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A PROPOSITIONAL CALCULUS FOR INCONSISTENT DEDUCTIVE SYSTEMS*

1. The origin of the problem

"The principle that two contradictory statements are not both true is the most certain of all." This is how Aristotle (quoted after Łukasiewicz [12], p. 10) formulates the opinion known as the logical principle of contradiction. Examples of convincing reasonings which nevertheless yield two contradictory conclusions were the reason why others sometimes disagreed with the Stagirite's firm stand. That was why Aristotle's opinion was not in the least universally shared in antiquity. His opponents included Heraclitus of Ephesus, Antisthenes the Cynic, and others (cf. Łukasiewicz [12], p. 1). In the early 19th century Heraclitus' idea was taken up by Hegel, who opposed to classical logic a new logic, termed by him dialectics, in which co-existence of two contradictory statements is possible. That opinion remains to this day as one of the theoretical foundations of Marxist philosophy, as the fol-

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It is the second English version of this paper. The first one — translated by Olgierd Wojtasiewicz — was published under the title "Propositional calculus for contradictory deductive systems", in *Studia Logica*, Vol. XXIV (1969), pp. 143–157. The present version is a small variation of the previous one. The chief difference is the change of the original Polish notation (done by A. Pietruszczak) into modern and standard one.

For further Editorial Notes see Notes (denoted in the text by natural numbers) at the end of the paper.

lowing authors refer: prof. Schaff ([16], pp. 113–121, 142–143), Wudel ([22]). Chwistek ([3], pp. 25ff) voices his doubts as to whether dialectics is necessary for that Weltanschauung, and prof. Ossowski ([14]) holds that people whose opinions differ widely from Marxism accept obvious contradictions (cf. Łukasiewicz [12], pp. 36–38). In a paper by the present author ([7]) the reader can find certain introductory explanations concerned with the issue here under consideration.

In the early 20th century the increasing precision of logical research known as logistics, mathematical logic, and symbolic logic, resulted in a revival, in a new, and more precisely formulated form, of some problems known to the ancients, and also in the discovery of many other reasonings which yield contradictions in theories which up to then had been accepted as correct. These reasonings were termed antinomies, the better known antinomies being those of Burali-Forti, Russell ([15], p. 102), Richard etc. (cf. Chwistek [3], pp. 18, 53, 54, 127). Russell's antinomy brought about a crisis in Cantor's set theory and also in Frege's deductive formalized logical system. Consistency in those theories could be restored only at the price of certain restrictions, such as the theories of logical types, the earliest of which is due to Russell ([15], 1st ed., pp. 523–528, [21], 2nd ed., pp. 37–65), and the simplest is Chwistek's simplified type theory (cf. [3], p. 129), which has the form of syntactical rules for a symbolic language and suffices to eliminate some of the logical antinomies, including that of Russell. The principle of making distinction between two (and sometimes more) languages, to which only one language corresponds in everyday usage, means a much greater deviation from the current use of language. That distinction is to be made between the language of a theory and the language in which we can discuss the properties of the former language. The latter language is termed the language of methodology or, as is done by Hilbert (cf. [6], vol. 1, p. 44), the language of a metasystem for the theory formulated in the former language. This distinction between languages is at variance with the natural striving synthetically to formulate all the truths we know in a single language, and thus renders a synthesis of our knowledge more difficult.

The transfer of Aristotle's principle of contradiction to contemporary logic risks a misunderstanding. As is known, in mathematical logic reference is made to sentences and terms, and not to judgements and concepts, as was done by Aristotle. The contemporary formal approach to logic increases the precision of research in many fields, but it would not be correct to formulate Aristotle's principle of contradiction as: "Two contradictory sentences are not both true." We have namely to add: "in the same language" or "if the words occurring in those sentences have the same meanings". This reservation is not always observed in every-day usage, and in science too we often use terms that are more or less vague (in the sense explained by Kotarbiński [10], pp. 26–29), as was noticed by Chwistek ([3], p. 12). Any vagueness of the term a can result in a contradiction of sentences, because with reference to the same object X we may say that "X is a" and also "X is not a", according to the meaning of the term a adopted for the moment.

Finally it is known that the evolution of the empirical disciplines is marked by periods in which the theorists are unable to explain the results of experiments by a homogenous and consistent theory, but use different hypotheses, which are not always consistent with one another, to explain the various groups of phenomena. This applies, for instance, to physics in its present-day stage. Some hypotheses are even termed "working" hypotheses when they result in certain correct predictions, but have no chance to be accepted for good, since they fail in some other cases. A hypothesis which is known to be false is sometimes termed a fiction. In the opinion of Vaihinger [19] fictions are characteristic of contemporary science and are indispensable instruments of scientific research. Regardless of whether we accept that extremist and doubtful opinion or not, we have to take into account the fact that in some cases we have to do with a system of hypotheses which, if subjected to a too consistent analysis, would result in a contradiction between themselves or with a certain accepted law, but which we use in a way that is restricted so as not to yield a self evident falsehood.

All these considerations raise the issue which shall be formulated precisely in terms of mathematical logic.

2. The formulation of the problem

Łukasiewicz's [13] parenthesis-free notation is used [in the original text, but not in the present translation; cf. Editorial Note at the beginning of the paper]:

$p \to q$	means "if p, then q",
$p \lor q$	"p or q",
$p \land q$	"p and q",
$p \leftrightarrow q$	"p if and only if q",
$\neg p$	"it is not true that q".

In any deductive system $\mathscr{S}$ under consideration, Leśniewski's usage of calling all formulae asserted in that system the theses of the system $\mathscr{S}$ is followed; this covers both the axioms and the theorems deduced from them or proved in any other way, specific for the given system, for instance those which satisfy a certain interpretation, adequate for that system. By the *assertion* of a formula is meant that which might be defined as acceptance as universally true or universally valid, although further analysis will cover systems to which this explanation does not apply.

In the two-valued sentential calculus, usually symbolized as $\mathbf{L_2}$, there is a well-known thesis which shall here be termed the implicational law of overfilling:

$\mathrm{L}_2 1$ $$p \to (\neg p \to q).$$

A deductive system $\mathscr{S}$ is called *inconsistent*, if its theses include two such which contradict one another, that is such that one is the negation of the other, e.g., $\mathcal{T}$ and $\neg\mathcal{T}$. If any inconsistent system is based on a two-valued logic, then by the implicational law of overfilling one can obtain in it as a thesis any formula $\mathcal{P}$ which is meaningful in that system. It suffices to substitute in $\mathrm{L}_2 1$ $\mathcal{T}$ for p and $\mathcal{P}$ for q and to apply the rule of *modus ponens* twice. A system in which any meaningful formula is a thesis shall be termed *overfilled*. This deviates from the terminology accepted so far: in the methodology of the deductive sciences such systems have so far been called inconsistent, but for the purpose of the analysis presented in this paper it is necessary to make a distinction between two different meanings of the term “an inconsistent system”, and to use it only in one sense, as specified above. The overfilled systems have no practical significance: no problem may be formulated in the language of an overfilled system, since every sentence is asserted in that system. Accordingly, the problem of the logic of inconsistent systems is formulated here in the following manner: the task is to find a system of the sentential calculus which: (1) when applied to the inconsistent systems would not always entail their overfilling, (2) would be rich enough to enable practical inference, (3) would have an intuitive justification. Obviously, these conditions do not univocally determine the solution, since they may be satisfied in varying degrees, the satisfaction of condition 3 being rather difficult to appraise objectively. [1]

3. The known solutions

In addition to the two-valued sentential calculus other systems of the sentential calculus are known, and some of them provide a solution of the problem formulated above.

A. KOLMOGOROV'S SYSTEM ([9], p. 651). It is a sentential calculus based on the four axioms of Hilbert's positive logic, which shall be formulated as:

K1 $p \to (q \to p)$,

K2 $\big(p \to (p \to q)\big) \to (p \to q)$,

K3 $\big(p \to (q \to r)\big) \to \big(q \to (p \to r)\big)$,

K4 $(q \to r) \to \big((p \to q) \to (p \to r)\big)$,

and on the axiom introduced by Kolmogorov:

K5 $(p \to q) \to \big((p \to \neg q) \to \neg p\big)$.

In this system $L_2 1$ cannot be proved, which becomes obvious as soon as Łukasiewicz's matrix method is applied (cf. [13], pp. 109–114). Kolmogorov's axioms [2] satisfy the well-known matrix (Łukasiewicz [13], p. 114)

(1)

$\to$	1	0	$\neg$
*1	1	0	1
0	1	1	1

in which 1 is the designated value; the formula $L_2 1$ does not satisfy that matrix in view of

$$1 \to (\neg 1 \to 0) \ = \ 0\,.$$

In matrix (1) $\neg$ is interpreted as the operator known as "verum". It is worth while mentioning that this interpretation was described by Łukasiewicz ([12], pp. 102ff), without recourse to a symbolic notation, as early as 1910 as an example of such a meaning of negation in which two inconsistent sentences may be asserted. [3] In Kolmogorov's system, however, a special case of the law $L_2 1$ may be obtained, namely that in which the variable q is replaced by its negation:

$$p \to (\neg p \to \neg q).$$

The proof is given below; the use of inference rules is marked in the way introduced by Łukasiewicz ([13], p. 67) [in the original text; for the present translation see the editorial note 4 on p. 55], i.e., by reference to proof lines.

$\mathrm{K3}[p/(q \to r), q/(p \to q), r/(p \to r)] = \mathrm{K4} \to \mathrm{K6}$ [4]

K6 $(p \to q) \to \big((q \to r) \to (p \to r)\big)$

$\mathrm{K6}[q/(q \to p)] = \mathrm{K1} \to \mathrm{K7}$

K7 $\big((q \to p) \to r\big) \to (p \to r)$

$$\mathrm{K7}[r/((q \to \neg\, p) \to \neg\, q] = \mathrm{K5}[p/q, q/p] \to \mathrm{K8}$$

K8 $\quad p \to \big((q \to \neg\, p) \to \neg\, q\big)$

$$\mathrm{K6}[q/((q \to \neg\, p) \to \neg\, q), r/(\neg\, p \to \neg\, q)] = \mathrm{K8} \to \big(\mathrm{K7}[p/\neg\, p, r/\neg\, q] \to \mathrm{K9}\big)$$

K9 $\quad p \to (\neg\, p \to \neg\, q)$ q.e.d.

Suppose that Kolmogorov's system is applied to an inconsistent system in which $\mathcal{T}$ and $\neg\,\mathcal{T}$ are theses and $\mathcal{P}$ is any meaningful formula. The substitutions $p/\mathcal{T}$ and $q/\mathcal{P}$ in K9 and the application of the rule of *modus ponens* yields the theorem $\neg\,\mathcal{P}$. Hence in any inconsistent system $\mathscr{S}$ any meaningful formula beginning with the symbol of negation can be obtained as a thesis, so that negation must be interpreted as verum in accordance with matrix (1). This is a state which comes close to the overfilling of the system $\mathscr{S}$.

B. LEWIS'S SYSTEM OF STRICT IMPLICATION. In [11] Lewis and Langford analyse, in addition to the ordinary material implication, which satisfies the theorems of two-valued logic, another kind of implication, which Lewis termed strict implication and which can be defined by means of the modal operator "it is possible that p". In that system "p strictly implies q" means the same as "it is not possible that both p and not-q" ([11], p. 124). If the symbol $\to$ is interpreted as the symbol of strict implication, then the implicational law of the overfilling $\mathrm{L}_2 1$ is not a theorem (cf. [11], p. 142). But the set of the theses which include strict implication only, and do not include material implication, is very limited, and Lewis and Langford often used both symbols of implication in one and the same theorem. For material implication the law $\mathrm{L}_2 1$ remains valid (cf. [11], p. 142). [[5]]

C. MANY-VALUED LOGICS. As far as those systems of the sentential calculus which can be defined by a many-valued finite matrix are concerned, no publications directly related to the problem in question are known to the present author, but prof. Łukasiewicz, in his personal communication to the present author in 1940 or so, stated that he knew an interpretation of implication and negation in three-valued logic such for which the law $\mathrm{L}_2 1$ does not hold. Reservation being necessary about the possible inexactitude of that reminiscence, it seems that the matrix involved was that given by prof. J. Słupecki ([17], p. 112) and symbolized L_3^2, the function there symbolized as R being interpreted as negation. This makes the system defined by the matrix:

(2)

$\to$	1	2	3	$\neg$
*1	1	2	3	2
*2	1	2	3	3
3	1	1	1	1

in which two values, 1 and 2, are designated, $L_2 1$ is not a thesis in L_3^2, but the thesis known to prof. Łukasiewicz [6]

$L_3^2 1$ $$p \to \big(\neg p \to (\neg\neg p \to q)\big)$$

holds and results in the overfilling of a system that includes the inconsistent triple of theses: $\mathcal{T}$, $\neg\mathcal{T}$, $\neg\neg\mathcal{T}$. All purely implicational theses of the two-valued calculus remain valid. The system includes certain theses which are not in L_2:

$L_3^2 2$ $$p \to \neg\neg\neg p\,,$$

$L_3^2 3$ $$\neg\neg\neg p \to p\,,$$

$L_3^2 4$ $$\neg p \to \neg(p \to p)\,.$$

4. The calculus of modal sentences: $\mathbf{M_2}$

Further analysis shall be concerned with a system including modal operators namely the system symbolized **S5** by Lewis and Langford ([11], Appendix II) and studied by Becker [1], Wajsberg [20] and Carnap [2]. It can be defined by an interpretation in Boolean algebra, due to Henle (cf. Lewis and Langford [11], p. 501). That system shall here be symbolized $\mathbf{M_2}$ and termed *two-valued* calculus of modal sentences, and that because of the definition given below, which uses exclusively concepts belonging to two-valued logic. The definition given in this paper is equivalent to Henle's [7], but the proof of that equivalence is omitted as irrelevant to further analysis.

Suppose that the truth of the sentence $\mathcal{P}$ depends on certain factors which cannot be determined strictly: for instance, a person is to toss a coin, and the sentence $\mathcal{P}$ means "during the game heads will turn up more times than tails will".

For a certain sequence of random events the sentence $\mathcal{P}$ will prove true, whereas for some other sequence it will prove false. Thus the sentence $\mathcal{P}$ may be assumed to be a function that takes on the values: truth and falsehood, according to the values of the variables that stand for random events. Since the functional relationship is not revealed by the notation, a sentence of this kind may be represented by the dependent sentential variable introduced by Heyting and discussed by the present author [8], in a way similar to that in which in mathematics the functions of the variable x are often represented by the letter y. The formula "it is necessary that p", symbolized by

$$\Box p\,,$$

will be supposed to mean the same as "p occurs for all the possible events". In order to obtain the logical laws that govern the operator $\Box$ interpreted in this way it suffices to formulate the foregoing explanations in a more precise manner. Let $\mathcal{Q}$ be any formula that includes the operators of the sentential calculus $\rightarrow$, $\vee$, $\wedge$, $\leftrightarrow$, $\neg$, and the symbol $\Box$, and also the sentential variables p, q, $\ldots$. Let those variables be replaced respectively by the predicates $p(x)$, $q(x)$, $\ldots$, and $\Box$ by the universal quantifiers "for every x". These replacements yield the formula $\mathcal{R}$; if the latter proves to be a thesis in the functional calculus, then $\mathcal{Q}$ shall be termed a thesis in the system $\mathbf{M_2}$. [8] The meaningfulness of the formulae of the functional calculus ought to be defined so that the equiform variables x might be bound by every quantifier.

Since the calculus of predicates of one argument is decidable by the Behmann method (cf. Hilbert and Ackermann [5], 1st ed., pp. 77–78), i.e., it can be decided about any meaningful expression in that calculus whether it is a theorem or not, it is accordingly possible to decide about every formula whether it is a thesis in the two-valued calculus of modal sentences $\mathbf{M_2}$ or not. Now that the theory of necessity is completed, the second modal formula

$$\Diamond p \quad - \quad \text{it is possible that } p,$$

can easily be introduced: $\Diamond p$ can be defined as "it is not necessary that not-p", in symbols:

$$\Diamond p := \neg \Box \neg p\,.$$

It would also not be difficult to define $\Diamond p$ by a method similar to that which was used for $\Box p$, namely by a comparison with the functional calculus.

The symbol $\Diamond$ then corresponds to the existential quantifier "for some x". The fact that variable of the only one form, namely x, is used does not reduce the general validity of the interpretation: should all the variables p, q, $\ldots$, be given more arguments than one x, y, $\ldots$ and should necessity be interpreted as "for all x, $\ldots$", and possibility as "for some x, y, $\ldots$", the result of the interpretation would be the same.

5. Definitions of discussive implication and discussive equivalence

As is known, even sets of those inscriptions which have no intuitive meaning at all can be turned into a formalized deductive system. In spite of this theoretical possibility, logical researches so far have been taking into consideration such deductive systems which are symbolic interpretations of consistent theories, so that theses in each such system are theorems in a theory formulated in a single symbolic language free from terms whose meanings are vague. But suppose that theses which do not satisfy those conditions are included into a deductive system. It suffices, for instance, to deduce consequences from several hypotheses that are inconsistent with one another in order to change the nature of the theses, which thus shall no longer reflect a uniform opinion. The same happens if the theses advanced by several participants in a discourse are combined into a single system, or if one person's opinions are so pooled into one system although that person is not sure whether the terms occurring in his various theses are not slightly differentiated in their meanings. Let such a system which cannot be said to include theses that express opinions in agreement with one another, be termed a *discussive system.* To bring out the nature of the theses of such a system it would be proper to precede each thesis by the reservation: "in accordance with the opinion of one of the participants in the discussion" or "for a certain admissible meaning of the terms used". Hence the joining of a thesis to a discussive system has a different intuitive meaning than has assertion in an ordinary system. *Discussive assertion* includes an implicit reservation of the kind specified above, which — out of the logical operators so far introduced in this paper — has its equivalent in possibility $\Diamond$. Accordingly, if a thesis $\mathcal{T}$ is recorded in a discussive system, its intuitive sense ought to be interpreted so as if it were preceded by the symbol $\Diamond$, that is, the sense: "it is possible that $\mathcal{T}$". This is how an impartial arbiter might understand the theses of the various participants in the discussion. [9]

Can a discussive system be based on ordinary two-valued logic? It can easily be seen that it is not so. Even such an elementary form of reasoning as the rule of *modus ponens* fails. If implication is interpreted so as it is done in two-valued logic, then out of the two theses one of which is

$$\mathcal{P} \to \mathcal{Q}\,,$$

and thus states: "it is possible that if $\mathcal{P}$, then $\mathcal{Q}$", and the other is

$$\mathcal{P}\,,$$

and thus states: "it is possible that $\mathcal{P}$", it does not follow that "it is possible that $\mathcal{Q}$", so that the thesis

$$\mathcal{Q}\,,$$

does not follow intuitively, as the rule of *modus ponens* requires.

The same can be proved in a strict form by demonstrating that the formula

$$\text{(non M}_2\text{) 1} \qquad \Diamond(p \to q) \to (\Diamond p \to \Diamond q)$$

is not a thesis in the system $\mathbf{M_2}$.

This is why in the search for a "logic of discussion" the prime task is to choose such a function which, when applied to discursive theses, would play the role analogous to that which in ordinary systems is played by implication. The problem, if formulated in this way, has a number of solutions, one of them being Lewis's strict implication, referred to above. Each solution would yield a different system of discussive logic. One such system is presented in this paper. It is chosen because of the variety of the theses that can be obtained in it, with a simultaneous rejection of the implicational law of overfilling and several of its special cases. The following definition is introduced into the system $\mathbf{M_2}$:

$$\text{M}_2\ \text{def. 1} \qquad p \to_{\mathrm{d}} q \ := \ \Diamond p \to q\,.$$

The formula $p \to_{\mathrm{d}} q$, as defined above, shall be termed discussive implication; it may be read: "if it is possible that p, then q", or, if applied of a discourse, "if anyone states that p, then q", or "if, for a certain admissible meaning of the terms, p, then q".

In every discussive system two theses, one of the form: $\mathcal{P} \to_{\mathrm{d}} \mathcal{Q}$, and the other of the form: $\mathcal{P}$, entail the thesis $\mathcal{Q}$, and that on the strength of the theorem

$$\text{M}_2 1 \qquad \Diamond(\Diamond p \to q) \to (\Diamond p \to \Diamond q)\,.$$

Thus the rule of *modus ponens* may be applied to discussive theses if discussive implication is used instead of ordinary implication. Discussive equivalence $\leftrightarrow_{\mathrm{d}}$ is defined in a similar way:

$$\text{M}_2\ \text{def. 2} \qquad p \leftrightarrow_{\mathrm{d}} q \ := \ (\Diamond p \to q) \wedge (\Diamond q \to \Diamond p),$$

i.e., "p is discussively equivalent to q" means the same as: "both: if it is possible that p, then q; and: if it is possible that q, then it is possible that p". The rule of *modus ponens* may be applied both ways to discussive equivalence defined in this manner. If $\mathcal{P} \leftrightarrow_{\mathrm{d}} \mathcal{Q}$ is a thesis in a discussive system and if either $\mathcal{P}$ or $\mathcal{Q}$ is a thesis, then the other side of that equivalence is a thesis, too. This follows from the theorems of the system $\mathbf{M_2}$, which, by making use of M_2 def. 2, may be given the abbreviated forms:

$\mathrm{M}_2 2$ $$\Diamond(p \leftrightarrow_{\mathrm{d}} q) \to (\Diamond p \to \Diamond q),$$

$\mathrm{M}_2 3$ $$\Diamond(p \leftrightarrow_{\mathrm{d}} q) \to (\Diamond q \to \Diamond p).$$

6. The two-valued discussive system of the sentential calculus: $\mathbf{D_2}$

By M_2 def. 1–2, the symbols $\to_{\mathrm{d}}$ and $\leftrightarrow_{\mathrm{d}}$ may be considered functors in the system $\mathbf{M_2}$. This fact is taken into account in defining the discussive system of the sentential calculus. The *system* $\mathbf{D_2}$ *of the two-valued discussive sentential calculus* is the set of formulæ $\mathcal{T}$, termed the theses of the system $\mathbf{D_2}$ and marked by the following properties:

1) $\mathcal{T}$ includes sentential variables and at the most the following functors: $\to_{\mathrm{d}}$, $\leftrightarrow_{\mathrm{d}}$, $\vee$, $\wedge$, $\neg$,
2) preceding $\mathcal{T}$ with the symbol $\Diamond$ yields a theorem in the two-valued sentential calculus of modal sentences M_2.

The system defined in this way is discussive, i.e., its theses are provided with discussive assertion which implicitly includes the functor $\Diamond$. This is an essential fact, since even such a simple law as $p \to p$, on the replacement of $\to$ by $\to_{\mathrm{d}}$, becomes

$\mathrm{D}_2 1$ $$p \to_{\mathrm{d}} p\,,$$

which is not a theorem in $\mathbf{M_2}$, and becomes such only when preceded by the symbol $\Diamond$:

$\mathrm{M}_2 4$ $$\Diamond(p \to_{\mathrm{d}} p).$$ [10]

Since the system $\mathbf{M_2}$ is decidable, the discussive sentential calculus $\mathbf{D_2}$, defined by an interpretation in $\mathbf{M_2}$, is decidable, too.

METHODOLOGICAL THEOREM 1. *Every thesis* $\mathcal{T}$ *in the two-valued sentential calculus* $\mathbf{L_2}$ *which does not include constant symbols other than* $\to$, $\leftrightarrow$, $\vee$, *becomes a thesis* $\mathcal{T}_{\mathrm{d}}$ *in the discursive sentential calculus* $\mathbf{D_2}$ *when in* $\mathcal{T}$ *the implication symbols* $\to$ *are replaced by* $\to_{\mathrm{d}}$, *and the equivalence symbols* $\leftrightarrow$ *are replaced by* $\leftrightarrow_{\mathrm{d}}$. [11]

PROOF. Consider a formula $\mathcal{T}_{\mathrm{d}}$ constructed so as the theorem to be proved describes. It is to be demonstrated that $\Diamond\mathcal{T}_{\mathrm{d}}$ is a thesis in $\mathbf{M_2}$. It is claimed that $\Diamond\mathcal{T}_{\mathrm{d}}$ is equivalent to some other formulae; the equivalences will be proved gradually. The following theorems will be referred to:

$M_2 5 \qquad \Diamond(p \to_{\mathrm{d}} q) \leftrightarrow (\Diamond p \to \Diamond q),$

$M_2 6 \qquad \Diamond(p \leftrightarrow_{\mathrm{d}} q) \leftrightarrow (\Diamond p \leftrightarrow \Diamond q),$

$M_2 7 \qquad \Diamond(p \vee q) \leftrightarrow (\Diamond p \vee \Diamond q).$

They may be described as the laws of distribution of the symbol $\Diamond$ with respect to implication, equivalence, and disjunction, with the replacement of $\to_{\mathrm{d}}$ and $\leftrightarrow_{\mathrm{d}}$ by $\to$ and $\leftrightarrow$, respectively. The replacement in $\Diamond\mathcal{T}_{\mathrm{d}}$ of the formulæ of the form $\Diamond(\mathcal{P} \to_{\mathrm{d}} \mathcal{Q})$ by the equivalent $\Diamond\mathcal{P} \to \Diamond\mathcal{Q}$, or of $\Diamond(\mathcal{P} \leftrightarrow_{\mathrm{d}} \mathcal{Q})$ by $\Diamond\mathcal{P} \leftrightarrow \Diamond\mathcal{Q}$, eliminates one of the symbols $\to_{\mathrm{d}}$, $\leftrightarrow_{\mathrm{d}}$, and at the same time the symbol $\Diamond$ is replaced by two such symbols placed to the right of the position of the original $\Diamond$. Iterated application of this procedure and the replacement of $\Diamond(\mathcal{P} \vee \mathcal{Q})$ by $\Diamond\mathcal{P} \vee \Diamond\mathcal{Q}$ yields the formula $\mathcal{W}$, which is equivalent to $\Diamond\mathcal{T}_{\mathrm{d}}$ and includes only the symbols $\to$, $\leftrightarrow$, $\vee$, variables, and the symbols $\Diamond$ in certain special positions, such that each variable is directly preceded by the symbol $\Diamond$, and each symbol $\Diamond$ directly precedes a variable. On considering the manner of forming $\mathcal{T}_{\mathrm{d}}$ from the thesis $\mathcal{T}$ belonging to $\mathbf{L_2}$ it can be seen that $\mathcal{W}$ can be obtained from $\mathcal{T}$ by preceding each variable by the symbol $\Diamond$, that is, by substituting $p/\Diamond p$, $q/\Diamond q$, ... This yields the following theorems in M_2:
a) $\mathcal{W}$ – as a result of the substitution in $\mathcal{T}$,
b) $\Diamond\mathcal{T}_{\mathrm{d}}$ – as equivalent to $\mathcal{W}$.
Hence $\mathcal{T}_{\mathrm{d}}$ is a thesis of $\mathbf{D_2}$. q.e.d.

The theorem proved above yields immediately that

$D_2 2 \qquad (p \leftrightarrow_{\mathrm{d}} q) \leftrightarrow_{\mathrm{d}} (q \leftrightarrow_{\mathrm{d}} p),$

$D_2 3 \qquad (p \to_{\mathrm{d}} q) \to_{\mathrm{d}} \big((q \to_{\mathrm{d}} p) \to_{\mathrm{d}} (p \leftrightarrow_{\mathrm{d}} q)\big),$

are theses in $\mathbf{D_2}$.

METHODOLOGICAL THEOREM 2. *If $\mathcal{T}$ is a thesis in the two-valued sentential calculus* L_2 *and includes variables and at the most the functors* $\vee$, $\wedge$, $\neg$, *then*

1) $\mathcal{T}$,
2) $\neg\mathcal{T} \to_{\mathrm{d}} q$,

are theses of $\mathbf{D_2}$.

PROOF. The proof is based on the fact that the symbols $\vee$, $\wedge$, $\neg$ retain their respective meanings in $\mathbf{M_2}$ and $\mathbf{D_2}$, and that

3) $\Box\mathcal{T}$

is a thesis in M_2. Hence (1) by

$\mathrm{M}_2 8$ $\Box p \to \Diamond p\,,$

and (2) by

$\mathrm{M}_2 9$ $\Box p \to \Diamond(\Diamond \neg p \to q).$ q.e.d.

The application of Methodological Theorem 2 to the thesis

$\mathrm{L}_2 3$ $$\neg(p \wedge \neg p),$$

which is termed the law of contradiction, yields — in view of the law of double negation — the following theorems of discussive logic:

$\mathrm{D}_2 4$ $\neg(p \wedge \neg p)$ (law of contradiction),

$\mathrm{D}_2 5$ $(p \wedge \neg p) \to q$ (conjunctional law of overfilling). [[12]]

In spite of its name which is adopted here $\mathrm{D}_2 4$ has no closer relation to the problem of the logic of contradictory systems. On the other hand, $\mathrm{D}_2 5$ results in the overfilling of every discussive system which includes at least one thesis of the type

$$\mathcal{P} \wedge \neg\mathcal{P}\,,$$

and which thus is internally inconsistent. By referring to the examples used so far it may be said that discussion becomes "overfilled" when one of the opinions held is contradictory with itself.

Computations show that the system $\mathbf{D_2}$ includes the following theses:

$\mathrm{D}_2 6$ $(p \wedge q) \to_{\mathrm{d}} p\,,$

$\mathrm{D}_2 7$ $p \to_{\mathrm{d}} (p \wedge p),$

$\mathrm{D}_2 8$ $(p \wedge q) \leftrightarrow_{\mathrm{d}} (q \wedge p),$

$\mathrm{D}_2 9$ $\big(p \wedge (q \wedge r)\big) \leftrightarrow_{\mathrm{d}} \big((p \wedge q) \wedge r\big),$

$\mathrm{D}_2 10$ $\big(p \to_{\mathrm{d}} (q \to_{\mathrm{d}} r)\big) \to_{\mathrm{d}} \big((p \wedge q) \to_{\mathrm{d}} r\big)$ (law of importation),

$\mathrm{D}_2 11$ $\big((p \to_{\mathrm{d}} q) \wedge (p \to_{\mathrm{d}} r)\big) \leftrightarrow_{\mathrm{d}} \big(p \to_{\mathrm{d}} (q \wedge r)\big),$

$\mathrm{D}_2 12$ $\big((p \to_{\mathrm{d}} r) \wedge (q \to_{\mathrm{d}} r)\big) \leftrightarrow_{\mathrm{d}} \big((p \vee q) \to_{\mathrm{d}} r\big),$

$D_2 13$ $\quad p \leftrightarrow_{\mathrm{d}} \neg\neg p,$

$D_2 14$ $\quad (\neg p \rightarrow_{\mathrm{d}} p) \rightarrow_{\mathrm{d}} p,$

$D_2 15$ $\quad (p \rightarrow_{\mathrm{d}} \neg p) \rightarrow_{\mathrm{d}} \neg p,$

$D_2 16$ $\quad (p \leftrightarrow_{\mathrm{d}} \neg p) \rightarrow_{\mathrm{d}} p\,,$

$D_2 17$ $\quad (p \leftrightarrow_{\mathrm{d}} \neg p) \rightarrow_{\mathrm{d}} \neg p\,,$

$D_2 18$ $\quad \big((p \rightarrow_{\mathrm{d}} q) \wedge \neg q\big) \rightarrow_{\mathrm{d}} \neg p\,.$

Certain laws of inference by reductio ad absurdum remain valid:

$D_2 19$ $\quad \big((p \rightarrow_{\mathrm{d}} q) \wedge (p \rightarrow_{\mathrm{d}} \neg q)\big) \rightarrow_{\mathrm{d}} \neg p\,,$

$D_2 20$ $\quad \big((\neg p \rightarrow_{\mathrm{d}} q) \wedge (\neg p \rightarrow_{\mathrm{d}} \neg q)\big) \rightarrow_{\mathrm{d}} p\,,$

$D_2 21$ $\quad \big(p \rightarrow_{\mathrm{d}} (q \wedge \neg q)\big) \rightarrow_{\mathrm{d}} \neg p\,,$

$D_2 22$ $\quad \big(\neg p \rightarrow_{\mathrm{d}} (q \wedge \neg q)\big) \rightarrow_{\mathrm{d}} p\,.$

Other theses include:

$D_2 23$ $\quad \neg(p \leftrightarrow_{\mathrm{d}} \neg p),$

$D_2 24$ $\quad \neg(p \rightarrow_{\mathrm{d}} q) \rightarrow_{\mathrm{d}} p\,,$

$D_2 25$ $\quad \neg(p \rightarrow_{\mathrm{d}} q) \rightarrow_{\mathrm{d}} \neg q\,,$

$D_2 26$ $\quad p \rightarrow_{\mathrm{d}} \big(\neg q \rightarrow_{\mathrm{d}} \neg(p \rightarrow_{\mathrm{d}} q)\big).$

The system of discussive logic could be completed by the introduction of the symbols $\rightarrow$ and $\leftrightarrow$ in addition to the symbols $\rightarrow_{\mathrm{d}}$ and $\leftrightarrow_{\mathrm{d}}$ by analogy to Lewis's system, in which both the symbols of strict implication and those of material implication are used. Material implication could be defined in the well-known way:

$$p \rightarrow q \;:=\; \neg p \vee q\,,$$

which would yield all those theses in which only the symbols of implication and negation occur, including the implicational law of overfilling $L_2 1$. This will not, however, result in the overfilling of every inconsistent system, because the system does not include the rule of *modus ponens* for ordinary (material) implication, as has been demonstrated in Section 5 above, where reference was made to the rejection in $\mathbf{M_2}$ of the formula (non M_2) 1. The formulation that the discussive sentential calculus is used in a system $\mathscr{S}$ means the application of the rule of *modus ponens* to discussive implication $\rightarrow_{\mathrm{d}}$ and to discussive equivalence $\leftrightarrow_{\mathrm{d}}$, but neither to material implication $\rightarrow$ nor to material equivalence $\leftrightarrow$.

7. Examples of formulæ that are not theses in $\mathbf{D_2}$

METHODOLOGICAL THEOREM 3. *If in a thesis that belongs to the discursive sentential calculus* $\mathbf{D_2}$ $\rightarrow_{\mathrm{d}}$ *is replaced by* $\rightarrow$, *and* $\leftrightarrow_{\mathrm{d}}$ *by* $\leftrightarrow$, *a thesis belonging to the sentential calculus* $\mathbf{L_2}$ *is obtained.*

The proof follows immediately, if it is noted that every theorem in $\mathbf{M_2}$ becomes a theorem in $\mathrm{L_2}$ as soon as all the symbols $\Diamond$ and $\Box$ are omitted. Methodological Theorem 3 shows that if $\rightarrow_{\mathrm{d}}$ is identified with $\rightarrow$, and $\leftrightarrow_{\mathrm{d}}$ with $\leftrightarrow$, then $\mathbf{D_2}$ becomes a subsystem of $\mathbf{L_2}$. Hereafter those meaningful formulae which are not theses in $\mathbf{D_2}$, that is, the formulae rejected in $\mathbf{D_2}$, shall be marked by the symbol (non $\mathrm{D_2}$), followed by the consecutive number. Several characteristic examples of such formulae are given below.

(non $\mathrm{D_2}$) 1 $$p \rightarrow_{\mathrm{d}} \big(q \rightarrow_{\mathrm{d}} (p \wedge q)\big).$$

The rejection of this formula can easily be justified on intuitive grounds: from the fact that a thesis $\mathcal{P}$ and a thesis $\mathcal{Q}$ have been advanced in a discourse it does not follow that the thesis $\mathcal{P} \wedge \mathcal{Q}$ has been advanced, because it may happen that $\mathcal{P}$ and $\mathcal{Q}$ have been advanced by different persons. And from the formal point of view, from the fact that p is possible and q is possible it does not follow that p and q are possible simultaneously. Thus the rejection in $\mathrm{M_2}$ of the formula

(non $\mathrm{M_2}$) 2 $$\Diamond p \rightarrow \big(\Diamond q \rightarrow \Diamond(p \wedge q)\big)$$

results in the rejection of (non $\mathrm{D_2}$) 1. In this connection

(non $\mathrm{D_2}$) 2 $$\big((p \wedge q) \rightarrow_{\mathrm{d}} r\big) \rightarrow_{\mathrm{d}} \big(p \rightarrow_{\mathrm{d}} (q \rightarrow_{\mathrm{d}} r)\big)$$ (law of exportation)

is rejected, too.

* * *

The rejection of the implicational law of overfilling

(non $\mathrm{D_2}$) 3 $$p \rightarrow_{\mathrm{d}} (\neg p \rightarrow_{\mathrm{d}} q)$$

is of essential importance. It is a consequence of the rejection in $\mathrm{M_2}$ of the formula

(non $\mathrm{M_2}$) 3 $$\Diamond\big(\Diamond p \rightarrow (\Diamond \neg p \rightarrow q)\big).$$

To prove the falsehood of (non M_2) 3 it suffices to take for p a sentence which is possible, but not true, and for q a sentence which is not possible. Then the antecedents $\Diamond p$ and $\Diamond \neg p$ are true, but the formula as a whole is false. The rejection of (non D_2) 3 makes the coexistence of inconsistent discussive theses without the overfilling of the discussive system in question possible. Moreover, it can be demonstrated that not only is the formula (non D_2) 3 rejected, but so are its various special cases, obtained by substitution.

(non D_2) 3a $$p \to_{\mathrm{d}} (\neg p \to_{\mathrm{d}} \neg q)$$

(analogon of K9 in Kolmogorov's system),

(non D_2) 3b $$(p \to_{\mathrm{d}} q) \to_{\mathrm{d}} \big(\neg(p \to_{\mathrm{d}} q) \to_{\mathrm{d}} r\big),$$

(non D_2) 3c $$(p \leftrightarrow_{\mathrm{d}} q) \to_{\mathrm{d}} \big(\neg(p \leftrightarrow_{\mathrm{d}} q) \to_{\mathrm{d}} r\big).$$

The definitions of $\to_{\mathrm{d}}$ and $\leftrightarrow_{\mathrm{d}}$ have been formulated with the intention that they enable the rejection of possibly many substitutions for (non D_2) 3. The formula (non D_2) 3b would be a thesis if instead of M_2 def. 1 another definition had been used, namely that which imposes itself in a natural manner and which defines $p \to_{\mathrm{d}} q$ as $\Diamond p \to \Diamond q$. Then the overfilling of the deductive system in question would be due to the coexistence of two theses one of which would be a discussive implication, and the other would be its negation. Likewise, should $p \leftrightarrow_{\mathrm{d}} q$ have been defined not in accordance with M_2 def. 2, but as $\Diamond p \leftrightarrow \Diamond q$, the formula (non D_2) 3c would be a thesis.

(non D_2) 3d $$p \to_{\mathrm{d}} \big(\neg p \to_{\mathrm{d}} (\neg\neg p \to_{\mathrm{d}} q)\big)$$

(counterpart of Theorem $\mathrm{L}_3^2 1$ in the system discussed in Section 3 §C above).

Further multiplication of antecedents including the variable p with the various numbers of negation symbols will not yield a thesis.

* * *

(non D_2) 4 $$(p \leftrightarrow_{\mathrm{d}} q) \to_{\mathrm{d}} \big((p \to_{\mathrm{d}} q) \wedge (q \to_{\mathrm{d}} p)\big).$$

The rejection of (non D_2) 4 becomes comprehensible when the definitions of the symbols $\to_{\mathrm{d}}$ and $\leftrightarrow_{\mathrm{d}}$ are compared; the rejection of that formula accounts for the fact that the discussive equivalence $p \leftrightarrow_{\mathrm{d}} q$ entails either of the implications $p \to_{\mathrm{d}} q$ and $q \to_{\mathrm{d}} p$, but does not entail their conjunction.

(non D_2) 5 $$(p \leftrightarrow_{\mathrm{d}} \neg p) \to_{\mathrm{d}} q,$$

(non D_2) 5a $$(p \leftrightarrow_{\mathrm{d}} \neg p) \to_{\mathrm{d}} (p \wedge \neg p),$$

(non D_2) 6 $$(p \to_{\mathrm{d}} \neg p) \to_{\mathrm{d}} \big((\neg p \to_{\mathrm{d}} p) \to_{\mathrm{d}} q\big),$$

(non D_2) 6a $$(p \to_{\mathrm{d}} \neg p) \to_{\mathrm{d}} \big((\neg p \to_{\mathrm{d}} p) \to_{\mathrm{d}} (p \wedge \neg p)\big).$$

Although $p \leftrightarrow_{\mathrm{d}} \neg p$ entails both p and $\neg p$ (theses D_2 16–17), yet a thesis in a discussive system $\mathscr{S}$ which is a discussive equivalence between two contradictory sentences, e.g., the thesis $\mathcal{P} \leftrightarrow_{\mathrm{d}} \neg \mathcal{P}$, does not necessarily entail the overfilling of the system $\mathscr{S}$. It suffices for $\mathcal{P}$ to be a possible, but not a necessary, sentence to yield in a discussive system the thesis

$$\mathcal{P} \leftrightarrow_{\mathrm{d}} \neg \mathcal{P}\,,$$

which by M_2 def. 2 is equivalent to the formula

$$\Diamond\big((\Diamond\mathcal{P} \to \neg \mathcal{P}) \wedge (\Diamond \neg \mathcal{P} \to \Diamond\mathcal{P})\big)$$

which follows from

$$\Diamond\mathcal{P} \wedge \Diamond \neg \mathcal{P}\,.$$

The rejection of the formulæ (non D_2) 5, 5a, 6, 6a can be useful in the study of antinomies. Antinomies result in the overfilling of a given system on the strength of the thesis

L_2 3 $$(p \leftrightarrow \neg p) \to q\,,$$

which is termed here the equivalential law of overfilling, or on the strength of the thesis

L_2 4 $$(p \to \neg p) \to \big((\neg p \to p) \to q\big).$$

Consider the antinomy of the liar, known already to Eubulides, which will here be formulated in a way which is also known, but different from the original wording. A person utters the sentence, which hereafter will briefly symbolized by $\mathcal{Z}$: "The sentence which I am uttering now is false." If it is assumed that the sentence $\mathcal{Z}$ is true, then in accordance with the classical definition of truth and falsehood it must be stated that $\mathcal{Z}$ is false. If, on the contrary, it is assumed that $\mathcal{Z}$ is false, then it must be concluded that it is true.

Thus two theses can be stated about the sentence $\mathcal{Z}$:

1) If $\mathcal{Z}$ is true, then $\mathcal{Z}$ is not true.

2) If $\mathcal{Z}$ is not true, then $\mathcal{Z}$ is true.

These two theses can be replaced by one:

3) $\mathcal{Z}$ is true if and only if $\mathcal{Z}$ is not true.

If the implications and the equivalence included in the theses 1), 2), 3) are interpreted as discussive, then by D_2 14–17 the following theses are obtained:

4) $\mathcal{Z}$ is true.

5) $\mathcal{Z}$ is not true.

But in view of the rejection of the formulae (non D_2) 3, 5, 5a, 6, 6a it is not evident that the theses 1)–5) should result in the overfilling of the system in question, and it can be stated with certainty that the ordinary procedure resulting in overfilling fails. These remarks do not prove that there exists a system which is not overfilled and such that the sentence $\mathcal{Z}$ can be formulated in it. If such a proof were to be made, such a formalized system would have to be defined, and that is a separate task. Similar issues can be raised with reference to other antinomies, e.g., that of Russell.

* * *

(non D_2) 7 $$\neg(p \rightarrow_{\mathrm{d}} p) \rightarrow_{\mathrm{d}} q\,.$$

This means that the negation of the law of identity $D_2$1 for a sentence $\mathcal{P}$ in a discussive system $\mathscr{S}$ does not necessarily result in the overfilling of $\mathscr{S}$. This fact seems to comply with the intuitions of the dialecticians who question the law of identity, though in a different form (cf. Chwistek [3, p. 28], Schaff [16, pp. 120–121], Łukasiewicz [12, pp. 43–49]). The rejection of the formula (non D_2) 7 results from the definition of the symbol $\rightarrow_{\mathrm{d}}$, as adopted here, since in $\mathbf{M_2}$ the formula

(non M_2) 4 $$\Diamond\big(\Diamond\neg(\Diamond p \rightarrow p) \rightarrow q\big)$$

is rejected. Indeed, the antecedent $\Diamond\neg(\Diamond p \rightarrow p)$ is equivalent to the formula $\Diamond p \wedge \Diamond\neg p$.

If a possible but not necessary sentence is substituted for p, and a not possible one is substituted for q, the incorrectness of the formula (non M_2) 4 is demonstrated. Moreover, if in a discourse the sentence $\mathcal{P}$ is meaningful and possible, but not necessary, so that inconsistent theses:

1) $\mathcal{P}$,

2) $\neg\mathcal{P}$,

are advanced, then by D_2 26 the thesis

3) $\neg(\mathcal{P} \rightarrow_{\mathrm{d}} \mathcal{P})$

is obtained in that discourse.

* * *

The law of transposition (also known as the law of contraposition) is rejected in all its forms, for instance:

(non D_2) 8 $\qquad (p \rightarrow_d q) \rightarrow_d (\neg q \rightarrow_d \neg p)$,

(non D_2) 9 $\qquad (\neg p \rightarrow_d \neg q) \rightarrow_d (q \rightarrow_d p)$.

Their rejection is not difficult to justify; e.g., when it comes to (non D_2) 8, its incorrectness is demonstrated by the example in which a necessarily true sentence is substituted for p, and a possible but not necessary one is substituted for q. Then the antecedents $p \rightarrow_d q$ and $\neg q$ are possible, and the consequent $\neg p$ is not possible. Also rejected are certain forms of inference by reductio ad absurdum:

(non D_2) 10 $\qquad (p \rightarrow_d q) \rightarrow_d \big((p \rightarrow_d \neg q) \rightarrow_d \neg p\big)$

(cf. Kolmogorov's axiom K5),

(non D_2) 11 $\qquad (\neg p \rightarrow_d q) \rightarrow_d \big((\neg p \rightarrow_d \neg q) \rightarrow_d p\big)$.

The rejection of (non D_2) 10 is justified by the substitution for p of a necessarily true sentence, and for q, of a sentence that is possible but not necessary.

References

[1] Becker, O., "Zur Logik der Modalitäter?", *Jahrbuch für Philosophie und Phänomenologische Forschung* 11 (1930), 497–548.

[2] Carnap, R., "Modalities and quantification". *Journal of Symbolic Logic* 11 (1946) 33–64.

[3] Chwistek, L., *Granice nauki. Zarys logiki i metetodologii nauk ścisłych*, Lwów-Warszawa.

[4] Hilbert, D., "Die logischen Grurtdlagen der Mathematik", *Mathematische Annalen* 88 (1922).

[5] Hilbert, D., and W. Ackermann, *Grundzüge der theoretischen Logik*, Berlin 1928, 2nd ed. Berlin 1938.

[6] Hilbert, D., and P. Bernays, *Grundlagen der Mathematik*. Bd. I Berlin 1934. Bd. II Berlin 1939.

[7] Jaśkowski, S., "Zagadnienia logiczne a matematyka", *Myśl Współczesna*, r. 1947, nr 7–8 (14–15), pp. 57–70.

[8] Jaśkowski, S., "Sur les variables propositionnelles dépendantes", *Studia Societatis Scientiarum Torunensis*, Secrio A, Vol. I, No 2 (1948), pp. 17–21.

[9] A. N. Kolmogorov, "O principie tertium non datur", *Matcmaticeskij Sbornik* 32 (1924–1925).

[10] Kotarbiński, T., *Elementy teorii poznania, logiki formalnej i metodologii nauk*, Lwów 1929.

[11] Lewis, C. I., C. H. Langford, *Symbolic Logic*, New York – London 1932.

[12] Łukasiewicz, J., *O zasadzie sprzeczności u Arystotelesa*, Kraków 1910.

[13] Łukasiewicz, J., *Elementy logiki matematycznej. Skrypt autoryzowany opracował M. Presburger*, Warszawa 1929.

[14] Ossowski, S., "Teoretyczne zadania marksizmu", *Myśl Współczesna*, r. 1948 nr 1 (20), 11–12 Logika.

[15] Russell, B., *The Principles of Mathematics*, Cambridge 1903, 2nd ed. London 1937 – New York 1938.

[16] Słupecki, J., "Dowód aksjomatyzowalności pełnych systemów wielowartościowych rachunku zdań", *Sprawozdania z pos. Tow. Nauk. Warsz.* 32 (1939), wydz. III.

[17] Schaff, A., *Wstęp do teorii marksizmu*, Warszawa 1948. [13]

[18] Tarski, A., *O logice matematycznej i metodzie dedukcyjnej*, Warszawa (1936).

[19] Vaihinger, H., *Philosophie des Als-Ob*, Berlin 1911.

[20] Wajsberg, M., "Ein erweiterter Klassenkalkül", *Monatsh. Math. Phys.* 40 (1933), 113—126.

[21] Whitehead, A. N., and B. Russell, *Principia Mathemtatica*, Vol. 1–3, 2nd ed. Cambridge 1925–1927.

[22] W. Wudel, "Powstanie logiki wyższej", *Myśl Współczesna*, r. 1947, nr 3 (10), 343–367.

(translated by Olgierd Wojtasiewicz
with corrections and notes by Jerzy Perzanowski)

Editorial Notes

0. The present translation is based on Olgierd Wojtasiewicz's one (cf. Editorial Note on p. 35). The chief difference is:

– the change of the notation from Polish one into more common, and

– the difference in the translation of few key terms: Polish 'sprzeczność' is translated as 'inconsistency', 'sprzeczny' as 'inconsistent', and Jaśkowski's original term 'przepełnienie' is translated *verbatim* as 'overfilling', not in a misleading way — as 'over-complete'. Also 'dyskusyjny' is translated as 'discussive', not 'discursive'.

Few words on Jaśkowski's names for calculi. The classical logic is named '$\mathbf{L_2}$' ('2' – for being two-valued, 'L' – for obvious reasons). Lewis' logic **S5** is named '$\mathbf{M_2}$' ("two-valued modal logic"). It is in fact equivalent to monadic part of the two-valued classical quantifier logic (cf. M. Wajsberg [20], R. Carnap [2]); whereas its discussive counterpart $\mathbf{D_2}$ is named in such a way for reasons obvious for everybody.

1. The conditions (1)–(3) from the last paragraph of Section 2 form well-known Jaśkowski's problem and criterion of paraconsistency.

2. Kolmogorov's calculus is the implicational-negation fragment of the calculus of J. Johanson.

3. In these remarks we find the written support for the well-known oral Polish tradition saying that:

(i) the interest in paraconsistency started in Poland with the famous book of J. Łukasiewicz *O zasadzie sprzeczności u Arystotelesa* [12], in particular in his well-known criticism of both the ontological and the logical law of non-contradiction. Cf. also examples like this emphasized by Jaśkowski in the comments concerning the matrix (1).

(ii) In textbook [13], concerning notes for Łukasiewicz's lectures in 1920-ties, we find in the implicit form the paraconsistent propositional logic defined by means of suitable matrix.

(iii) Last but not least, Jaśkowski himself, working under influence of Łukasiewicz [12] and [13], was trying in early 1940-ties to find an acceptable solution for problem of Section 2.

4. Łukasiewicz's notation for detachment-substitutional proofs has to be understood as follows:

$\mathrm{K3}[p/(q \to r), q/(p \to q), r/(p \to r)] = \mathrm{K4} \to \mathrm{K6}$ means that suitable substitution of K3 equals to the implication $\mathrm{K4} \to \mathrm{K6}$, which by detachment gives K6.

5. Observe that the present criticism in comparison with the previous one, is rather weak. Some calculi of the strict implication can thereby be treated as paraconsistent ones.

6. Cf. A. S. Karpenko "Jaśkowski's criterion and three-valued paraconsistent logics" in this volume, pp. 81–86.

7. Henle defined a family of subdivetly irreducible **S5**-algebras built up of an infinite sequences of two-values 0 and 1.

8. Cf. the previous remark in the note 0.

9. Quite basic assumption about the discussive meaning of possibility!

10. The formula D_21 occurs to be a theorem of $\mathbf{D_2}$ by the condition 2) of definition of $\mathbf{D_2}$.

11. Quite essential strengthening of the metatheorem 1 is given in the note following the paper which introduced discussive conjunction (cf. this volume, pp. 57–59).

12. It can also be eliminated in the strengthening of the system $\mathbf{D_2}$ mentioned above in note 11.

13. Reference to A. Schaff's work occurrent in the original Jaśkowski's paper, but not in its 1969 translation. Adam Schaff during the period 1946–1968 was the official leader of Polish Communist Party' philosophers. He lost his position in 1968. Thus the reader can see that Communist Censorship had influence even on logical journals. In the present translation the reference to Schaff's paper is back, like in the original paper.

J.P.