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A SIMPLE HENKIN-STYLE COMPLETENESS PROOF FOR GÖDEL 3-VALUED LOGIC G3

Abstract. A simple Henkin-style completeness proof for Gödel 3-valued propositional logic G3 is provided. The idea is to endow G3 with an under-determined semantics (u-semantics) of the type defined by Dunn. The key concept in u-semantics is that of "under-determined interpretation" (u-interpretation). It is shown that consistent prime theories built upon G3 can be understood as (canonical) u-interpretations. In order to prove this fact we follow Brady by defining G3 as an extension of Anderson and Belnap's positive fragment of First Degree Entailment Logic.

Keywords: many-valued logic; Gödel 3-valued logic; bivalent under-determined and over-determined semantics

1. Introduction

The aim of this paper is to present a simple Henkin-style completeness proof for Gödel 3-valued propositional logic G3. As it is well-known, Gödel logics were introduced in [9]. The aim of Gödel was simply to show that intuitionistic logic does not have a characteristic finite matrix, but Gödel logics are currently important, from several points of view, non-classical logics (see, e.g., [2] on Gödel logics). The strategy of the completeness proof we present is the following. Firstly, G3 is endowed with an under-determined semantics (u-semantics, for short) of the type defined by Dunn in [5, 7] (this semantics is briefly discussed below). The key concept in this semantics is that of "under-determined interpretation" (u-interpretation, for short). A u-interpretation is a function from the set of wffs to the set of proper subsets of the set $\{T, F\}$ (T and F represent truth and falsity in the classical sense; cf. Definition 7). It is then

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shown that this u-semantics is equivalent to the standards G3-semantics based upon the 3-valued matrix MG3 (cf. Definition 5). Finally, consistent prime theories are used as canonical u-interpretations, and it is shown that each non-theorem fails to belong to a consistent prime theory (to a canonical u-interpretation). In order to prove that consistent prime theories can be understood as u-interpretations, Brady's method in [3] is followed by axiomatizing G3 as an extension of Anderson and Belnap's positive fragment, FD_{+}, of First Degree Entailment Logic (FD) (cf. [1]).

As it is known, Dunn provided long ago a bivalent semantics (with "gaps" and "gluts") for the logic FD (see [5, 7]). (These semantics go back to Dunn's doctoral dissertation [4], but, as remarked by Dunn himself ([5, p. 150]) essentially equivalent semantics are defined in [15] and [18].) Then, after considerable time, Dunn resumed this semantics (now with "gaps" and/or "gluts") in [7] where he investigates some logics in the vicinity of intuitionistic logic. The u-semantics in this paper only allows "gaps", not "gluts", and differs from that in [7] in the following aspects: (1) the interpretation of the conditional (missing, of course, in [5]); (2) the interpretation of negation (cf. Remark 6 below); and, most of all, in the following point: Dunn notes in [7, p. 5], concerning part of the aims of his paper: "I investigate twelve natural extensions containing nested implications, all of which can be viewed as coming from *natural variations on Kripke's semantics for intuitionistic logic"*. (The italics are ours.) A similar remark could be applied to the results in [19] (where G3 is precisely one of the logics treated), but not to the developments in the present paper, that are instead related to the standard semantics of propositional classical logic: the bivalent set of truth values and a set of functions from the set of all wffs to this bivalent set. (The Deduction Theorem is used nowhere in the sequel.)

The present paper connects G3 with relevant logics trough Dunn's under-determined semantics and Anderson and Belnap's FD. And although it is known since long ago that such connection exists (cf. [8], [1, §29.4]), our results take it back to the most basic foundations of relevant logics, FD and Dunn's semantics for it.

The structure of the paper is as follows. In Section 2, the logic $\mathrm{G3}_{(\mathrm{FD}_{+})}$ is defined and we prove some of its theorems to be used in the following sections of the paper. In Section 3, we prove some facts (essential in the completeness proofs) about prime theories and consistent prime theories built upon $\mathrm{G3}_{(\mathrm{FD}_{+})}$. In Section 4, u-semantics for

$G3_{(FD_+)}$ (uG3-semantics) is defined. It is proved that this semantics is equivalent to the standard semantics definable on the 3-valued matrix MG3. In Section 5, simple (i.e., w.r.t. theoremhood) soundness and completeness of $G3_{(FD_+)}$ are proved. Finally, in Section 6, we prove strong (i.e., w.r.t. deducibility) soundness and completeness of $G3_{(FD_+)}$. The paper is ended with a couple of remarks. The first one is on the Routley and Meyer semantics; the second one, on the extension of the present semantics.

2. The logic $G3_{(FD_+)}$

As remarked out in the preceding section, G3 can be axiomatized as an extension of FD_+, namely, the logic $G3_{(FD_+)}$. In this section, we shall define this logic and will record some of its theorems to be used in the following sections. Firstly, we shall specify the logical language used in the paper and then we shall define the logic FD_+, the positive fragment of the logic *First Degree Entailments* FD (see [1]).

Remark 1 (Language, logics). The propositional language consists of a denumerable set of propositional variables $p_0, p_1, \ldots, p_n, \ldots$, and some or all of the following connectives: $\to$ (conditional), $\wedge$ (conjunction), $\vee$ (disjunction) and $\neg$ (negation). The biconditional ($\leftrightarrow$) and the set of wffs are defined in the customary way. A, B (possibly with subscripts $0, 1, \ldots, n$), etc., are metalinguisitic variables; we shall refer by $\mathcal{P}$ and $\mathcal{F}$ to the set of all propositional variables and all formulas, respectively. Except for the last section of the paper, we shall consider, from the proof-theoretical point of view, propositional logics formulated in the Hibert-style way. That is, formulated by means of a finite set of axioms (actually, axiom schemes) and a finite set of rules of derivation. The notions of "proof" and "theorem" are understood as it is customary in Hilbert-style axiomatic systems. That is, a *proof* is a sequence of formulas each one of which is an axiom or the result of applying any of the rules to one or two more previous formulas in the sequence. And a *theorem* is a proved formula. Let S be a logic. By $\vdash_S A$ it is indicated that A is a theorem of S.

Definition 1 (The logic FD_+). The logic FD_+ is axiomatized as follows:

Axioms:

A1. $A \to A$

A2. $(A \wedge B) \to A$ and $(A \wedge B) \to B$
A3. $A \to (A \vee B)$ and $B \to (A \vee B)$
A4. $[A \wedge (B \vee C)] \to [(A \wedge B) \vee (A \wedge C)]$

Rules

(Adj) $A \ \& \ B \Rightarrow A \wedge B$ — Adjunction
(MP) $A \to B \ \& \ A \Rightarrow B$ — Modus Ponens
(Trans) $A \to B \ \& \ B \to C \Rightarrow A \to C$ — Transitivity
(CI$\wedge$) $A \to B \ \& \ A \to C \Rightarrow A \to (B \wedge C)$ — Conditioned introduction of conjunction
(E$\vee$) $A \to C \ \& \ B \to C \Rightarrow (A \vee B) \to C$ — Elimination of disjunction

Remark 2 (On the definition of FD_+). We have defined FD_+ following [17, §4.3, pp. 51–52], but dropping T, F, t, $\circ$ and the rules establishing the antisymmetry of $\to$. The present definition of FD_+ differs from the classical one in [1, p. 158] in the addition of the rules Adj and MP. These rules are added (as in [17]) in order to extend FD_+.

The following, provable in FD_+ will be useful:

T1. $[A \wedge (B \wedge C)] \to [(A \wedge B) \wedge (A \wedge C)]$
T2. $[(A \vee B) \wedge (C \wedge D)] \to [(A \wedge C) \vee (B \wedge D)]$
R$\wedge$. $A \to C \ \& \ B \to D \Rightarrow (A \wedge B) \to (C \wedge D)$

Consider now the axioms that follow:

A5. $[(A \to B) \wedge A] \to B$
A6. $A \to (B \to A)$
A7. $(A \to \neg B) \to (B \to \neg A)$
A8. $\neg(A \wedge B) \to (\neg A \vee \neg B)$
A9. $\neg A \to (A \to B)$
A10. $(A \vee \neg B) \vee (A \to B)$

The logic $\mathrm{G3}_{(\mathrm{FD}_+)}$ (Gödel 3-valued logic G3 built upon FD_+) is defined as follows:

Definition 2 ($\mathrm{G3}_{(\mathrm{FD}_+)}$). The logic $\mathrm{G3}_{(\mathrm{FD}_+)}$ is axiomatized by adding A5–A10 to FD_+.

On the axiomatization of $\mathrm{G3}_{(\mathrm{FD}_+)}$, we have the following remark and proposition:

Remark 3 (On the axiom A10). The axiom A10 originates in [11], as a referee of the LLP has called to our attention.

Proposition 1 (On the axiomatization of $G3_{(FD_+)}$). *Given the logic* FD_+, *the axioms* A5–A10 *are independent from each other.*

Proof. See Appendix 1. ⊣

Therefore, $G3_{(FD_+)}$ is well-axiomatized w.r.t. FD_+. Next, we shall record some theorems and rules of this logic. We note that the following are provable in $G3_{(FD_+)}$ (a proof is sketched to the right of each one of them).

T3.	$A \to \neg\neg A$	A1, A7
Con	$A \to B \Rightarrow \neg B \to \neg A$	A7, T3
T4.	$A \to (\neg A \to B)$	A9, T3
T5.	$(A \wedge \neg A) \to B$	A5, A9
T6.	$\neg\neg(A \to A)$	A1, T3
T7.	$\neg(A \wedge \neg A)$	T5, T6, Con
T8.	$\neg A \vee \neg\neg A$	T7, A8
T9.	$\neg(A \to B) \to \neg\neg A$	A9, Con
T10.	$\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$	Con, A7
T11.	$\neg(A \wedge B) \leftrightarrow (\neg A \vee \neg B)$	A8, Con
T12.	$\neg(A \to B) \to \neg B$	A6, Con
T13.	$\neg B \to [\neg A \vee \neg(A \to B)]$	A5, T11, Con

3. Theories, primeness, consistency

In this section, we shall prove some facts about prime and consistent prime theories built upon the logic $G3_{(FD_+)}$. We begin by defining the notion of a theory.

Definition 3 (Theories). A theory is a set of formulas closed under Adjunction (Adj) and $G3_{(FD_+)}$-implication ($G3_{(FD_+)}$-imp). That is, a is a theory iff if whenever $A, B \in a$, then $A \wedge B \in a$, and if whenever $A \to B$ is a theorem of $G3_{(FD_+)}$ and $A \in a$, then $B \in a$.

The following definition classifies $G3_{(FD_+)}$-theories into different special classes.

Definition 4 (Prime and consistent prime theories). Let a be a theory. We set: (1) a is prime iff if $A \vee B \in a$, then $A \in a$ or $B \in a$; (2) a is inconsistent iff for some wff A, $A \wedge \neg A \in a$. Then, a is consistent iff it is not inconsistent.

Next, we prove an easy but very useful proposition, immediate by using A5 and A6.

PROPOSITION 2 (Closure of theories under MP and Veq). *Any theory a is closed under* MP *and* Veq. *That is, for any A, $B \in \mathcal{F}$,* (1) *if $A \to B \in a$ and $A \in a$, then $B \in a$;* (2) *if $A \in a$, then $B \to A \in a$.* (Veq *is an abbreviation for* "Verum e quodlibet" — *"a true proposition follows from any proposition".*)

Moreover, by T5, we obtain:

PROPOSITION 3 (Closure under Ecq). *Any theory is closed under* Ecq. *That is, if $A \wedge \neg A \in a$, then $B \in a$.* (Ecq *is an abbreviation for* "E contradictione quodlibet" — *"any proposition follows from a contradiction".*)

Remark 4 (Regular, complete theories). Notice that any non-empty theory contains all theorems of $G3_{(FD_+)}$, that is, it is regular (by closure under Veq); but, on the other hand, non-empty theories are not, in general, complete since the "Principium of Excluded Middle" is provable in $G3_{(FD_+)}$ only in the restricted form $\neg A \vee \neg\neg A$ (T8).

Next, we prove some properties of prime theories. Firstly, we shall prove the primeness lemma.

LEMMA 1 (Extension to consistent prime theories). *Let a be a theory and A a wff such that $A \notin a$. Then, there is a consistent prime theory x such that $a \subseteq x$ and $A \notin x$.*

PROOF. We prove Lemma 1 for any extension S of FD_+ with T5 as a theorem. Assume the hypothesis of Lemma 1. Next, extend a to a maximal S-theory x such that $a \subseteq x$ and $A \notin x$. Now, suppose that x is not prime. Then, $B \vee C \in x$, $B \notin x$, $C \notin x$ for some wffs B, C. Define the set $[x, B] = \{D : \exists F[F \in x$ and $\vdash_S (B \wedge F) \to D]\}$; define $[x, C]$ similarly. Then, we have: (1) $[x, B]$ and $[x, C]$ are closed under S-imp (cf. Definition 3): by Trans. (2) $[x, B]$ and $[x, C]$ are closed under (Adj): by (R$\wedge$), T1 and Trans. Therefore, $[x, B]$ and $[x, C]$ are S-theories. Moreover: (3) $x \subset [x, B]$ and $x \subset [x, C]$: by A2 and the supposition that $B \notin x$ and $C \notin x$. Now, as x is the maximal S-theory without A, we are entitled to conclude: (4) $A \in [x, B]$ and $A \in [x, C]$. But then: $A \in x$ (by T2 and Trans), which is impossible. Consequently, x is prime. Finally, x is -consistent: immediate by Proposition 3. ⊣

It what follows we shall prove the basic properties of conjunction, disjunction and negation in prime theories. By T3 and T8, respectively, we obtain:

LEMMA 2 (Theories, prime theories and double negation). *Let A be a wff. Then,* (1) *for any theory a, if $A \in a$, then $\neg\neg A \in a$.* (2) *For any prime theory a, $\neg A \in a$ or $\neg\neg A \in a$.*

LEMMA 3 (Conjunction and disjunction in prime theories). *Let a be a prime theory and A, $B \in \mathcal{F}$. Then,* (1) (a) *$A \wedge B \in a$ iff $A \in a$ and $B \in a$;* (b) *$\neg(A \wedge B) \in a$ iff $\neg A \in a$ or $\neg B \in a$;* (2) (a) *$A \vee B \in a$ iff $A \in a$ or $B \in a$;* (b) *$\neg(A \vee B) \in a$ iff $\neg A \in a$ and $\neg B \in a$.*

PROOF. The case (1a) follows by A2 and the fact that a is closed under (Adj). The case (1b) follows by T11 and the fact that a is prime. The case (2a): by A3 and the fact that a is prime. The case (2b): by T10 and the fact that a is closed under (Adj). ⊣

Finally, in the lemma that follows we shall establish the behavior of the conditional in consistent prime theories.

LEMMA 4 (The $\to$ in consistent prime theories). *Let a be a consistent prime theory and A, $B \in \mathcal{F}$. Then,* (1) *$A \to B \in a$ iff $\neg A \in a$ or $B \in a$ or ($A \notin a$ and $\neg B \notin a$);* (2) *$\neg(A \to B) \in a$ iff $\neg A \notin a$ and $\neg B \in a$.*

PROOF. Let a be a consistent prime theory.

(1) (a) $A \to B \in a \Rightarrow \neg A \in a$ or $B \in a$ or $(A \notin a \ \ \& \ \ \neg B \notin a)$:

Suppose that A and B are wffs such that $A \to B \in a$ and, for reductio, (i) $\neg A \notin a \ \ \& \ \ B \notin a \ \ \& \ \ A \in a$ or (ii) $\neg A \notin a \ \ \& \ \ B \notin a \ \ \& \ \ \neg B \in a$. Now the first alternative is impossible: as a is closed under MP (Proposition 2(1)), $B \in a$. But $B \notin a$. Then, the second one is impossible as well: by T13, $\neg B \to [\neg A \vee \neg(A \to B)]$. So, $\neg A \vee \neg(A \to B) \in a$, whence $\neg A \in a$ or $\neg(A \to B) \in a$. By hypothesis, $\neg A \notin a$. So, $\neg(A \to B) \in a$, contradicting the consistency of a.

(b) $[(\neg A \in a$ or $B \in a)$ or $(A \notin a \ \ \& \ \ \neg B \notin a)] \Rightarrow A \to B \in a$:

Let $\neg A \in a$. By A9, $\neg A \to (A \to B)$. So, $A \to B \in a$. Next, let $B \in a$. Then, $A \to B \in a$, as a is closed under Veq (Proposition 2(2)). Finally, let $A \notin a$ and $\neg B \notin a$. Then, by A10, $(A \vee \neg B) \vee (A \to B)$. So, $A \to B \in a$.

(2) (a) $\neg(A \to B) \in a \Rightarrow (\neg A \notin a \ \ \& \ \ \neg B \in a)$:

Suppose that A and B are wffs such that $\neg(A \to B) \in a$. By T12, $\neg(A \to B) \to \neg B$. So, $\neg B \in a$. Now, suppose for reductio $\neg A \in a$. By

T9, $\neg(A \to B) \to \neg\neg A$. Then $\neg\neg A \in a$ contradicting the consistency of a. Therefore, $\neg A \notin a$, and thus $\neg A \notin a$ and $\neg B \in a$, as was to be proved.

(b) $(\neg A \notin a \;\; \& \;\; \neg B \in a) \Rightarrow \neg(A \to B) \in a$:

Suppose that A and B are wffs such that $\neg A \notin a$ and $\neg B \in a$. By T13, $\neg B \to [\neg A \vee \neg(A \to B)]$. So, $\neg(A \to B) \in a$. $\dashv$

4. U-semantics for $\mathbf{G3}_{(\mathbf{FD}_+)}$

In this section we provide an underdetermined semantics for G3. We begin by recalling the 3-valued matrix MG3.

DEFINITION 5 (The 3-valued matrice MG3). Let S_3 be the set $\{0, \frac{1}{2}, 1\}$ and 1 is the only designated value. Then, the 3-valued matrix MG3 is defined by the following truth tables:

$\to$	0	$\frac{1}{2}$	1
0	1	1	1
$\frac{1}{2}$	0	1	1
1	0	$\frac{1}{2}$	1

$\wedge$	0	$\frac{1}{2}$	1
0	0	0	0
$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$
1	0	$\frac{1}{2}$	1

$\vee$	0	$\frac{1}{2}$	1
0	0	$\frac{1}{2}$	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1
1	1	1	1

	$\neg$
0	1
$\frac{1}{2}$	0
1	0

Given MG3, interpretations and validity are defined as follows.

DEFINITION 6 (Interpretations, validity). A G3-interpretation, I, is a function from $\mathcal{F}$ to S_3 according to the truth tables in MG3. Then, a wff is G3-valid (in symbols $\vDash_{G3} A$) iff $I(A) = 1$ for all G3-interpretations I. A rule of derivation $A_1, ..., A_n \Rightarrow B$ preserves G3-validity iff B is G3-valid if each A_i $(1 \leqslant i \leqslant n)$ is G3-valid.

Next, we define the general concept of u-semantics. Consider the following definitions where T and F represent the logical truth values truth and falsity in the classical sense.

DEFINITION 7 (U-interpretations). Let K be the set $\{T, F\}$ and K_{u} be the set of all proper subsets of K; that is, K_{u} is the set $\{\{T\}, \{F\}, \emptyset\}$. A u-interpretation (under-determined interpretation) I is a function from $\mathcal{F}$ to K_{u}.

DEFINITION 8 (U-structures). A u-structure (under-determined structure), S_{u}, is a pair $(K_{\mathrm{u}}, I_{\mathrm{u}})$ where K_{u} is as in Definition 7 and I_{u} is a class of interpretations.

Definition 9 (U-semantics). A u-semantics (under-determined semantics), Σ_{u}, is a pair $(S_{\mathrm{u}}, \vDash_{\mathrm{u}})$ where S_{u} is a u-structure and $\vDash_{\mathrm{u}}$ is a (valuation) relation such that for any $A \in \mathcal{F}$, A is u-valid (in symbols, $\vDash_{\mathrm{u}} A$) iff $T \in I(A)$ for all I in I_{u}.

Remark 5 (On the value of wffs in u-semantics). Let Σ_{u} be a u-semantics and $I \in I_{\mathrm{u}}$. Notice that if $T \in I(A)$ ($F \in I(A)$), then $F \notin I(A)$ ($T \notin I(A)$), but the converse does not hold generally: A can be assigned neither $\{T\}$ nor $\{F\}$.

We shall provide u-semantics (uG3-semantics) for $\mathrm{G3}_{(\mathrm{FD}_{+})}$. More precisely, it will be proved that uG3-validity (as defined in this uG3-semantics) and G3-validity (defined on the matrix MG3. Cf. Definition 6) are coextensive in the sense that a wff A is uG3-valid iff A is G3-valid. Then, in the next section it will be proved that $\mathrm{G3}_{(\mathrm{FD}_{+})}$ is sound and complete w.r.t. uG3-validity and, consequently, w.r.t. G3-validity.

Firstly, uG3-semantics is defined:

Definition 10 (UG3-semantics). A uG3-semantics is the u-semantics $(K_{\mathrm{u}}, I_{\mathrm{uG3}}, \vDash_{\mathrm{uG3}})$ where I_{uG3} is the set of all uG3-interpretations, a uG3-interpretation I being defined according to the following conditions for each $p \in \mathcal{P}$ and A, $B \in \mathcal{F}$, (1) $I(p) \in K_{\mathrm{u}}$; (2) (a) $T \in I(\neg A)$ iff $F \in I(A)$; (b) $F \in I(\neg A)$ iff $T \in I(A)$ or $F \notin I(A)$; (3) (a) $T \in I(A \wedge B)$ iff $T \in I(A)$ and $T \in I(B)$; (b) $F \in I(A \wedge B)$ iff $F \in I(A)$ or $F \in I(B)$; (4) (a) $T \in I(A \vee B)$ iff $T \in I(A)$ or $T \in I(B)$; (b) $F \in I(A \vee B)$ iff $F \in I(A)$ and $F \in I(B)$; (5) (a) $T \in I(A \to B)$ iff $F \in I(A)$ or $T \in I(B)$ or ($T \notin I(A)$ and $F \notin I(B)$); (b) $F \in I(A \to B)$ iff $F \notin I(A)$ and $F \in I(B)$.

Then uG3-validity is defined following Definition 9: A is uG3-valid (in symbols $\vDash_{\mathrm{uG3}} A$) iff $T \in I(A)$ for all uG3-interpretations I. Finally, a rule A_1 & A_2 & ... & $A_n \Rightarrow B$ preserves uG3-validity iff B is uG3-valid if each A_i ($1 \leqslant i \leqslant n$) is uG3-valid.

Remark 6 (Alternative interpretation of negation). We note that negation is interpreted in [5, 7, 13] as follows: by clause (2a) and the following clause (2b′) instead of (2b): $F \in I(\neg A)$ iff $T \in I(A)$.

Next, we shall put in correspondence uG3-interpretations and G3-interpretations.

Definition 11 (Corresponding uG3- and G3-interpretations). Let I_{G3} be a G3-interpretation (cf. Definition 6). Then, a uG3-interpretation

I_{u} is defined as follows: for each $p \in \mathcal{P}$, we set: (1) $I_{\mathrm{u}}(p) = \{T\}$ iff $I_{\mathrm{G3}}(p) = 1$; (2) $I_{\mathrm{u}}(p) = \emptyset$ iff $I_{\mathrm{G3}}(p) = \frac{1}{2}$; (3) $I_{\mathrm{u}}(p) = \{F\}$ iff $I_{\mathrm{G3}}(p) = 0$.

Next, I_{u} assigns $\{T\}$, $\{F\}$ or $\emptyset$ to each $A \in \mathcal{F}$ according to conditions 2-5 in Definition 10. It is said then that I_{u} is the corresponding uG3-interpretation to I_{G3}.

On the other hand, suppose given a uG3-interpretation I_{u}. The G3-interpretation I_{G3} corresponding to I_{u} is defined in a similar way. Therefore, given a uG3-interpretation (G3-interpretation) it is always possible to define the corresponding G3-interpretation (uG3-interpretation).

Now, it can be proved that uG3-validity and G3-validity are coextensive concepts by leaning on:

LEMMA 5 (Isomorphism of uG3- and G3-interpretations). *Let I_{G3} (I_{u}) be a G3-interpretation (uG3-interpretation) and I_{u} (I_{G3}) its corresponding uG3-interpretation (G3-interpretation) as defined in Definition 11. Then, for each wff A, it is proved*: (1) $I_{\mathrm{u}}(A) = \{T\}$ *iff* $I_{\mathrm{G3}}(A) = 1$; (2) $I_{\mathrm{u}}(A) = \emptyset$ *iff* $I_{\mathrm{G3}}(A) = \frac{1}{2}$; (3) $I_{\mathrm{u}}(A) = \{F\}$ *iff* $I_{\mathrm{G3}}(A) = 0$.

PROOF. By an easy induction on the length of A (the proof is left to the reader). ⊣

Immediate by definitions 6 and 10, and Lemma 5 we obtain:

THEOREM 1 (Coextensiveness of uG3-validity and G3-validity).
For every wff A, $\vDash_{\mathrm{uG3}} A$ iff $\vDash_{\mathrm{G3}} A$.

5. Soundness and completeness of $\mathbf{G3_{(FD_+)}}$

In this section we will prove the soundness and completeness of $\mathrm{G3}_{(\mathrm{FD}_+)}$ w.r.t. G3-validity. Actually, we shall prove soundness w.r.t. G3-validity and completenes w.r.t. uG3-validity, which is easier than a direct axiomatization of MG3. Then, leaning on the coextensiveness theorems in the preceding section, it will be shown that $\mathrm{G3}_{(\mathrm{FD}_+)}$ is sound and complete w.r.t., equivalently G3-validity and uG3-validity. As pointed out above, the strategy of the completeness proofs is to define u-interpretations via consistent prime theories by using the results in Section 3.

We begin by proving soundness.

THEOREM 2 (Soundness of $\mathrm{G3}_{(\mathrm{FD}_+)}$ w.r.t. G3-validity). *For every wff A, if $\vdash_{\mathrm{G3}_{(\mathrm{FD}_+)}} A$, then $\vDash_{\mathrm{G3}} A$.*

Proof. It is easy to check that the axioms of $G3_{(FD_+)}$ are G3-valid and that the rules of derivation of $G3_{(FD_+)}$ preserve G3-validity (in case a tester is needed, the reader can use MaTest (cf. [10]). ⊣

We now proceed into proving the completeness of $G3_{(FD_+)}$. We begin by defining the concept of an interpretation induced by a prime theory (a $\mathcal{T}$-interpretation). Then, we shall define the class of $\mathcal{T}$-interpretations we are interested in in this paper.

Definition 12 ($\mathcal{T}$-interpretations). Let K be the set $\{T, F\}$, as above, and $\mathcal{T}$ be a prime theory. A $\mathcal{T}$-interpretation, I, is a function from $\mathcal{F}$ to K such that for each $A \in \mathcal{F}$, (1) $T \in I(A)$ iff $A \in \mathcal{T}$; (2) $F \in I(A)$ iff $\neg A \in \mathcal{T}$.

Definition 13 ($\mathcal{T}^{u}_{G3}$-interpretations). Let K_u be the set $\{\{T\}, \{F\}, \emptyset\}$, as in Definition 7 and let $\mathcal{T}$ be a consistent prime $G3_{(FD_+)}$-theory. A $\mathcal{T}^{u}_{G3}$-interpretation, I, is a function from $\mathcal{F}$ to K_u defined as follows. For each $p \in \mathcal{P}$, we set: (a) $T \in I(p)$ iff $p \in \mathcal{T}$; (b) $F \in I(p)$ iff $\neg p \in \mathcal{T}$. Next, I assigns $\{T\}, \{F\}$ or $\emptyset$ to each $A \in \mathcal{F}$ according to conditions 2–5 in Definition 10.

Next, we shall prove that $\mathcal{T}^{u}_{G3}$-interpretations are actually $\mathcal{T}$-interpretations.

Lemma 6 ($\mathcal{T}^{u}_{G3}$-interpretations are $\mathcal{T}$-interpretations). *Let I be a $\mathcal{T}^{u}_{G3}$-interpretation. Then, I is indeed a $\mathcal{T}$-interpretation.*

Proof. Let I be a $\mathcal{T}^{u}_{G3}$-interpretation. We prove, for any wff A, conditions 1 and 2 in Definition 12. The proof is by induction on the length of A (the clauses cited in points 2–5 below refer to clauses in Definition 10. "H.I." abbreviates "hypothesis of induction").

(a) *A is a propositional variable*: By conditions a and b in Definition 13.

(b) *A is of the form $\neg B$*: (i) $T \in I(\neg B)$ iff (clause 2a) $F \in I(B)$ iff (H.I.) $\neg B \in \mathcal{T}$. (ii) $F \in I(\neg B)$ iff (clause 2b) $T \in I(B)$ or $F \notin I(B)$ iff (H.I.) $B \in \mathcal{T}$ or $\neg B \notin \mathcal{T}$. Now if $B \in \mathcal{T}$ or $\neg B \notin \mathcal{T}$, then $\neg\neg B \in \mathcal{T}$ follows by Lemma 2(1) and Lemma 2(2); and if $\neg\neg B \in \mathcal{T}$ then $B \in \mathcal{T}$ or $\neg B \notin \mathcal{T}$ follows by the consistency of $\mathcal{T}$. Consequently, $F \in I(\neg B)$ iff $\neg\neg B \in \mathcal{T}$, as was to be proved.

(c) *A is of the form $B \wedge C$*: (i) $T \in I(B \wedge C)$ iff (clause 3a) $T \in I(B)$ and $T \in I(C)$ iff (H.I.) $B \in \mathcal{T}$ and $C \in \mathcal{T}$ iff (Lemma 3(1a)) $B \wedge C \in \mathcal{T}$.

(ii) $F \in I(B \wedge C)$ iff (clause 3b) $F \in I(B)$ or $F \in I(C)$ iff (H.I) $\neg B \in \mathcal{T}$ or $\neg C \in \mathcal{T}$ iff (Lemma 3(1b)) $\neg(B \wedge C) \in \mathcal{T}$.

(d) *A is of the form $B \vee C$*: Similar to c by using clause 4a, 4b and Lemma 3(2).

(e) *A is of the form $B \to C$*: (i) $T \in I(B \to C)$ iff (clause 5a) $F \in I(B)$ or $T \in I(C)$ or ($T \notin I(B)$ and $F \notin I(C)$) iff (H.I.) $\neg B \in \mathcal{T}$ or $C \in \mathcal{T}$ or ($B \notin \mathcal{T}$ and $\neg C \notin \mathcal{T}$) iff (Lemma 4(1)) $B \to C \in \mathcal{T}$. (ii) $F \in I(B \to C)$ iff (clause 5b) $F \notin I(B)$ and $F \in I(C)$ iff (H.I.) $\neg B \notin \mathcal{T}$ and $\neg C \in \mathcal{T}$ iff (Lemma 4(2)) $\neg(B \to C) \in \mathcal{T}$. ⊣

Once that it has been proved that $\mathcal{T}^{\mathrm{u}}_{\mathrm{G3}}$-interpretations are in fact $\mathcal{T}$-interpretations, we have to explain how $\mathcal{T}$-interpretations are related to uG3-interpretations. Then, we shall prove completeness.

Lemma 7 (uG3-interpretations and $\mathcal{T}$-interpretations). *If I is a $\mathcal{T}^{\mathrm{u}}_{\mathrm{G3}}$-interpretation, then I is a uG3-interpretation.*

Proof. Suppose that I is a $\mathcal{T}^{\mathrm{u}}_{\mathrm{G3}}$-interpretation. Firstly, notice that as $\mathcal{T}$ is consistent, each propositional variable p is assigned $\{T\}$ or $\{F\}$ but not both. Then, it follows by definitions 10 and 13 that I is a uG3-interpretation. ⊣

We can now prove the completeness theorems.

Theorem 3 (Completeness of $\mathrm{G3}_{(\mathrm{FD}_+)}$). *For any wff A, if $\vDash^{\mathrm{u}}_{\mathrm{G3}} A$, then $\vdash_{\mathrm{G3}_{(\mathrm{FD}_+)}} A$. (Completeness of $\mathrm{G3}_{(\mathrm{FD}_+)}$ w.r.t. uG3-validity).*

Proof. Suppose $\nvdash_{\mathrm{G3}_{(\mathrm{FD}_+)}} A$ for some wff A and let $\mathrm{G3}_{(\mathrm{FD}_+)}$ be the set of its theorems. By Lemma 1, there is a (regular) consistent prime theory $\mathcal{T}$ such that $\mathrm{G3}_{(\mathrm{FD}_+)} \subseteq \mathcal{T}$ and $A \notin \mathcal{T}$. By Definition 13 and Lemma 6, we have a $\mathcal{T}^{\mathrm{u}}_{\mathrm{G3}}$-interpretation I such that for any wff B, $T \in I(B)$ iff $B \in \mathcal{T}$. So, $T \notin I(A)$, and since I is a uG3-interpretation (Lemma 7), $\nvDash^{\mathrm{u}}_{\mathrm{G3}} A$, by Definition 10. ⊣

To end this section we shall record a couple of corollaries.

Corollary 1 (Soundness and completeness w.r.t. G3-validity). *For any wff A, $\vDash_{\mathrm{G3}} A$ iff $\vdash_{\mathrm{G3}_{(\mathrm{FD}_+)}} A$.*

Proof. By Theorem 1, Theorem 2 and Theorem 3. ⊣

Corollary 2 (Soundness and completeness w.r.t. $\vDash_{\mathrm{uG3}}$). *For any wff A, $\vDash_{\mathrm{uG3}} A$ iff $\vdash_{\mathrm{G3}_{(\mathrm{FD}_+)}} A$.*

Proof. Immediate. It follows by Theorem 1 and Corollary 1. ⊣

6. Strong soundness and completeness of $\mathbf{G3}_{(\mathbf{FD}_+)}$

So far, we have been concerned with G3 viewed as the set of its theorems. Or, given the completeness theorems in the precedent section, viewed as the set of all valid formulas. In this section, however, G3 shall be understood in a more general way as a logic determined by a consequence relation. Firstly, we define this (standard) relation. Unless otherwise stated, let Γ and A refer to any set of wffs and a wff, respectively, throughout this section.

DEFINITION 14 (Deriving consequences from premises in $\mathrm{G3}_{(\mathrm{FD}_+)}$).
$\Gamma \vdash_{\mathrm{G3}_{(\mathrm{FD}_+)}} A$ ("A is derivable from the set of premises Γ in $\mathrm{G3}_{(\mathrm{FD}_+)}$") iff there is a finite sequence of wffs $B_1, ..., B_n$ such that B_n is A and for each B_i ($1 \leqslant i \leqslant n$) one of the following is the case: (1) $B_i \in \Gamma$; (2) B_i is an axiom of $\mathrm{G3}_{(\mathrm{FD}_+)}$; (3) B_i is the result of applying any of the primitive rules of derivation of $\mathrm{G3}_{(\mathrm{FD}_+)}$ to one or more previous formulas in the sequence.

Next, we shall prove the "strong soundness" of $\mathrm{G3}_{(\mathrm{FD}_+)}$.

THEOREM 4 (Strong soundness of $\mathrm{G3}_{(\mathrm{FD}_+)}$ w.r.t. $\vDash_{\mathrm{G3}}$). *If* $\Gamma \vdash_{\mathrm{G3}_{(\mathrm{FD}_+)}} A$, *then* $\Gamma \vDash_{\mathrm{G3}} A$.

PROOF. It is easy by induction on the length of the proof of A from Γ. If $A \in \Gamma$, the proof is trivial, and if A is an axiom, A is G3-valid by Theorem 2. Next, it is clear that Adj is truth-preserving, and, on the other hand, it is easily checked that the theses corresponding to the rest of the rules, (that is, $[(A \to B) \wedge A] \to B$ (MP), $[(A \to B) \wedge (B \to C)] \to (A \to C)$ (Trans), $[(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)]$ (CI$\wedge$), $[(A \to C) \wedge (B \to C)] \to [(A \vee B) \to C]$ (E$\vee$)) are G3-valid. So, each one of these rules preserves truth. Consequently, for any G3-interpretation I, if $I(\Gamma) = 1$, then $I(A) = 1$, as it was to be proved. $\dashv$

In the sequel, we prove strong completeness of $\mathrm{G3}_{(\mathrm{FD}_+)}$ w.r.t. the u-determined relation $\vDash^{\mathrm{u1}}_{\mathrm{G3}}$ that will be defined below. We begin by defining two different consequence relations on MG3. Then, u-interpretations of sets of wffs are also defined. Finally, we prove that uG3-interpretations and G3-interpretations of sets of wffs are isomorphic.

DEFINITION 15 (U-interpretations of sets of wffs). Let Σ_{u} be a u-semantics and I a u-interpretation in I_{u} (cf. Definition 10). Then, (1) $T \in I(\Gamma)$ iff $\forall A \in \Gamma(T \in I(A))$; (2) $F \in I(\Gamma)$ iff $\exists A \in \Gamma(F \in I(A))$.

In the case of many-valued logics, there are essentially two ways of defining consequence relations: truth-preserving and degree of truth-preserving relations, which regarding the logic treated in this paper can be defined as shown below. Firstly, G3-interpretations of sets of wffs are defined.

DEFINITION 16 (G3-interpretations of sets of wffs). Let I be an arbitrary G3-interpretation. Then, $I(\Gamma) = \inf\{I(A) : A \in \Gamma\}$.

DEFINITION 17 (Truth-preserving consequence relation). $\Gamma \vDash^1_{\mathrm{G3}} A$ iff if $I(\Gamma) = 1$, then $I(A) = 1$ for each G3-interpretation I.

DEFINITION 18 (Degree of truth-preserving consequence relation). $\Gamma \vDash^{\leqslant}_{\mathrm{G3}} A$ iff $I(\Gamma) \leqslant I(A)$ for each G3-interpretation I.

We shall refer by the symbols $\vDash^1_{\mathrm{G3}}$, $\vDash^{\leqslant}_{\mathrm{G3}}$ to the relations just defined. These two ways of understanding the notion of semantical consequence in many-valued logics are not in general equivalent. For example, they are not equivalent in the case of Łukasiewicz logics; but G3 is a remarkable case in which $\vDash^1_{\mathrm{G3}}$ and $\vDash^{\leqslant}_{\mathrm{G3}}$ are equivalent relations. This fact is recorded in the proposition that follows.

PROPOSITION 4 (Equivalence of $\vDash^1_{\mathrm{G3}}$ and $\vDash^{\leqslant}_{\mathrm{G3}}$). $\Gamma \vDash^1_{\mathrm{G3}} A$ *iff* $\Gamma \vDash^{\leqslant}_{\mathrm{G3}} A$.

PROOF. Cf., e.g., [2, Proposition 2.15]. ⊣

Given that $\vDash^1_{\mathrm{G3}}$ and $\vDash^{\leqslant}_{\mathrm{G3}}$ are equivalent, we shall dispense from now on with, say, $\vDash^{\leqslant}_{\mathrm{G3}}$.

PROPOSITION 5 (Isomorphism of G3- and uG3-interpr. of sets of wffs). *Let I_{G3} (I_{u}) be a G3-interpretation (uG3-interpretation) and I_u (I_{G3}) its corresponding uG3-interpretation (G3-interpretation). Then, we have,* (1) $I_{\mathrm{u}}(\Gamma) = \{T\}$ *iff* $I_{\mathrm{G3}}(\Gamma) = 1$; (2) $I_{\mathrm{u}}(\Gamma) = \emptyset$ *iff* $I_{\mathrm{G3}}(\Gamma) = \frac{1}{2}$; (3) $I_{\mathrm{u}}(\Gamma) = \{F\}$ *iff* $I_{\mathrm{G3}}(\Gamma) = 0$.

PROOF. Immediate by Lemma 5, Definition 15 and Definition 16. ⊣

We can now define an under-determined relation that is immediately proved to be coextensive with $\vDash^1_{\mathrm{G3}}$.

DEFINITION 19 (Under-determined $\vDash^{\mathrm{u1}}_{\mathrm{G3}}$-relation). $\Gamma \vDash^{\mathrm{u1}}_{\mathrm{G3}} A$ iff if $T \in I(\Gamma)$, then $T \in I(A)$ for all uG3-interpretations I.

PROPOSITION 6 (Coextensiveness of $\vDash^{\mathrm{u1}}_{\mathrm{G3}}$ and $\vDash^1_{\mathrm{G3}}$). $\Gamma \vDash^1_{\mathrm{G3}} A$ *iff* $\Gamma \vDash^{\mathrm{u1}}_{\mathrm{G3}} A$.

PROOF. Immediate by definitions 17 and 19, and Proposition 5. ⊣

Turning to the proof-theoretical side, we need the standard concept of "set of consequences of a set of wffs" that is defined as follows.

Definition 20 (The set of consequences in G3 of a set of wffs). The set $\mathrm{Cn}\Gamma[\mathrm{G3}]$ ("The set of all consequences of Γ in G3") is defined as follows: $\mathrm{Cn}\Gamma[G3] = \{A : \Gamma \vdash_{\mathrm{G3}_{(\mathrm{FD}_+)}} A\}$.

The most useful fact concerning $Cn\Gamma[\mathrm{G3}]$ is recorded in:

Proposition 7 ($Cn\Gamma[\mathrm{G3}]$ is a theory). *The set* $\mathrm{Cn}\Gamma[G3]$ (*i.e., the set* $\{A : \Gamma \vdash_{\mathrm{G3}_{(\mathrm{FD}_+)}} A\}$) *is a G3-theory.*

Proof. It is trivial that $\mathrm{Cn}\Gamma[\mathrm{G3}]$ is closed under (Adj) and (MP), and it is clear, by definitions 2 and 14, that $\mathrm{Cn}\Gamma[\mathrm{G3}]$ contains all theorems of $\mathrm{G3}_{(\mathrm{FD}_+)}$. Finally, $\mathrm{Cn}\Gamma[\mathrm{G3}]$ is closed under $\mathrm{G3}_{(\mathrm{FD}_+)}$-imp, since it contains all theorems and is closed under MP. Consequently, $\mathrm{Cn}\Gamma[\mathrm{G3}]$ is a theory. ⊣

The facts recorded so far in this section suffice to prove completeness.

Theorem 5 (Strong completeness of $\mathrm{G3}_{(\mathrm{FD}_+)}$ w.r.t. $\vDash^{\mathrm{u1}}_{\mathrm{G3}}$). *If* $\Gamma \vDash^{\mathrm{u1}}_{\mathrm{G3}} A$, *then* $\Gamma \vdash_{\mathrm{G3}_{(\mathrm{FD}_+)}} A$.

Proof. Suppose, for some set of wffs Γ and wff A, $\Gamma \nvdash_{\mathrm{G3}_{(\mathrm{FD}_+)}} A$. Then, $A \notin Cn\Gamma[\mathrm{G3}]$ (by Definition 20). And since $Cn\Gamma[\mathrm{G3}]$ is a G3-theory, by Lemma 1, there is a consistent prime theory $\mathcal{T}$ such that $Cn\Gamma[\mathrm{G3}] \subseteq \mathcal{T}$ and $A \notin \mathcal{T}$. By Definition 13, Lemma 6 and Lemma 7, $\mathcal{T}$ induces a uG3-interpretation I such that for any wff B, $T \in I(B)$ iff $B \in \mathcal{T}$. Now, $\Gamma \subseteq \mathcal{T}$ ($\Gamma \subseteq Cn\Gamma[\mathrm{G3}]$). So, $T \in I(\Gamma)$ (cf. Definition 15). And, on the other hand, $T \notin I(A)$. Therefore, $\Gamma \nvDash^{\mathrm{u1}}_{\mathrm{G3}} A$, by Definition 19, as it was to be proved. ⊣

We note a couple of corollaries.

Corollary 3 (Strong sound. and compl. of $\mathrm{G3}_{(\mathrm{FD}_+)}$ w.r.t. $\vDash^{1}_{\mathrm{G3}}$ and $\vDash^{\mathrm{u1}}_{\mathrm{G3}}$). (1) $\Gamma \vDash^{1}_{\mathrm{G3}} A$ *iff* $\Gamma \vdash_{\mathrm{G3}_{(\mathrm{FD}_+)}} A$; (2) $\Gamma \vDash^{\mathrm{u1}}_{\mathrm{G3}} A$ *iff* $\Gamma \vdash_{\mathrm{G3}_{(\mathrm{FD}_+)}} A$.

Proof. (1) immediate by theorems 4 and 5, and Proposition 6. (2) By (1) and Proposition 6. ⊣

Corollary 4 (Strong sound. and compl. of $\mathrm{G3}_{(\mathrm{FD}_+)}$ w.r.t. $\vDash^{\leq}_{\mathrm{G3}}$).
$\Gamma \vDash^{\leq}_{\mathrm{G3}} A$ *iff* $\Gamma \vdash_{\mathrm{G3}_{(\mathrm{FD}_+)}} A$.

Proof. Immediate by Proposition 4 and Corollary 3(2). ⊣

The paper is ended with a couple of remarks. The first one is on the Routley and Meyer semantics; the second one, on the extension of the present semantics. As it is well-known, the Routley-Meyer semantics (RM-semantics) were devised in the early seventies of the past century for interpreting relevant logics (cf. [16] and references therein). The RM-semantics can modelize a wealth of logics; actually, a wide spectrum of logics provided the basic positive logic B_+ is included in each one of them. The logic B_+ is axiomatized as follows: A1–A4, (Adj) and (MP) (cf. Section 2) and the following axioms and rules:

$[(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)]$

$[(A \to C) \wedge (B \to C)] \to [(A \vee B) \to C]$

$A \to B \Rightarrow (B \to C) \to (A \to C)$ Suffixing

$B \to C \Rightarrow (A \to B) \to (A \to C)$ Prefixing

Now, since G3 can be axiomatized by extending FD_+, this logic can be axiomatized by extending B_+. G3 can in particular be formulated with independent axioms by adding A5, A6, A7, A9 and A10 (A8 is not independent) to B_+. Then, it seemed worth it to try and provide a RM-semantics for G3. This aim was fulfilled and the results recorded in [12], but we do not have room here to discuss it. Let us only point out that the (simple) RM-semantics for G3 has, we think, some interest because it provides means for comparing G3 and logics of similar structure to relevant logics from the perspective of the latter, the RM-semantics.

Turning to the extension of the semantics, we remark that it can be used for modelling other logics by varying the interpretation of the conditional and/or the negation. Actually, as pointed out above, in [7] some logics in the vicinity of intuitionistic logic are interpreted; in [6], the quasi-relevant logic R-Mingle and in [3], the relevant logic BN4 and the quasi relevant logic RM3 (an extension of R-Mingle); in [14], Łukasiewicz 3-valued logic Ł3. Finally, in [13], the logic $G3^{<}_{Ł}$, a paraconsistent logic akin to G3. But, in none of the aforementioned papers (except in [13]), the logics under consideration have been defined from FD_+. However, it is to be expected that the pattern set in [13] and in this paper can be used for defining different logics by introducing interpretations of the conditional and/or the negation in bivalent semantics with "gaps" and/or "gluts" of the type we have considered here.

A. Appendix. Independence in $\mathbf{G3}_{(\mathbf{FD}_+)}$

All matrices that follow are such that:

1. A total order $0 \leqslant 1 \leqslant \cdots \leqslant n$ is defined on the set of truth values $\mathcal{V} = \{0, 1, ..., n\}$. (Except in one case as pointed out below.)
2. For all $a, b \in \mathcal{V}$, $a \vee b$ and $a \wedge b$ are understood as $\max\{a, b\}$ and $\min\{a, b\}$, respectively.
3. Designated values are starred.
4. The six matrices verify FD_+ plus five of the six axioms A5–A10 whereas falsifying the sixth one.
5. Each one of the matrices is the simplest one supporting its respective claim. These matrices have been found by using MaGIC, the matrix generator developed by John Slaney (see [17]). In case a tester is needed, the reader may use that in [10].

Independence of A5–A10 w.r.t. FD_+:
Matrix I. Independence of A5:

$\to$	0	1	2	$\neg$
0	2	2	2	2
1	1	2	2	1
*2	0	1	2	0

Falsifies A5 ($v(A) = 1$, $v(B) = 0$).
Matrix II. Independence of A6:

$\to$	0	1	2	$\neg$
0	2	2	2	2
*1	0	1	2	0
*2	0	0	2	0

Falsifies A6 ($v(A) = 1$, $v(B) = 2$).
Matrix III. Independence of A7:

$\to$	0	1	$\neg$
0	1	1	0
*1	0	1	0

Falsifies A7 ($v(A) = 0$, $v(B) = 1$).

Matrix IV. Independence of A8:

$\to$	0	1	2	3	$\neg$
0	3	3	3	3	3
1	0	3	2	3	0
2	0	1	3	3	0
*3	0	1	2	3	0

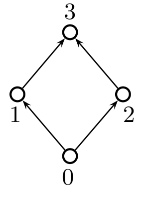
Falsifies A8 ($v(A) = 2$, $v(B) = 1$).
Matrix V. Independence of A9:

$\to$	0	1	$\neg$
0	1	1	1
*1	0	1	1

Falsifies A9 ($v(A) = 1$, $v(B) = 0$).
Matrix VI. Independence of A10:

$\to$	0	1	2	3	$\neg$
0	3	3	3	3	3
1	0	3	3	3	0
2	0	1	3	3	0
*3	0	1	2	3	0

Falsifies A10 ($v(A) = 2$, $v(B) = 1$).
The structure of Matrix IV is as follows:



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