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# LOGIC MAY BE SIMPLE
# Logic, Congruence and Algebra

**Abstract.** This paper is an attempt to clear some philosophical questions about the nature of logic by setting up a mathematical framework. The notion of congruence in logic is defined. A logical structure in which there is no non-trivial congruence relation, like some paraconsistent logics, is called simple. The relations between simplicity, the replacement theorem and algebraization of logic are studied (including MacLane-Curry's theorem and a discussion about Curry's algebras). We also examine how these concepts are related to such notions as semantics, truth-functionality and bivalence. We argue that a logic, which is simple, can deserve the name logic and that the opposite view is connected with a reductionist perspective (reduction of logic to algebra).

Key words: algebraic logic, paraconsistent logic, universal logic.
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## 1. Introduction: is a good logic necessarily intricate?

One of the most famous and well-known paraconsistent logic, the logic C1 (see [da Costa 1963]) has been strongly criticized because in C1 the replacement theorem (or intersubstitutivity of equivalents) does not hold; the fact that moreover C1 admits no non-trivial congruence relation (see [Mortensen 1980]) was used to strengthen this negative appraisal.

As a typical example of these criticisms, we can quote the following remark by R. Sylvan: "the systems (C1 and some related logics) appear to lack natural and elegant algebraic and semantical formulations, largely because they fail to guarantee intersubstitutivity of equivalents" ([Sylvan 1990], p. 48).

This kind of discussion is related to a more general problem: to know what is a logic, or a "good" logic. Following a famous declaration of Quine (cf. *Philosophy of Logic*, Chapter 6), some people used to say that classical logic is the only logic, and that the other "logics" are in fact algebras, games, fictions or whatever. However nowadays people are more liberal and they are ready to consider that many "deviant" logics are ¡seal" logics, even if they still want to control the meaning of the word, as expressed by J.-Y. Girard:

> by the end of the century we are faced with an incredible number of logics — some of them only named "logic" by antiphrasis, some of them introduced on serious grounds. ([Girard 1993], p. 201–202)

One question is to know what kind of properties must have logic to be a "good" or "real" logic, for example, *pour revenir à nos moutons*, must a logic necessarily obeys the replacement theorem or admitting congruence relations?

R. Sylvan in [Sylvan 1990] seems to argue like this:

(S1) A logic in which the replacement theorem does not hold lacks natural and elegant algebraic and semantical formulations.

(S2) A logic which lacks natural and elegant algebraic and semantical formulations is not a good logic.

(S3) Therefore a logic in which the replacement theorem does not hold is not a good logic.

To analyze this appreciative argument objectively we must answer the following questions:

(Q1a) What is an algebraic formulation of logic?

(Q1s) What is a semantical formulation of logic?

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(Q2a) Is it necessary for logic that the replacement theorem holds to be algebraizable (i.e. to have an algebraic formulation)?

(Q2s) Is it necessary for logic that the replacement theorem holds to have a semantical formulation?

(Q3a) Must logic necessarily be algebraizable?

(Q3s) Must a logic necessarily have a semantics?

In this paper we will concentrate on the a-questions and deal with semantics only in connection with algebraization (section 7), the only s-question that will be directly examined will be (Q2s).

To deal with these questions we will first need to define what is a logical structure (Section 2), and then the notion of congruence in a logical structure (Section 3). A logic will be said simple if it has no non-trivial congruence relation. A logic will be said Fregean if the theorem of replacement holds. A logic which is Fregean is not simple, but a non-Fregean logic is not necessarily simple (case of the paraconsistent logic $C1^+$). Depending on how we answer (Q1a) we can get the two following answers to (Q2a):

(A2) A logic is algebraizable iff it is Fregean.

(A2*) A logic is algebraizable iff it is not simple.

In both cases a logic which is algebraizable is not simple. Thus to answer (Q3a) is to answer the question:

Must a logic necessarily be intricate (i.e. not simple)?

## 2. Logic as structure

One reason why many people think that a logic which is not algebraizable is not a good logic seems to be related to a misconception about algebra. The confusion can be decomposed in two parts:

(U1) All mathematical structures are algebraic structures.

(U2) A logic, which has no algebraic formulation, cannot be conceived as a mathematical structure.

The confusion (U1) can be explained by historical considerations and even justified. Bourbaki is known to have reconstructed and reorganized all the mathematics from the point of view of the notion of structure, and in particular all structures are constructed starting with three fundamental distinct mother structures (order, algebra, topology), which are mixed up to compound cross-structures, the canonical example of Bourbaki being the

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structure of the real numbers which is a mix of the three fundamental mother structures (see [Cartan 19431, Bourbaki 1950]).

However the "structuralist" approach of Bourbaki was inspired mainly by the development of abstract algebra (see [Dieudonné 1982], p. 619). It is worth noting that there was a time when "structure" was used to name what is now called "lattice" (see [Ore 1936, Glivenko 1938]). Even nowadays there is a strong tendency to consider universal algebra as a general theory of structures. Some people think that algebraic structures are more fundamental than the other ones (see [Papert 1967]) and that they are the prototypes of all structures.

But if we follow the initial idea of Birkhoff (see [Birkhoff 1946]) according to which an algebra is a "set with operations", it seems rather artificial to consider that all structures are algebras. Maybe, for example, a structure of order can be considered as an algebra (see [Cohn 1965]. p. 63), but at first it is not. In the best case, what can be said is that a structure of order "equivalent" to an algebra.

Even the people who do not accept explicitly (U1) have a strong tendency to accept (U2). We can also see here a historical origin: the first way to conceive logic as a mathematical structure was to conceive it as an algebra.

The whole confusion (U1) and (U2) can furthermore being explained by the fact that Boolean logic was fundamental to the development of universal algebra (see [Whitehead 1898, Birkhoff 1976, Birkhoff 1987]), that therefore Boolean logic is indirectly one of the origin of structuralism in mathematics, being one of the main origin of universal algebra which is the main origin of structuralism in mathematics.

There are many ways to consider logics as mathematical structures (see for example [Porte 1965]). Here we will consider the following structure which has emerged from the Polish school (see for example [Wójcicki 1988]):

A logic $\mathfrak{L}$ is a structure $\langle \mathscr{F}, \vdash \rangle$ where:

(i) $\mathscr{F}$ is an absolute free algebra, $\mathscr{F} = \langle \mathsf{F}, \tau \rangle$;

(ii) $\vdash$ is a relation on $\mathcal{P}(\mathsf{F}) \times \mathsf{F}$ obeying the three usual Tarskian axioms (reflexivity, monotonicity, transitivity).

Our presentation, and our conclusions can be adapted in the particular case where $\vdash$ is taken as a monadic predicate, i.e. when a logic is taken as a particular case of matrix, or in the case where $\vdash$ is taken as a relation on $\mathcal{P}(\mathsf{F}) \times \mathsf{F}$) like in [Scott 1974, Shoesmith/Smiley 1978, Zygmunt 1984].

Following Bourbaki we can see that this structure is not an algebraic structure and that it is not a fundamental structure. It is a cross-structure

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between an algebra and a structure of type $\langle \mathsf{L}, \vdash \rangle$ where $\vdash$ is a relation on $\mathcal{P}(\mathsf{L}) \times \mathsf{L}$.

In another paper (see [Béziau 1995]) we have argued that logic should be consider as a fundamental structure of this last type obeying no axiom. However to simplify the discussion here we will call a logical structure a structure of type $\langle \mathscr{F}, \vdash \rangle$ (what we have called "Polish logic" in [Béziau 1995]) and we will study the relation between this kind of logical structures and algebra.

For many people:

(U3) to consider logic as a structure like the one presented above is already to consider logic as an algebra.

The confusion (U3) is due not only to the confusion (U1) but also to the fact that this structure involves an algebraic structure. Let us insist first that if we consider for example the classical prepositional logic as a structure $\mathfrak{K} = \langle \mathscr{F}, \vdash \rangle$ we are still far from the so-called Lindenbaum-Tarski algebra. The idea of Lindenbaum to consider the set of formulas $\mathscr{F}$ (usually called the language) as an algebra does not yield directly to the Lindenbaum-Tarski algebra which we get by factoring the structure (see Sections 3 and 4 below and [Surma 1982]).

Another reason of the confusion (U3) is due to the fact that this approach is different from the usual linguistic one that does not mention explicitly the structuralist background and where the set of formulas is presented as a language involving strings of symbols. But the gap between the two approaches is not so big. In fact the concept of absolute free algebra is the structure closely corresponding to the notion of language, often called in fact "word algebra", as it is explain by D. Barnes and J. Mack: "to say precisely what is meant by a formal expression [...] is tantamount to constructing the free algebra" ([Barnes/Mack 1975], p. 4).

The aim of this paper is not to argue directly in favor of the structuralist approach in logic. but to emphasize that the structuralist approach does not necessarily imply the reduction of logic to algebra; this is not immediately clear for the language oriented logician who puts in the same bag structure and algebra.

## 3. Congruence in logic

We will not recall here the notion of replacement of an occurrence of the formula $B$ by the formula $A$ in the formula $C$, noted $C(A/B)$ (see for example [Kleene 1952]). This notation is ambiguous because it doesn't say which

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is the occurrence of $B$ which is replaced. But for the present purpose this ambiguity is not fundamental.

**Definition of Congruence.** For any $A, B \in \mathsf{F}$:

$$A \cong B \overset{\text{df}}{\Longleftrightarrow} \bigwedge_{\Gamma \subseteq \mathsf{F}} (\Gamma \vdash C \Leftrightarrow \Gamma \vdash C(A/B)),$$

A binary relation $\cong$ over the set of formulas is called a congruence relation if it obeys the above condition. The notation is ambiguous because only one occurrence of $B$ is replaced, in a formula of the set $\Gamma$ or in $C$.

This definition is conform to the general definition of congruence given by Bourbaki (cf. [Bourbaki 19681).

A congruence is a kind of identity, the "real" identity, i.e. the diagonal relation, being the finest one, also called the *trivial* congruence.

**Definition of Logical Equivalence.** For any $A, B \in \mathsf{F}$:

$$A \dashv\vdash B \overset{\text{df}}{\Longleftrightarrow} A \vdash B \text{ and } B \vdash A.$$

It is easy to see that, due to the Tarskian axioms, logical equivalence is an equivalence. The question to know whether it is a congruence will be touch upon in the next section.

**Proposition.** *If two formulas are congruent they are it logically equivalent.*

The following definition and theorem permit to give a simpler view on congruence.

**Definition of Quasi-Congruence.** For any $A, B \in \mathsf{F}$:

$$A \simeq B \overset{\text{df}}{\Longleftrightarrow} \bigwedge_{C \in \mathsf{F}} C \dashv\vdash C(A/B).$$

**Theorem.** *A relation is a congruence iff it is a quasi-congruence.*

Now we can give the central definition adapting the existing terminology (see for example [Cohn 1965], p. 61 – "simple algebra"; [Wójcicki 1988], p. 198 – "simple matrix").

**Definition of Simplicity.** A logic is *simple* iff all congruencies are trivial.

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We thus give the following versions of Mortensen's theorem and its proof (compare with [Mortensen 1980] and Lewin/Mikenberg/Schwarze 1991]):

**Theorem.** (Mortensen) *The persistent logic* C1 *is simple.*

Proof. Suppose $A \cong B$, then $p$ being an atomic formula we have: $\neg(p \wedge A) \dashv\vdash \neg(p \wedge B)$, but it is easy to show by semantics means (semantics of valuation; see [da Costa/Alves 1977]) or proof-theoretical methods (sequent calculus; see [Béziau 1993]) that $\neg(p \wedge A) \dashv\vdash \neg(p \wedge B)$ iff $A = B$ (in fact to show that $\neg(p \wedge A) \dashv\vdash \neg(p \wedge B)$ iff $A = B$ or $\neg(p \wedge A) \dashv\vdash \neg(p \wedge B)$ iff $A = B$. □

## 4. Algebraization of logic

Roughly speaking a logical structure is *algebraizable* if there is a correspondence with this structure and an algebraic structure; to be a little more precise: if it can be "transformed" or "represented" somehow by an algebraic structure.

There are in fact two points here:
- the nature of this correspondence,
- conditions for a logic to have an algebraic correlate.

First we shall examine the canonical example, the case of classical prepositional logic. Following the Polish framework we shall consider here classical logic as a structure $\mathfrak{K} = \langle \mathscr{F}, \vdash \rangle$.

The structure is factored by the relation of logical equivalence. This is possible because this relation is a congruence. Now in which sense the factor structure $\mathfrak{K}/_{\dashv\vdash} = \langle \mathscr{F}/_{\dashv\vdash}, \vdash /_{\dashv\vdash} \rangle$ is a Boolean algebra? (*nb* the original construction by Tarski was not presented like this; see [Tarski 1935]).

If we are aware of the confusion (U1) according to which any structure is called an algebra, we must be careful with the notion of "Boolean algebra". It will be better in fact to speak about "Boolean structure". A Boolean structure can be considered from various point of view, two of the most famous being the following ones:
- an idempotent ring,
- a distributive complemented lattice.

The notion of idempotent ring gives a purely algebraic formulation of a Boolean structure: this structure is conceived as a set with operations.

The fact that a Boolean structure can be formulated as an idempotent ring is not necessarily obvious, depending from which conception of Boolean structure we are starting with. Historically this observation. due to M. Stone. was painstaking (see [Johnstone 1982] p. xv, and [MacLane 1981] p. 8–9).

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We must thus distinguish the fact that a structure
– is an algebra,
– can be conceived as an algebra.

A distributive complemented lattice is strictly speaking a cross-structure involving an algebraic structure and a structure of order, but it can be conceived as a pure algebraic structure, in particular because the relation of order is definable in terms of functions; however there are some order-algebraic structures which are not reducible to algebras.

It is clear that the factor structure $\mathfrak{K}/_{\mathsf{H}}$ is not an algebra, but we can ask the following questions
– can it be conceived as an algebra?
– can it be conceived as a Boolean structure?

According to what has been said before, it is sufficient to answer positively the second question to have also a positive answer to the first question.

By small transformations, it is very easy to see that $\mathfrak{K}/_{\mathsf{H}}$ can be conceived as a distributive complemented lattice. To do so we just have to observe that in this structure the relation $\vdash/_{\mathsf{H}}$ can be transformed into a binary relation. this is possible because CLASSICAL logic is finite, i.e., $\Gamma \vdash B$ iff there exists a finite subset $\Gamma_0$ of $\Gamma$ such that $\Gamma_0 \vdash B$, and has a conjunction, so that $\Gamma_0 \vdash B$ is expressible as $A_1 \wedge \ldots \wedge A_n \vdash B$, where $\Gamma_0 = \{A_1, \ldots, A_n\}$.

The difficult point is to know in which finite-conjunctive logic:
– the relation $\vdash$ can be interpreted as a relation of order.
– the relation of logical equivalence can be interpreted as a identity.

In general a finite-conjunctive logic appears as a kind of order-algebraic structure, even if we don't quotient it, but it is not so clear in which sense this kind of structure is really an order-algebraic structure in the case we CANNOT quotient it (this point will be discussed in the next section); in the other case we will get an order-algebraic structure, which is not necessarily reducible to a purely algebraic structure, but the fact that the factor relation of logical equivalence is a congruence and can be treated as identity plays in favor of saying that even in this case we are algebraizing. In some cases the resulting algebra keeps all the information about the structure, in other cases it does not catch all the aspects of the order-algebraic structure.

These important remarks being made, we can now go further on with our process of systematization.

**Definition of Fregean Logic.** A logic is *Fregean* iff the relation of logical equivalence is a congruence.

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About the use of the name Frege in this context see [Suszko 1975, Suszko 1977, Font/Jansana 1993]. Wójcicki uses the expression "self-extensional" (cf. [Wójcicki 1988]), which is a good terminology if we want to emphasize the semantical aspect of the problem (see Section 7 below).

**Definition of Replacement Law.** If $A \dashv\vdash B$ then $C(A/B) \dashv\vdash C$.

Due to the fact that quasi-congruences are congruences, logics in which the law of replacement is valid (i.e. the replacement theorem holds) are exactly Fregean logics, because the replacement law says that the relation of logical equivalence is a quasi-congruence.

Using this fact and with some obvious adaptations, we have the following version of the MacLane-Curry's general formulation of replacement theorem:

**Theorem.** (MacLane-Curry) *If a Logic is monotonic then it is Fregean.*

For details, see [MacLane 19341, Curry 1952a, Curry 1952b]. Here the notion of "monotony" has nothing to do with the use of the word in the context of non-monotonic logics.

If we define algebraizable logics as Fregean logics, thus the above theorem gives a sufficient condition to algebraize logics.

It is interesting to compare this result with the more recent work of [Blok/Pigozzi 1989]. It seems that in fact J. W. Blok and D. Pigozzi don't know the work of Curry, for example in [Blok/Pigozzi 1991] they say that "the specific phrase algebraic Logic originated with Halmos in his 1956 expository paper *The basic concepts of algebraic logic*" (p. 365), however the book of Curry *Leçons de logique algébrique* was published in 1952 and there is no doubt that the expression "logique algébrique" is the French equivalent of "algebraic logic".

Now the question arises whether a logic which is not simple and not Fregean (admits congruencies other than logical equivalence) can also said to be algebraizable. An example of this kind of logic is the logic $C1^+$ an extension of the paraconsistent logic C1 proposed by the Author (see [Béziau 1990, da Costa/Béziau/Bueno 1995]).

In this case the relation of logical equivalence is not a congruence, thus we are back to the more general problem to know if in this case we can really speak of an order-algebraic structure, problem we will treat in the next section.

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### 5. Are Curry's algebras algebras?

N. C. A. da Costa has introduced the notion of Curry's algebras in [da Costa 1966a], following a suggestion of M. Guillaume (cf. [da Costa 1966b], p. 432). These algebras, are supposed to be the algebraic counterparts of his paraconsistent logics, in particular there is a C1-Curry-algebra corresponding to the logic C1. As we can imagine, due to the fact that C1 is simple and not Fregean (the fact that the replacement theorem does not hold in C1 was noticed in [da Costa/Guillaume 1965]), this notion will not be usual.

A C1-Curry-algebra is a structure $\mathfrak{C}\mathbf{1} = \langle \mathscr{A}, \angle \rangle$ where $\mathscr{A}$ is an algebra $\langle \mathsf{A}, \tau \rangle$ of the same type $\tau$ as $\mathscr{F}$ and $\angle$ a binary relation on $\mathsf{A}$.

What is the basic difference between $\mathrm{C1} = \langle \mathscr{F}, \Vdash \rangle$, where $\Vdash$ is the binary reduction of $\vdash$ (using the fact that C1 is a finite-conjunctive logic), and the C1-Curry-algebra $\mathfrak{C}\mathbf{1} = \langle \mathscr{A}, \angle \rangle$?

In fact the only difference is that the underlying structures are not the same, in the case of the logic C1, $\mathscr{F}$ is an absolute free algebra and in case of the Curry-algebra $\mathfrak{C}\mathbf{1}$, $\mathscr{A}$ is not an absolute free algebra. However there is the following correspondence between C1 and $\mathfrak{C}\mathbf{1}$:

$$A_1 \wedge \ldots \wedge A_n \Vdash B \text{ is a valid schema in C1 iff} \\ \bigwedge_{a_1,\ldots,a_n,b\in\mathsf{A}} a_1 \wedge \ldots \wedge a_n \angle b \text{ in } \mathfrak{C}\mathbf{1}\,.$$

(In one case $\wedge$ is the conjunction of the absolute free algebra $\mathscr{F}$, in the other case it is the corresponding operator in the algebra $\mathscr{A}$).

In fact we have this kind of correspondence between any finite-conjunctive logic $\langle \mathscr{F}, \vdash \rangle$ and the related order-algebraic structure $\langle \mathscr{A}, \angle \rangle$ which is the same structure as $\langle \mathscr{F}, \Vdash \rangle$ except that the domain $\mathscr{A}$ is not supposed to be an absolute free algebra, but is any algebra. By this consideration it is possible in fact to transform any order-algebraic structure into a logic. In this context it appears that the difficulty is not to "logicize" algebra but to algebraize logic.

In case of the Curry-algebra $\mathfrak{C}\mathbf{1} = \langle \mathscr{A}, \angle \rangle$, the relation $\stackrel{\perp}{=}$, defined by $a \stackrel{\perp}{=} b$ iff $a \angle b$ and $b \angle a$, is not a congruence relation, in particular we have $\bigwedge_{ab}\ a \wedge b \stackrel{\perp}{=} b \wedge a$, a but not $\bigwedge_{ab}\ \neg(a \wedge b) \stackrel{\perp}{=} \neg(b \wedge a)$.

However da Costa used originally the symbol $\leq$ for denoting the "order" relation $\angle$. But in general in a order-algebraic structure $\mathfrak{S} = \langle \mathscr{A}, \leq \rangle$ the relation $\doteq$, defined by $a \doteq b$ iff $a \leq b$ and $b \leq a$, is a congruence relation even if it is not the identity (see [Glivenko 1938], p. 50–51 and [Curry 1952], p. 40–41).

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Due to the fact that in a C1-Curry-algebra $\doteq$ is not a congruence, it cannot be interpreted as an identity even though the structure is not simple, i.e. if it is possible to define another congruence (case of a C1$^+$-Curry-algebra), thus a C1-Curry-algebra, or a C1$^+$-Curry-algebra, is in fact a order-algebraic structure where the notion of order is strongly irreducible. It seems therefore not appropriate to call such kind of structures, simply "algebras", and even if we classify them as order-algebraic structures, we must keep in mind that they differ strongly from other order-algebraic structures in the sense that the equivalence induced by the relation of order is not a congruence.

It will be good then to distinguish this kind of structures and to call them for example "Curry-order-algebraic-structure". Among these structures, we can distinguish the case where the underlying algebra is an absolute free algebra. Now if we consider a structure of this last kind in which the central relation is not a binary order relation, but a relation of the type $\vdash$, there is no good reason to still used the expression order-algebraic, thus it will be better to call it a "Curry-logical-algebraic-structure", or simply a "Curry-logical-structure", or a "Curry-logic". But "Curry-logics" are exactly what we have called non-Fregean-logics.

Our conclusion thus is that Curry-algebras are not algebras but nevertheless they are mathematical structures that arise in the structuralist approach of logic. In fact it seems that da Costa reaches recently a similar conclusion, using the expression "Curry systems" to speak about any kind of structures useful for the mathematization of logic ([Barros/da Costa/Abe 1995], p. 3).

## 6. Is logic reducible to algebra?

It appears clearly that logics are structures that are not in general reducible to algebras.

There i s no doubt that classical prepositional logic (presented in a Frege-Hilbert fashion) is reducible to the notion of Boolean algebra, as Tarski was the first to show (cf. [Tarski 19353]) and contrary to the opinion of the people who thought that the Fregean approach was entirely different and irreducible to the Boolean approach.

Intuitionistic prepositional logic, a modal prepositional logic such as S5, first order classical logic and many others are equivalent to some algebraic structures.

But they are logics that are not reducible to algebras, like simple logics, or more generally non-Fregean logics, not to speak about logics which are not finite-conjunctive.

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Funny enough we can say, to cope with Quinians and other classicists that in fact classical logic is rather an algebra than a logic, but that a paraconsistent logic like C1 is really a logic, a logic which cannot be reduced to an algebra.

However even if a structure can be conceived from a double viewpoint, as an algebraic structure and as a logical structure, it is worth studying it logically, i.e. taken as, a logical structure.

In [Béziau 1995] we have called "Universal Logic", the general study of logical structures, insisting on the fact that "Universal Logic" is not reducible to "Universal Algebra" as many people seem to think. For example S. L. Bloom says that "Roman [Suszko] taught us the Polish view of logic — as a branch of universal algebra [a novel outlook for us]" ([Bloom 1984], p. 313); see also [Andréka/Gergely/Németi 1977]).

This belief seems based on the fact that, as we have already pointed out, usually logic is not presented from the structuralist viewpoint and that the language-oriented logician has a tendency to consider any structuralist approach as an algebraic approach, thinking that any structure is an algebra or that universal algebra is a general theory of structures.

Against this reductionist point of view in mathematics we can recall again the Bourbaki illuminating example: the structure of the real is not reducible to an algebra, the notion of field does not catch all the properties of the structure of the reals. This kind of consideration permits to understand the following paradox:

**Gödel-Tarski Paradox.** The theory of real closed fields is a complete theory (Tarski) but Peano arithmetic is not (Gödel), at the same time the structure of real numbers is a more complex structure than the structure of natural numbers.

Similarly logical structures are more complex structures, the algebraic aspects of a logical structure catch usually only one aspect of the structure, thus we can conclude in agreement with M. Eytan: "L'utilisation de techniques algébriques en logique donnent l'impression d'appauvrir en quelque sorte la structure étudiée et d'être insuffisantes pour tout dire." ([Eytan 1974], p. 211)

## 7. Do simple logics have semantics?

Now we will say a little word on semantics, because there is a connection between algebraization of logic and semantical formulation of logic.

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In the case of a logic in which the replacement theorem holds (Fregean logic), the quotient structure is a characterizing matrix for this logic, a matrix in fact where most of the time the set of designated values is a singleton. This kind of matrix is called an "algebraic matrix", however it is good to emphasize that a matrix, even an algebraic one is not an algebra.

Does this mean that Fregean logics are truth-functional?

It depends what is the use of the expression "truth-functional". But if this expression means that there is a finite characterizing matrix (this is the natural meaning of the expression, see [da Costa/Béziau/Bueno 1996] for a more detail account on the subject), the answer is "no".

In the case of classical prepositional logic (with a infinite set of atomic formulas), the corresponding characteristic matrix is not finite. Of course there is also a finite characteristic matrix (the Boolean algebra on $\{0, 1\}$ with $\{1\}$ as the set of designated values) but it is not necessarily the case as the example of intuitionistic logic shows very well.

In fact intuitionistic logic is e extensional logic (or self-extensional, using Wójcicki's terminology) which is not truth-functional. This is an important point because generally people tend to confuse truth-functionality and extensionality (see Béziau 1994]).

We can say that a Fregean logic has an "algebraic semantics", because it can be characterized by a matrix, using the expression of Suszko (see [Suszko 1977]), even if it has no truth-functional semantics. But in fact the extended version of Lindenbaum's theorem due to Wójcicki shows that any structural logic has such kind of algebraic semantics. in the sense that it can be characterized by a bundle of matrices (a logic is structural if the theorem of substitution holds, see [Łoś/Suszko 1958]).

Thus from the point of view of algebraic semantics, i.e. matrix semantics, the important point is not the replacement theorem but the substitution theorem. Even a simple logic, if it is structural, has an algebraic semantics in the sense of Suszko.

A truth-functional logic is not necessarily Fregean, but we have the following result:

**Theorem.** *All bivalent truth-functional logics are Fregean.*

Now let us turn to a paradox:

**Da Costa's Paradox.** C1 is a simple logic which has a bivalent semantics.

The solution of the paradox lies in the distinction between truth-functional bivalent semantics and non-truth-functional bivalent semantics. The

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bivalent semantics of C1 is non-truth-functional, thus there is no contradiction between the fact that C1 is simple therefore neither algebraizable, nor Fregean, and that it has a bivalent semantics.

The elaboration of the semantics of C1 was in fact the origin of the theory of valuation, developed by da Costa (see [da Costa,/Béziau 1994]) which is a method that generalized matrix theory, taking in consideration semantics not necessarily truth-functional.

This kind of semantics was also known by Suszko who presented a bivalent non-truth-functional semantics for the three-valued logic of Łukasiewicz (cf. [Suszko 1975]), and in [Suszko 1977]. he calls bivalent non-truth-functional semantics, "logical semantics", in order to distinguish them from "algebraic semantics".

It seems however that the fact that the Polish school has mainly developed algebraic or matrix semantics is due to its tendency to reduce logic to algebra (because a matrix even if it is not exactly an algebra is a structure close to an algebra).

The theory of valuation presents an alternative, showing that interested non-matricial semantics can be provided. like the semantics for C1.

## 8. Conclusion: simplicity is not a defect

As we have seen, C1 is a simple logic. It is in no acceptable way possible to say that it is reducible to an algebra, without distorting considerably the word "algebra".

Nevertheless it can be treated in a properly mathematical way, as a mathematical structure, more specifically as a logical structure.

We can take advantage of its specificity of logical structure, for example using the fact that the underlying structure is an absolute free algebra, this permits to work some inductive proofs on the complexity of the formulas, like the Cut-Elimination theorem.

We can also notice that the sequent-treatment of C1 does not suffer of its simplicity. The connection between the semantics of valuation of C1 and its sequent formulation is establish without difficulty. In fact it is possible to establish a general ABSTRACT (abstract in particular in the sense in which we are making abstraction of the structure of the set of formulas) version of the completeness theorem connecting sets of bivaluations and rules of sequent calculus which does not depend on Fregeanity and on any algebraic features (see [Béziau 1995b]).

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From this point of view we can give a proof of the completeness theorem for the classical prepositional logic which does not depend on any algebraic aspects of this logic, on the fact for example that the set of formulas is an absolute free algebra (Gödel in [Gödel 1932] already saw that this feature was not necessary) or that the characterizing matrix for classical logic is the Boolean algebra on $\{0, 1\}$. This is a symmetrical alternative to the algebraic proof of J. Łoś [Łoś 1951]. But if, in the case of classical logic, we can have a double perspective, we must keep in mind that this is not generally the case.

Many people think that simplicity is a defect because a logic that is simple is non-Fregean, or non-extensional, and this last feature is seen as an insidious disease. This is the idea of R. Sylvan in [Sylvan 1990] and many other people including for example R. Wójcicki who wrote: “Although there is no generally accepted definition of a ‘good’ or ‘interesting’ logic, we incline to consider certain properties of logical calculi as desirable whereas some others are not”, selfextensionality is a property that “an adequately defined logic might be expected to have” ([Wójcicki 1988], p. 200).

The reason why non-Fregeanity is seen as a defect is that non-Fregean logics are in no acceptable way algebraizable. But we must insist again, to conclude, that the non-algebraizability of a logic is not a defect.

It is a defect only from a reductionist perspective according to which logic must be reducible to algebra. This reductionist perspective is just an aspect of a well-known psychological method, the tendency of reducing the unknown to the already known in order to understand it. But it is easy to see that this method is not necessarily a good method.

In the case of paraconsistent logic for example some people prefer to deal with a more “normal” logic, “normal” from the point of view of the algebraic reductionism, like dual-intuitionistic logic (about this logic see [Urbas 1996]). But even if this logic is algebraizable, it has also some “abnormalities”, which are even worse. from the point of view of paraconsistency at least, the fact for example that $\neg(A \wedge \neg A)$ is a theorem.

If we think that Logic is a subject different, from Algebra, it is clear that to understand logic through algebra will give only a partial comprehension of logic.

Thus it is of very high interest to develop a “Universal Logic”, a general theory of logical structures, if we want to understand the very nature of logic, and according to which logic may be simple.

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