\omega\tau}(\mathsf{G}B[a/x]) = 1$ iff for all τ' such that $\tau < \tau'$, $v_{\omega\tau'}(B[a/x]) = 1$ iff for all τ' such that $\tau < \tau'$, $v_{\omega\tau'}(B[b/x]) = 1$ (IH) iff $v_{\omega\tau}(\mathsf{G}B[b/x]) = 1$.

Note that $(\Box B)[a/x] = \Box(B[a/x])$. So, the ambiguity in $\Box B[a/x]$ is harmless. The same goes for the ambiguity in $\mathsf{G}B[a/x]$. The arguments for the other primitive positive modal operators are similar.

(Π) Let A be of the form $\Pi y B$. If $x = y$, then $A[a/x] = A[b/x] = A$, so the result is trivial. Accordingly, suppose that x and y are distinct. Then, $(\Pi y B)[b/x] = \Pi y(B[b/x])$ and $(B[b/x])[a/y] = (B[a/y])[b/x]$. $v_{\omega\tau}((\Pi y B)[a/x]) = 1$ iff $v_{\omega\tau}(\Pi y(B[a/x])) = 1$ iff for all $d \in D$, $v_{\omega\tau}((B[a/x])[k_d/y]) = 1$ iff for all $d \in D$, $v_{\omega\tau}((B[k_d/y])[a/x]) = 1$ iff for all $d \in D$, $v_{\omega\tau}((B[k_d/y])[b/x]) = 1$ (IH) iff for all $d \in D$, $v_{\omega\tau}((B[b/x])[k_d/y]) = 1$ iff $v_{\omega\tau}(\Pi y(B[b/x])) = 1$ iff $v_{\omega\tau}((\Pi y B)[b/x]) = 1$.

The case for the particular quantifier is similar. ⊣

6.1.1. Soundness theorem

Let $\mathcal{M}$ be any (constant domain) model and $\mathcal{B}$ any branch of a tableau. Then $\mathcal{B}$ is satisfiable in $\mathcal{M}$ iff there is a function f from $w_0, w_1, w_2, \ldots$ to W, and a function g from $t_0, t_1, t_2, \ldots$ to T such that (i) A is true in $f(w_i)$

at $g(t_j)$ in $\mathcal{M}$, for every node A, w_it_j on $\mathcal{B}$, (ii) if $rw_iw_jt_k$ is on $\mathcal{B}$, then $Rf(w_i)f(w_j)g(t_k)$ in $\mathcal{M}$, (iii) if $sw_iw_jt_k$ is on $\mathcal{B}$, then $Sf(w_i)f(w_j)g(t_k)$ in $\mathcal{M}$, (iv) if $t_i < t_j$ is on $\mathcal{B}$, then $g(t_i) < g(t_j)$ in $\mathcal{M}$, (v) if $t_i = t_j$ is on $\mathcal{B}$, then $g(t_i) = g(t_j)$ in $\mathcal{M}$. If these conditions are fulfilled, we say that f and g show that $\mathcal{B}$ is satisfiable in $\mathcal{M}$.

Lemma 13 (Soundness Lemma). *Let $\mathcal{B}$ be any branch of a tableau and $\mathcal{M}$ be any (constant domain) model. If $\mathcal{B}$ is satisfiable in $\mathcal{M}$ and a tableau rule is applied to it, then there is a (constant domain) model $\mathcal{M}'$ and an extension of $\mathcal{B}$, $\mathcal{B}'$, such that $\mathcal{B}'$ is satisfiable in $\mathcal{M}'$.*

Proof. As usual the proof is an induction. Let f and g be functions that show that the branch $\mathcal{B}$ is satisfiable in $\mathcal{M}$.

Connectives and the modal operators. See [37].

($\neg\Pi$) Since $\mathcal{B}$ is satisfiable in $\mathcal{M}$, $\neg\Pi xA$ is true in $f(w_i)$ at $g(t_j)$ in $\mathcal{M}$. Thus, $\Pi xA(x)$ is false in $f(w_i)$ at $g(t_j)$ in $\mathcal{M}$. Accordingly, there is some $d \in D$ such that $A[k_d/x]$ is false in $f(w_i)$ at $g(t_j)$ in $\mathcal{M}$, i.e., $\neg A[k_d/x]$ is true in $f(w_i)$ at $g(t_j)$ in $\mathcal{M}$. Consequently, $\Sigma x\neg A$ is true in $f(w_i)$ at $g(t_j)$ in $\mathcal{M}$. We can therefore take $\mathcal{M}'$ to be $\mathcal{M}$. The argument for ($\neg\Sigma$) is similar.

(Π) $\mathcal{M}$ makes ΠxA true in $f(w_i)$ at $g(t_j)$. For $\mathcal{B}$ is satisfiable in $\mathcal{M}$. Hence, $A[k_d/x]$ is true in $f(w_i)$ at $g(t_j)$ in $\mathcal{M}$, for all $d \in D$. Let d be such that $v(a) = v(k_d)$. By the Denotation Lemma, $A[a/x]$ is true in $f(w_i)$ at $g(t_j)$ in $\mathcal{M}$. Accordingly we can take $\mathcal{M}'$ to be $\mathcal{M}$.

(Σ) Since $\mathcal{B}$ is satisfiable in $\mathcal{M}$, ΣxA is true in $f(w_i)$ at $g(t_j)$ in $\mathcal{M}$. Hence, there is some $d \in D$ such that $\mathcal{M}$ makes $A[k_d/x]$ true in $f(w_i)$ at $g(t_j)$. Let $\mathcal{M}' = \langle D, W, T, <, R, S, v'\rangle$ be the same as $\mathcal{M}$ except that $v'(c) = d$. Since c does not occur in $A[k_d/x]$, $A[k_d/x]$ is true in $f(w_i)$ at $g(t_j)$ in $\mathcal{M}'$, by the Locality Lemma. By the Denotation Lemma and the fact that $v'(c) = d = v'(k_d)$, $A[c/x]$ is true in $f(w_i)$ at $g(t_j)$ in $\mathcal{M}'$. Furthermore, $\mathcal{M}'$ makes all other formulas on the branch true at their respective world-moment pairs as well, by the Locality Lemma. For c does not occur in any other formula on the branch. ⊣

Theorem 14 (Soundness Theorem). *CS is strongly sound with respect to its semantics.*

Proof. The proof is as in [37]. ⊣

6.1.2. Completeness theorem

DEFINITION 15 (Induced Model). Let $\mathcal{B}$ be an open complete branch of a tableau, let t_i, t_j and t_k be temporal constants, and let I be the set of numbers on $\mathcal{B}$ immediately preceded by "t" in a temporal constant. We shall say that $i \rightleftharpoons j$ just in case $i = j$, or "$t_i = t_j$" or "$t_j = t_i$" occur on $\mathcal{B}$. $\rightleftharpoons$ is an equivalence relation and $[i]$ is the equivalence class of i. Furthermore, let C be the set of all non-temporal, rigid constants on $\mathcal{B}$. The (constant domain) model, $\mathcal{M} = \langle D, W, T, <R, S, v\rangle$, induced by $\mathcal{B}$ is defined as follows. $D = \{o_a : a \in C\}$ (or if C is empty, $D = \{o\}$, for some arbitrary o). For all non-temporal, rigid constants, a, on $\mathcal{B}$, $v(a) = o_a$. For every n-place predicate on $\mathcal{B}$ $\langle o_{a_1}, \ldots, o_{a_n}\rangle \in v_{\omega_i\tau_{[j]}}(P)$ iff $Pa_1 \ldots a_n, w_it_j$ is on $\mathcal{B}$. (o is not in the extension of anything.) $W = \{\omega_i : w_i \textit{ occurs on } \mathcal{B}\}$, $T = \{\tau_{[i]} : i \in I\}$, $\tau_{[i]} < \tau_{[j]}$ iff $t_i < t_j$ occurs on $\mathcal{B}$, $R\omega_i\omega_j\tau_{[k]}$ iff $rw_iw_jt_k$ occurs on $\mathcal{B}$, $S\omega_i\omega_j\tau_{[k]}$ iff $sw_iw_jt_k$ occurs on $\mathcal{B}$. If a temporal constant, t_i, occurs on $\mathcal{B}$, then $v(t_i) = \tau_{[i]}$. If our tableau system neither includes T-FC, T-PC nor T-C, $\rightleftharpoons$ is reduced to identity and $[i] = i$. Hence, in such systems, we may take T to be $\{\tau_i : t_i \textit{ occurs on } \mathcal{B}\}$ and dispense with the equivalence classes.

LEMMA 16 (Completeness Lemma). *Let $\mathcal{B}$ be an open branch in a complete tableau and let $\mathcal{M}$ be a (constant domain) model induced by $\mathcal{B}$. Then, for every formula A:*
(i) If A, w_it_j is on $\mathcal{B}$ then $v_{\omega_i\tau_{[j]}}(A) = 1$, and (ii) If $\neg A, w_it_j$ is on $\mathcal{B}$ then $v_{\omega_i\tau_{[j]}}(A) = 0$.

PROOF. The proof is by induction on the complexity of A.

Atomic formulas. $Pa_1...a_n, w_it_j$ is on $\mathcal{B} \Rightarrow \langle o_{a_1}, \ldots, o_{a_n}\rangle \in v_{\omega_i\tau_{[j]}}(P)$ $\Rightarrow \langle v(a_1), \ldots, v(a_n)\rangle \in v_{\omega_i\tau_{[j]}}(P) \Rightarrow v_{\omega_i\tau_{[j]}}(Pa_1 \ldots a_n) = 1$.

$(\neg)$ $\neg Pa_1 \ldots a_n, w_it_j$ is on $\mathcal{B} \Rightarrow Pa_1 \ldots a_n, w_it_j$ is not on $\mathcal{B}$ ($\mathcal{B}$ open) $\Rightarrow \langle o_{a_1}, \ldots, o_{a_n}\rangle \notin v_{\omega_i\tau_{[j]}}(P) \Rightarrow \langle v(a_1), \ldots, v(a_n)\rangle \notin v_{\omega_i\tau_{[j]}}(P)$ $\Rightarrow v_{\omega_i\tau_{[j]}}(Pa_1 \ldots a_n) = 0$.

Other truth-functional connectives and modal operators. The argument is as in [37].

(Σ) Suppose that $\Sigma xA, w_it_j$ is on the branch. Since the tableau is complete (Σ) has been applied. Accordingly, for some c, $A[c/x], w_it_j$ is on the branch. Hence, $v_{\omega_i\tau_{[j]}}(A[c/x]) = 1$, by (IH). For some $d \in D$, $v(c) = d$. However, $v(k_d) = d$. Consequently, $v_{\omega_i\tau_{[j]}}(A[k_d/x]) = 1$, by the Denotation Lemma. It follows that $v_{\omega_i\tau_{[j]}}(\Sigma xA) = 1$. Suppose that $\neg\Sigma xA, w_it_j$ is on the branch. Since the tableau is complete $(\neg\Sigma)$ has been applied. So, $\Pi x \neg A, w_i t_j$ is on the branch. Again, since the tableau is complete (Π) has been applied. Thus, for all $c \in C$, $\neg A[c/x], w_i t_j$ is on the branch. Consequently, $v_{\omega_i \tau_{[j]}}(A[c/x]) = 0$ for all $c \in C$, by (IH). If $d \in D$, then for some $c \in C$, $v(c) = v(k_d)$. By the Denotation Lemma, for all $d \in D$, $v_{\omega_i \tau_{[j]}}(A[k_d/x]) = 0$. Consequently, $v_{\omega_i \tau_{[j]}}(\Sigma x A) = 0$.

The case for Π is similar. ⊣

THEOREM 17 (Completeness Theorem). *CS is strongly complete with respect to its semantics.*

PROOF. The proof is as in [37]. ⊣

6.1.3. General correctness theorem

THEOREM 18 (General Correctness Theorem). *Let S be any of the constant domain tableau systems discussed in this essay (without identity). Then S is (strongly) sound and complete with respect to its semantics.*

PROOF. The proof is as for *CS*, with some minor modifications. There are new cases for the various accessibility rules in the Soundness Lemma. In the proof of the Completeness Theorem, we have to check that the induced model is a model of the appropriate kind. This is as in [37]. ⊣

6.2. Variable domain logics

In this section, we will prove soundness and completeness theorems for variable domain systems. We start with the weakest system, *VS*, and then consider all stronger systems. We also consider the addition of the domain-inclusion (Barcan) rules.

LEMMA 19 (Locality). *The same as in the constant domain case, except that we replace "constant domain" with "variable domain".*

PROOF. We use the actualist quantifiers. However, the proof is essentially as in the constant domain case. The only difference is that clauses of the form "$d \in D$" are replaced by ones of the form "$d \in D_{\omega\tau}$". ⊣

LEMMA 20 (Denotation). *The same as in the constant domain case, except that we replace "constant domain" with "variable domain".*

PROOF. Again, we use the actualist quantifiers. And again, the proof is essentially as in the constant domain case. The only difference is that clauses of the form "$d \in D$" are replaced by ones of the form "$d \in D_{\omega\tau}$". ⊣

6.2.1. Soundness theorem

THEOREM 21 (Soundness Theorem). *VS is strongly sound with respect to its semantics.*

PROOF. The proof is as in the constant domain case. However, now we are using the actualist quantifiers. So, we have to add steps for them in the Soundness Lemma. Let f and g be functions that show that the branch $\mathcal{B}$ is satisfiable in $\mathcal{M}$.

($\forall$) Since $\mathcal{B}$ is satisfiable in $\mathcal{M}$, $\forall xA$ is true in $f(w_i)$ at $g(t_j)$ in $\mathcal{M}$. Hence, $A[k_d/x]$ is true in $f(w_i)$ at $g(t_j)$ in $\mathcal{M}$, for all $d \in D_{f(w_i)g(t_j)}$. Consequently, for any $d \in D$, either $\neg Ek_d$ or $A[k_d/x]$ is true in $f(w_i)$ at $g(t_j)$ in $\mathcal{M}$. Let d be such that $v(a) = v(k_d)$. By the Denotation Lemma, either $\neg Ea$ or $A[a/x]$ is true in $f(w_i)$ at $g(t_j)$ in $\mathcal{M}$. Consequently, at least one branch is satisfiable in $\mathcal{M}$, and we can take $\mathcal{M}'$ to be $\mathcal{M}$.

($\exists$) Since $\mathcal{B}$ is satisfiable in $\mathcal{M}$, $\exists xA$ is true in $f(w_i)$ at $g(t_j)$ in $\mathcal{M}$. Accordingly, for some $d \in D_{f(w_i)g(t_j)}$, $A[k_d/x]$ is true in $f(w_i)$ at $g(t_j)$ in $\mathcal{M}$. Hence, $\mathcal{M}$ makes $\mathrm{E}k_d$ and $A[k_d/x]$ true in $f(w_i)$ at $g(t_j)$. Let $\mathcal{M}' = \langle D, W, T, <, R, S, v'\rangle$ be the same as $\mathcal{M}$, except that $v'(c) = d$. By the Locality Lemma, $\mathcal{M}'$ makes $\mathrm{E}k_d$ and $A[k_d/x]$ true in $f(w_i)$ at $g(t_j)$, since c does not occur in $A[k_d/x]$. Furthermore, $v'(c) = d = v'(k_d)$. So, $\mathrm{E}c$ and $A[c/x]$ are true in $f(w_i)$ at $g(t_j)$ in $\mathcal{M}'$, by the Denotation Lemma. By the Locality Lemma, $\mathcal{M}'$ makes all other formulas on the branch true in $f(w_i)$ at $g(t_j)$ as well. For c does not occur in any other formula on the branch. $\dashv$

6.2.2. Completeness theorem

THEOREM 22 (Completeness Theorem). *VS is strongly complete with respect to its semantics.*

PROOF. The proof is a modification of that for *CS*. In the Completeness Lemma we add steps for the actualist quantifiers. The Completeness Theorem then follows from the Completeness Lemma as usual. Furthermore, in the induced model $D_{\omega_i\tau_{[j]}} = v(\omega_i\tau_{[j]}) = v_{\omega_i\tau_{[j]}}(\mathrm{E}) = \{o_a : \mathrm{E}a, w_it_j \text{ occurs on } \mathcal{B}\}$.

($\exists$) Assume that $\exists xA, w_it_j$ is on the branch. Since the tableau is complete ($\exists$) has been applied. Hence, for some $c \in C$, $\mathrm{E}c, w_it_j$ and $A[c/x]$, w_it_j are on the branch. By (IH), $v_{\omega_i\tau_{[j]}}(\mathrm{E}c) = 1$ and $v_{\omega_i\tau_{[j]}}(A[c/x]) = 1$. For some $d \in D$, $v(c) = d = v(k_d)$. Accordingly, by the Denotation Lemma, $v_{\omega_i\tau_{[j]}}(\mathrm{E}k_d) = v_{\omega_i\tau_{[j]}}(A[k_d/x]) = 1$. It follows that

$v_{\omega_i\tau_{[j]}}(\exists x A) = 1$. Suppose that $\neg\exists x A, w_i t_j$ is on the branch. Since the tableau is complete $(\neg\exists)$ has been applied. So, $\forall x\neg A, w_i t_j$ is on the branch. Again, since the tableau is complete $(\forall)$ has been applied. Hence, for every $c \in C$, either $\neg Ec, w_i t_j$ or $\neg A[c/x], w_i t_j$ is on the branch. Accordingly, for all $c \in C$, if $\mathrm{E}c, w_i t_j$ is on the branch, then $\neg A[c/x], w_i t_j$ is so too. For the branch is open. By (IH), if $v_{\omega_i\tau_{[j]}}(Ec) = 1$, $v_{\omega_i\tau_{[j]}}(A[c/x]) = 0$. If $d \in D$, then for some $c \in C$, $v(c) = v(k_d)$. By Denotation Lemma, for all $d \in D$, such that $v_{\omega_i\tau_{[j]}}(Ek_d) = 1$ $(d \in D_{\omega_i\tau_{[j]}})$, $v_{\omega_i\tau_{[j]}}(A[k_d/x]) = 0$. It follows that $v_{\omega_i\tau_{[j]}}(\exists x A) = 0$. The case for $\forall$ is similar. ⊣

6.2.3. Systems with domain-inclusion (Barcan) rules

THEOREM 23 (Soundness with Barcan Rules). *VS + any subset of the domain-inclusion (Barcan) rules is (strongly) sound with respect to its semantics.*

PROOF. In the Soundness Lemma we have some new cases. Assume that f and g show that the branch $\mathcal{B}$ is satisfiable in $\mathcal{M}$.

(T-ACBF) Suppose that $\mathcal{B}$ contains $rw_i w_j t_k$ and $\mathrm{E}a, w_i t_k$. Then $Rf(w_i)f(w_j)g(t_k)$ and $v(a) \in v_{\omega_i\tau_{[k]}}(\mathrm{E}) = D_{\omega_i\tau_{[k]}}$. By (C-ACBF), $D_{f(w_i)g(t_k)} \subseteq D_{f(w_j)g(t_k)}$. Hence, $v(a) \in D_{\omega_j\tau_{[k]}} = v_{\omega_j\tau_{[k]}}(\mathrm{E})$. Consequently, $\mathrm{E}a$ is true in $f(w_j)$ at $g(t_k)$ in $\mathcal{M}$, and we can take $\mathcal{M}'$ to be $\mathcal{M}$.

(T-TBF) Suppose that $\mathcal{B}$ contains $t_i < t_j$ and $\mathrm{E}a, w_i t_j$. Then $g(t_i) < g(t_j)$ and $v(a) \in v_{\omega_i\tau_{[j]}}(\mathrm{E}) = D_{\omega_i\tau_{[j]}}$. By (C-TBF), $D_{f(w_i)g(t_j)} \subseteq D_{f(w_i)g(t_i)}$. Therefore, $v(a) \in D_{\omega_i\tau_{[i]}} = v_{\omega_i\tau_{[i]}}(\mathrm{E})$. It follows that $\mathrm{E}a$ is true in $f(w_i)$ at $g(t_i)$ in $\mathcal{M}$, and we can take $\mathcal{M}'$ to be $\mathcal{M}$.

The other cases are proved similarly. ⊣

THEOREM 24 (Completeness with Barcan Rules). *VS + any subset of the domain-inclusion (Barcan rules) is (strongly) complete with respect to its semantics.*

PROOF. In the relevant Completeness Theorem, we have to check that the induced model is of the appropriate kind.

(T-ACBF) Suppose that $R\omega_i\omega_j\tau_{[k]}$ and that $o_a \in D_{\omega_i\tau_{[k]}}$. Then $rw_i w_j t_k$ and $\mathrm{E}a, w_i t_k$ are on the branch. Since the branch is complete (T-ACBF) has been applied. Accordingly, $\mathrm{E}a, w_j t_k$ is on the branch. Consequently, $o_a \in D_{\omega_j\tau_{[k]}}$.

(T-TBF) Suppose that $\tau_{[i]} < \tau_{[j]}$ and that $o_a \in D_{\omega_i\tau_{[j]}}$. Then $t_i < t_j$ and $\mathrm{E}a, w_i t_j$ are on the branch. Since the branch is complete (T-TBF) has been applied. Hence, $\mathrm{E}a, w_i t_i$ is on the branch. It follows that $o_a \in D_{\omega_i \tau_{[i]}}$. The other cases are proved similarly. $\dashv$

6.2.4. General correctness theorem

THEOREM 25 (General Correctness Theorem). *Let S be any of the variable domain tableau systems discussed in this essay (without identity). Then S is (strongly) sound and complete with respect to its semantics.*

PROOF. The proof is similar to the proof in the constant domain case. $\dashv$

6.3. Constant and variable domain logics

6.3.1. General correctness theorem

THEOREM 26 (General Correctness Theorem). *Let S be any constant and variable domain tableau system discussed in this essay (without identity). Then S is (strongly) sound and complete with respect to its semantics.*

PROOF. Combine the proofs for the constant systems and the variable systems. $\dashv$

6.4. Systems with necessary identity

So far we have assumed that the identity predicate is not part of our language. In this section we will prove soundness and completeness for systems with necessary identity. In Subsection 6.4.3 we will see what happens when we add descriptors to our language. In the next section we turn to systems with contingent identity.

Adding the identity predicate does nothing to affect the proofs of the Locality and Denotation Lemmas; they still hold.

6.4.1. Soundness, necessary identity

THEOREM 27 (Soundness Necessary Identity). *Let S be any system in this essay (without identity). Then S + the rules for necessary identity is strongly sound with respect to its semantics (variable, constant or constant and variable).*

PROOF. There are three new cases in the Soundness Lemma.

(T-R=) Trivial.

(T-S=) At this stage the only (non-temporal) constants in our language are rigid. So, let $s = a$ and $t = b$. Since f and g show that the branch is satisfiable in $\mathcal{M}$, $v(a) = v(b)$ and $\langle v(a_1), \ldots, v(a), \ldots, v(a_n)\rangle \in v_{\omega_i\tau_{[j]}}(P)$. Accordingly, $\langle v(a_1), \ldots, v(b), \ldots, v(a_n)\rangle \in v_{\omega_i\tau_{[j]}}(P)$. Consequently, $Pa_1 \ldots b \ldots a_n$ is true in $f(w_i)$ at $g(t_j)$ in $\mathcal{M}$. So, we may take $\mathcal{M}'$ to be $\mathcal{M}$.

(T-N=) Since f and g show that the branch is satisfiable in $\mathcal{M}$, $a = b$ is true in $f(w_i)$ at $g(t_j)$. Accordingly, $v(a) = v(b)$. Hence, $a = b$ is true in $f(w_k)$ at $g(t_l)$, and we may take $\mathcal{M}'$ to be $\mathcal{M}$.

The Soundness Theorem then follows as usual. ⊣

6.4.2. Completeness, necessary identity

DEFINITION 28 (Induced Model). We define the induced model as before, but with the following modification. Define $a \sim b$ to mean that $a = b, w_0t_0$ is on the branch. This is obviously an equivalence relation. Let $[a]$ be the equivalence class of a under $\sim$. $D = \{[a] : a \in C\}$ (or, if $C = \emptyset$, $D = \{o\}$ for an arbitrary o). $v(a) = [a]$, and $\langle [a_1], \ldots, [a_n]\rangle \in v_{\omega_i\tau_{[j]}}(P)$ iff $Pa_1 \ldots a_n, w_it_j$ is on $\mathcal{B}$, given that P is any n-place predicate other than identity. (Due to (T-N=) and (T-S=) this is well defined.) For the variable domain case, $D_{\omega_i\tau_{[j]}} = v_{\omega_i\tau_{[j]}}(\mathrm{E})$.

THEOREM 29 (Completeness Necessary Identity). *Let S be any system in this essay (without identity). Then $S+$ the rules for necessary identity is strongly complete with respect to its semantics (variable, constant or constant and variable).*

PROOF. Here are the modified cases in the Completeness Lemma.

$Pa_1...a_n, w_it_j$ is on $\mathcal{B} \Rightarrow \langle [a_1], ..., [a_n]\rangle \in v_{\omega_i\tau_{[j]}}(P) \Rightarrow \langle v(a_1), ..., v(a_n)\rangle \in v_{\omega_i\tau_{[j]}}(P) \Rightarrow v_{\omega_i\tau_{[j]}}(Pa_1...a_n) = 1$.

$\neg Pa_1...a_n, w_it_j$ is on $\mathcal{B} \Rightarrow Pa_1...a_n, w_it_j$ is not on $\mathcal{B}$ ($\mathcal{B}$ open) $\Rightarrow \langle [a_1], ..., [a_n]\rangle \notin v_{\omega_i\tau_{[j]}}(P) \Rightarrow \langle v(a_1), ..., v(a_n)\rangle \notin v_{\omega_i\tau_{[j]}}(P) \Rightarrow v_{\omega_i\tau_{[j]}}(Pa_1...a_n) = 0$.

$a = b, w_it_j$ is on $\mathcal{B} \Rightarrow a \sim b$ (T-N=) $\Rightarrow [a] = [b] \Rightarrow v(a) = v(b) \Rightarrow v_{\omega_i\tau_{[j]}}(a = b) = 1$.

$\neg a = b, w_it_j$ is on $\mathcal{B} \Rightarrow a = b, w_0t_0$ is not on $\mathcal{B}$ ($\mathcal{B}$ open) $\Rightarrow$ it is not the case that $a \sim b \Rightarrow [a] \neq [b] \Rightarrow v(a) \neq v(b) \Rightarrow v_{\omega_i\tau_{[j]}}(a = b) = 0$.

The proof of the Completeness Theorem then goes through as before. ⊣

6.4.3. Soundness and completeness with descriptors and necessary identity

In this section we show that systems including descriptors are sound and complete. The proofs of the Locality and Denotation Lemmas go through as before. The only modification is that we replace anything of the form $v(t)$ by $v_{\omega_i\tau_{[j]}}(t)$, where t is a non-temporal term. Note that the co-referring constants are rigid in the Denotation Lemma. (Descriptors that co-refer at a world-moment pair do not necessarily co-refer at all world-moment pairs.)

THEOREM 30 (Soundness Descriptors). *Let S be any system in this essay* (*with necessary identity*). *Then $S+$ the rule for descriptors is strongly sound with respect to its semantics* (*variable, constant or constant and variable*).

PROOF. There is one novel case in the Soundness Lemma. The rest is as in the necessary identity case.

(T-D=) Suppose that f and g show that the branch, $\mathcal{B}$, to which we apply the rule, is satisfiable in $\mathcal{M}$. In world $f(w_i)$ at time $g(t_j)$, α has some denotation, d. Thus, $v(k_d) = v_{f(w_i)g(t_j)}(\alpha)$. Let $\mathcal{M}'$ be the same as $\mathcal{M}$, except that $v'(c) = d$. $v'(c) = d = v(k_d) = v_{f(w_i)g(t_j)}(\alpha)$. Hence, $c = \alpha$ is true in $f(w_i)$ at $g(t_j)$ in $\mathcal{M}'$. And the rest of the branch is satisfiable in $\mathcal{M}'$ as well, by the Locality Lemma. This is shown by f and g, since c does not occur in any formula on $\mathcal{B}$. ⊣

THEOREM 31 (Completeness Descriptors). *Let S be any system in this essay* (*with necessary identity*). *Then $S+$ the rule for descriptors is strongly complete with respect to its semantics* (*variable, constant or constant and variable*).

PROOF. We want to show that the Completeness Lemma still holds. So, we extend the definition of the induced model to descriptors. For any α, w_i and t_j, on the branch, there is a line of the form $a = \alpha, w_i t_j$. Take any one such a (it does not matter which, because of (T-S=)), and let this be $\widehat{\alpha}$. Let $\widehat{b}$ be b itself, for any rigid designator, b. In the induced model, we define $v_{\omega_i\tau_{[j]}}(\alpha)$ to be $[\widehat{\alpha}]$. Here are the necessary modifications in the Completeness Lemma.

$Pt_1...t_n, w_i t_j$ is on $\mathcal{B} \Rightarrow P\widehat{t_1}...\widehat{t_n}, w_i t_j$ is on $\mathcal{B}$ (T-S=) $\Rightarrow \langle[\widehat{t_1}], ..., [\widehat{t_n}]\rangle \in v_{\omega_i\tau_{[j]}}(P) \Rightarrow \langle v_{\omega_i\tau_{[j]}}(t_1), ..., v_{\omega_i\tau_{[j]}}(t_n)\rangle \in v_{\omega_i\tau_{[j]}}(P) \Rightarrow v_{\omega_i\tau_{[j]}}(Pt_1...t_n) = 1$.

$\neg Pt_1 \ldots t_n, w_i t_j$ is on $\mathcal{B} \Rightarrow P\widehat{t_1}...\widehat{t_n}, w_i t_j$ is not on $\mathcal{B}$ ((T-S=), $\mathcal{B}$ open) $\Rightarrow \langle[\widehat{t_1}], ..., [\widehat{t_n}]\rangle \notin v_{\omega_i \tau_{[j]}}(P) \Rightarrow \langle v_{\omega_i \tau_{[j]}}(t_1), ..., v_{\omega_i \tau_{[j]}}(t_n)\rangle \notin v_{\omega_i \tau_{[j]}}(P) \Rightarrow v_{\omega_i \tau_{[j]}}(Pt_1 \ldots t_n) = 0$.

$t_1 = t_2, w_i t_j$ is on $\mathcal{B} \Rightarrow \widehat{t_1} = \widehat{t_2}, w_i t_j$ is on $\mathcal{B}$ (T-S=) $\Rightarrow \widehat{t_1} = \widehat{t_2}, w_0 t_0$ is on $\mathcal{B}$ (T-N=) $\Rightarrow \widehat{t_1} \sim \widehat{t_2} \Rightarrow [\widehat{t_1}] = [\widehat{t_2}] \Rightarrow v_{\omega_i \tau_{[j]}}(t_1) = v_{\omega_i \tau_{[j]}}(t_2) \Rightarrow v_{\omega_i \tau_{[j]}}(t_1 = t_2) = 1$.

$\neg t_1 = t_2, w_i t_j$ is on $\mathcal{B} \Rightarrow \widehat{t_1} = \widehat{t_2}, w_i t_j$ is not on $\mathcal{B}$ ((T-S=), $\mathcal{B}$ open) $\Rightarrow \widehat{t_1} = \widehat{t_2}, w_0 t_0$ is not on $\mathcal{B}$ ((T-N=), $\mathcal{B}$ open) $\Rightarrow$ it is not the case that $\widehat{t_1} \sim \widehat{t_2} \Rightarrow [\widehat{t_1}] \neq [\widehat{t_2}] \Rightarrow v_{\omega_i \tau_{[j]}}(t_1) \neq v_{\omega_i \tau_{[j]}}(t_2) \Rightarrow v_{\omega_i \tau_{[j]}}(t_1 = t_2) = 0$.

The Completeness Theorem then follows as usual. $\dashv$

6.5. Contingent identity logic

In this section we will prove soundness and completeness for systems with contingent identity. In Subsection 6.5.3 we add descriptors to our systems and prove soundness and completeness.

Our models now have one new component, H. With this exception, the Locality and Denotation Lemmas are as in the constant or variable domain cases.

THEOREM 32 (Denotation and Locality). *The Denotation and Locality Lemmas hold in contingent identity semantics.*

PROOF. The proofs are as in the constant or variable domain cases, except for the atomic formulas.

Locality: $v_{1\omega\tau}(Pa_1...a_n) = 1$ iff $\langle|v_1(a_1)|_{\omega\tau}, ..., |v_1(a_n)|_{\omega\tau}\rangle \in v_{1\omega\tau}(P)$ iff $\langle|v_2(a_1)|_{\omega\tau}, ..., |v_2(a_n)|_{\omega\tau}\rangle \in v_{2\omega\tau}(P)$ iff $v_{2\omega\tau}(Pa_1 \ldots a_n) = 1$.

Denotation: $v_{\omega\tau}(Pa_1...a...a_n) = 1$ iff $\langle|v(a_1)|_{\omega\tau}, ..., |v(a)|_{\omega\tau}, ..., |v(a_n)|_{\omega\tau}\rangle \in v_{\omega\tau}(P)$ iff $\langle|v(a_1)|_{\omega\tau}, ..., |v(b)|_{\omega\tau}, ..., |v(a_n)|_{\omega\tau}\rangle \in v_{\omega\tau}(P)$ iff $v_{\omega\tau}(Pa_1...b...a_n) = 1$. $\dashv$

6.5.1. Soundness, contingent identity

THEOREM 33 (Soundness Contingent Identity). *Let S be any system in this essay (without identity). Then $S+$ the rules for contingent identity is strongly sound with respect to its semantics (variable, constant or constant and variable).*

PROOF. The proof is as in the necessary identity case with some minor modifications. E.g. here is the step for (T-S=). Since f and g show that the branch is satisfiable in $\mathcal{M}$, $|v(a)|_{\omega_i \tau_{[j]}} = |v(b)|_{\omega_i \tau_{[j]}}$ and

$\langle |v(a_1)|_{\omega_i\tau_{[j]}}, ..., |v(a)|_{\omega_i\tau_{[j]}}, ..., |v(a_n)|_{\omega_i\tau_{[j]}}\rangle \in v_{\omega_i\tau_{[j]}}(P)$. So, $\langle |v(a_1)|_{\omega_i\tau_{[j]}}, ..., |v(b)|_{\omega_i\tau_{[j]}}, ..., |v(a_n)|_{\omega_i\tau_{[j]}}\rangle \in v_{\omega_i\tau_{[j]}}(P)$. Accordingly, $Pa_1...b...a_n$ is true in $f(w_i)$ at $g(t_j)$ in $\mathcal{M}$, and so we may take $\mathcal{M}'$ to be $\mathcal{M}$. ⊣

6.5.2. Completeness, contingent identity

Definition 34 (Induced Model). Given an open complete branch $\mathcal{B}$ of a tableau, the induced model is defined as usual, but with the following modifications. If there are no constants on the branch, $D = \{o\}$, for an arbitrary o; $H = \{h\}$, for an arbitrary h; and for every $\omega \in W$ and $\tau \in T$, $|o|_{\omega\tau} = h$. Otherwise, $D = \{o_a : a \textit{ occurs on } \mathcal{B}\}$ as usual. The objects in D are now functions from $W \times T$ to H. We shall say that $a \sim_{\omega_i\tau_{[j]}} b$ iff $a = b, w_it_j$ occurs on $\mathcal{B}$. $\sim_{\omega_i\tau_{[j]}}$ is an equivalence relation. Let $[a]_{\omega_i\tau_{[j]}}$ be the equivalence class of a under $\sim_{\omega_i\tau_{[j]}}$. $H = \{[a]_{\omega_i\tau_{[j]}} : \textit{for all } a, w_it_j \textit{ on } \mathcal{B}\}$. Let $|o_a|_{\omega_i\tau_{[j]}} = [a]_{\omega_i\tau_{[j]}}$, for $\omega_i \in W$ and $\tau_{[j]} \in T$. For each (non-temporal, rigid) constant, a, $v(a) = o_a$. For each n-place predicate, P, other than identity: $\langle [a_1]_{\omega_i\tau_{[j]}}, ..., [a_n]_{\omega_i\tau_{[j]}}\rangle \in v_{\omega_i\tau_{[j]}}(P)$ iff $Pa_1...a_n, w_it_j$ is on $\mathcal{B}$. Any n-tuple that contains a substratum that is not of the form $[a]_{\omega_i\tau_{[j]}}$ is not in $v_{\omega_i\tau_{[j]}}(P)$. Because of (T-S=), it does not matter which member of an equivalence class we chose. If the model is a variable domain model, $D_{\omega_i\tau_{[j]}} = \{d \in D : |d|_{\omega_i\tau_{[j]}} \in v_{\omega_i\tau_{[j]}}(\mathrm{E})\}$.

Theorem 35 (Completeness Contingent Identity). *Let S be any system in this essay (without identity). Then $S+$ the rules for contingent identity is strongly complete with respect to its semantics (variable, constant or constant and variable).*

Proof. The proof of the Completeness Lemma is as in the non-identity case, except for the following steps.

$Pa_1 \ldots a_n, w_it_j$ is on $\mathcal{B} \Rightarrow \langle [a_1]_{\omega_i\tau_{[j]}}, \ldots, [a_n]_{\omega_i\tau_{[j]}}\rangle \in v_{\omega_i\tau_{[j]}}(P) \Rightarrow \langle |o_{a_1}|_{\omega_i\tau_{[j]}}, ..., |o_{a_n}|_{\omega_i\tau_{[j]}}\rangle \in v_{\omega_i\tau_{[j]}}(P) \Rightarrow \langle |v(a_1)|_{\omega_i\tau_{[j]}}, \ldots, |v(a_n)|_{\omega_i\tau_{[j]}}\rangle \in v_{\omega_i\tau_{[j]}}(P) \Rightarrow v_{\omega_i\tau_{[j]}}(Pa_1 \ldots a_n) = 1$.

$\neg Pa_1 \ldots a_n, w_it_j$ is on $\mathcal{B} \Rightarrow Pa_1 \ldots a_n, w_it_j$ is not on $\mathcal{B} \Rightarrow \langle [a_1]_{\omega_i\tau_{[j]}}, \ldots, [a_n]_{\omega_i\tau_{[j]}}\rangle \notin v_{\omega_i\tau_{[j]}}(P) \Rightarrow \langle |o_{a_1}|_{\omega_i\tau_{[j]}}, \ldots, |o_{a_n}|_{\omega_i\tau_{[j]}}\rangle \notin v_{\omega_i\tau_{[j]}}(P) \Rightarrow \langle |v(a_1)|_{\omega_i\tau_{[j]}}, \ldots, |v(a_n)|_{\omega_i\tau_{[j]}}\rangle \notin v_{\omega_i\tau_{[j]}}(P) \Rightarrow v_{\omega_i\tau_{[j]}}(Pa_1 \ldots a_n) = 0$.

$a = b, w_it_j$ is on $\mathcal{B} \Rightarrow a \sim_{\omega_i\tau_{[j]}} b \Rightarrow [a]_{\omega_i\tau_{[j]}} = [b]_{\omega_i\tau_{[j]}} \Rightarrow |o_a|_{\omega_i\tau_{[j]}} = |o_b|_{\omega_i\tau_{[j]}} \Rightarrow |v(a)|_{\omega_i\tau_{[j]}} = |v(b)|_{\omega_i\tau_{[j]}} \Rightarrow v_{\omega_i\tau_{[j]}}(a = b) = 1$.

$\neg a = b, w_it_j$ is on $\mathcal{B} \Rightarrow a = b, w_it_j$ is not on $\mathcal{B}$, ($\mathcal{B}$ open) $\Rightarrow$ it is not the case that $a \sim_{\omega_i\tau_{[j]}} b \Rightarrow [a]_{\omega_i\tau_{[j]}} \neq [b]_{\omega_i\tau_{[j]}} \Rightarrow |o_a|_{\omega_i\tau_{[j]}} \neq |o_b|_{\omega_i\tau_{[j]}} \Rightarrow |v(a)|_{\omega_i\tau_{[j]}} \neq |v(b)|_{\omega_i\tau_{[j]}} \Rightarrow v_{\omega_i\tau_{[j]}}(a = b) = 0$.

The rest of the Completeness Theorem then follows as usual. ⊣

6.5.3. Soundness and completeness with descriptors and contingent identity

In this section we consider the addition of descriptors to contingent identity systems. The Locality and Denotation Lemmas are established as in the constant or variable domain cases. The proofs of the soundness and completeness theorems are as in the necessary identity case, with some minor modifications.

Theorem 36 (Soundness Descriptors). *Let S be any system in this essay* (*with contingent identity*). *Then $S+$ the rule for descriptors is strongly sound with respect to its semantics* (*variable, constant or constant and variable*).

Proof. Suppose that f and g show that the branch $\mathcal{B}$ is satisfiable in $\mathcal{M}$. Here are the new interesting cases.
(T-S=) For the sake of illustration, assume that there is only one occurrence of s. Accordingly, $|v_{f(w_i)g(t_j)}(s)|_{f(w_i)g(t_j)} = |v_{f(w_i)g(t_j)}(t)|_{f(w_i)g(t_j)}$ and $\langle|v_{f(w_i)g(t_j)}(t_1)|_{f(w_i)g(t_j)}, ..., |v_{f(w_i)g(t_j)}(s)|_{f(w_i)g(t_j)}, ..., |v_{f(w_i)g(t_i)}(t_n)|_{f(w_i)g(t_j)}\rangle \in v_{f(w_i)g(t_j)}(P)$. So, $\langle|v_{f(w_i)g(t_j)}(t_1)|_{f(w_i)g(t_j)}, ..., |v_{f(w_i)g(t_j)}(t)|_{f(w_i)g(t_j)}, ..., |v_{f(w_i)g(t_j)}(t_n)|_{f(w_i)g(t_j)}\rangle \in v_{f(w_i)g(t_j)}(P)$. It follows that $Pt_1...t...t_n$ is true in $f(w_i)$ at $g(t_j)$ in $\mathcal{M}$, and we may take $\mathcal{M}'$ to be $\mathcal{M}$.

(T-D=) In $f(w_i)$ at $g(t_j)$, α has some denotation, $d \in D$. Thus $|v(k_d)|_{f(w_i)g(t_j)} = |v_{f(w_i)g(t_j)}(\alpha)|_{f(w_i)g(t_j)}$. Let $\mathcal{M}'$ be the same as $\mathcal{M}$, except that $v(c) = d$. $|v(c)|_{f(w_i)g(t_j)} = |d|_{f(w_i)g(t_j)} = |v(k_d)|_{f(w_i)g(t_j)} = |v_{f(w_i)g(t_j)}(\alpha)|_{f(w_i)g(t_j)}$. Hence, $c = \alpha$ is true in $f(w_i)$ at $g(t_j)$ in $\mathcal{M}'$. Furthermore, f and g show that the rest of the branch is satisfiable in $\mathcal{M}'$, by the Locality Lemma. For c does not occur in any formula on $\mathcal{B}$. $\dashv$

Theorem 37 (Completeness Descriptors). *Let S be any system in this essay* (*with contingent identity*). *Then $S+$ the rule for descriptors is strongly complete with respect to its semantics* (*variable, constant or constant and variable*).

Proof. We modify the definition of an induced model so that it applies to descriptors as well. Then we check that the Completeness Lemma holds. For any descriptor, α, and any w_i and t_j on the branch, there is a line of the form $a = \alpha, w_i t_j$. Let any one such a (it does not matter which, because of (T-S=)) be $\widehat{\alpha}$. If b is a rigid designator, let $\widehat{b}$ be b itself.

Finally: $v_{\omega_i\tau_{[j]}}(\alpha) = o_{\widehat{\alpha}}$. Here are the modified steps in the Completeness Lemma.

$Pt_1...t_n, w_it_j$ is on $\mathcal{B} \Rightarrow P\widehat{t_1} \ldots \widehat{t_n}, w_it_j$ is on $\mathcal{B}$ (T-S=) $\Rightarrow \langle[\widehat{t_1}]_{\omega_i\tau_{[j]}}, \ldots, [\widehat{t_n}]_{\omega_i\tau_{[j]}}\rangle \in v_{\omega_i\tau_{[j]}}(P) \Rightarrow \langle|o_{\widehat{t_1}}|_{\omega_i\tau_{[j]}}, \ldots, |o_{\widehat{t_n}}|_{\omega_i\tau_{[j]}}\rangle \in v_{\omega_i\tau_{[j]}}(P) \Rightarrow \langle|v_{\omega_i\tau_{[j]}}(t_1)|_{\omega_i\tau_{[j]}}, ..., |v_{\omega_i\tau_{[j]}}(t_n)|_{\omega_i\tau_{[j]}}\rangle \in v_{\omega_i\tau_{[j]}}(P) \Rightarrow v_{\omega_i\tau_{[j]}}(Pt_1...t_n) = 1$.

$\neg Pt_1...t_n, w_it_j$ is on $\mathcal{B} \Rightarrow Pt_1...t_n, w_it_j$ is not on $\mathcal{B}$ ($\mathcal{B}$ open) $\Rightarrow P\widehat{t_1}...\widehat{t_n}, w_it_j$ is not on $\mathcal{B}$ ((T-S=), $\mathcal{B}$ open) $\Rightarrow \langle[\widehat{t_1}]_{\omega_i\tau_{[j]}}, \ldots, [\widehat{t_n}]_{\omega_i\tau_{[j]}}\rangle \notin v_{\omega_i\tau_{[j]}}(P) \Rightarrow \langle|o_{\widehat{t_1}}|_{\omega_i\tau_{[j]}}, \ldots, |o_{\widehat{t_n}}|_{\omega_i\tau_{[j]}}\rangle \notin v_{\omega_i\tau_{[j]}}(P) \Rightarrow \langle|v_{\omega_i\tau_{[j]}}(t_1)|_{\omega_i\tau_{[j]}}, \ldots, |v_{\omega_i\tau_{[j]}}(t_n)|_{\omega_i\tau_{[j]}}\rangle \notin v_{\omega_i\tau_{[j]}}(P) \Rightarrow v_{\omega_i\tau_{[j]}}(Pt_1 \ldots t_n) = 0$.

$t_1 = t_2, w_it_j$ is on $\mathcal{B} \Rightarrow \widehat{t_1} = \widehat{t_2}, w_it_j$ is on $\mathcal{B}$ (T-S=) $\Rightarrow \widehat{t_1} \sim_{\omega_i\tau_{[j]}} \widehat{t_2} \Rightarrow [\widehat{t_1}]_{\omega_i\tau_{[j]}} = [\widehat{t_2}]_{\omega_i\tau_{[j]}} \Rightarrow |o_{\widehat{t_1}}|_{\omega_i\tau_{[j]}} = |o_{\widehat{t_2}}|_{\omega_i\tau_{[j]}} \Rightarrow |v_{\omega_i\tau_{[j]}}(t_1)|_{\omega_i\tau_{[j]}} = |v_{\omega_i\tau_{[j]}}(t_2)|_{\omega_i\tau_{[j]}} \Rightarrow v_{\omega_i\tau_{[j]}}(t_1 = t_2) = 1$.

$\neg t_1 = t_2, w_it_j$ is on $\mathcal{B} \Rightarrow t_1 = t_2, w_it_j$ is not on $\mathcal{B}$ ($\mathcal{B}$ open) $\Rightarrow \widehat{t_1} = \widehat{t_2}, w_it_j$ is not on $\mathcal{B}$, ((T-S=), $\mathcal{B}$ open) $\Rightarrow$ it is not the case that $\widehat{t_1} \sim_{\omega_i\tau_{[j]}} \widehat{t_2} \Rightarrow [\widehat{t_1}]_{\omega_i\tau_{[j]}} \neq [\widehat{t_2}]_{\omega_i\tau_{[j]}} \Rightarrow |o_{\widehat{t_1}}|_{\omega_i\tau_{[j]}} \neq |o_{\widehat{t_2}}|_{\omega_i\tau_{[j]}} \Rightarrow |v_{\omega_i\tau_{[j]}}(t_1)|_{\omega_i\tau_{[j]}} \neq |v_{\omega_i\tau_{[j]}}(t_2)|_{\omega_i\tau_{[j]}} \Rightarrow v_{\omega_i\tau_{[j]}}(t_1 = t_2) = 0$.

The Completeness Theorem now follows in the usual fashion. $\dashv$

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References

[1] Bailhache, P., *Les normes dans le temps et sur l'action, Essai de logique déontique*, Université de Nantes, 1986.

[2] Bailhache, P., *Essai de logique déontique*, Paris, Librarie Philosophique, Vrin, Collection Mathesis, 1991.

[3] Barcan (Marcus), R. C., "A functional calculus of first order based on strict implication", *Journal of Symbolic Logic*, 11 (1946): 1–16. DOI: 10.2307/2269159

[4] Barcan (Marcus), R. C., "The identity of individuals in a strict functional calculus of second order", *Journal of Symbolic Logic*, 12, 1 (1947): 12–15. DOI: 10.2307/2267171

[5] Bartha, P., "Moral preference, contrary-to-duty obligation and defeasible oughts", pages 93–108 in *Norms, Logics and Information Systems: New Studies in Deontic Logic and Computer Science*, P. McNamara, and H. Prakken (eds.), Amsterdam, IOS Press, 1999.

[6] Belnap, N., Perloff, M. and Xu, M., *Facing the Future: Agents and Choices in Our Indeterminist World*, Oxford, Oxford University Press, 2001.
[7] Bowen, K., *Model Theory for Modal Logic*, Reidel, Dordrecht, 1979.
[8] Bressan, A., *A General Interpreted Modal Calculus*, Yale University Press, 1973.
[9] Brown, M. A., "Conditional obligation and positive permission for agents in time", *Nordic Journal of Philosophical Logic* 5, 2 (2000): 83–112.
[10] Brown M. A., "Rich deontic logic: a preliminary study", *Journal of Applied Logic*, 2 (2004): 19–37. DOI: 10.1016/j.jal.2004.01.002
[11] Carnap, R., "Modalities and quantification", *Journal of Symbolic Logic*, 11, 2 (1946): 33–64. DOI: 10.2307/2268610
[12] Carnap, R., *Meaning and Necessity*, Chicago, Chicago University Press, 1947.
[13] Chellas, B. F., *The Logical Form of Imperatives*, Stanford, Perry Lane Press, 1969.
[14] D'Agostino, M., D. M. Gabbay, R. Hähnle, and J. Posegga (eds.), *Handbook of Tableau Methods*, Dordrecht, Kluwer Academic Publishers, 1999.
[15] Eck, J. E. v., *A System of Temporally Relative Modal and Deontic Predicate Logic and its Philosophical Applications*, Department of Philosophy, University of Groningen, The Netherlands, 1981.
[16] Fitting, M., and R. L. Mendelsohn, *First-Order Modal Logic*, Kluwer Academic Publishers, 1998.
[17] Gabbay, D. M., *Investigations in Modal and Tense Logics with Applications to Problems in Philosophy and Linguistics*, Reidel, Dordrecht, 1976.
[18] Garson, J. W., "Quantification in modal logic", in *Handbook of Philosophical Logic*, vol. 2, D. M. Gabbay and F. Guenthner (eds.), 1984.
[19] Garson, J. W., *Modal Logic for Philosophers*, New York, Cambridge University Press, 2006.
[20] Hilpinen, R. (ed.), *New Studies in Deontic Logic: Norms, Actions, and the Foundation of Ethics*, Dordrecht, D. Reidel Publishing Company, 1981.
[21] Hintikka, J., "Quantifiers in deontic logic", *Societas Scientarum Fennica, Commentationes Humanarum Literarum*, Helsinki 23, 4 (1957).
[22] Hintikka, J., "Modality as referential multiplicity", *Ajatus*, 20 (1957): 49–64.
[23] Hintikka, J., "Existential presuppositions and existential commitments", *Journal of Philosophy*, 56 (1959): 125–137. DOI: 10.2307/2021988
[24] Hintikka, J., "Modality and quantification", *Theoria*, 27 (1961): 117–128. DOI: 10.1111/j.1755-2567.1961.tb00020.x
[25] Horty, J. F., *Agency and Deontic Logic*, Oxford, Oxford University Press, 2001.
[26] Hughes, G. E., and M. J. Cresswell, *An Introduction to Modal Logic*, London, Routledge, 1968.

[27] Kanger, S., *Provability in Logic*, Stockholm, 1957.
[28] Kripke, S. A., "A completeness theorem in modal logic", *Journal of Symbolic Logic*, 24 (1959): 1–14. DOI: 10.2307/2964568
[29] Kripke, S. A., "Semantical considerations on modal logic", *Acta Philosophica Fennica*, 16 (1963): 83–94. DOI: 10.1007/978-3-0346-0145-0_16
[30] Kripke, S. A., "Semantical analysis of modal logic I. Normal propositional calculi", *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 9 (1963): 67–96. DOI: 10.1002/malq.19630090502
[31] Kripke, S. A., "Semantical analysis of modal logic II. Non-normal modal propositional calculi", pages 206–220 in *The Theory of Models (Proceedings of the 1963 International Symposium at Berkeley)*, J. W. Addison, L. Henkin, and A. Tarski (eds.), North-Holland, Amsterdam, 1965.
[32] Lewis, C. I., *Survey of Symbolic logic*, Berkeley, University of California Press, 1918.
[33] Lewis C. I., and C. H. Langford, *Symbolic Logic*, New York, The Century Company, 1932.
[34] Montague, R., "Logical necessity, physical necessity, ethics and quantifiers", *Inquiry*, 4 (1960): 259–269. DOI: 10.1080/00201746008601312
[35] Parks, Z., "Investigations into quantified modal logic I", *Studia Logica*, 35 (1976): 109–125. DOI: 10.1007/BF02120875
[36] Priest, G., *An Introduction to Non-Classical Logic*, Cambridge, Cambridge University Press, 2008.
[37] Rönnedal, D., "Temporal alethic-deontic logic and semantic tableaux", *Journal of Applied Logic*, 10 (2012): 219–237.
DOI: 10.1016/j.jal.2012.03.002
[38] Rönnedal, D., *Extensions of Deontic Logic: An Investigation into some Multi-Modal Systems*, Department of Philosophy, Stockholm University, 2012.
[39] Thomason, R., "Deontic logic as founded on tense logic", pages 165–176 in [20].
[40] Thomason, R., "Deontic logic and the role of freedom in moral deliberation", pages 177–186 in [20].
[41] Thomason, R., "Combinations of tense and modality", pages 135–165 in *Handbook of Philosophical Logic*, vol. 2, D. M. Gabbay and F. Guenthner (eds.), 1984 (2nd edition, vol. 7, 2002, pp. 205–234).
[42] Thomason, R., "Some completeness results for modal predicate calculi", in *Philosophical Problems in Logic*, K. Lambert (ed.), D. Reidel, Dordrecht, 1970.
[43] Wölfl, S., "Combinations of tense and modality for predicate logic", *Journal of Philosophical Logic*, 28 (1999): 371–398.
DOI: 10.1023/A:1004359325754

[44] Åqvist, L., "The logic of historical necessity as founded on two-dimensional modal tense logic", *Journal of Philosophical Logic*, 28 (1999): 329–369. DOI: 10.1023/A:1004425728816

[45] Åqvist, L., "Combinations of tense and deontic modality: On the Rt approach to temporal logic with historical necessity and conditional obligation", *Journal of Applied Logic*, 3 (2005): 421–460.

[46] Åqvist, L., and J. Hoepelman, "Some theorems about a 'tree' system of deontic tense logic", pages 187–221 in [20].

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