Logic and Logical Philosophy
Volume 23 (2014), 133–170
DOI: 10.12775/LLP.2014.003

Pavel Materna

CONCEPTS AS HYPERINTENSIONAL OBJECTS

Abstract. The author defends the view that the notion of *concept*, if used in the logical (not cognitivist) tradition, should be explicated procedurally (i.e., not set-theoretically). He argues that Tichý's Transparent Intensional Logic is an apt tool for such an explication and derives the respective definition. Some consequences of this definition concern the notions of emptiness, simple concepts, empirical concepts and algorithmic concepts.

Keywords: constructions; TIL; ontological definition; meaning; denotation; hierarchy of types (simple, ramified); hyperintensional

Introduction

The present paper should show that any plausible explication of the notion of *concept* has to take into account that

a) as soon as concepts are regarded to be *universals*, i.e., sets/classes to be handled by *extensional systems*, the notion of *concept* becomes a superfluous notion that can be easily replaced by the notion of set/class;
b) if concepts are regarded to be *intensions* then either intensions are regarded to be functions from possible worlds (classical PW intensions as used by Kripke, Montague, Tichý) and can be handled by *intensional systems*: then the result is the same as in the case a), or intensions are defined in some other way (see, e.g., Bealer's [2] and then the term *intension* has to be first explicated, which may result in some more plausible explication that is no more dependent on the dilemma *extension* vs. *intension*.

Received February 25, 2013. *Revised* August 14, 2013. *Published online* February 7, 2014

Thus if we accept that the notion of *concept* should not be superfluous (see [32]). i.e., if using the notion of concept makes it possible to solve such (logical) problems that cannot be solved (and even formulated) when concepts are handled as intensions (let alone extensions) then we have got the situation, which made Carnap in his [5] admit that his method of intensions and extensions was not efficient enough. This negative result of Carnap's attempt at doing intensional logic led A. Church to his own Alternatives (see Anderson [1] but intensional logic has got some good years (Kripke, Hintikka) independently of this problem.

In his [10] Cresswell handled the problem using the term *hyperintensionality* and defining it negatively:

> "Hyperintensional contexts are simply contexts which do not respect logical equivalence." [10, p. 25]

I will return to Cresswell in Section 1.2. Now I only state that it was Pavel Tichý, who recognized in [40] and [41] that hyperintensionality should be based on (abstract) procedures.

Tichý's Transparent Intensional Logic (TIL) shows how a logical system can be hyperintensional when based on procedures, and the present state-of-affairs as for TIL is summarized in [12].

I am convinced that every way to obtaining hyperintensionality starts with procedures. If a convincing argument shows that other ways are possible then at least some claims in the present paper must be corrected. But now I will use the system described in [12] to justify my claim that *concepts should be explicated as hyperintensional objects.*

In the following sections particular topics are presented:

- *logical tradition* that takes concepts to be non-mental (abstract) objects in contrast to cognitivism (Section 1);
- *concepts and intensions* (Section 2);
- *procedural complexity* (Section 3);
- *hierarchy of types* (Section 4);
- *concepts* (Section 5).

While sections 1–3 are rather informal and make up something like a philosophical background the last two sections are 'more technical' and result in a procedural definition of concepts.

Naturally, much is here taken over from the basic literature on TIL, the stress is however put on the specific character of *concepts* and justification of our proposal of procedural explication of *concept*.

1. Logical notions of *concept*

1.1. Concepts in cognitivist sense

Since our analyses concern *logical* explications of *concept* we will not take into account the way in which the term "concept" is used by cognitivists. Therefore, what follows is just a brief commentary to this cognitivist use.

One of the most well-known representatives of this use is J. A. Fodor (see, e.g. [16]). In his Representative Theory of Mind (RTM) he defends the view that concepts are *mental objects*, i. e., they are in the head of the given individual, and thus they are *concrete*, obeying causal laws.

Interestingly, some 160 years before Fodor it was Bernard Bolzano, who argued that concepts (*Begriffe*) are a kind of *Vorstellungen an sich*, which are *abstract*, and so not existent[1], whereas *Vorstellungen* as mental objects are subjective and concrete. Bolzano has convincingly argued that concepts cannot be subjective: they are *shared* by distinct individuals unlike the subjective *Vorstellungen*. Now the contemporary cognitivists cannot solve this problem of shareability. It was Fodor, who tried to save the shareability of concepts and at the same time his claim that concepts are mental particulars. He used the argument that sharing a concept means that the respective mental particulars are *tokens* of a *type* and the type (abstract) is shared. That this argument is logically untenable (because fallacious) has been proved by H.-J. Glock [21].

The main thesis of cognitivists concerning concepts is that concepts are mental objects and this thesis has not been justified. Moreover, Glock's analysis convincingly showed that this thesis could not probably be ever justified. The cognitivists' notion of *concept* cannot be rid of subjectivism and thus cannot be accepted as a *logical* notion. RTM can be regarded to be a modern version of psychologism.

1.2. Logical tradition

Theories of concept either assume that concepts are mental objects, in particular that they are a kind of *image, mental idea* (*Vorstellung*), or follow some *objectivist* characteristics, such as are articulated, e.g. by Glock:

[1] In this connection *abstract* means *not localizable* (*temporarily or spatially*), therefore not existent; *concrete* means *temporarily and spatially localizable*, therefore existent. Cf. also the footnote on p. 224 of Bolzano ([3, vol. I]).

> "According to *objectivist* or *logical* conceptions, concepts exist independently of individual human minds, e.g. as self-subsistent abstract entities or as abstractions from linguistic practices." [21, pp. 5–6]

We have stated (see Section 1.1) that we will not take into account theories of the former kind, so it is just the objectivist tradition which is of interest for us.

Within this tradition one should be aware of a contrast between *a complex* and *a set-theoretical object*, a contrast, which in general belongs to the most important problems in semantics. A widespread notion of concept has it that concepts are simply *universals* (we will illustrate this conception when talking about Frege). The contrast has been sometimes guessed and sometimes ignored in the history of semantics.[2]

Aristotle. Trying to find the oldest sources that can be considered to be something like a germ of a theory of concept we probably come to *Aristotle's theory of definition* (*Metaphysics*, *Topics*, *Posterior Analytics*). Aristotle's ${}^{c}o\rho(\iota\sigma\mu)o\zeta$ means *definition or definiens* and should signify a thing's essence. Two points, whose importance will be clear later, are to be registered:

a) A definiens is always *complex*, *structured*.
b) No object can have more than one definition. (See *Topics VI*, cf. [33, p. 6]).

Ad a): Quine famously criticized Carnap's attempt at defining intensional semantics and showed that neither using notion *synonymy* nor using notion *analyticity* suffices to define *meaning*[3] So far so good. Quine however deduced from his arguments a not following conclusion, viz. that intensional semantics was impossible: a kind of 'pragmatization' of semantics had to replace futile attempts at vindicating intensional semantics. Quine as if forgot the possibility to define *meaning* independently of synonymy and analyticity and, on the contrary, to define *synonymy* and *analyticity* in terms of *meaning*. Quine did not forget this

[2] Such historically significant guesses can be observed in Aristotle (his definition of definitions), in Bolzano (his theory of concepts in his [3]), Frege (just implicitly), Bealer ([2], two kinds of 'intension'), David Lewis ([27]), Cresswell ([11]), Heijenoort ([22, 23]), Church ([6, 9]) and, of course, Tichý (not only guesses, [41, 45]). See also Materna ([28, 29]).

[3] See Materna [31]. I use *meaning* or, equivalently, *sense*.

possibility: he rejected it because *meaning* was for him from the very beginning an obscure (word? entity?).

Tichý in his [40], translated in [48], using *sense* instead of *meaning*, says:

> "In current logic there is a strong tendency to define the sense by means of the notion of synonymy or analytical identity of expressions. It stems from the assumption that the relation of synonymy or analytical identity is definable without the notion of sense. [...] [It follows the description of the way such definitions are realizable.] This approach is formally correct, but from the semantic-content point of view we can object that this method of defining is quite opposite to our intuition. [...] in both cases [meaning postulates, possible worlds] defining the sense by means of the relation of analytical identity is either to turn over the natural logical sequence of these notions, or to fall into a circular definition." [48, p. 81]

Tichý shows however that semantic notions like *analyticity* or *synonymy* are definable in terms of *sense*. As early as in 1968 Tichý proposes a definition of *sense* (*meaning*) as an *abstract procedure* and so independently of *analyticity* or *synonymy*, which are then easily definable. In this connection Tichý appreciates Aristotle's way of defining. He says:

> "It is noteworthy that from this viewpoint classical logic treats these notions in a more adequate way, at least concerning the terms. The sense of a term (in classical terminology rather the "content of concept" of a term) is understood as a collection or a family of features, i.e. properties, which is something that does not logically depend on any semantic notion, in particular not on the notion of truth." (*ibidem*)

(Remember the structure of definitions *per genus proximum et differentias specificas*.)

Tichý admits, of course, that some features of classical logic are revisable from the viewpoint of contemporary logic but

> "the opinion that the notion of intension[4] logically precedes the notions of truth, analyticity and synonymy, and not vice versa, is in our opinion quite justified, [...]." (*ibidem*)

Tichý recognized a feature of classical definitions that is essential from the viewpoint of the above mentioned contrast *complex* vs. *simple* (*set-theoretical*). As early as in Aristotle's works the basic intuition connected

[4] Here by "intension" Tichý means "content", surely not Possible-World-intension.

with the notion of *concept* was realized: *a concept should determine* (*the essence of*) *an object by proposing the way how to get the object, how to* ***construct*** *the object using other concepts. Concepts are complex.*

Ad b): This claim follows, of course, from Aristotle's conception of definition, which determines the *essence* of the defined object. What is interesting is the fact that more than 2000 years after Aristotle it is George Bealer (see later) who characterizes the distinction between intensionality and hyperintensionality as follows:

> "[...] there have been two fundamentally different conceptions of properties, relations, and propositions. On the first conception intensional entities are considered to be identical if and only if they are *necessarily equivalent* [...] On the second conception [...] each definable entity is such that, when it is defined completely, it has a *unique, non-circular definition*." [2, p. 2]

Bealer's example: consider two definitions:

> (c) x is a trilateral iff x is a closed plane figure having three sides.
> (d) x is a trilateral iff x is a closed plane figure having three angles.
> On the first conception both (c) and (d) count as correct definitions since they both express necessary truths. On the second conception [...] (d) does not count as a correct definition; only (c) does.
> [2, p. 3]

This is really an interesting comparison. True, Bealer's characteristic does not mention the notion of *essence*, but what he could say would be that the second conception makes it possible to distinguish *concepts* (and he explicitly classifies concepts with the second conception). Thus being a closed plane figure having three sides is *the same property* as being a closed plane figure having three angles, but this necessary equivalence has been reached in virtue of two *distinct concepts*.

Thus the intuition that led Aristotle to his *unique definition* claim was the same as Bealer's in this respect: *both have considered concepts as entities that cannot be identified just in terms of necessary equivalence.*

Summing up: *Aristotle can be considered to be the founder of a theory of concepts* (*definitions*); *to say that his concepts have been hyperintensional entities means to translate his conception into the contemporary logical language. This is no shallow anachronism.*

Bolzano. First of all, we should appreciate that Bolzano was one of the first explicit adversaries of psychologism in logic. In his [3] he has shown that it is *images* (*representations*) *in themselves* ("*Vorstellungen an sich*") what is explored in logic, where being an sich means being an *abstraction*, which is not localizable in time and space.[5] *Concepts* ("*Begriffe*") are just a kind of *Vorstellungen an sich* and are therefore abstract, extra-linguistic (non-mental) entities.

But besides, concepts are not simple: In general, they are structured. Traditionally, what concepts *are* was not clear or even was psychologically explained, but every student knew that concepts possess *content* and *extent* (*Inhalt*, *Umfang*). Talking about content Bolzano suggests that while the content of a concept consists of some components[6] it does not determine the way in which these components combine[7] (p. 244). Thus the concept is just this *way* (we comment). That this interpretation is right can be justified, e.g. by a remarkable place in §148 of [3] where[8] "Bolzano distinguishes between the concept, say, TRIANGLE$_1$, as defined in terms of having three sides, and the concept, say, TRIANGLE$_2$, as defined in terms of having the sum of its angles equal to 2R. Now Bar-Hillel (a famous logician!) says about Bolzano's reasoning: "[i]ts uncritical acceptance may lead to strange, even contradictory formulations. [...] the two occurrences of the word 'triangle' [...] though differently defined, express both the property Triangle as their intension, so that the property Triangle is different from the property Triangle." If Bolzano had used our terminology he would, of course, have replied along the following lines: I do not speak about the *property* (being a) triangle: I speak about the *concepts* TRIANGLE$_1$, TRIANGLE$_2$: these are mutually distinct, for [...] they possess distinct structures."

The quoted formulations justify our opinion that *Bolzano became a (premature) pioneer of the modern theories of structured meaning/ concepts*.

Frege–Church. Frege's controversial theory of concepts (especially in [17, 18]) shares some problems with his controversial but ingenious theory of *sense* and *reference* (*denotation*) [19]. Here we will be also brief because the relevant literature is vast. The most relevant source is — from our

5 See the seemingly enigmatic footnote, p. 224: "Die Nominalisten hatten [...] richtig bemerkt, dass ein Begriff an sich nichts Existierendes; die Realisten, dass er kein blosser Name sey."

6 Die *Summe* der Bestandteile, aus denen die Vorstellung bestehet.

7 Nicht aber die *Art*, *wie* diese Theile untereinander verbunden sind.

8 The following quotation is from Materna [28, p. 107].

viewpoint — Tichý's [45], where Frege's oscillation between a procedural and a set-theoretical conception of *functions* is analyzed. This oscillation can be stated also in Frege's famous definition of concepts in [17] and [18]. A concept should be — according to this definition — a *function* from objects to truth-values. Defined in this way concepts would be just (one-place) predicates. Further: If concepts were these functions, which position would they occupy in Frege's semantic triangle? Surprisingly, it looks like if they were *denotations* (*Bedeutungen*)! This interpretation is evidently dubious: concepts would be identical with objects (at least in the mathematical case) and — as *Bedeutungen* — they could not play the role they should according to our intuition: concepts should be — in general — concepts of something. Not only that: if concepts were objects then the respective *concept word* (*Begriffswort*) could occur in a sentence as a subject but concepts are *functions* as well (!) and then the concept word could *not* occur as a subject. Frege tried to explain his position in [18] in his famous discussion with Kerry but one can see that the problem stems from Frege's conception of function. On the one hand, concept as a function is the *sense* of a concept word. On the other hand, the denotation of a concept word is a concept in the sense of a graph of function (*Wertverlauf*), which is, of course, a Fregean object.

The great Fregean Alonzo Church was evidently disappointed with this Fregean chaos and radically corrected the notion of concept using Frege's semantic triangle as follows (see [6, p. 6] and [8, p. 41]):

[†] "Of the sense we say that it *determines* the denotation, or *is a concept* of the denotation." [6, p. 6]

[‡] "Anything which is capable of being the sense of some name in some language, actual or possible, is a concept." [8, p. 41]

We can see that Church's proposal of defining concepts is most general. Let us compare the [‡] quotation with Bolzano and Frege:

> For Bolzano every expression of a language expresses a concept with one exception: (declarative) sentences express *sentences in themselves* (*Sätze an sich*). For Church also sentences express senses, which are concepts of truth-values (or propositions, we say).

For Frege only predicates (one-place but this is not important) express (denote? see above) concepts. Names (Frege cannot distinguish names and descriptions) name 'objects', sentences express 'thoughts' (*Gedanken*).

Church is a realist (Platonist), he can imagine any language ("actual or possible") and *anything* which can be the sense of an expression[9] is a concept. We can express this idea saying: *Concepts are potential meanings*.

Now let us observe the quotation from [6]. In a sense it is so precise that one can hardly imagine a more precise formulation. We can reformulate it so that something can be emphasized: **The** *sense* of an expression is **a** *concept* of the denotation. The definite vs. the indefinite article play an important role here: Church indicates that an expression *E* has just one sense (*the sense*) but that the denoted object can be given not only by that concept that is the sense of *E* (*a concept*). We can illustrate this proposal by a simple example.

Consider two expressions *E1* and *E2*:

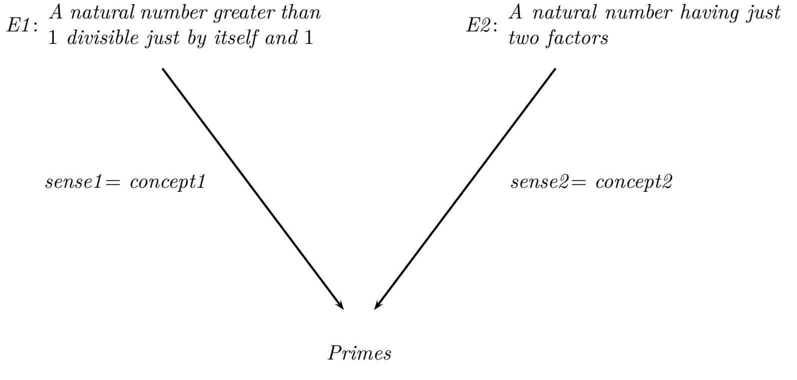

Figure 1.

This conception makes it possible to interpret such cases as *two concepts determine one and the same object*.

Observe that if the senses (concepts) were set-theoretical objects (e.g. functions) we would not be able to explain such cases. We will return to this in Section 2.

Bealer. George Bealer has emphasized some points essential for the transition to a hyperintensional conception of concepts in [2]. He has dis-

[9] I suppose that by *name* Church means an expression: for example a sentence is also a name (of a truth-value or a proposition).

tinguished two levels of analysis: the coarse-grained level and the fine-grained level. We have seen (see *Aristotle*) that Bealer's concepts are unambiguously hyperintensional. The way he logically handles such entities differs from ours in that he remains in first-order and is distrustful of possible worlds. As for a more detailed critical analysis of his theory see [12].

Carnap–Church. When Rudolf Carnap discovered the charm of semantics in [5] he at the same time once again discovered the problem with intentional context — the problem which Frege tried to solve by giving up independence of meaning of the context.[10]

Here I would like to decide that I will use the term *meaning* as Frege would use the term *Sinn*. So I will speak about meaning and denotation. The term *reference* (Black, Geach) will be used in another sense.

Carnap's *intensional isomorphism*, which should have solved the problem of a too coarse-grained approach to equivalent expressions, has attracted Church's attention. Church's writings between [7] and [9] are attempts at correcting Carnap's proposal and finding a positive solution. This has been finally found in [9], where Church defines his *synonymous isomorphism*, based on λ-convertibility and essentially similar to our *procedural isomorphism* (see Section 5).

This interesting story is described in details by C. A. Anderson in [1].

Cresswell. Max J. Cresswell began to explicitly talk about *hyperintensionality* in his [10] and about structured meaning [11]. Cresswell has felt the lack of structure in the meaning of an expression E when this meaning is defined as a function so that the meanings of the particular subexpressions of the expression E are lost. His proposal in [10, p. 30] is:[11]

Let $\alpha, \alpha_1, \ldots$ be expressions of the given languages, δ a functor, $I(\alpha)$ be the intension of α, V the value assignment, $M(\alpha)$ the meaning of α. Then

for α a simple expression: $I(\alpha) = M(\alpha) = V(\alpha)$

for $\alpha = \langle \delta, \alpha_1, \ldots, \alpha_n \rangle$: $I(\alpha) = I(\delta)(I(\alpha_1, \ldots, I(\alpha_n))$

$M(\alpha) = \langle M(\delta), M(\alpha_1), \ldots, M(\alpha_n) \rangle$, and says:

[10] Pavel Tichý offers a precise analysis of Frege's struggle for solving such problems in [45].

[11] In what follows I have exploited my [29, p. 22].

> "The point is that the intension of a complex expression is obtained by allowing the intension of its functor to operate on the intensions of the arguments of its functor. *The meaning however is simply the* $n+1$*-tuple consisting of the meaning of the functor together with the meanings of its arguments.*" (Emphasis P.M.) [10, p. 30]

This is how Cresswell wants to save meanings of the components of the expression E. He says (*ibidem*, p. 32): "Truth-conditional semantics is sufficient to determine meaning."

So we get the *set-theoretical paradigm*: *What counts is always the result of applying a procedure rather than the procedure itself.*

Cresswell's tuple-theory of meaning has been criticized by Tichý ([46]), Jespersen ([25]) and Duží, Jespersen, Materna ([12]). The point is that it is not the case that meaning of the expression E equals the set of meanings of the subexpressions of E (Bolzano knew it in 1837, see above) but it does not equal the ordered tuple of them as well. We get simply a *list* of those meanings but we do not know how they combine to become one meaning.

Cresswell simply has not solved the older problem: Bolzano's *way of combining* (see above) but also Russell's in [38], where Russell, building up a conception of structured propositions, states that "every proposition has a unity which renders it distinct from the sum of its constituents" (p. 52).

This *problem of unity*, precisely formulated in King ([26, p. 6]) as *What Binds Together the Constituents of Structured Propositions?* and generalized as *What Binds Together the Constituents of a Concept?* (see Materna [29, p. 23]) is exactly what Cresswell has not solved and what makes up the core of the following problem:

In which way is meaning/concept structured?

Tichý. Pavel Tichý has shown in [40] and [41] that this problem can be solved as soon as *meaning/concept are considered to be abstract procedures*. In both papers this view is supported by associating the main semantic notions with Turing machines. The problem with empirical expressions is solvable when O-machines (using oracles) are applied. Later Tichý founded TIL (see Section 3).

2. Concepts and intensions

In this section we will argue that if *intensions* are defined (rather standardly) as functions from possible worlds then *concepts cannot be intensions*.

First of all, what does it mean to say that a (logical, theoretical) system is *extensional*. A general *principle of extensionality* (**PE**) can be formulated as Leibniz's rule of substitution

$$\frac{a = b,\ \Phi(\ldots a \ldots)}{\Phi(\ldots b \ldots)} \tag{\textbf{L}}$$

where identity given by the expression "$a = b$" justifies substitution of one member of this identity for the other one.

Among other formulations of (**PE**) we find the extensionalist definition of identity of *functions*:

$$\forall f \forall g (\forall x (f(x) = g(x)) \supset f = g). \tag{Fu}$$

Using classical first-order predicate logic we preserve (**PE**).

Considering (**L**) Gamut writes:

> Its extensionality is both the strength and the weakness of standard propositional and predicate logic. It shows that in studying the validity of inferences in either of these systems, it suffices to consider the references of expressions and the principle of compositionality.[12]
>
> [20, p. 5]

Gamut reminds us however that we may need richer semantics and adduces some well-known cases where it seems that principles of extensionality do not hold any more. Now there are two ways how to cope with such anomalies. One of them sacrifices universal extensionality (Montague's Intensional Logic), the other one preserves all rules of extensionality (Gamut adduces Two-Sorted Type Theory, where s becomes another type besides e, t, and we will see that TIL is such an extensional system; see Tichý [42]).

[12] **Principle of compositionality**: "Let E be a set of expressions, m a meaning-assignment, M a set of 'available' meanings: Consider F, a k-ary syntactic operation on E. m is F-*compositional* just in case there is a k-ary partial function G on M such that whenever $F(e_1, \ldots, e_k)$ is defined, $m(F(e_1, \ldots, e_k)) = G(m(e_1), \ldots, m(e_k))$." [39, p. 5]

This problem arose because principles of extensionality were interpreted on the assumption that what we call *meaning* can be identified with *denotation* (a commonly accepted terminology has *reference* instead). Thus the problems with extensionalism were connected with the problem *What is meaning?* And the reduction of meaning to denotation, i.e., repudiating Frege's category of *sense* has punished the authors of this reduction of semantics to *denotational semantics* by confronting them with anomalies.

Consider the following *anomaly*.

$$\textit{Charles calculates } 2+3 \tag{1}$$

$$2+3=+\sqrt{25} \tag{2}$$

From (1) and (2) we get according to (**L**)

$$\textit{Charles calculates } +\sqrt{25},$$

which is, of course false if (1) is true. What happened?

(**L**) is valid but our analysis of premises is wrong. Thus (**L**) cannot be applied.

The reason why (**L**) cannot be applied is connected with the reduction of meaning to denotation. Indeed, the meaning of $2+3$ according to the reductionism is the number 5. So the premise (1) is true as well as (2). But let us try to show that (1) is not true (the first possibility of arguing that (**L**) cannot be applied) or that (1) can be true and (2) false (!).

The first possibility: The meaning of $2+3$ is the number 5 (denotation): then it is not the case that Charles calculates 5, it is rather a kind of nonsense.

The second possibility: The meaning of $2+3$ is not the number 5 but the procedure consisting in identifying the meanings of $+$, 2, 3, and applying the meaning of $+$ to the meanings of 2, 3. Then (1) may be true but (2) is false: the procedure of adding 2 and 3 is not identical with the procedure of extracting the square root of 25.

The solution offered by TIL consists in the claim that the meaning of an expression is never its denotation. In our case the meaning of $2+3$ is the procedure described in "the second possibility" above. We can say equivalently that $2+3$ expresses a *concept* (one of the concepts) of the number 5, while $+\sqrt{25}$ expresses another such concept. (The meaning of $2+3$ is the same in both (1) and (2), the distinction consists in the fact that in (1) this meaning is *mentioned* while in (2) it is *used* so that (2) is true even on this analysis. All this is 'technically' ensured in TIL.)

Another anomaly arises when notional attitudes are analyzed. Consider the wrong argument:

- *The President of Czech Republic is the husband of Livia Klaus.*
- *Charles wants to be the President of Czech Republic.*
- *Charles wants to be the husband of Livia Klaus.*

Again, we have applied (**L**) where it is not applicable. The meaning of *The President of Czech Republic* is the same in both premises, it is a concept of an intension, viz. of an 'individual role', i.e., a function that associates with a given possible world (and time) at most one individual. In the first premise however the *value* of this function in the given world and time is constructed whereas what is constructed in the second premise is just this function: Charles' attitude concerns the role, so the function, he is not interested in the actual value of the role.

Extensionalists such as those who use predicate logics cannot handle anomalies ('puzzles') of this kind because they do not admit *intensions.* Intensionalists like Montague are able to avert the threat stemming from the puzzles of this kind, but neither they can do anything with the puzzles of the preceding kind because they do not know *constructions* (in the sense of objective abstract procedures).

Since *concepts* can play the role of *meanings* (so that denotation is that object — if any — which is determined (constructed) by meaning) and a widespread opinion has it that meaning could be an *intension* let us prove that *on the assumption that intensions are functions from possible worlds concepts cannot be intensions.*

Consider

Proposition 1. "that Sun is greater then Moon",

Proposition 2. "that Sun is greater then Moon and whales are mammals".

Propositions are functions that associate every world and time with at most one truth-value. Let *F1*, *F2* be such functions that correspond respectively to Proposition 1, Proposition 2. Observe that the proposition that whales are mammals is true in all possible worlds, which means that the graph of *F1* is identical with the graph of *F2*. According to (Fu) it means however that *F1* = *F2* and therefore PROPOSITION 1 = PROPOSITION 2. This means however that we get a puzzle:

F1 = *F2*, so *F2* can be substituted for *F1* in the analysis of the sentence "Charles knows that Sun is greater than Moon" but Charles may know that Sun is greater than Moon and, at the same time, not know that whales are mammals. Contradiction.

The moral is: To be happy with intensions means that we cannot distinguish between objects that should be distinguished. We are surely not content with claiming that there is just one proposition here, so we could say that there are two distinct *concepts* of one proposition.

If concepts were intensions it would mean that we could not explain why distinct entities can be logically or analytically equivalent. There must be a more fine-grained criterion of diversity. We will show that such a criterion is definable and that concepts satisfy this criterion.

There is another reason why concepts cannot be intensions. We certainly believe that it is meaningful to talk about mathematical concepts. Mathematical concepts cannot be defined in terms of intensions because no mathematical construction is dependent as for its value on possible worlds[13]. But we will surely talk about such concepts as *a concept of primes, concepts of irrational numbers* etc. etc.

3. Procedural complexity

Let us return to comparison of two expressions:

E1: "*a natural number greater than* 1 *divisible just by itself and* 1"

and

E2: "*a natural number having just two factors*".

Intuitively, *E1* and *E2* express two (equivalent) concepts.

Let us suppose that concepts are *functions*. Then we get the same result as in our example with propositions. Let *F1* be a function defined on natural numbers and returning **T**(rue) iff the argument is a number satisfying the criterion given by *E1*, and *F2* be the function that returns **T** iff the argument satisfies the criterion given by *E2*. Clearly, *F1* returns

[13] Fitting in [14, p. 24] suggests some option of introducing possible worlds into mathematics, which is however rather artificial.

the same graph as *F2*, so that according to (Fu) *there is just one function here*. We have got *two concepts of one and the same function*.

This negative conclusion has to be supplemented by a positive conjecture that would *define* concept. More precisely, we should now attempt at an *explication* of the notion of concept.

We have argued that concepts cannot be intensions, now we have shown that in general concepts cannot be (functions as) set-theoretical objects. They should be *complex* (*structured*) rather than *simple*. In the brief historical remark (see Section 1) we have suggested that neither Cresswell's tuple theory can be considered to explain what "structured" means because it is not able to answer the question: *What Binds Together the Constituents of a Concept?* We have also indicated the answer saying that to be complex essentially means to be a *procedure*; we have stressed that what we mean by *procedures* are *abstract* procedures: their concrete counterparts are *processes*, i.e. time consuming entities. Thus while two tokens of a record of a program P are concrete (remember: localizable in time and space) just as two realizations of P on a computer (i.e. processes), the *algorithm* itself prescribed by P is *abstract*.[14]

The term that was chosen by TIL for explication of abstract objective procedures is *construction*. This choice may be criticized but probably every choice of this kind may be considered controversial. Nevertheless the first question that can be evidently asked concerns the relation between the term *construction* in TIL and the 'same term — let it be named *construction*' here — in various intuitionist versions. Some intuitionists emphasize the common features of the TIL notion and the intuitionist notion of construction.[15] I will adduce some formulations from Fletcher:

> "The word 'construction' is a metaphor. Clearly it is supposed to make us think of building houses or machines by *connecting components* together. For any sort of construction, components come in certain *basic types* (bricks, wheels, pistons, [...]); we take as many instances as we like of each type and connect them together with certain *combination procedures* (cementing, gluing, soldering, screwing, [...]), subject to certain constraints (only two things can be glued together at once, bricks have to be cemented not soldered. [...]) to form an arbitrarily large construction." ([15, p. 51]; emphasis P.M.)

[14] Remember Moschovakis' [37]: *Sense and denotation as algorithm and value.* So Frege's *sense* (surely an abstract entity) is understood as an algorithm.

[15] TIL itself is never mentioned.

Fletcher adduces this characteristic of constructions because he wants to compare it with Brouwer's views. We can immediately see one distinction: unlike Brouwer[16] Fletcher does not require that constructions were *mental*. Neither are here all constructions reduced to *proofs*. A general summarization follows (*ibidem*):

[*a*] *construction is a recursive structure*,

where such structures are dealt with in mathematical arguments and are idealizations stemming from "iterative processes and recursively structured objects in the physical world" (see [15, p. 50]).

(This 'transition' from real world to idealized objects is well illustrated as follows:

> "A *mathematical* construction is an *abstract* recursive structure. [...] Each abstract atom corresponds to a *type* of physical atom. [...] Equality of abstract constructions is defined by: $x = y$ iff x and y could be instantiated by the same physical constructions. Equivalently, $x = y$ iff x and y are built out of the same atoms using the combination rules in the same way." [15, pp. 51, 52]

Fletcher's conception is in many points compatible with the notion of construction as defined in TIL. Other intuitionists (like Per Martin-Löf) develop other conceptions, which share some features with TIL but differ from it more essentially.

Constructions in TIL are *abstract procedures* (see Section 4), not necessarily recursive. In the next section they will be defined. Important features of constructions can be found in Tichý [44].

The reason of my choice to base the explication of *concept* on TIL (rather than, e.g., on intuitionistic notions of construction) is double: first, my philosophy of logic is not an intuitionist one, second, I share with Tichý his conception of explication ("epistemic framework") as formulated in his ([45, pp. 194–200]). Now I quote the important pregnant formulation of the core of this conception:

> "The purpose of theoretical explication is to represent intuitions in terms of rigorously defined entities. It is to Frege that we owe the insight that the mathematical notion of function is a universal medium of explication not just in mathematics but in general. To explicate

[16] See Brouwer [4]. This does not mean that such a view must be shared by every philosopher / logician who calls himself intuitionist.

> a system of intuitive, pre-theoretical, notions is to assign to them, as surrogates, members of the functional hierarchy over a definite objectual base. Relations between the intuitive notions are then represented by the mathematically rigorous relationships between the functional surrogates. [...] By representing intuitions with functional surrogates we can throw light on their logical interdependence and show how some of them can be defined in terms of others." (*ibidem*)

This approach is essentially similar to the way chosen by Fletcher when he was explaining his idea of construction.

The functional character of our explication is now clear: Including nullary functions (particular objects like numbers) we can state that TIL deals with *functions* (functional surrogates) and *constructions*, i.e. the ways these functions are given. In another paper Tichý formulates this fact as follows: Logic studies

> "[l]ogical objects (individuals, truth-values, possible worlds, propositions, classes, properties, relations, and the like) and [...] ways such objects can be constructed from other such objects."
> [46], [48, p. 295]

Both functions and constructions are type-theoretically classified (similarly as in Montague's School). Types of order one are types of functions whose values and arguments are not constructions, higher-order types are types of constructions and functions whose values or arguments are constructions. Thus a *simple hierarchy of types* as well as a *ramified hierarchy of types* has to be defined.

4. Hierarchy of types

To compare:

Russell's hierarchy of types is based on properties and relations: TIL (simple) hierarchy classifies functions.

Martin-Löf's theory defines types based on the notion of proof: TIL hierarchy is more general.

Montague: atomic types e, t correspond approximately to ι, o types, respectively, in TIL, in the two-sorted theory s corresponds to ω in TIL, no type corresponds to τ in TIL, there are no higher-order types.

4.1. Types of order 1

Base[17] **B**

$\{o, \iota, \tau, \omega\}$, the set of the following types:

- o – the set of truth-values **T**, **F**;
- ι – the set of individuals;
- τ – the set of real numbers / the set of time moments;
- ω – the set of possible worlds.

i) Every member of **B** is a *type of order* 1. (atomic types)

ii) Let α, β_1, ..., β_m be *types of order* 1. Then $(\alpha\beta_1 \dots \beta_m)$ is a *type of order* 1: it is a set of partial functions with α the type of the value and β_i the types of the arguments. (functional types).

iii) Only [...]

In general, the choice of a base for any type-theoretical system is motivated by some tasks that should be performed. The present choice can be justified by the task of analyzing expressions of *natural language* ("NL expressions"). Naturally, it would be naïve to claim that no better choice could be found, but one point supports our choice: many well-known problems articulated in semantics of NL expressions have been solved when the present base has been used (see for example Tichý [48, 45], Duží, Jespersen, Materna [12]).

Let us illustrate the above definition of first-order types:

a) *Atomic types*:

The choice of $\boldsymbol{o}$ is clear: There are just two truth-values, *TIL is not a many-valued logic*. The cases where a sentence is neither true nor false are explained due to *partiality*. Thus the sentence *The greatest prime is odd* cannot be true or false in virtue of the fact that the function that associates any class of numbers with at most one number (the greatest one) is partial and takes no value at the class of primes. Absence of a truth-value is not the same as being a third value.

We have to comment the choice of $\boldsymbol{\iota}$. Individuals, which are inhabitants of $\boldsymbol{\iota}$, are *bare individuals*. Briefly, no individual possesses an empirical property by necessity.[18]

[17] We can change the Base. For analyses, e.g., of the expressions of arithmetic the Base can contain just two atomic types.

[18] The apparent exceptions (like: the empirical property *being the size of Aristotle* is possessed by necessity by just one individual, viz. Aristotle) are systematically explained in [12, 1.4.2.1].

As for $\boldsymbol{\tau}$, it plays (innocuously) double role. It is the type of real numbers, and since our natural assumption has it that time is continuum of moments, any non-empty interval of moments can be mapped onto the set of real numbers. No essential ambiguities arise.

Finally $\boldsymbol{\omega}$ is interpreted as follows: the chain of definitions that determine a property cannot be infinite. Thus we assume an *intensional base* as containing *intensions*, i.e., intuitively, pre-theoretically given 'determiners', empirical traits. Each member of $\boldsymbol{\omega}$ determines a unique "combinatorical possibility as to what objects are determined [...] by what intensions at what times" (Tichý [45, 46].) These combinatorically possible distributions of traits over objects are just called *possible worlds*.

b) *Functional types*:

Some important types of *extensions*:

- truth functions (oo) (negation), (ooo) (conjunction, disjunction etc.)
- quantifiers $(o(o\alpha))(\forall, \exists, \alpha$ any type)
- singularizer $(\alpha(o\alpha))(\iota,$ the only x such as)
- mathematical functions $(\tau\tau\tau)$ (for example *adding*, *dividing* ...)

Types of some *intensions*:

Intensions in general: $((\alpha\tau)\omega)$, α any type. Abbreviation $\alpha_{\tau\omega}$. Intensions (surrogates for objects pre-theoretically defined over the *intensional base*, see above) are dealt with as functions from possible worlds, frequently to *chronologies* $(\alpha\tau)$ of some type.

For example:

$o_{\tau\omega}$ is the type of *propositions*, which for any world and time return at most one truth-value,

$(o\iota)_{\tau\omega}$ is the type of (empirical) *properties of individuals* (like *being blue*, *being a table*, *to kill the President*, etc.). In general, $(o\alpha)_{\tau\omega}$ is the type of a property of objects of the type α. Thus *being an interesting proposition* is of the type (= belongs to the type) $(oo_{\tau\omega})_{\tau\omega}$.

$(o\beta_1 \dots \beta_m)_{\tau\omega}$ is the type of m-ary relations-in-intension, β_i types of arguments.

We can see that classes (relations-in-extension) are treated as the respective *characteristic functions*. Thus properties and relations-in intension are functions that associate with every world and time some class (relation-in-extension). So *being a table* is a function that associates with every world W and time T the class of individuals that are tables in W at T, and similarly for relations.

4.2. Constructions

Higher-order types, which make it possible to make a 'jump' into hyperintensionality, are defined in the *ramified hierarchy of types*. The latter can be however defined only after *constructions* have been defined. To justify this claim we now return to our example with the invalid argument:

- "*Charles calculates* $2+3$",
- $\underline{\text{“}2+3=+\sqrt{25}\text{”}}$,
- "*Charles calculates* $+\sqrt{25}$.

We have said that the meaning of "$2+3$" is the procedure consisting in identifying the meanings of "+", "2", "3", and applying the meaning of "+" to the meanings of "2", "3". We have also said that *constructions* in TIL are abstract procedures. To determine the type of *calculate*, we can first state that it is a relation (-in-intension) between an individual and a construction. Thus the type of the relation denoted by *calculate* would be

$$(o\iota?)_{\tau\omega},$$

where the question mark indicates the type of the construction. Indeed, we know the type of the object constructed by a construction C, which can be encoded by an arrow but up to now we do not know the type of the construction itself, which would be encoded by a slash. So we can write, e.g., *table* / $(o\iota)_{\tau\omega}$, $+/(\tau\tau\tau)$ but if x is a numerical variable[19] we cannot write x/τ, since we do not know the type of x. Instead we write $x \to \tau$.

In what follows I will reproduce the definition of TIL constructions, as they are formulated in [12, p 45].

i) The *Variable* x is a ***construction*** that constructs an object O of the respective type dependently on a valuation v; it v-constructs O.
ii) *Trivialization*: Where X is an object whatsoever (an extension, an intension or a *construction*), ${}^0X$ is the ***construction*** *Trivialization*. It constructs X without any change.
iii) The *Composition* $[X\ Y_1 \ldots Y_m]$ is the following ***construction***. If X *v-constructs* a function f of a type $(\alpha\ \beta_1 \ldots \beta_m)$, and $Y_1, \ldots, Y_m$ *v-construct* entities $B_1, \ldots, B_m$ of types $\beta_1, \ldots, \beta_m$, respectively, then the *Composition* $[X\ Y_1 \ldots Y_m]$ *v-constructs* the value (an entity, if any, of type α) of f on the tuple-argument $\langle B_1, \ldots, B_m\rangle$. Otherwise

[19] Variables are constructions, as we will immediately learn.

the *Composition* $[X\ Y_1 \dots Y_m]$ does not v-*construct* anything and so is v-*improper*.

iv) The *Closure* $[\lambda x_1 \dots x_m\ Y]$ is the following ***construction***. Let x_1, x_2, …, x_m be pairwise distinct variables v-constructing entities of types β_1, …,β_m and Y a construction v-constructing an α-entity. Then $[\lambda x_1 \dots x_m\ Y]$ is the ***construction*** λ-*Closure* (or *Closure*). It v-*constructs* the following function $f/(\alpha\ \beta_1 \dots \beta_m)$. Let $v(B_1/x_1, \dots, B_m/x_m)$ be a valuation identical with v at least up to assigning objects B_1/β_1, …, B_m/β_m to variables x_1, …, x_m. If Y is $v(B_1/x_1, \dots, B_m/x_m)$-improper (see iii), then f is undefined on $\langle B_1, \dots, B_m\rangle$. Otherwise the value of f on $\langle B_1, \dots, B_m\rangle$ is the α-entity $v(B_1/x_1, \dots, B_m/x_m)$-constructed by Y.

v) The *Execution* ${}^1X$ is the ***construction*** that either $v-$ *constructs* the entity v-constructed by X or, if X v-constructs nothing, is v-improper.

vi) The *Double Execution* ${}^2X$ is the following ***construction***. Let X be any entity; the Double Execution ${}^2X$ is v-*improper* (yielding nothing relative to v) if X is not itself a construction, or if X does not v-construct a construction, or if X v-constructs a v-improper construction. Otherwise, let X v-construct a construction X' and X' v-construct an entity Y. Then ${}^2X$ v-constructs Y.

Nothing is a ***construction***, unless it so follows from (i) through (vi).

Comments. All constructions are extra-linguistic objective abstract procedures. To understand properly this fact let us compare a λ-term with a construction. (Let us assume that τ is the type of natural numbers this time.)

λ-*term*:	$\lambda x(x+1)$
Construction:	$\lambda x[{}^0{+}\ x\ {}^01]$

The ***λ-term*** is an expression in an artificial language. It contains two occurrences of the variable x, parentheses, and is *interpreted* as the function *Successor*. As an expression, it does not construct anything.

The ***construction*** is not an expression:

- it is the abstract procedure encoded by the inscription above;
- it contains just one occurrence of x (λx is only our instruction "abstract over x");
- it does not contain any parentheses (any symbols for that matter);
- it *constructs* the function *Successor*.

Ad ii): Trivialization ^0X is a construction of key importance. It *mentions* the object X. Any object becomes a component of a construction either as a value of a variable, or as an object mentioned by Trivialization. Therefore the objects + and 1 cannot be directly components of a construction: in our example they are components of the construction due to being constructed (here by Trivialization).

Now imagine that X is a *construction*. The *type of* X is then determined by *mentioning* X, i.e. by ^0X.

In this way higher-order types are definable. Simple hierarchy is replaced by *Ramified hierarchy of types.*

4.3. Ramified hierarchy of types (RHT)

$\mathbf{T_1}$ (***types of order* 1**): defined above.

$\mathbf{C}_{\boldsymbol{n}}$ (***constructions of order n***):

(i) Let x be a variable ranging over a type of order n. Then x is a *construction of order n over B.*

(ii) Let X be a member of a type of order n. Then ^{0}X, ^{1}X, ^{2}X are *constructions of order n over B.*

(iii) Let X, X_1, …, X_m ($m > 0$) be constructions of order n over B. Then $[X\ X_1 \ldots X_m]$ is a *construction of order n over B.*

(iv) Let x_1, …. x_m, X ($m > 0$) be constructions of order n over B. Then $[\lambda x_1 \ldots x_m\ X]$ is a *construction of order n over B.*

(v) Nothing is a construction of order n over B unless it so follows from $\mathbf{C}_{\boldsymbol{n}}$ (i)–(iv).

$\mathbf{T}_{\boldsymbol{n+1}}$ (**types of order $n+1$**).

Let $*_n$ be the collection of all constructions of order n over B:

(i) $*_n$ and every type of order n are *types of order* $n + 1$.

(ii) If $m > 0$ and α, β_1, …, β_m are types of order $n + 1$ over B, then $(\alpha\ \beta_1 \ldots \beta_m)$ is a *type of order* $n + 1$ *over* B (see $\mathbf{T_1}$ (ii)).

(iii) Nothing is a *type of order* $n+1$ *over* B unless it so follows from (i) and (ii).

Comments. Let us present some examples[20], which show the way of increasing order of constructions.

[20] The examples are taken over from [12, p. 53].

(a) The constructions ${}^0{+}$, $[{}^0{+}\,x\ {}^01]$, $\lambda x[{}^0{+}\,x\ {}^01]$, $[\lambda x[{}^0{+}\,x\ {}^01]\ {}^05]$, $[{}^0{:}\ x\ {}^00]$, $\lambda x[{}^0{:}\ x\ {}^00]$, construct objects of types of order 1. They are constructions of order 1 (see definition of $\mathbf{C}_{\boldsymbol{n}}$), and belong, thus, to the type $*_1$ (see definition of $\mathbf{T}_{\boldsymbol{n+1}}$); i.e., to the type of order 2 (see definition of $\mathbf{T}_{\boldsymbol{n+1}}$ i).

(b) Let *Improper* be the set of constructions of order 1 that are v-improper for all valuations v; then *Improper* is an object belonging to (o^*_1), the type of order 2. (See definition of $\mathbf{T}_{\boldsymbol{n+1}}$ ii.)

(c) The Composition $[{}^0Improper\ {}^0[{}^0{:}\ x\ {}^00]]$ is a member of $*_2$, the type of order 3. It constructs the truth-value $\mathbf{T}$. The constituent ${}^0[{}^0{:}\ x\ {}^00]$ of this Composition is a member of $*_2$; it is an atomic *proper* construction that constructs $[{}^0{:}\ x\ {}^00]$, a member of $*_1$. It is atomic, because the construction $[{}^0{:}\ x\ {}^00]$ is not used here as a constituent but only mentioned as an input object.

The way **RHT** is built up guarantees that no such dangerous events like collision of variables can happen.

Now we can state (or at least plausibly claim) that as soon as *meaning* (or what Frege meant by *sense*) of an expression E is explicated as *construction expressed by* E we get a fine-grained semantics that makes it possible to solve many puzzles which cannot be satisfactorily solved by intensional logic (some such puzzles are adduced in previous sections, many others in Tichý's writings and [12]. An especially remarkable feature of this approach to Logical Analysis of NL expressions is the fact that principles of extensionality are obeyed: introducing a new, hyperintensional level does not cancel set-theoretical objects as intensions (we do not say that constructions replace functions: they just *construct* functions).

Another remarkable consequence of defining meanings as constructions consists in the fact that meanings defined in this way are *independent* of any context. We will briefly return to this point in the next section.

5. Concepts

5.1. Concepts and meanings

We have already stated that Church in [6] revising Frege's semantic triangle proposed the following semantic principle: **The** *sense* of an expression is **a** *concept* of the denotation. According to this principle any expression E of a language possesses just one meaning (*sense*) and this meaning is one of concepts of the denotation of E.[21] So — if the meaning is a construction — the construction that is the sense (meaning) *constructs the denotation* (if any). But there are other expressions, distinct from E, that construct the same object O, which means that while an expression possesses just one meaning there may be (always are) various distinct concepts of O. In the part "Frege – Church" of Section 1 we have shown one such example (two definitions of primes). Here we show the particular constructions.

a natural number greater than 1 divisible just by itself and 1

Types:

$Natn/(o\tau)$, $>/(o\tau\tau)$, $1/\tau$, $Div/(o\tau\tau)$, $\wedge, \supset, \vee/(ooo)$, $x, y \to \tau$, $\forall/(o(o\tau))$ $=/(o\tau\tau)$

$C1$ $\lambda x[{}^0\wedge[{}^0Natn\ x]\,[{}^0\wedge[{}^0>x\ {}^01]\,[{}^0\forall\ \lambda y\ [{}^0\supset\ [{}^0Divxy][{}^0\vee[{}^0=y\ x]\ [{}^0=y\ {}^01]]]\ ]]]$

a natural number having just two factors

Types: as above, $Card/(\tau(o\tau))$, $2/\tau$
$C2$ $\lambda x\ [{}^0\wedge\ [{}^0Natn\ x]\ [{}^0=[{}^0Card\ [\lambda y[{}^0Div\ x\ y]]\ {}^02]]]$

Imagine that *C1* as well as *C2* define a *function*. We have seen that then we get just *one* function. If concepts were functions then we would have *one* concept here whereas our intuition shouts: There are *two* concepts here.

Yet we can talk about *C1* and *C2* as follows: *C1* is the *meaning* of

a natural number greater than 1 *divisible just by itself and* 1

and *C2* is the meaning of

a natural number having just two factors

and meanings are constructions, i.e., *procedures*. Then the procedure prescribed by *C1* differs from the procedure prescribed by *C2*. The function constructed by both *C1* and *C2* is, of course, the same function

[21] In some cases there is no denotation. Then the meaning cannot be a concept of something. Consider the expression *the greatest prime*. We understand this expression: we know what to do in order to get the denoted object. The meaning is the respective construction (it is a concept in our sense, as we will soon learn) but the construction does not construct anything, so the concept is not a concept of some number.

(the type: $(o\tau)$) (viz. the characteristic function of the class of primes) but the procedures, which are the meanings (senses) of the respective definitions, are distinct (which we can see when performing the steps prescribed by *C1* and *C2*). The type of both procedures is $*_1$. Thus we can write *C1*, *C2* $/ *_1 \to (o\tau)$.

In this example everything what we say about meanings holds of concepts as well. *Can we identify constructions with concepts?*

5.2. Constructions and concepts

Every explication should take into account the way the *explicandum* is actually used. Let us try to test the hypothesis that every construction is a concept.

Consider following expressions. Which of them you think do express a concept?

(1) *(a) teacher*
(2) *the teacher of Alexander the Great*
(3) *the teacher of my son*
(4) *the highest mountain*
(5) *The highest mountain is in Asia*
(6) *the smallest prime number*
(7) *the smallest real number*
(8) *It rains*
(9) *Sherlock Holmes' pipe*

(1) expresses a concept, it is a 'simple concept' that constructs a property.

(2) expresses a concept, which constructs an 'individual role' i.e. an intension that associates every world-time with at most one individual.

(3) does not express any concept: the 'my' refers to an individual that will be identified after the situation of the utterance of the sentence.

(4) expresses a concept, similarly as (2).

Ad (5): here we can see that our answer will be counterintuitive: this expression does express a concept, which identifies (constructs) a proposition. Church (unlike Bolzano) would accept this answer (see [6]). The objection to this answer stems from the right conviction that concepts — unlike propositions or sentences — cannot be true or false. Yet the concept that constructs a proposition (our case) is not true or false: it only identifies a proposition, whether true or false).

(6) is one of the concepts of the number 2.

(7) is an *empty concept*: it defines a procedure that ends in a blind alley (there is nothing like the smallest number).

Ad (8): no proposition is constructed. A hidden parameter (variable) of the place is missing.

Ad (9): this is a problem. It could be a concept of Sherlock Holmes' pipe but there is no such individual that could play the role of Sherlock Holmes. To know what, e.g. Napoleon said on December 1^{st}, 1812 is practically impossible but only because we cannot detect this expression — no documents are here, and so. Theoretically however we are convinced that Napoleon said something at that time, only we have got not data enough. The case with Sherlock Holmes is different. Here there cannot be any data (unless Doyle referred to it): the expressions used by Sherlock Holmes are not a part of the state of affairs of our world. Thus I think that the fictive names are not connected with concepts (see [45, §49]).

If my answers are acceptable then the following provisional definition of concepts seems to be acceptable as well.

A construction is *closed* iff it does not contain any free variable.[22]

Definition (Provisional). *Concept* is a closed construction.

Justification: We have decided that (3) and (8) do not express concepts: the respective constructions would contain free variables (in (3) for *my*, in (8) for the localization.

As for (9), no Sherlock Holmes as an individual role can be constructed because no predication about 'him' is in principle verifiable or falsifiable: states of the world are not defined by fairy-tales or other kind of literature. See [12, p. 286–287] for a more detailed justification.

Possible objections

1. Let τ by the type of natural numbers. Consider following constructions:

$\lambda x_1[{}^0{+}\ x_1\ {}^0 1]$, $\lambda x_2[{}^0{+}\ x_2\ {}^0 1]$, $\lambda x_3[{}^0{+}\ x_3\ {}^0 1]$, …, $\lambda x_{56}[{}^0{+}\ x_{56}\ {}^0 1]$, …, $\lambda x_{3333}[{}^0{+}\ x_{3333}\ {}^0 1]$, …

[22] In TIL there are two kinds of boundness. A variable x is ^{0}bound iff it occurs in a trivialized construction. It is λ-bound iff it occurs in a closure $\lambda x_1 \dots x_m C$ and is one of the variables $x_1, \dots, x_m$ and is there not ^{0}bound. Thus, e.g., x is not free in the construction ${}^0[{}^0{+}\ x\ {}^0 1]$, $x \to \tau$, and it could be considered a concept if our definition were accepted.

Which of them is a concept of the Successor function?

This problem cannot be explained away by saying that more than one concept can construct one and the same object. All such cases that are relevant share one property: the particular (equivalent) concepts are meanings of distinct expressions (see the example with two definitions of primes). Here it is unthinkable (rather nonsensical) to associate the particular members of the above sequence with distinct NL expressions. The sequence as if represents one concept, expressed by the (English) expression *successor*.

2. What about the sequence

$$((Bel(ieve)/(o\iota o_{\tau\omega})_{\tau\omega}, x \to \iota, p \to (o_{\tau\omega})), w \to \omega, t \to \tau)$$

${}^0Bel$, $\lambda w {}^0Bel_w$, $\lambda w \lambda t\ [{}^0Bel_{wt}]$, $\lambda w\, \lambda t\, \lambda xp\ [{}^0Bel_{wt}xp]$, ..., where a construction differs from its neighbor just by η-reduction (expansion)? The procedure itself is essentially the same, the way of encoding differs. No distinct NL expressions can be found.

These two objections can be refuted as soon as *procedural isomorphism is defined*:

DEFINITION (*procedural isomorphism*). Let C and D be constructions. Then C and D are *α-equivalent*, denoted '${}^0C \approx_\alpha {}^0D$', $\approx_\alpha /(o*_n*_n)$, iff they differ at most by using different λ-bound variables. C and D are *η-equivalent*, denoted '${}^0C \approx_\eta {}^0D$', $\approx_\alpha /(o*_n*_n)$, iff one arises from the other by η-reduction or η-expansion. C and D are *procedurally isomorphic* iff there are constructions C_1, ..., C_n ($n > 1$) such that ${}^0C = {}^0C_1$, ${}^0D = {}^0C_n$, and each C_i, C_{i+1} are either α- or θ-equivalent.

EXAMPLES. ${}^0[\lambda x\ [{}^0\!> x\ {}^00]] \approx_\alpha {}^0[\lambda y[{}^0\!> y\ {}^00]]$; ${}^0[\lambda xy\ [{}^0\!+ x\ y]] \approx_\eta {}^{00}+$.

The relation *procedural isomorphism* is provably reflexive, symmetric and transitive so that it induces equivalence classes. Materna in [28] called these classes of pairs of constructions *Quid* ("quasi-identical" constructions). Clearly, every closed construction C is a member of the infinite class of constructions Quid-related (= procedurally isomorphic) with C. This class, let it be C^* , represents one and the same concept. It would be, however, incorrect to identify C^* with a concept:[23] We have strongly emphasized that concepts are not sets. Thus the problem arose

[23] This identification has been Materna's error in [28].

how to select for any such set of procedurally isomorphic constructions one of them that would be considered a *concept*. (The following text is a quotation from [12, p. 155].)

Remark. The solution that Horák puts forward in [24] is based on exploiting the *Quid* relation to define a *normalization* procedure resulting in the unique *normal form* of a *construction* C: $NF(C)$. If this procedure is applied to a closed construction C, the result, $NF(C)$, is the simplest member of the *Quid* equivalence class generated by C. The simplest member is defined as the alphabetically first, non-η-reducible construction. For every closed construction C it holds that $NF(C)$ is the *concept induced by* C, the other members of the same equivalence class *pointing to* this concept. Thus Horák's solution makes it possible to define *concepts as normalized closed constructions*. Their type is always $*_n$ for some n, $n \geqslant 1$.

For instance, the following constructions are procedurally isomorphic and thus belong to the same *Quid* class (a Materna-style concept of the successor function):

$$\lambda x[{}^0\!+ x\, {}^0 1];\ \lambda y[{}^0\!+ y\, {}^0 1];\ \lambda z[{}^0\!+ z\, {}^0 1];\ \lambda x[\lambda x[{}^0\!+ x\, {}^0 1]x];\ \lambda y[\lambda x[{}^0\!+ x\, {}^0 1]y], \ldots$$

The normal form of these constructions is $\lambda x[{}^0\!+ x\, {}^0 1]$. Thus, $\lambda x[{}^0\!+ x\, {}^0 1]$ is a Horák-style concept of the successor function, the other constructions of this class pointing to this concept. $\dashv$

Definition (Concept). *Concept* is a normalized closed construction.

In general, it holds that the meaning of an expression is a construction. The meaning of an expression that contains some free variables is an open construction. Such an expression does not have any definite denotation: its meaning only v-constructs an object, where v is a parameter of valuation (see the definition of constructions).[24]

Does it mean that every concept constructs some object? No, we have seen in the example (7) above that any concept of the smallest real number constructs nothing: any such concept is an improper construction. Indeed, it would be incorrect to claim that there is no concept of the smallest real number: the fact that we *understand* the expression *the smallest real number* suggests that there is a concept here, viz. a procedure that would identify an object (number) if there were such an object.

[24] Expressions that express open constructions use some indexical subexpressions (mostly pronouns).

Our definition makes it possible to derive some classifications of concepts.

A. A construction C is an *empirical concept* iff it constructs a non-trivial intension.[25]

A construction C is a *non-empirical concept* iff it either constructs a trivial intension or an extension.

Thus the sentence "*Charles calculates* $2+3$" expresses an empirical concept

$$\lambda w \lambda t[{}^0 Calc_{wt}\ {}^0 Ch\ {}^0[{}^0{+}\ {}^0 2\ {}^0 3]],$$

which constructs a proposition that is for its value certainly dependent on the current state of the world and, therefore, is not a constant function.

The sentence "*Every professional pianist is a musician*" expresses the non-empirical concept

(types: $Every/((o(o\iota))(o\iota))$, $Profpian$, $Mus/(o\iota)_{\tau\omega}$)):

$$\lambda w \lambda t[[{}^0 Every\ {}^0 Profpian_{wt}]\,{}^0 Mus_{wt}]$$

The procedure abstracts over worlds and times, so it is an intension (a proposition). This time however no world-time can be found where a truth-value other than **T** comes into consideration. We have got a trivial intension.[26]

Finally the expressions like "2 *is a prime*" or "*the greatest divisor of* 60" do not contain empirical subexpressions. They denote — i. e., the respective concepts construct — extensions, mostly mathematical objects.

B. A construction is an *empty concept* iff it is improper (see the definition of constructions).

A construction is a *quasi-empty concept* iff it constructs an empty class/relation.

A construction is an *empirically empty concept* iff the value of the constructed intension in the actual world-time either is missing or is an empty class/relation.

Thus the following constructions are respectively: (a) empty, (b) quasi-empty and (c) empirically empty concepts:

---

[26] There are three trivial propositions: propositions true in all world-times, false in all world-times and undefined in all world-times.

(a) *The smallest number* (Type: $the/(\tau(o\tau))$)

$$[{}^0the\ \lambda x\ [{}^0\forall\ \lambda y[{}^0\leq x\ y]]]$$

(b) *The class of smallest numbers*

$$\lambda x[{}^0\forall\ \lambda y[{}^0\leq x\ y]]]$$

(c) (i) *The man who is taller than the Eiffel tower*
(Types: $Taller/(o\iota\iota)_{\tau\omega}$, $Et/\iota_{\tau\omega}$)

$$\lambda w\lambda t[{}^0the\ \lambda x\ [{}^0\wedge\ [{}^0Man_{wt}x][{}^0Taller\ x\ {}^0Et_{\tau\omega}]]]$$

(ii) *to be a man who is taller than the Eiffel tower* (a property)

$$\lambda w\lambda t[\lambda x[{}^0\wedge\ [{}^0Man_{wt}x][{}^0Taller\,x\ {}^0Et_{\tau\omega}]]]$$

Thus the empirical concepts are never empty or quasi-empty. So that empirical expressions *always denote* (viz. an intension); what they often miss is *reference* (better: *referent*), i.e. the value of the denoted intension in the actual world-time. (This value is contingent, i.e., never conceptually determined.)

C. *Simple concepts*. Let X be an object of a type of order 1 (i.e., a non-construction). Then ${}^0X$ is a *simple concept of* X. Let $x/*_n$ be a variable. Then $[\lambda x\ x]$ is a *simple concept of an identity function* ($x \to \alpha$, α an arbitrary type).

The general criterion is: A simple concept is such a concept that no its proper subconstruction is a concept. This criterion is satisfied by (${}^0X$ because X is a non-construction, and by $\lambda x\,x$ because x as a free variable is no concept. The former case is more interesting because the important notion of *conceptual system* (see [29]) is based on this kind of simple concept.

(A conceptual system arises as a finite set of simple ("primitive") concepts, which unambiguously determines an infinite set of complex concepts, i.e., such constructions whose simple subconcepts are members of the set of primitive concepts. Thus *simple concepts* are always simple just w.r.t. some conceptual systems.)

There are some problems with simple concepts (see **D**).

D. *Algorithmic constructions.* Constructions are abstract procedures involving some *steps* just like *algorithms*. The question whether constructions are algorithms can be answered as soon as the definition of constructions is taken into account. Explicitly is the answer given in Tichý

[40] (also [48, pp. 599–621]). Constructions are rather algorithmic computations but moreover:

> "[n]ot every construction is an algorithmic computation. An algorithmic computation is a sequence of *effective* steps, steps which consist in subjecting a manageable object [...] to a feasible operation. A construction, on the other hand, may involve steps which are not of this sort. The application of any function to any argument, for example, counts as a legitimate constructional step; it is not required that the argument be finite or the function effective. Neither is it required that the function constructed by a closure have a finite domain or be effective. As distinct from an algorithmic computation, a construction is an *ideal* procedure, not necessarily a mechanical routine for a clerk or a computing machine." [48, p. 613]

Thus we can distinguish *algorithmic* and *non-algorithmic concepts*.

While algorithmic concepts are concepts of recursive functions *we cannot claim that, in general, non-algorithmic concepts are not concepts of recursive functions*.

EXAMPLES. (instead of properly formed constructions a mathematically correct way of encoding them has been chosen; the way how to get the respective constructions is clear but the result is a little long [...]):

Let a, b, c and n be variables that v-construct (= range over) natural numbers. Observe the constructions (expressed by)

(a) $$\lambda abcn(n > 2 \supset \neg(a^n + b^n = c^n)),$$

(b) $$\forall abcn(n > 2 \supset \neg(a^n + b^n = c^n)).$$

The respective constructions, composed from simple concepts ${}^0{>}$, ${}^0{\supset}$, ${}^0\neg$, ${}^0{+}$, ${}^0\forall$, and say, ${}^0Exp$,[27] are: (a) algorithmic ("effective steps", see above) and (b) non-algorithmic. We know however that (b) constructs a recursive function since the truth-value **T** has been born in a computer (a very hard birth, by the way).

Now as for the problems with simple concepts: Let M be any infinite set. In which way does the simple concept ${}^0M$ lead us to the set M? It simply offers M without any change, but on our assumption that M is infinite it means that there is no algorithm here which would lead to

[27] The expression x^n will encode what $[{}^0Exp\, x n]$ in a regularly written construction encodes.

actual infinity. Let us apply this situation to our example with the set Prime.

The simple concept ${}^0Prime$ confronts us with an impossible 'way' to the set of primes. It offers the actually infinite set of primes without any change, which is inaccessible to our knowledge. But after all, we know how to decide of any natural number whether it is a prime.[28] An accessible way is, e.g., the already mentioned construction

$$\lambda x[{}^0\wedge\,[{}^0Natn\,x][{}^0{=}\,[{}^0Card[\lambda y[{}^0Div\,x\,y]]{}^02]]]$$

This construction (a concept of the set of primes) contains also some simple concepts of infinite objects: ${}^0Natn$, the simple concept of natural numbers, ${}^0Card$, the simple concept of the set of pairs $\langle \text{set}, \text{number}\rangle$, ${}^0{=}$, the simple concept of all pairs of natural numbers $\langle x, x\rangle$, $\langle y, y\rangle, \dots,$ ${}^0Div$, the simple concept of the set of all pairs of natural numbers such that the first member is divisible by the second. This time however we are not asked to present these actual infinities: we have just to apply these infinite functions to a (de)finite argument: the actual infinity is replaced by potential infinity.

Consider now any conceptual system. Its finite set of simple concepts may contain such simple concepts which construct infinite sets. Then the operations of *refinement* (see below) become remedies.

Before explaining what a refinement is let us introduce a useful notion of *ontological definition*.

While verbal 'equational' definitions (Aritostotelian, Russellian, even explications) consist of two parts, viz. *definiendum* and *definiens*, the essence of definitions is compatible with the situation where there is no *definiendum*. In my [28] describe a dialogue with Tichý:

> "I explained my motivation: an essential feature of (explicit) definitions consists in assigning the *definiendum* with a meaning (i.e., with a concept). Tichý did not assent. He said: "Imagine a function having as its domain the set of natural numbers, and returning 1, 2 or 3 depending on whether the given argument is divisible by at most two numbers, or by more than two and less than six numbers, or by more than six numbers, respectively. Have I defined this function?" I admitted that he had. Tichý: "But I have not introduced any new term for denoting this function: there is no *definiendum* here."" [28, p. 7]

[28] We cannot take into account such factors as the length of human life or even the duration of cosmos etc., of course.

Yes, having at our disposal some simple concepts $\{C_1, \ldots, C_k\}$ we can *define* ***objects*** in terms of those simple concepts, viz. creating compound concepts whose simple subconcepts are all from that set of simple concepts. This is what Tichý did when he defined that 'nameless' function. In general:

An *ontological definition* is any compound concept that is not empty.

Now we can define *refinement*[29]:

Let C_1, C_2, C_3 be constructions. Let ${}^0X$ be a simple concept of X, let ${}^0X$ occur as a constituent of C_1, and let C_2 differ from C_1 only by containing in lieu of ${}^0X$ an ontological definition of X. Then C_2 is a *refinement of* C_1. If C_3 is a refinement of C_2 and C_2 is a refinement of C_1, then C_3 is a *refinement of* C_1.

Our 'paradigmatic' example of refinement is just our replacement of ${}^0Prime$ by various equivalent compound concepts (e.g. $\lambda x[{}^0\wedge[{}^0Natn\, x][{}^0= [{}^0Card[\lambda y[{}^0Div\, xy]]{}^02]]]$, see above). Thus one way how a non-algorithmic concept can be transformed to an equivalent algorithmic one consists in finding a refinement. That this is by far not a trivial task can be testified by the case *Fermat's Last Theorem*.

6. Conclusion

Explicating our intuitions concerning the world around us we exploit such exactly definable notions like *classes*, *properties*, *relations*, in general: *functions*.[30] We have to use, of course, *language*, linguistic expressions, and we meet a strange phenomenon: on the one hand, all expressions of the respective language are, in general, complex while functions are 'flat', simple set-theoretical objects. How come that *complex expressions are about simple objects?*

This question seems to support the views according to which any form of correspondence theory is doomed to breaking down.

The present article shows that a logical analysis of natural language can associate complex expressions of a natural language with complex extra-linguistic *meanings*, which can be viewed as *abstract procedures*.

[29] The first definition of refinement can be found in Duží [13].

[30] Let me remind you of the fact that classes, relations, and properties can be viewed as (characteristic) *functions*.

All those 'functional surrogates' (see [45]) like classes, properties, relations are then results of applying such abstract procedures. But then our problem disappears: the product of a procedure is not necessarily *similar* to the procedure.

Abstract procedures, here defined as (TIL) *constructions*, which do not contain free variables serve here as *explicans* for the notion of *concept*. Thus the complexity of concepts is mirrored by the complexity of expressions. We can say that expressions encode their meaning, in the case of expressions that do not contain indexical subexpressions, *expressions can be said to encode concepts*.

Explicating concepts as (abstract) procedures is in harmony with our (often hidden) intuitions. This explication can be used wherever we see that some objects are certainly distinct while the set-theoretic semantics says that they are not distinct. Typically, wherever we are told by set-theoretic semantics[31] that the two obviously distinct objects are actually one and the same function we can say: well, but *this function is created by two distinct concepts*, and we know what we mean.

The present paper argues that this procedural definition of concepts was implicitly suggested by some great philosophers, in particular already by Aristotle, not surprisingly by Bolzano, and that some suggestions came with recognizing [5] that neither intensionality can solve all logical puzzles connected with the need of making a jump to hyperintensionality, suggested and propagated especially by Cresswell and strictly realized by Pavel Tichý.

Our explication is based on Transparent Intensional Logic (TIL) founded by Pavel Tichý. All newly introduced expressions needed for the formulations are defined and the final definition is extremely simple. Whoever understands the essence of the basic idea does not need to study and remember all auxiliary definitions articulated in the paper.

Acknowledgements. First of all, the comments of my anonymous reviewer were certainly helpful. Further, I began to realize the present paper enjoying the support of Grant Agency of Czech Republic (Project No P401/10/1279).

[31] Don't forget that it is not only an extensional logic but also an intensional logic, whose semantics is set-theoretic, at any rate if intensions are defined as functions from possible worlds.

References

[1] Anderson, C. A., "Alonzo Church's contributions to philosophy and intensional logic", *The Bulletin of Symbolic Logic*, 4 (1998), 2: 129–171. DOI: 10.2307/421020
[2] Bealer, G., *Quality and Concept*, Clarendon Press, Oxford 1982.
[3] Bolzano, B., *Wissenschaftslehre*, vols. I, II, Sulzbach, 1837.
[4] Brouwer, L. E. J., *Over de grondslagen der wiskunde. Academisch proefschrift*, Maas et van Suchtelen, Amsterdam 1907.
[5] Carnap, R., *Meaning and Necessity*, Chicago University Press, Chicago 1947.
[6] Church, A., *Introduction to Mathematical Logic*, Princeton University Press, Princeton 1956.
[7] Church, A. "Intensional isomorphism and identity of belief", *Philosophical Studies*, 5 (1954): 65–73. DOI: 10.1007/BF02221771
[8] Church, A.: "Intensional semantics", pages 40–47 in [34].
[9] Church, A., "A revised formulation of the logic of sense and denotation. Alternative (1)", *Noûs*, 27 (1993): 141–157. DOI: 10.2307/2215752
[10] Cresswell, M. J., "Hyperintensional logic", *Studia Logica*, 34 (1975): 25–38. DOI: 10.1007/BF02314421
[11] Cresswell, M. J. *Structured Meanings*, MIT Press, Cambridge, 1985.
[12] Duží, M., B. Jespersen and P. Materna, *Procedural Semantics for Hyperintensional Logic*, Springer, 2010.
[13] Duží, M., "The paradox of inference and the non-triviality of analytic information", *Journal of Philosophical Logic*, 39 (2010), 5: 473–510. DOI: 10.1007/s10992-010-9127-5
[14] Fitting, M., "Intensional Logic", *Stanford Encyclopedia of Philosophy*, rev. January 27, 2011.
[15] Fletcher, P., *Truth, Proof and Infinity*, Dordrecht: Kluwer, 1998.
[16] Fodor, J. A., *Concepts. Where Cognitive Science Went Wrong*, Clarendon Press, Oxford 1998.
[17] Frege, G., *Funktion und Begriff*, H. Pohle, Jena 1891. (*Vortrag, gehalten in der Sitzung vom 9. Januar 1891 der Jenaischen Gesellschaft für Medizin und Naturwissenschaft*, Jena.)
[18] Frege, G., "Über Begriff und Gegenstand", *Vierteljahrschrift für wissenschaftliche Philosophie*, 16 (1892): 192–205.
[19] Frege, G., "Über Sinn und Bedeutung", *Zeitschrift für Philosophie und philosophische Kritik*, 100 (1892): 25–50.
[20] Gamut, L. T. F., *Logic, Language and Meaning*, vol. II., Chicago, London, The University of Chicago Press, 1991.
[21] Glock, H. J., "Concepts: Where subjectivism goes wrong", *Philosophy*, 84 (2009): 5–29. DOI: 10.1017/S0031819109000011

[22] van Heijenoort, J., "Sense in Frege", *Journal of Philosophical Logic*, 6 (1977): 93–102. DOI: 10.1007/BF00262050
[23] van Heijenoort, J., "Frege on Sense Identity", *Journal of Philosophical Logic*, 6 (1977): 103–108. DOI: 10.1007/BF00262051
[24] Horák, A., *The Normal Translation Algorithm in Transparent Intensional Logic for Czech*, PhD Thesis, Masaryk University, Brno, 2002. http://www.fi.muni.cz/~hales/disert/
[25] Jespersen, B., "Why the tuple theory of structured propositions isn't a theory of structured propositions", *Philosophia*, 31 (2003): 171–183. DOI: 10.1007/BF02380932
[26] King, J. C., "Structured propositions", http://plato.stanford.edu/entries/propositions-structured/, version as of 8 August 2001.
[27] Lewis, D., "General semantics", in *Semantics of Natural Language*, D. Davidson and G. Harman (eds.), Reidel, Dordrecht 1972, pp. 169–218.
[28] Materna, P., "Concepts and objects", *Acta Philosophica Fennica*, vol. 63, Helsinki 1998.
[29] Materna, P., *Conceptual Systems*, Logos, Berlin 2004.
[30] Materna, P., "The so-called myth of museum", *Organon F*, 11 (2004): 229–242.
[31] Materna, P., "Once more on analytic vs. synthetic", *Logic and Logical Philosophy*, 6 (2007): 3–43. DOI: 10.12775/LLP.2007.001
[32] Materna, P. "A notion of concept is either superfluous or procedural", *The Logica Yearbook 2010*, College Publications, London 2011, pp. 125–142.
[33] Materna, P., and J. Petrželka, "Definition and concept. Aristotelian definition vindicated", *Studia Neoaristotelica*, 5 (2008): 1, 3–37.
[34] Martinich, A. P. (ed.), *The Philosophy of Language*, Oxford University Press, 1985.
[35] Martin-Löf, P., *Intuitionistic Type Theory*, Bibliopolis, Naples 1984.
[36] Montague, R., *Formal Philosophy: Selected papers of R. Montague*, R. Thomason (ed.), Yale University Press, New Haven 1974.
[37] Moschovakis, Y. N., "Sense and denotation as algorithm and value", in *Lecture Notes in Logic*, vol. 2, J. Väänänen and J. Oikkonen (eds.), Berlin: Springer, 1994, pp. 210–249.
[38] Russell, B., *The Principles of Mathematics*, University Press, Cambridge 1903.
[39] Szabó, Z., "Compositionality", in *Stanford Encyclopedia of Philosophy*, E. Zalta (ed.), 2005, http://plato.stanford.edu/archives/spr2005/entries/compositionality
[40] Tichý, P., "Smysl a procedura", *Filosofický èasopis*, 16 (1968): 222–232. Translated as "Sense and procedure" in [48], pp. 77–92.
[41] Tichý, P., "Intensions in terms of Turing machines", *Studia Logica*, 26 (1969): 7–25. DOI: 10.1007/BF02134290. Reprinted in [48], pp. 93–109.

[42] Tichý, P. "Two kinds of intensional logic", *Epistemologia* 1 (1978): 143–164. Reprinted in [48], pp. 305–328.

[43] Tichý, P., "Questions, answers and logic", *American Philosophical Quarterly*, 15 (1978): 275–284. Reprinted in [48], pp. 293–304.

[44] Tichý, P., "Constructions", *Philosophy of Science*, 53 (1986): 514–534. DOI: 10.1086/289338. Reprinted in [48], pp. 599–621.

[45] Tichý, P., *The Foundations of Frege's Logic*, De Gruyter, Berlin – New York 1988.

[46] Tichý, P., "The analysis of natural language", 1994. Reprinted in [48], pp. 801–841.

[47] Tichý, P., "Constructions as the subject-matter of mathematics", in *The Foundational Debate: Complexity and Constructivity in Mathematics and Physics*, W. DePauli-Schimanovich, E. Köhler and F. Stadler (eds.), Kluwer, Dordrecht, Boston, London, and Vienna 1995, pp 175–185. Reprinted in [48], pp. 873–85.

[48] Tichý, P., *Collected Papers in Logic and Philosophy*, V. Svoboda, B. Jespersen, C. Cheyne (eds.), Filosofia, Czech Academy of Sciences — Prague and University of Otago Press — Dunedin, 2004.

Pavel Materna
Institute of Philosophy
Academy of Sciences of Czech Republic
Prague, Czech Republic
`maternapavel@seznam.cz`