Logic and **L**ogical **P**hilosophy
Volume 5 (1997), 93–120

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CHARTING THE LABYRINTH OF BELL-TYPE THEOREMS

Abstract. The objective of the paper is to present a comprehensive picture of Bell-type theorems, by giving both the theorems and the proofs of them. Special care is given to specifying the assumptions of the arguments and their physical or metaphysical significance. Taking the EPR argument as a point of departure, the paper discusses four probabilitic Bell-type theorems, which are then followed by two versions on non-probailitic (GHZ) arguments. The final section provides the reader with a classification of the assumptions, which specifies which assumption is used in which proof.

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Received March 3, 1998

1. Introduction

Since the inception of Bell theorem in 1964[1], which proves that no deterministic local model can reproduce (theoretical) predictions of quantum mechanics (QM), there has been a profusion of related proofs, all of them attempting to put a limitation on the possibility of reconstructing quantum mechanical predictions along classical lines. A typical proof of this kind derives a testable proposition from some or other classical assumptions (possibly taken together with a fact about QM systems), and shows that the proposition is violated by a *theoretical prediction* of QM. Such a proposition can have the form of either a statistical inequality or a testable equation. Now, a significant fact is that most of these propositions have been tested, the wide consensus being that the they are also *empirically violated*. This empirical violation is readily interpreted as the indication that quantum mechanics is correct and classical completions of it leads to a clash with experimental data. The consensus mentioned above lives happily with a minority trend that sees a loophole in the tests, and suggests how to improve the experiments.[2] The minority rights already asserted, we will only scarcely draw on the issue of the decisiveness of the tests, our main goal here being the charting of a map of the Bell-type arguments. It is perhaps worth explaining why we think this charting is needed. The answer points to a logical feature common to Bell-type arguments, namely the structure of an indirect proof. A proposition, shown successively violated, is deduced from assumptions that have classical underpinnings, the violation clearly indicating that at least one of these assumptions must go. To find out which one, one may set out the task of deriving the proposition from some other set of assumptions, hoping that by repeating this procedure a single culprit responsible for the violation will be found. This vision is surely too optimistic

[1] Bell J., "On the Einstein-Podolski-Rosen paradox", *Physics* 1 (1964), 195–200.

[2] To get acquainted with the arguments of those skeptical about the actual outcome of the tests, the reader is advised to consult Brody T.: *The Philosophy Behind Physics*, Springer Verlag, Berlin 1993, chs. 15–18 and Huelga S., Ferrero M., Santos E.: "Loophole-free test of the Bell inequality", *Phys Rev A* 51(6), 1995, p. 5008. The tests are described in Aspect A., Grangier P., Roger G., "Experimental tests of realistic local theories via Bell's Theorem", *Phys. Rev. Lett.* 47, 460–467 and Aspect A., Grangier P., Roger G., "Experimental realization of Einstein-Podolsky-Rosen-Bohm Gedankenexperiment", *Phys. Rev. Lett.* 48, 91–94. A more philosophically oriented survey of the tests is in Readhead M., *Incompleteness Nonlocality and Realism*, Clarendon Press, Oxford 1987.

since what all the present proofs yield, is rather distinct sets of assumptions that result in the troublesome proposition, than a single assumption. Nevertheless, the fact that we have various sets of the premises sheds light on *why* the propositions are violated and, consequently, what the world is like. For this reason, investigations of this sort are sometimes believed to belong to *empirical metaphysics*. The research is also of utmost importance for the minority mentioned above since by obtaining clarity about the premises, we gain more insight into whether or not the test of a Bell-type proposition had a loophole, for we may ask whether the setup of the test ensured the premises to hold.

Since a work on Bell-type theorems that does not draw on Einstein, Podolski, Rosen (EPR) argument of 1935[3], is like Hamlet without Ophelia, we shall start with a sketch of their reasoning that attempted to show that QM is incomplete. In order to avoid heavy weapons of the theory, we will, however, work with Bohm's version of the EPR argument.[4] This should prepare the stage for the presentation of Bell-type theorems. In what follows, we will analyze four statistical inequalities, and later on two versions of the so-called GHZ proofs that end with deriving a non-statistical observable equation. Needless to say, we will give extra care to spelling out the premises of the derivations.

2. EPR argument – Bohm's version

Let us consider a pair of 1/2 spin particles prepared in the singlet spin state φ,

$$\varphi = \frac{1}{\sqrt{2}}(\mid +\rangle \mid -\rangle - \mid -\rangle \mid +\rangle), \tag{1}$$

where $\mid +\rangle$ and $\mid +\rangle$ are eigenstates of the operator representing the measurement of the spin projection on a given direction, say z-direction, of a *single* particle and the states correspond to outcomes $+1$ and -1 of the measurement, respectively and no outcomes other than $+1$ and -1 are possible (see Fig. 1). Now, the salient feature of (1) is that, given that measurement of the spin projection on *any* direction of one particle yields some outcome, the measurement of the spin projection on this very direction, as performed on the other particle, will certainly yield the opposite outcome. In other

[3] Einstein A., Podolski B., Rosen N.: *Phys. Rev. A* 47 (1935), p. 777.

[4] Bohm A.: *Quantum Theory*, Prentice-Hall, Englewood Cliffs NJ, 1951.

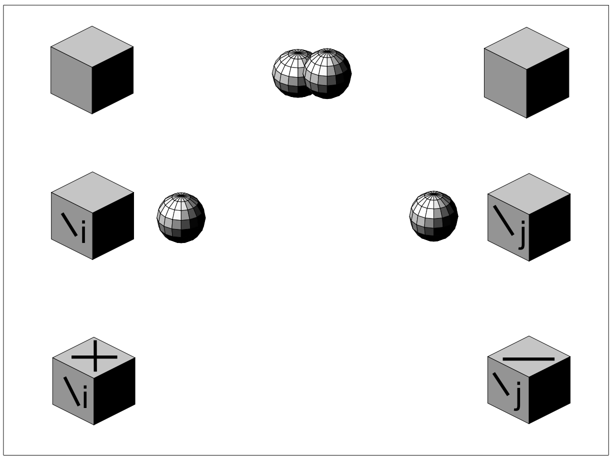

Fig. 1. The successive phases of the setup of EPR-Bohm and (Bell 1)–(Bell 4). The source emits pairs of particles, with each pair in the spin singlet state. When the particles from a given pair are widely separated, the direction i and j of the spin measurement are set. (In the EPR-Bohm *Gedanken experiment* the directions are the same, i.e., $i = j$). Finally, each counter yields a definite signal, either $+1$ or -1. If $i = j$, right signals and left signals are perfectly anti-correlated.

words, knowing the result of measurement on the first particle, we may with certainty predict the result of respective measurement on the other particle. Moreover, we may think of these measurements as being performed when the particles are widely separated in space, so that a result at one wing of the setup could not be communicated with a subliminal velocity to the result at the other wing. Now, the wisdom of classical physics has it that a particle has a definite property, including a property corresponding to 'the projection of spin on a given direction', *before* the measurement takes place, and this is precisely what quantum mechanics denies. The argument of EPR attempts to show, taking as an example the setup above, that a particle has a property corresponding to the spin projection long before measurement is carried out, and since QM refuses to ascribe this property to the particle before the measurement, it is deemed incomplete. To get hold on the argument we need to elucidate its two crucial premises, which are as follows:

Einsteinian separability

If two space-time region are such that no signal travelling with speed no greater than the speed of light can be transmitted between them, the factual situation at one region is independent from the factual situation at the other region.

Einsteinian reality criterion
If one can predict with certainty a value of a physical quantity pertaining to a system without by any means disturbing the system, then there exists an element of reality corresponding to the quantity in question.

To set these premises in train in the example considered, we need to introduce a particular timing between the event of measuring the spin projection of the particle on the left hand side and the event of measuring the spin projection on the right hand side. These two events are held to be spatially separated so that the act of setting a direction on which the spin projection is to be measured and the decision of the experimenter to choose this particular setting cannot be transmitted to the event of registering an outcome in the other wing of the setup. Accordingly, the two events “in the left wing, the experimenter’s decision plus the act of setting a direction plus an outcome” and “in the right wing, the outcome of the measurement of the spin projection on some direction” are spatially separated, and hence (Einsteinian) separable. Now, the correlations between the measured outcomes guarantee that the experimenter can advance certainly true predictions of the following sort:

Given that I obtained result $+1$ of the measurement of the spin projection on direction i, the result of measuring the spin projection on direction i in the other wing of the setup must be -1.

And, since both the events are spatially separated, the prediction above is made without disturbing the system in the right wing of the setup, so that the reality criterion allows us to pass to:

Given that I obtained result $+1$ of the measurement of the spin projection on direction i, there is an element of reality corresponding to outcome -1 of the measurement of the spin projection on direction i in the other wing of the setup.

Logically speaking, to obtain from this a statement to the effect that there is such-and-such an element of reality, we need to argue that the clause beginning with “given that” of the above sentence is irrelevant to the truth-value of the main clause. After all, the sentence above as well as Einsteinian separability is compatible with a somehow conspiratory vision of pre-established (Melabranchean?) harmony according to which the experimenter is “made to choose” a given direction in one region and the respective element of reality, pertaining to the other subsystem, is brought to being in the second region, both region being spatially separated. This vision makes also clear that the existence of the element of reality *before* the measurement is not proved yet. An attempt to eliminate this lofty possibility is to stress that the experimenter is essentially free to measure the spin projection on whatever direction she pleases. Thus, imagine him announcing the counterfactual statements below, "counterfactual" since he is not making any observation.

If I measured spin projection S_i on direction i and result $+1$ were observed, an element of reality would exist that corresponds to outcome -1 of the measurement of S_i, as performed in the other wing of the setup.

If I measured spin projection S_j on direction j and result -1 were observed, an element of reality would exist that corresponds to outcome $+1$ of the measurement of S_j, as performed in the other wing of the setup.

Now, since the experimenter's decision is essentially free and the event of his setting a given direction with the subsequent observation of an outcome cannot be signaled to the event of observing the outcome in the other wing of the setup, similarity of this situation to some everyday reasoning suggests that long before measurements on the left wing of the setup, there exist elements of reality corresponding to outcomes of the measurements (pertaining to the other wing of the setup) of spin projection on *any* direction. That is, we have come to the conclusion that is contrary to the wisdom of QM, i.e., a particle has a full set of definite properties of the form: the spin projection on i, the spin projection on j, the spin projection on k, and so on. It should be clear, however, that the reasoning above is not a logical derivation. As a matter of fact, it is done in terms of counterfactual statements, which is rather troublesome, since such statements are known for the propensity to generate a logical paradox. The reasoning may nevertheless be justified by its already mentioned similarity to a more down-to-earth counterfactual arguments. Consider for instance the following set of counterfactuals.

If I had exercised in the morning, there would have been (an element of reality of) a crash at Tokyo stock exchange today.

If I had eaten scrambled eggs in the morning, there would have been a supernova's blast in OXZ galaxy.

If I had read a local newspaper in the morning, Steamsters would have gone on strike,

with the understanding that I have been essentially free in choosing the actions stated in the antecedent, and these actions have had no bearing whatsoever on the states of affairs expressed by the implicants. If this is granted, you are likely to conclude that Tokyo stock exchange collapsed today, a supernova has exploded in OXZ galaxy and Steamsters have gone on strike. So much for the earthy misfortunes — we had better return to intricacies of QM.

Since we use the EPR argument only to prepare the stage for the Bell--type theorems, we do not attempt here a deeper analysis of it, referring the reader to the existing extensive literature. At this point it is worth stressing that the argument and its conclusion of incompleteness of QM divided the founding fathers of the 20th century physics. Nevertheless, already in the 40's, as physicists witnessed an overwhelming empirical adequacy of QM and did not suspect that the debate over elements of reality can have an experimentally testable consequence, the EPR argument was relegated to pure philosophy, with only a few physicists in the foundational studies working on it. It required a genus of one of them, John S. Bell, to show that Einstein's position, with elements of reality ascribed to a system prior to measurement, has a testable consequence.

3. Bell 1

In the EPR-Bohm setup, we will be considering measurements of the spin projection on some directions that are performed on two distant particles, the compound system "particle 1 and particle 2" being in singlet state φ from (1). Contrary to the EPR situation, however, we will investigate measurements along different directions, not necessarily parallel. To fix the notation, by $A(i)$ we understand the result of the measurement on particle 1 of the spin projection on i, and by $B(j)$ the result of the measurement on particle 2 of the spin projection on j, where a result can be either $+1$ or -1. We will calculate correlation factors (expectation values) of the following kind:

$$E(i,j) = \lim_{N\to\infty} \frac{\sum_N A(i)B(j)}{N} \tag{2}$$

Turning now to the premises, we first bring in the idea that an actual outcome of the measurement is *determined* by both the setting of the direction and the value of some parameter, say λ, so that

$$A(i,\lambda) = \pm 1 \quad \text{and} \quad B(j,\lambda) = \pm 1\,. \tag{3}$$

Since this assumption boils down to eradicating chance from QM, parameters that satisfy it are called *deterministic*. As soon should become clear, the question what mathematical objects, real numbers, vectors or others, λ stands for, is irrelevant. We will assume after Bell, however, that λ has a continuous range. One may argue that results of left and right measurements should be rather determined by two different parameters, say λ_1 and λ_2, nevertheless this case is already covered by a single parameter, since λ can stand for a two-dimensional vector, with each dimension responsible for results in a single wing.

The second assumption, clear from the form of $A = A(i, \lambda)$ is that an actual outcome is determined by both a direction set and a value of the parameter, but does *not* directly depend on a direction setting in the other wing of the setup. We will refer to this as *locality* requirement. In symbols,

$$A(i, j, \lambda) = A(i, \lambda), \qquad B(i, j, \lambda) = B(j, \lambda).$$

We need further to put some constraints on the functioning of the parameter. In fact, by saying that a result should be independent from the remote setting, we already put a limitation on λ, namely that a value it has at a given measurement should be independent from direction settings. We need to exclude an even more *outré* possibility, of there being some conspiracy between the experimenter' s decision to measure what she pleases and a value taken by λ on this particular occasion. A value of the parameter is independent of this decision and the decision is not by any means influenced by the value of λ. In other words, we require that neither experimenter's decision nor a setting of a direction tamper with values of the parameter, which we call *non-contextuality* assumption.

Finally, we will take advantage of one physical assumption, that is, strict anti-correlations, which, if QM is correct, obtain between results of the measurements of the spin projection on pairs of particles in the singlet state, if the same direction is set in both the wings of the setup. To recall, the strict anti-correlations means that measurements of the spin projection on both the particles, given the same directions are set, yield opposite outcomes.

We will now present the proof of (Bell 1) inequalities. We assume that $\sigma(\lambda)$ is the probability density of λ, that is, it satisfies:

$$\int_\Omega \sigma(\lambda)\, d\lambda = 1\,, \tag{4}$$

where Ω is the set of values that λ can take. Expressed in terms of the probability density, the correlation factor (2) reads now as:

$$E(i, j) = \int_\Omega A(i, \lambda) B(j, \lambda) \sigma(\lambda)\, d\lambda\,. \tag{5}$$

Eqs. (3) and (4) imply that $E(i,j)$ cannot be less than 1. Taking advantage of the strict anti-correlations,

$$E(i,j) = -1 \quad \text{for same direction } i = j \text{ only if} \quad A(i,\lambda) = -B(i,\lambda).^5$$

Given the assumption of non-contextuality, this allows one to re-write (5) as follows:[6]

$$E(i,j) = -\int_\Omega A(i,\lambda)A(j,\lambda)\sigma(\lambda)\,d\lambda\,. \tag{6}$$

Consider now the following expression:

$$E(i,j) - E(i,k) = -\int_\Omega [A(i,\lambda)A(j,\lambda) - A(i,\lambda)A(k,\lambda)]\sigma(\lambda)\,d\lambda.$$

Since $A(j,\lambda)A(j,\lambda) = 1$, by multiplying by it the integral above, we have:

$$E(i,j) - E(i,k) = \int_\Omega A(i,\lambda)A(j,\lambda)[A(j,\lambda)A(k,\lambda) - 1]\sigma(\lambda)\,d\lambda.$$

And, since factor $A(i,\lambda)A(j,\lambda)$ in this integral is equal to ± 1 and the second factor is not positive, we have the following inequality:

$$|E(i,j) - E(i,k)| \leqslant \int_\Omega [1 - A(j,\lambda)A(k,\lambda)]\sigma(\lambda)\,d\lambda$$

which, given (4), (5) and (6) simplifies to:

$$|E(i,j) - E(i,k)| \leqslant 1 + E(j,k). \tag{Bell 1}$$

This is the final form of (Bell 1). We add a few comments. First, the inequality is not satisfied by quantum predictions, since QM estimation of the correlation factors, if the directions are set in the plane orthogonal to the path of the particles, is:

$$E(i,j) = -\cos \sphericalangle(i,j),$$

where $\sphericalangle(i,j)$ is the angle between directions i and j. Setting i, j and k such that $\sphericalangle(i,j) = 30^\circ$, $\sphericalangle(i,k) = 30^\circ$ and $\sphericalangle(j,k) = 60^\circ$ we have:

$$\sqrt{3} = |-\cos 30^\circ - \cos 30^\circ| > 1 - \cos 60^\circ = 1/2$$

which contradicts (Bell 1).

[5] More precisely, the equality may not hold, but then the set of λ for which it does not hold, has zero measure.

[6] The last result comes from considering measurement with same directions, and now we are applying this result to any possible combination of directions.

It is instructive to give more thought to the parameter. Perhaps a natural way to construe the parameter is to claim that a quantum mechanical state is not a complete description of a physical state, since it refers to rather an (idealized) ensemble of identically prepared systems than a single system. On this construal, to fix a state of an individual pair in the quantum mechanical singlet state φ, we need to additionally specify a value of parameter λ. Moreover, the specification is thought of as restoring determinism: a value of λ taken together with the state of standard QM determines results of the measurement of spin projections. This vision is shattered by (Bell 1) that rules out the ensemble interpretation with local deterministic and non-contextual parameters. However, it is clear that there is no logical link between (Bell 1) and the ensemble interpretation. If you take a now more standard view that a quantum state refers to a single system, but add that the state is not a complete description, (Bell 1) puts a heavy constraint on ways of making the description complete by concluding that no local deterministic and non-contextual parameters will do the job.

To see how non-contextuality works it is advantageous to put the proof in a different perspective. In the presence of the strict anti-correlations, non-contextuality allows one to argue as follows. If the same direction, say j, were selected at both the wings of the setup, parameter λ must be such that $B(j, \lambda) = -A(j, \lambda)$ holds. As a matter of fact, however, we have different settings i and j. Nevertheless, the fact that in the left we have rather i than j should not influence the value of λ and accordingly, we should still have $B(j, \lambda) = -A(j, \lambda)$. If that is so, the correlation factor from (5) is determined by one function only, that is, either $A(i, \lambda)$ or $B(j, \lambda)$. Thus, if we deal with three directions i, j and k, we may divide values of λ in eight classes in respect to what results of measuring $A(i)$, $A(j)$ and $A(k)$ are. For instance, [+ + -] is the class of those values of λ that yield $A(i, \lambda) = A(j, \lambda) = +1$ and $A(k, \lambda) = -1$. We may now ask what fraction of all possible values of λ a given class, say [+ + -], makes, and designate this fraction by η_{++-}.[7] With this notation, using (6), the correlation factor is expressed as:

$$\begin{aligned} E(i,j) &= -(+1)(+1)(\eta_{+++} + \eta_{++-}) - (+1)(-1)(\eta_{+-+} + \eta_{+--}) \\ &\quad -(-1)(+1)(\eta_{-++} + \eta_{-+-}) - (-1)(-1)(\eta_{--+} + \eta_{---}). \end{aligned}$$

[7] More precisely, $\eta_{++-} = \int_{\Omega^*} \sigma(\lambda)\, d\lambda$, where

$$\Omega^* = \{\lambda : A(i, \lambda) = A(j, \lambda) = +1 \text{ and } A(k, \lambda) = -1\}.$$

Now, calculating by 'brute force' $|E(i,j) - E(i,k)|$ and $1 - E(j,k)$ and noting that

$$1 = \eta_{+++} + \eta_{++-} + \eta_{+-+} + \eta_{-++} + \eta_{--+} + \eta_{-+-} + \eta_{+--} + \eta_{---},$$

we arrive at (Bell 1).

Bell's 1964 paper contains other results of note. First, it shows that local deterministic non-contextual parameters work in some cases, notably, in the cases of the measurement of the spin projection of a single spin 1/2 particle and the measurement of spin projection on *parallel* directions of pairs of particles in the singlet state (that is, EPR case). Second, it indicates how (Bell 1) can be reproduced in other physical situations, and which conditions should be satisfied for this to be possible.

4. Bell 2

What we call (Bell 2) is a technical variation of (Bell 1), the difference being that the assumption of the strict anti-correlations is skipped in (Bell 2). The inequality was first derived by J. F. Clauser, M. A. Horne, A. Shimony and R. A. Holt.[8] The motivation for this derivation comes from a need to block a trivial answer to the violation of (Bell 1) that claims the inequality is violated because in nature there *are no* perfect anti-correlations. The answer appears to be unbeatable, since testing perfect anti-correlations is excessively hard. To put this in other words, ingenious as (Bell 1) is, its testing is not much experimentally feasible. (Bell 2) is a substantial improvement since its proof does not appeal to the perfect anti-correlations.

As for the remaining assumptions of (Bell 1), they stay intact and the proof goes quite similarly. We start with the following estimation:

$$\begin{aligned} |E(i,j) - E(i,k)| &= \int_\Omega |A(i,\lambda)B(j,\lambda) - A(i,\lambda)B(k,\lambda)|\sigma(\lambda)\, d\lambda \\ &= \int_\Omega |A(i,\lambda)B(j,\lambda)|[(1 - B(j,\lambda)B(k,\lambda)]\sigma(\lambda)\, d\lambda \\ &\leqslant 1 - \int_\Omega B(j,\lambda)B(k,\lambda)\sigma(\lambda)\, d\lambda. \end{aligned} \tag{7}$$

Suppose now that there is some other direction, say j', such that

$$E(j',j) = 1 - \delta\,, \tag{8}$$

[8] Clauser J. F., Horne M. A., Shimony A., Holt R. A., "Proposed experiment to test local hidden-variable theories", *Phys Rev Lett* 23 (15), 1969, p. 880.

where $0 \leqslant \delta \leqslant 1$. We may divide set Ω into two sets, Ω_+ and Ω_- according to this prescription:

$$\begin{array}{lll} \lambda \in \Omega_+ & \text{iff} & B(j,\lambda) = A(j',\lambda); \\ \lambda \in \Omega_- & \text{iff} & B(j,\lambda) = -A(j',\lambda). \end{array}$$

And, since the above are exhaustive possibilities, $\Omega_+ \cup \Omega_- = \Omega$.

In the next step we need to calculate $\int_{\Omega_-} \sigma(\lambda)\, d\lambda$:

$$\begin{aligned} 1-\delta &= E(j',j) = \int_\Omega A(j',\lambda)B(j,\lambda)\sigma(\lambda)\, d\lambda \\ &= \int_{\Omega_+} A(j',\lambda)B(j,\lambda)\sigma(\lambda)\, d\lambda + \int_{\Omega_-} A(j',\lambda)B(j,\lambda)\sigma(\lambda)\, d\lambda \\ &= \int_{\Omega_+} \sigma(\lambda)\, d\lambda - \int_{\Omega_-} \sigma(\lambda)\, d\lambda \\ &= \int_{\Omega} \sigma(\lambda)\, d\lambda - 2\int_{\Omega_-} \sigma(\lambda)\, d\lambda \\ &= 1 - 2\int_{\Omega_-} \sigma(\lambda)\, d\lambda\,, \end{aligned}$$

and hence,

$$\int_{\Omega_-} \sigma(\lambda)\, d\lambda = \delta/2\,. \tag{9}$$

Given this, the second factor in inequality (7) can be estimated as below:

$$\begin{aligned} &\int_\Omega B(j,\lambda)B(k,\lambda)\sigma(\lambda)\, d\lambda \\ &\quad = \int_{\Omega_+} A(j',\lambda)B(k,\lambda)\sigma(\lambda)\, d\lambda - \int_{\Omega_-} A(j',\lambda)B(k,\lambda)\sigma(\lambda)\, d\lambda \\ &\quad = \int_{\Omega} A(j',\lambda)B(k,\lambda)\sigma(\lambda)\, d\lambda - 2\int_{\Omega_-} A(j',\lambda)B(k,\lambda)\sigma(\lambda)\, d\lambda \\ &\quad \geqslant E(j',k) - 2\int_{\Omega_-} |A(j',\lambda)B(k,\lambda)|\sigma(\lambda)\, d\lambda = E(j',k) - \delta\,, \end{aligned} \tag{10}$$

where we used (9). Tying inequalities (7) and (10) and equation (8) together, we obtain (Bell 2) inequality:[9]

$$|E(i,j) - E(i,k)| \leqslant 2 - E(j',k) - E(j',j). \tag{Bell 2}$$

[9] The inequality is sometimes written as $|E(i,j) - E(i,k)| + |E(j',k) + E(j',j)| \leqslant 2$.

Recalling now that for directions set in the plane perpendicular to the path of the particles, QM yields $E(i,j) = -\cos \sphericalangle(i,j)$, we find that (Bell 2) contradicts the prediction of QM, for instance, for the following settings: $\sphericalangle(i,j) = 60°$, $\sphericalangle(i,k) = 120°$, $\sphericalangle(j',k) = 150°$ and $\sphericalangle(j',j) = 210°$. Thus, the assumption of the perfect anti-correlations being eliminated, the ramifications of (Bell 2) are twofold:

1. no deterministic local and non-contextual parameters permit a reproduction of QM predictions;
2. the conjunction of assumptions: deterministic dependence of outcomes on parameters, locality and contextuality, is false, as it entails an empirically falsified proposition.

The two next types of Bell inequalities will dig deeper in the meaning of the assumptions and try to remove the strict determinism of (Bell 1) and (Bell 2).

5. Bell 3

What we call (Bell 3), is an argument of van Fraassen[10] in which it is initially assumed that the parameters do not work deterministically but have only a statistical bearing on outcomes of measurements. A salient feature of this model is that in the presence of the assumption of the strict anti-correlations, this stochastic dependence turns into full-fledged determinism, so we are again in the framework of (Bell 1). The initially stochastic character of the argument requires a more complicate notation, however. To set the stage, we have two classes of outcomes, $\{A\}$ and $\{B\}$ of measurements on the left and right wings of the setup, respectively, each outcome being either $+1$ or -1. We will derive a testable constraint on joint probabilities of the form $P(LiA\,\&\ RjB|Li;Rj)$, that is, the probability of the outcome on the left (L) being A and the outcome on the right (R) being B, given the settings on the left (L) and on the right (R) are i and j, respectively. Now, we think of two outcomes of a given pair as being influenced by both, the settings of measuring devices and parameter λ that describes the situation in the common past of two outcomes. The values of λ vary from occasion to occasion, and statistically influence outcomes.

Coming to the central assumption of the argument, it is the so-called principle of common cause, introduced by Reichenbach[11] to save causation

[10] van Fraassen B.: *Quantum Mechanics. An Empiricist View*, Oxford, 1991.

[11] Reichenbach H.: *The Direction of Time*, ed. by M. Reichenbach, Univ. of California Press, 1956.

in relativistic settings. It applies to stochastically correlated events, which for some reasons can not be thought of as directly influencing each other, where stochastic correlation between two classes of events, say $\{X\}$ and $\{Y\}$, means that $P(X|Y) \neq P(X)$. For such correlated events, a common cause, say λ, is postulated that in general has only a statistical influence on events. This means that $P(X|\lambda)$ and $P(Y|\lambda)$ can be neither 0 nor 1 and $P(X|\lambda) > P(X)$ and $P(Y|\lambda) > P(Y)$. The correlation between X and Y means that knowing that an event Y has occurred, you may estimate the probability of the occurrence of Y. You may now ask: if I know that such-and-such a common cause occurred and event X took place, will my estimation of the probability of Y be any different from the estimation of this probability as obtained merely on the basis of my knowledge that this common cause occurred? The common cause principle answers this query in the negative, which in probabilistic terms is rendered as:

$$P(X|Y;\lambda) = P(X|\lambda) \quad \text{or, alternatively} \quad P(X\,\&\,Y|\lambda) = P(X|\lambda)P(Y|\lambda).$$

The usual story that attempts to elucidate the principle invokes an attempt to estimate the life expectancy of a smoker. Suppose that you spotted a person smoking a cigarette, and since you know that smoking history and life expectancy are correlated, your estimation of the probability that this person will live longer than 70 years went down, from the respective probability for non-smokers to the probability calculated for smokers. In doing so, you took your seeing the person with a cigarette as an indication of his having been a smoker for some time. However, if you knew in advance that the person has a smoking history, you estimation of his life expectancy would not change as a result of your seeing him with a cigarette.

Returning to the argument, we begin with spelling out its premises. The first is the statement of the perfect anti-correlations (and correlations):

$$\text{COR} \qquad P(Li\pm\ \&\ Ri\pm|Li;Ri) = 0, \qquad P(Li\pm\ \&\ Ri\mp|Li;Ri) = 1.$$

The status of the next statement is more delicate, since it is not directly used in the derivation, though it provides a motivation for accepting some other assumptions. It says that an outcome at one wing of the setup is statistically independent of a remote setting; van Fraassen calls it *surface locality.*

$$\text{SL} \quad P(LiA|Li;Rj) = P(LiA|Li), \qquad P(RjB|Li;Rj) = P(RjB|Rj).$$

The next assumption is an instance of the principle of the common cause and it identifies common causes of outcomes of A and B with parameters from a set Λ. It states that the probability of obtaining an outcome at one wing of the setup, given the settings of directions and the parameter, is independent from an outcome at the other wing of the setup.[12] In symbols, for the left wing:

CC $$P(LiA|RjB; Li; Rj; \lambda) = P(LiA|Li; Rj; \lambda),$$

or alternatively

CC′ $$P(LiA \,\&\, RjB|Li; Rj; \lambda) = P(LiA|Li; Rj; \lambda)P(RjB|Li; Rj; \lambda).$$

The next assumption is motivated by SL, as it states that given a value of the parameter, the remote setting has no influence on an outcome. For this reason, van Fraassen dubs it *hidden locality*:[13]

HL $$\begin{aligned} P(LiA|Li; Rj; \lambda) &= P(LiA|Li; \lambda), \\ P(RjB|Li; Rj; \lambda) &= P(RjB|Rj; \lambda). \end{aligned}$$

The final assumption is to exclude a kind of conspiracy mentioned in section 3: the setting of directions have no bearing on the parameters, and the other way round, the choice of settings is no influenced by the parameters. This is called *hidden autonomy*

HA $$P(\lambda|Li; Rj) = P(\lambda).$$

We already mentioned a rather peculiar status of surface locality SL. It will not be used in the proof, and is regarded here as a well-confirmed fact based on special relativity. SL can be derived from hidden locality HL and hidden autonomy HA, but this does not exclude the possibility of HL or HA being false. To comment on the proof that will follow, it splits into two parts, the first establishing that parameters work deterministically and the other deducing the inequality in a way similar to that sketched in the last lines of section 3.

The proof starts with a statement of the perfect anti-correlations:

COR $$P(LiA \,\&\, RiA|Li; Ri) = 0$$

from which it follows:

(11) $$\begin{aligned} 0 &= P(LiA \,\&\, RiA|Li; Ri; \lambda) = P(LiA|Li; Ri; \lambda)P(RiA|Li; Ri; \lambda) \\ &= P(LiA|Li; \lambda)P(RiA|Ri; \lambda) = 0\,, \end{aligned}$$

[12] In literature it bears different names: factorizability (Shimony) outcome independence (Jarett), causality (van Fraassen).

[13] Jarett's term is *parameter independence.*

where we used CC and HL. The last equation shows that if one multiplicand is 0, the other is 1 and *vice versa*. Suppose for example that $P(LiA|Li;\lambda) = 0$. Then $P(Li-A|Li;\lambda) = 1$. By the last line of the equation above, $P(Ri-A|Ri;\lambda) = 0$, which leads to $P(RiA|Ri;\lambda) = 1$.

Given this fact, we may argue like at the end of section 3, dividing parameters from Λ into classes, each class being responsible for pairs of outcomes of a given kind. First, however, we make the following observation:

$$\begin{aligned} P(LiA\,\&\, RjB|Li;Rj) &= \int_\Omega P(LiA\,\&\, RjB|Li;Rj;\lambda)\sigma(\lambda|Li;Rj)\,d\lambda \\ &= \int_\Omega P(LiA|Li;\lambda)P(Lj-B|Lj;\lambda)\sigma(\lambda)\,d\lambda, \end{aligned}$$

where we used COR, CC, HL and HA. In the next step we calculate:

$$\begin{aligned} P(L1+\ \&\ R2+|L1;R2) &= \int_\Omega P(L1+|L1;\lambda)P(L2-|L2;\lambda)\sigma(\lambda)\,d\lambda \\ &= \eta(+--)+\eta(+-+)\,, \end{aligned} \tag{12}$$

where we used fact (11) about determinism and expression $\eta(+--)$ stands for the fraction of those parameters from Λ that yield $P(L1+|L1;\lambda) = P(L2-|L2;\lambda) = P(L3-|L3;\lambda) = 1$. In a similar vein, we obtain:

$$P(L2+\ \&\ R3+|L2;R3) = \eta(++-)+\eta(-+-)\,, \tag{13}$$
$$P(L1+\ \&\ R3+|L1;R3) = \eta(++-)+\eta(+--)\,. \tag{14}$$

Now, by adding the sides of (12) and (13) and observing that the result contains both the summands of (14), we have: (Bell 3) inequality:

$$\begin{aligned} P(L1+\ \&\ R2+|L1;R2)+P(L2+\ \&R3+|L2;R3) \\ \geqslant P(L1+\ \&\ R3+|L1;R3). \end{aligned} \tag{Bell 3}$$

To assess the argument, it fails to spin out the implications of stochastic dependence of outcomes on the parameter, since in the presence of the strict anti-correlations, the dependence turns into deterministic one. And, although this reduction to determinism is worth noting, the argument does not shed more light on the issue of why the inequality is violated than the proof of (Bell 1). We need an argument in which the statistical functioning of the parameter is irreducible.

6. Bell 4

What we call (Bell 4), is the inequality equivalent in form to (Bell 2), the proof of which can be extracted from later papers of J. Bell.[14] The already "classic" philosophical analysis of the derivation was given by J. Jarrett.[15] Our derivation draws heavily on the argument of M. Redhead.[16] As before, we will deal with the EPR-Bohm setup and consider measurements of the spin projection on some directions of distant particles, each pair of the particles being in the singlet state φ. For a given pair of particles, the events of registering outcomes A and B of the measurements are taken to be spatially separated and the outcome can be either $+1$ or -1. There is some correlation between classes $\{A\}$ and $\{B\}$ of outcomes; however, we do not assume that these are strict correlations (anti-correlations). We are aiming at deriving a constraint on observable probabilities, the probabilities having form $P(LiA \,\&\, RjB|Li;Rj)$ and read as "the probability of obtaining outcome A on the left and outcome B on the right, given that the directions on the left and the right are i and j, respectively".

Now, we need to be more cautious as to what might produce the correlations. To address this query it is helpful to draw the backward light cones of the outcome-events for a pair of particles, and focus our attention on three regions: the region of the common past of the events, say R_{AB}, the region R_A obtained by removing R_{AB} from the backward cone of A and the region R_B obtained by removing R_{AB} from the backward cone of B (see Fig. 2).

There are many things that may be believed to exert influence on the outcomes. First, there are directions, i and j, set when the particles are so close to the respective measuring devices that the setting on one device cannot be communicated with subliminal velocity to the remote outcome-event; this is why i and j are located in R_A and R_B, respectively. There are also many other physical entities that belong to those regions: (a part of) a measuring device at the moment of, and immediately before, the measurement, electrical currents and magnetic fields in the vicinity of an outcome-event at the moment of, and immediately before, the measurement, and others. Thus, we may describe a physical situation in R_A by giving two factors: setting i

[14] See Bell J. S., *Speakable and Unspeakable in Quantum Mechanics*, Cambridge UP, 1987.

[15] Jarrett J., "On the physical significance of the locality conditions in the Bell arguments", *Nous* 18, p. 569.

[16] Readhead M., *Incompleteness Nonlocality and Realism*, Clarendon Press, Oxford, 1987.

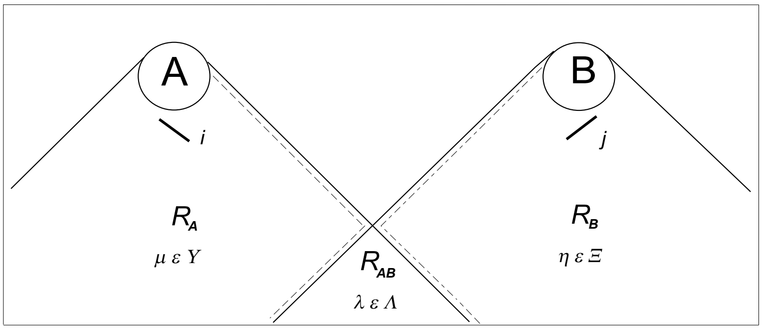

Fig. 2. The outcome events A and B and their backward light cones. Region R_{AB} is the common past of the two outcomes. Region R_A (R_B) transmits to A (B) causal influences from from R_{AB}. In the past of each outcome there is the event of setting the corresponding direction (i or j). The remaining physical situation in the regions are described by parameters: $\mu \in \Upsilon$ for R_A, $\eta \in \Xi$ for R_B and $\lambda \in \Lambda$ for R_{AB}.

of direction and parameter $\mu \in \Upsilon$ that accounts for the physical situation (except from a setting of direction) in R_A. Similarly, a physical situation in R_B is described by specifying a setting j of direction and parameter $\eta \in \Xi$ that accounts for the physical situation (except from a setting of direction) in R_B. As before, we do not commit ourselves as to what sort of mathematical objects parameters $\mu \in \Upsilon$ and $\eta \in \Xi$ may stand for. Turning now to the common past of A and B, it comprises the source of the two particles, the quantum mechanical state of the source, the physical situation in one wing and the other wing of the setup in the appropriate pasts of outcome-events. For each pair of particles, we will describe the physical situation in the common past of the outcome-events by giving a parameter $\lambda \in \Lambda$, again without specifying what mathematical object λ is capable of standing for. To avoid cumbersome notation we assume, however, that parameters from Λ encode also quantum mechanical states of the source.

With this machinery, a typical (not testable) probability has the form

$$P(LiA \,\&\, RjB|Li; \mu; Rj; \eta; \lambda),$$

which is to be read as: the probability of left outcome being A and right outcome being B given that the left direction is i, the parameter for region R_A is μ, the right direction is j, the parameter for region R_B is η, and the parameter for R_{AB} is λ.

With the division of space-time into regions R_A, R_B and R_{AB}, the crux of the argument consists in the claim that an outcome at one wing of the setup should be influenced by neither the other outcome-event nor a physical situation in the other spatially separated region. This constraint, which is motivated by the prohibition of superluminal signalling, lies behind premises of locality that follow.

The first premise arises from the observation that whatever the outcome at one wing is, say A at the left wing, it is produced by the physical situation in the backward light cone of it, here $R_A + R_{AB}$. Thus, if the influences are transmitted locally, that is, continuously from point to point with a subliminal velocity, correlations between outcome-events should be traceable to physical situations in common pasts of pairs of events A and B. That is, although we may expect a correlation $P(LiA|RjB; Li; Rj; \mu; \eta) \neq P(LiA|Li; \mu; Rj; \eta)$, if we take into account the common pasts, the correlation should vanish, i.e.,

LC $$P(Lia|RjB; Li; \mu; Rj; \eta; \lambda) = P(Lia|Li; \mu; Rj; \eta; \lambda).$$

In other words, whatever statistical influences of left outcomes on right outcomes and vice versa are, they are screened off by the common pasts of pairs of outcome-events. The condition goes by different names, like screening-off condition, factorizability, or local causality, from which we choose the last one (LC). The reader may recognize in it an instance of the familiar principle of the common cause, although the argument for LC is rather diffrent from the argument for CC.

The second premise says that, in the presence of hidden parameters, outcomes in one wing of the setup should be statistically independent from direction settings at the other wing. In symbols:

DL $$P(Lia|Li; \mu; Rj; \eta; \lambda) = P(Lia|Li; \mu; \eta; \lambda).$$

There is a rigorous proof, due to Jarrett[17], that if this condition is violated, experimenters at two wings could communicate instantaneously.[18] In essence, the requirement is the condition of deep locality from section 5.

The final assumptions shed light on the parameters. Being a more elaborate version of HA of section 5, the first is the claim that there are no statistical influences between the parameters and the settings of directions.

HA $$\sigma(\mu\,\&\,\eta\,\&\,\lambda|Li; Rj) = \sigma(\mu\,\&\,\eta\,\&\,\lambda).$$

[17] *Op. cit.*

[18] The proof may nevertheless contain a loophole; see Jones M., "What locality isn't", in: Kafatos M. (ed.) *Bell's Theorem, Quantum Theory, and Conceptions of Universe*, Kluwer, 1989, p. 77.

HA says that we may separate the description of the physical situation into a description of directions settings and a description of the rest, so that the choice of direction (and the agent or a device that makes it) at a wing may be thought of as independent from the physics that is captured by a relevant parameter (μ or η). Accordingly, it has also the meaning that the experimenter (or a semi-random device that does the job) is in this sense free to choose a direction setting. Once this condition had the status of a condition of the possibility of experimental knowledge, at present, however, it is seen somehow dubious.

To proceed to the proof, we start with the following transformations that are permitted by the probability calculus:

$$\begin{aligned} &P(LiA\,\&\; RjB|Li;Rj) \\ \text{(15)}\quad &= \int_\Lambda d\lambda \int_\Upsilon d\mu \int_\Xi d\eta P(LiA\,\&\; RjB|Li;Rj;\mu;\eta;\lambda)\sigma(\mu\,\&\;\eta\,\&\;\lambda|Li;Rj) \\ &= P(LiA|Li;\mu;\eta;\lambda)P(RjB|Rj;\mu;\eta;\lambda)\sigma(\mu\,\&\;\eta\,\&\;\lambda), \end{aligned}$$

where we used LC, DL and HA.

Now, let us focus our attention on an arithmetical fact that for any real numbers x, x', y and y' from interval [0,1] the following holds:

$$-1 \leqslant xy + x'y + xy' - x'y' - x - y \geqslant 0\,. \tag{16}$$

Making now substitutions:

$$\begin{aligned} x &= P(LiA|Li;\mu;\eta;\lambda), \\ y &= P(RjB|Rj;\mu;\eta;\lambda), \\ x' &= P(Li'A|Li';\mu;\eta\lambda), \\ y' &= P(Rj'B|Rj';\mu;\eta;\lambda), \end{aligned}$$

multiplying the sides of Inequality (16) by $\sigma(\mu\,\&\;\eta\,\&\;\lambda)$ and integrating over respective ranges of λ, μ and η, we obtain the following Bell-type inequality:

$$\begin{aligned} -1 \leqslant{}& P(LiA\,\&\; RjB|Li;Rj) + P(Li'A\,\&\; RjB|Li';Rj) \\ \text{(Bell 4)}\quad &+ P(LiA\,\&\; Rj'B|Li;Rj') - P(Li'A\,\&\; Rj'B|Li';Rj') \\ &-P(LiA|Li) - P(RjB|Rj) \geqslant 0\,, \end{aligned}$$

where we used (15) and the fact that probability density $\sigma(\mu\,\&\;\eta\,\&\;\lambda)$ is normalized to unity. Reflecting on the proof, the appeal to the three sets of parameters (Λ, Υ and Ξ) representing physical situations in the three regions is an unnecessary complication. The proof will go with a single set of parameters, each parameter describing jointly the situation in the sum of the three regions. What is important, however, the description of the overall physical situation is assumed to be given by two separate things: directions settings and the rest being parametrized by a parameter. We will use this observation in the next proofs of Bell-type theorems.

(Bell 4) is at present the most mature type of Bell-type theorems, and is most often used in the present debate over the violation of Bell-type inequalities. Although this paper is not intended to adjudicate between positions involved, a brief survey of them seems to be in place. A prevailing opinion sees the violation of (Bell 4) to be evidence for the break of local causality LC. This means that the outcomes at one wing, say left, of the setup depend stochastically not only on physical situations in the common pasts of these outcome-events, but also on physical situations in spatially separated regions. The break of local causality is nevertheless very subtle as it is not in conflict with the special relativity's prohibition of superluminal signalling. In other words, the experimenters at the two wings cannot exploit the break of LC to communicate instantaneously.[19] The remaining problems is to understand what this violation of LC means, a majority of researchers being in agreement that it reflects the holistic feature of QM systems. This answer ties in nicely with a mathematical feature of quantum theory, namely its holistic treatment of composite systems.

To say that LC is the main suspect does not mean that the other premises are sacrosanct. We have already mentioned the problematic loophole in Jarett's proof of DL. However, only very few people are ready to reject this locality condition.

There is also a growing discontent with HA, where we seem to be facing a dilemma. On the one hand, a meaningfully talk about experimental science requires a certain degree of freedom of experimenters to measure what they like to measure, to set knobs of their devices in the way they please and so on. Without this freedom, nature would be hiding things that we could have no chance whatsoever to discover. On the other hand, however, the experimenter that chooses a setting (or a device that does this) is a part of nature and it is hard to see, at least in physicalist ontology, how the choice can be made independently of the physical situation in the respective space-time region.

[19] Mermin N. D., "Hidden quantum non-locality", in: Clifton R. (ed.), *Perspectives on Quantum Reality*, Kluwer, 1996, p. 57.

Finally, since (Bell 4) as well as its predecessors have the form of statistical inequalities, there is always a simple (not to say, simplistic) way out of the predicament. This relies on stubbornly rejecting the experimental violation of the inequalities by claiming that experimental data, that is, the obtained statistics, does not faithfully represent the (theoretical) probabilities involved in the inequalities. As was said, this position requires stubbornness, since its proponent must sustain his skepticism no matter how big a sample involved or how small statistical deviations are. There is still a more serious motivation to eschew probability from arguments for Bell-type theorems, that arises from a doubt about the applicability of standard (Kolmogorovian) probability calculus in quantum mechanical discourse. Considering a rather special language of the so-called experimental quantum propositions, if we read the algebraic structure of this language from the underlying formalism of Hilbert spaces, this structure will be non-Boolean, the consequence being that the probability assignable to these propositions is not Kolmogorovian.[20] And in this new generalized probability, some laws that we have been using above, like the law of conditional probability, does not in general hold. Now, this objection is removed and the skepticism mentioned above alleviated by a more recent version of Bell-type theorems, called by initials of its authors GHZ,[21] that deduces a testable non-statistical equation. Thus, we turn now to these "Bell's inequalities without inequalities", as they are sometimes called.

7. GHZ-Bell 1

Although this argument uses a setup different from that of EPR-Bohm, it shares with (Bell 1) the same set of premises, the most important being the assumption of deterministic hidden parameters. We start with sketching the setup.[22] The source provides spin 1 particles, the spin projection of these particles being 0 (see Fig. 3). Subsequently, the particle decays into two spin 1 particle, the state of the pair being:

$$\varphi_{I,II} = \frac{1}{\sqrt{2}}(|+\rangle_I\,|-\rangle_{II} - \;|-\rangle_I\,|+\rangle_{II}),$$

[20] Gudder S., *Quantum Probability*, Academic Press, 1988

[21] Greenberger D., Horne M. A., Zeilinger A., "Going beyond Bell's Theorem", p. 69, in: Kafatos M. (ed.), *Bell's Theorem, Quantum Theory, and Conceptions of Universe*, Kluwer, Dordrecht, 1989.

[22] *Op. cit.*

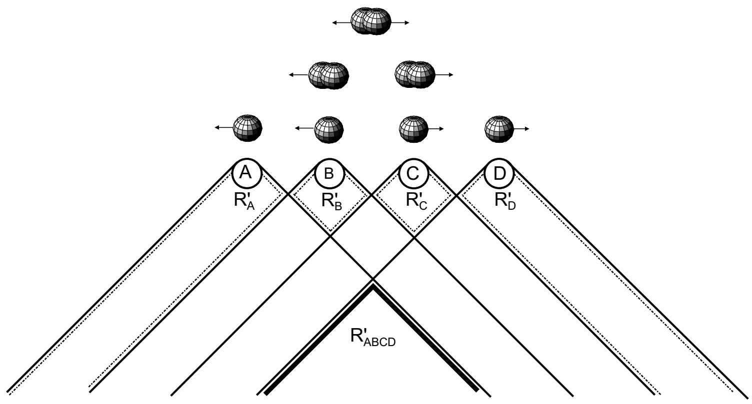

Fig. 3. The setup of the GHZ theorems. A source provides a sipin -1 partice that decays into two spin -1 particles, the quantum mechanical state of the pair being $\varphi_{I,II}$. Subsequently each particle decays into two spin $-1/2$ aprticles. The events of settings the directions as well as outcome events are spatially separated. Regions R'_A, R'_B, R'_C and R'_D shed teh corresponding outcomes from the physical situation in R'_{ABCD}.

where $|+\rangle_I$ and $|-\rangle_I$ are eigenstates of the observable of the spin projection on z-direction of particle I corresponding to result $+1$ and -1, respectively, and $|+\rangle_{II}$ and $|-\rangle_{II}$ are eigenstates of the observable of the spin projection on z-direction of particle II corresponding to result $+1$ and -1, respectively.

Next, each particle decays into two spin 1/2 particles. We assume that geometry of the setup guarantees that there is no exchange between orbital momentum and spin, so that spin is conserved. With this assumption, the state of the quadruple system is:

$$\varphi_{1,2;3,4} = \frac{1}{\sqrt{2}}(|+\rangle_1\,|+\rangle_2\,|-\rangle_3\,|-\rangle_4 - \,|-\rangle_1\,|-\rangle_2\,|+\rangle_3\,|+\rangle_4),$$

where $|\pm\rangle_k$ is the eigenstate of the operator representing measurements of the spin projection on z-direction of k-th particle that correspond to outcomes $+1/2$ and $-1/2$, respectively. To guarantee the satisfaction of the assumption above, the beams carrying particles 1 and 2 move in direction $-z$, whereas those carrying particles 3 and 4 move in $+z$ direction. Moreover, beam 1 moves faster than beam 2 and beam 4 faster than beam 3, which is to guarantee that if the devices measuring the spin projection of the particles are appropriately located, then the four events of registering the outcomes are spatially separated.

Let the outcomes of the measurements performed on particles 1, 2, 3 and 4 be symbolised by A, B, C and D, respectively. To simplify the calculations, although quantum mechanical spin projections of the particles are $\pm 1/2$ in units of Planck constant, we also assume that each outcome is either -1 or $+1$. Much in the spirit of (Bell 1), we assume that an outcome is *determined*, given a direction setting, by a physical situation in the sum of the pasts of the four outcome events, the description of a physical situation being provided by a parameter $\lambda \in \Lambda$. The second claim is that an outcome of measurement performed on one particle is independent from a direction set in the measurement on any other particle from a given quadruple. Both the assumptions are captured by the following notation:

$$A(i,\lambda) = \pm 1, \qquad B(j,\lambda) = \pm 1, \quad C(k,\lambda) = \pm 1, \qquad D(l,\lambda) = \pm 1.$$

Consider now the measurement of the spin projection performed on all the four particles from a quadruple, directions i, j, k and l being set in the plane perpendicular to z-axis and characterized by angles α, β, γ and δ, respectively. For this geometry, the quantum mechanical expectation value $E(i,j,k,l)$ is:

$$E(i,j,k,l) = -\cos(\alpha + \beta - \gamma - \delta),$$

the implication being that

$$E(i,j,k,l) = \pm 1 \quad \text{for} \quad (\alpha + \beta - \gamma - \delta) = \pi, 0.$$

In the framework of deterministic parameters, this translates, for instance, into the following constraint:

$$A(i,\lambda)B(j,\lambda)C(k,\lambda)D(l,\lambda) = 1 \quad \text{given that} \quad (\alpha + \beta - \gamma - \delta) = \pi.$$

This constraint cannot be satisfied, however. To see this, we may keep the two direction, say k and l fixed, while varying the remaining directions, with the angles satisfying $\alpha + \beta - \gamma - \delta = \pi$. Given that the product $A(i,\lambda)B(j,\lambda)C(k,\lambda)A(l,\lambda) = 1$, it follows that $A(i,\lambda)B(j,\lambda)$ must be independent from the settings i and j. Repeating this argument for other pairs of directions, we obtain that $A(i,\lambda)$, $B(j,\lambda)$, $C(k,\lambda)$ and $A(l,\lambda)$ are independent from directions i, j, k and l, respectively, which is patently false. Accordingly, the assumptions of deterministic parameters and locality cannot be both true.

A student of the proof of GHZ-(Bell 1) may fall under a misapprehension that the argument does not utilize the assumption of non-contextuality (or hidden autonomy). This is not so since we assumed above that the varying of directions has no impact on parameters. Thus, the argument indeed has the same set of assumptions as that of (Bell 1). Another delicate point is the (allegedly) non-stochastic character of it. In principle, GHZ-(Bell 1) pertains to a single quadruple system, and accordingly it may be tested by performing the described measurements on four particles produced in a two-step decay of a spin 1 particle. In reality, however, it should be tested by performing a series of joint measurements on a sample of quadruple systems. This bring us to another experimental problem, of ensuring the occurrence of the prefect correlations. To recall, this problem gave a motivation for the development of (Bell 2). Nevertheless, in the early 90's a new technique, called parametric down conversion technique, was developed that permits the production of pairs of photons, photons from each pair exhibiting sharply correlated momenta. A proposal of how to use these correlation to test a version of GHZ-(Bell 1) that involves positions instead of spin projections was put forward by Horne, Shimony and Zeilinger.[23] The experiment was carried out by Rarity and Tapster, [24] convincingly showing that GHZ-(Bell 1) is violated.

8. GHZ-Bell 3

The argument utilizes the premises of (Bell 3) to derive a testable equation in the framework of a quadruple system described above. Like (Bell 3), it starts with replacing determinism by stochastic dependence, however, in the presence of the strict correlations, this stochastic dependence of outcomes on parameters turns into the deterministic dependence.[25] With A, B, C and D standing as before for outcomes of measurements performed on the respective particles, let R'_{ABCD} designate the common past of these outcome-events and R'_A, R'_B, R'_C and R'_D stand for those regions of the backward light cones of A, B, C and D, respectively that sheds the respective outcomes from influences of a physical situation in R'_{ABCD} (see Fig. 3). The possible physical situations in $R'_A + R'_B + R'_C + R'_D + R'_{ABCD}$ are described by set Λ of parameters and again we do not specify what sort of mathematical objects $\lambda \in \Lambda$ can stand for. We will use two familiar assumptions: a version of local causality (or the common cause principle) and the deep locality, which in this setup take the following form:

[23] Horne M. A., Shimony A., Zeilinger A., *Phys. Rev. Lett.* 62 (1989), 2209.

[24] Rarity J. D., Tapster P. R., *Phys. Rev. Lett.* 64 (1990), 2495.

[25] d'Espagnat B., *Veiled Reality*, Adison-Wesley, Reading Mass., 1995.

LC $\quad P(1iA|2jB;3kC;4lD;1i;2j;3k;4l;\lambda) = P(1iA|1i;2j;3k;4l;\lambda),$

DL $\quad P(1iA|1i;2j;3k;4l;\lambda) = P(1iA|1i;\lambda).$

This notation is an extension of the notation introduced in section 5, two outcomes at two wings L and R of the setup being replaced by four outcomes at four wings (1, 2, 3 and 4) of the setup, and symbols of outcomes and directions being changed appropriately. In order to have the full statement of LC and DL, the above conditions should be rewritten for the remaining regions.

We will now prove that strict correlations with DL and LC entail the deterministic dependence of outcomes on parameters. Note first that LC, DL and HA authorize the following factorization of probabilities:

$$\begin{aligned} & P(1iA \,\&\, 2jB \,\&\, 3kC \,\&\, 4lD|1i;2j;3k;4l;\lambda) \\ & \quad = \quad P(1iA|1i;\lambda)P(2jB \,\&\, 3kC \,\&\, 4lD|2j;3k;4l;\lambda). \end{aligned}$$

Recalling the quantum mechanical expectation value for the joint measurement of spin projections in the system considered,

$$E(i,j,k,l) = 1 \quad \text{for} \quad \alpha + \beta - \gamma - \delta = \pi$$

the probability of obtaining a set of outcomes with an odd number of -1 outcomes is 0.

Consider now the following factors:

$$\begin{aligned} p_\pm &= P(1i \pm |1i;\lambda), \\ q_\pm &= P(2j \pm \,\&\, 3k \pm \,\&\, 4l \pm |2j;3k;4l;\lambda) \\ &\quad + P(2j \pm \,\&\, 3k \mp \,\&\, 4l \mp |2j;3k;4l;\lambda) \\ &\quad + P(2j \mp \,\&\, 3k \mp \,\&\, 4l \pm |2j;3k;4l;\lambda) \\ &\quad + P(2j \mp \,\&\, 3k \pm \,\&\, 4l \mp |2j;3k;4l;\lambda). \end{aligned}$$

Note that the expression $p_+q_- + p_-q_+$ captures probabilities of all set of outcomes with an odd number of -1 outcomes. Thus, on the assumption of hidden autonomy:

$$0 = \int_\Lambda [p_+q_- + p_-q_+]\sigma(\lambda)\,d\lambda\,.$$

Since factors p_+, q_-, p_- and q_+ are non-negative and $p_+ + p_- = 1 = q_- + q_+$, we obtain

$$p_- = 0,1 \qquad p_+ = 0,1 \qquad q_- = 0,1 \qquad p_+ = 0,1\,.$$

Repeating the argument for the remaining three combinations, we arrive at the deterministic result:

$$P(1iA|1i;\lambda) = 0,1 \qquad P(2jB|2j;\lambda) = 0,1$$
$$P(3kC|3k;\lambda) = 0,1 \qquad P(4lD|4l;\lambda) = 0,1\,.$$

Given the determinism of outcomes, the rest of the proof goes like the proof of GHZ-(Bell 1).

9. Conclusions

In a sense, charting the labyrinth of Bell-type theorems is like charting a geographical map: new geographical facts are still emerging and proofs of new Bell-type theorems are likely to come; a map can always include more details and a survey of Bell-type theorems can always be improved by including still another versions of proofs that people have come up with. Thus, both the tasks must be brought to an end somehow arbitrarily. Best to our knowledge, the set of the six arguments presented here is exhaustive in the sense that at present there are no proofs of Bell-type theorems that draw on different sets of premises than were used above. This is not to say that we described all known versions of the proofs. To give an example, we did not even mention the three-particles variants of GHZ proofs.[26]

Basically, the theorems split into four categories, generated by two dividing lines, the first being the distinction between deterministic or merely statistical impact of the parameters on outcomes. The second distinction is whether or not an argument assumes the strict (anti-)correlations.

Granted that a given premise slightly changes the meaning from argument to argument (and relying on the reader's ability to work out the differences), there are essentially four sorts of premises:

C-1 The causality conditions that put a constraint on the working of parameters. These are: the condition of the deterministic dependence of outcomes on parameters of (Bell 1), (Bell 2), (Bell 1-GHZ), the principle CC of the common cause of (Bell 3), the local causality LC of (Bell 4) and (Bell 3-GHZ).

C-2 The locality conditions requiring that, given a value of the parameter, a setting in a spatially separated region has no influence on the outcome observed.

[26] See Mermin N. D, "Quantum mysteries revisited", *Am. J. Phys.*, August 1990, p. 731–734.

C-3 The requirement of hidden autonomy (HA) or non-contextuality, as it is sometimes called.

C-4 Finally, (Bell 1)–(Bell 4) rely on classical (Kolmogorovian) probability calculus.

As was said before, none of the sets of the assumptions is hold sacrosanct. Making a not too impartial assessment, it is rather unlikely that assumptions C-2 or C-3 went wrong. Since I am not too resistant to the project of generalizing the probability calculus, I might try to put the blame on probability. But this is not going to work, since GHZ arguments are not in this sense probabilistic.[27] Thus, something must be wrong with our way of thinking about the functioning of parameters. But the possible reasons of why conditions C-1 must go point in opposite directions. One possibility is that we put too much determinism in these conditions and make the causes too deterministic, meaning the world is more chancy than LC or CC permit. The other option is the claim that conditions C-1 fail because they capture only a part of the truth. In this vein, an outcome is partially determined by the parameter, the other part of the determination being provided by a non-local influence of a spatially separated region. Similarly, local causality fails because besides local causes there are also some non-local influences.

Be that as it may, both the options leave us with the hard task to understand how non-local causation may work.

Acknowledgments. Supported by KBN grant 191/H01/97/12 and Furgal Fund.

[27] GHZ are probabilistic in respect to their testing; no probability calculus is needed to derive them, however.

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