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INFERENTIAL PARACONSISTENCY*

Paraconsistent logics settle the basis for inconsistent but not trivial theories. Usually they are constructed as systems of theorems and rules, defining *non explosive* consequence relations $\models$: a consequence $\models$ is *explosive* whenever from any set $\{\alpha, \neg\alpha\}$ of contradictory formulas any other formula β follows, $\{\alpha, \neg\alpha\} \models \beta$, cf. [6]. Most aften paraconsistency is involved by the set of theorems or tautologies of a logic constructed or "rediscovered" for that purpose. Thus, it reflexes only in internal quality of the logic.

The paper aims to show that paraconsistency can be generated outside of a system as a property of the inference. The gist of the proposal is the application of the so-called q-consequence, a special generalisation of Tarski consequence, cf. [3]. The actual q-extension of the three-valued Łukasiewicz logic results in a paraconsistent version of $Ł_3$. Moreover, it leads to a natural dualisation of the q-consequence and thus the concept of paraconsistency. The discussion on the last question will perhaps also subscribe to the dispute whether the term *parainconsistency* is not more appropriate than *paraconsistency* in reference to the property so widely considered in the literature.

1. Logical matrices

Let $Var = \{p, q, r, \ldots\}$ be a denumerable set of propositional variables and $F = \{F_1, \ldots, F_m\}$ a finite set of sentential connectives (we follow the custom-

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ary assumption that at least one among them has a non-zero arity, $a(F_i) \neq 0$. The set of formulæ, *For*, is defined inductively, putting

1. $Var \subseteq For$,
2. for any $F_i \in F$ such that $F_i(\alpha_1, ..., \alpha_k) \in For$, whenever $\alpha_1, \ldots, \alpha_k \in For$.

And, the *sentential language* is identified with an algebra of formulas

$$L = (For, F_1, \ldots, F_m).$$

L is an absolutely free algebra in its similarity class and *Var* is the set of its free generators. This means that for any mapping $s\colon Var \to A$ to the universe A of an algebra

$$A = (A, f_1, \ldots, f_m)$$

similar to L may be uniquely extended to the homomorphism $h_s\colon L \to A$, $h_s \in \mathrm{Hom}(L, A)$. Algebras similar to L are its interpretation structures. When equipped with a subset D of A they are called logical matrices. Thus, a pair

$$M = (A, D),$$

with A being an algebra similar to a language L and $D \subseteq A$ a subset of the universe of A, will be referred to as a *matrix* for L. Elements of D will be called *designated* elements of M and when the universe of the base algebra is a set of possible semantic correlates of sentences, the elements of D are then correlates of "true" sentences.

With each matrix M for L there is associated a set of formulae which take designated values only,

$$\mathrm{E}(M) = \{\alpha \in For : h\alpha \in D \text{ for any } h \in \mathrm{Hom}(L, A)\},$$

called its *content or logical system* determined by M. In turn, a relation $\models_M$ is said to be a matrix consequence of M provided that for any $X \subseteq For$, $\alpha \in For$

$$X \models_M \alpha \quad \text{iff} \quad \text{for every } h \in \mathrm{Hom}(L, A)(h\alpha \in D \text{ whenever } hX \subseteq D).$$

The content of a given matrix is a counterpart of the set of tautologies and the $\models_M$ is a natural generalization of the classical consequence. Everywhere in the sequel we shall refer to $\mathrm{E}(M)$ as to the "set of tautologies of M" or the "set of tautologies of the logic defined by M".

2. Quasi-matrices and q-consequences

The notion of a *quasi-matrix*, a *q-matrix* for short, was created as a base for the semantics of acceptance and rejection, cf. [3]. The leading assumption was that though acceptance and rejection should be disjoint, they need not be complementary. Thus, a q-matrix for L is a triple

$$M^* = (A, D^*, D),$$

with A being an algebra similar to a language L and D^*, D two disjoint subsets of the universe of A, $D^* \cap D \neq \emptyset$. Due to the primary motivation, the elements of D^* and D are referred to as *rejected* and *accepted* values, respectively.

A relation $\vDash_{M^*}$ is said to be a matrix *q-consequence* of M^* provided that for any $X \subseteq For$, $\alpha \in For$

$$X \vDash_{M^*} \alpha \quad \text{iff} \quad \text{for every } h \in \mathrm{Hom}(L, A)(h\alpha \in D \text{ whenever } hX \cap D^* = \emptyset).$$

Intuitively, a formula α is a q-consequence of a set of formulas X if it is accepted under any valuation for which all formulas in X are not rejected.

The q-concepts coincide with usual concept of logical matrix and consequence only if $D^* \cup D = A$, i.e. when the sets D^* and D are complementary. Then, the set of rejected elements coincides with the set of non-designated elements of M and the appearance of D^* in the structure of a (q-)matrix is unessential. Moreover, for any $M^* = (A, D^*, D)$

$$\mathrm{E}(M^*) = \{\alpha : \emptyset \vDash_{M^*} \alpha\} = \{\alpha \in For : h\alpha \in D \text{ for any } h \in \mathrm{Hom}(L, A)\}$$

and thus the q-consequence $\vDash_{M^*}$ is potentially as much appropriate inferential extension of the system of theorems determined by the matrix M as the consequence $\vDash_M$. For then we have

$$\mathrm{E}(M^*) = \mathrm{E}(M).$$

This means that any logical system may equally well be extended to the consequence $\vDash_M$ or to the quasi-consequence $\vDash_{M^*}$.

3. Paraconsistent three-valued Łukasiewicz logic

Let us consider the system of the three-valued Łukasiewicz logic $Ł_3$ defined on the standard propositional language:

$$L_k = (For, \neg, \rightarrow, \vee, \wedge, \equiv)$$

with negation ($\neg$), implication ($\rightarrow$), disjunction ($\vee$), conjunction ($\wedge$), and equivalence ($\equiv$). Ł$_3$ is the content of the three-valued matrix $M_3 = (A_3, \{1\})$ based on the algebra

$$A_3 = (\{0, {}^1\!/\!_2, 1\}, \neg, \rightarrow, \vee, \wedge, \equiv)$$

with the operations defined by the following tables:

x	$\neg x$
0	1
1/2	1/2
1	0

$\rightarrow$	0	1/2	1
0	1	1	1
1/2	1/2	1	1
1	0	1/2	1

$\vee$	0	1/2	1
0	0	1/2	1
1/2	1/2	1/2	1
1	1	1	1

$\wedge$	0	1/2	1
0	0	0	0
1/2	0	1/2	1/2
1	0	1/2	1

$\equiv$	0	1/2	1
0	1	1/2	0
1/2	1/2	1	1/2
1	0	1/2	1

The consequence of the three-valued Łukasiewicz logic is defined as follows:

$$X \models_{M_3} \alpha \quad \text{iff} \quad \text{for every } h \in \mathrm{Hom}(L, A_3)(\text{if } hX \subseteq \{1\}, \text{ then } h\alpha = 1).$$

One may easily verify that the formula

$$\text{(T1)} \qquad \alpha \rightarrow (\neg\alpha \rightarrow \beta)$$

belongs to Ł$_3$, which in turn is closed on the *Modus Ponens* being also a rule of the three-valued Łukasiewicz logic, i.e. for any $\alpha, \beta \in For$

$$\text{(MP)} \qquad \{\alpha, \alpha \rightarrow \beta\} \models_{M_3} \beta.$$

Therefore, $\{\alpha, \neg\alpha\} \models_{M_3} \beta$ for any $\alpha, \beta \in For$, which means that the logic in question is not paraconsistent.

Now, retaining 1 as the accepted value and taking 0 as the only rejected element we get the following *Łukasiewicz q-matrix*

$$M_3^1 = (A_3, \{0\}, \{1\}).$$

The q-consequence determined by is M_3^1 such that

$$X \models_{M_3^1} \alpha \quad \text{iff} \quad \text{for every } h \in \mathrm{Hom}(L, A_3)(\text{if } hX \subseteq \{{}^1\!/\!_2, 1\}, \text{ then } h\alpha = 1,$$

if all premises are not rejected (i.e. not false), then the conclusion is accepted (i.e. true).

(P) $\vDash_{M_3^1}$ is paraconsistent.

This holds true since there are α, β such that $\{\alpha, \neg\alpha\} \nvDash_{M_3^1} \beta$. One may simply take $\alpha = p$ and $\beta = q$ and see that then any valuation sending both variables p, q into $1/2$ falsifies the inference.

Remark. It is interesting to note that neither (MP) nor any of its "ascending" non-classical versions are the rules of $\vDash_{M_3^1}$. Therefore, the presence of (T1) in Ł$_3$ does not break the paraconsistency, as in the standard Łukasiewicz case.

4. Dualization

Now, let us consider the q-matrix

$$M_3^0 = (A_3, \{1\}, \{0\})$$

resulting from M_3^1 after the mutual exchange the sets of accepted and rejected values — in that case each set takes the role of the other. The logic thus arising in this way has the following properties: the logical system $\mathrm{E}(M_3^0)$ is the set of all formulas false under every valuation in A. Hence, this set consists of all formulas logically false (countertautologies) in the three-valued Łukasiewicz logic. On its side, the q-conseqence $\vDash_{M_3^0}$ admits inferences from non-accepted premises to the rejected conclusions:

$$X \vDash_{M_3^0} \alpha \quad \text{iff} \quad \text{for every } h \in \mathrm{Hom}(L, A_3)(\text{if } hX \subseteq \{0, 1/2\}, \text{ then } h\alpha = 0).$$

The dualisation just made is natural. However, in the present case the similarities of the two q-logics go much further. First, the sets of logically valid formulas of the "positive" logic and the set of logically false formulas of its dual, due to the properties of the Łukasiewicz negation, there is a direct "translatability":

$$\alpha \in \mathrm{E}(M_3^1) = \mathrm{E}(M_3) \quad \text{if and only if} \quad \neg\alpha \in \mathrm{E}(M_3^0),$$

and

$$\neg\alpha \in \mathrm{E}(M_3^1) = \mathrm{E}(M_3) \quad \text{if and only if} \quad \alpha \in \mathrm{E}(M_3^0).$$

Next, some characteristic formulas for both sides are on a par. Using definability properties of Łukasiewicz algebra it is even possible to translate dual q-consequences one into another.

Now, let us remark that for the “negative” Łukasiewicz q-consequence one may also formulate "paraconsistency" problem and to get a satisfying answer to it. Also in this case some sets $\{\alpha, \neg\alpha\}$ are not explosive. As before, it is not true that $\{p, \neg p\} \models_{M_3^0} q$, even though the roles of the truth (1) and the falsity (0) dually changed. The two following formulas

$$p \equiv \neg p \quad \text{and} \quad \neg(p \equiv \neg p)$$

exhibit these properties for $\models_{M_3^1}$ and $\models_{M_3^0}$. Accordingly, we have

$$\{p, \neg p\} \models_{M_3^1} p \equiv \neg p \qquad\qquad \{p, \neg p\} \not\models_{M_3^1} \neg(p \equiv \neg p),$$

and

$$\{p, \neg p\} \not\models_{M_3^0} p \equiv \neg p \qquad\qquad \{p, \neg p\} \models_{M_3^0} \neg(p \equiv \neg p).$$

The above considerations suggest the need for a distinction between two modes of "paraconsistency", at least for q-logics with a suitable $\equiv$. Thus, a system related to a non-explosive q-inference relation $\models$ may be called:

(i) *parainconsistent*,[1] when $\{p, \neg p\} \models p \equiv \neg p$ and $\{p, \neg p\} \not\models \neg(p \equiv \neg p)$,

(ii) *paraconsistent*, when $\{p, \neg p\} \models \neg(p \equiv \neg p)$ and $\{p, \neg p\} \not\models p \equiv \neg p$.

Adopting the terminology, we may say that the “positive” Łukasiewicz q-logic $\models_{M_3^1}$ is parainconsistent and its “negative” version, $\models_{M_3^0}$, is paraconsistent.

5. Final remarks

5.1. The constructions of paraconsistency in the paper are somewhat on the line of basing the paraconsistency on many-valued logic, see e.g. [1], [2] and [5]. In [5] we find a solution of the so-called Jaśkowski's problem in the three-valued logic and in [2] its generalisation onto the class of all finite-valued Łukasiewicz systems. In [1] there is a discussion of relations holding between paraconsistency and the finite many-valued logics of Rosser and Turquette. In the present approach many-valuedness is not only present in matrices, but also on the level of inference. As shown in [4], any matrix q-consequence is logically three-valued since in general only classes of three-element valuations may describe it.

[1] A time ago J. Perzanowski was first to use the term “parainconsistency” instead of “paraconsistency”.

5.2. The procedure of construction and dualisation of matrix quasi-logics is general and may be applied to a wide range of the known propositional logics. In particular one may, in a natural way, construct two versions of quasi-extensions for any finite and infinite Łukasiewicz system, just choosing $\{0\}$ and $\{1\}$ as "distinguished" sets in positive and negative q-logics. The results will be fairly similar to these for $Ł_3$. One may also distinguish other pair(s) of subsets in each case and thus to single out a wide family of paraconsistent and parainconsistent extensions of Łukasiewicz logics. The distinguishing feature of all these extensions would be that both properties were assured by inference tools outside the system. Concluding, one may say that deal with *inferential* paraconsistency and parainconsistency.

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