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ADMISSIBILITY OF ACKERMANN'S RULE δ IN RELEVANT LOGICS

Abstract. It is proved that Ackermann's rule δ is admissible in a wide spectrum of relevant logics satisfying certain syntactical properties.

Keywords: admissible rules; rule δ; rule assertion; substructural logics; relevant logics.

1. Introduction

As it is known, Ackermann's logics Π and Π' are the origin of relevant logics (see [1, 2, 3]). One of the rules of both Π and Π' is the rule Ackermann labels δ, which reads:

δ. From B and $A \to (B \to C)$ to infer $A \to C$
"den Schluß von B und $A \to (B \to C)$ auf $A \to C$"; cf. [1], p. 119).

After a strong criticism of δ (cf. [2], §8.2), Anderson and Belnap's conclusion is (cf. [3], §45.2) that "δ can be dispensed with" the "same effect" (as that produced by δ) being obtained in the Logic of Entailment E by adding the axiom:

a_0. $[[(A \to A) \wedge (B \to B)] \to C] \to C$

The aim of this paper is not to discuss Anderson and Belnap's criticism of δ, but to provide a basis for the affirmation that the axiom a_0 "produces the same effect" as δ. More specifically, the aim of this paper is to prove that δ is admissible in a wide spectrum Σ of relevant logics (the logic E being one of them) having a_0 as an axiom. It certainly

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follows from this fact that a_0 and δ "produce the same effect" in any logic belonging to the spectrum Σ.

I have found no proof of this fact in the two volumes of "Entailment" ([2] and [3]), but it easily follows from the sketched proof (cf. [2], p. 236) of the following theorem:

if A is a theorem of E, then $\Box A$ is a theorem of E,

where $\Box$ is a necessity operator.

I will generalize this sketched proof in order to show that the rule Assertion:[1]

(Asser) $\quad A \;/\; (A \to B) \to B$

is admissible in any logic in Σ, regardless of the fact that it is derivable in none.

It can easily be shown that δ and (Asser) are equivalent in almost any logic. Consequently, δ is admissible in any logic S belonging to Σ.

Commenting on Gentzen's system for intuitionistic implication, $LJ_{\to}$, Anderson and Belnap remark ((ER) abbreviates "Elimination Rule"):

> We think of $LJ_{\to}$ as a system which never had (ER) at all, and then show that (ER) holds for the system anyway. [2], p. 53

Then, they go on to remark that the expression "holds for the system anyway" can be understood as referring either to a "derivable rule" or else to an "admissible rule" (cf. §2 below).

Similar considerations are applicable in the present case: any logic S in Σ never had δ at all, but δ holds for S anyway. In particular E does not have δ, but nevertheless δ holds for E.

Before describing the structure of the paper, we first make a couple of remarks. Let us say that we cannot pause here to discuss Ackermann's interesting remarks concerning the use of δ in theories with non-logical axioms (cf. [1], pp. 119–120, 125–126). Secondly, let us briefly comment on the results in [7]. In this paper it is shown that (Asser) is admissible in a series of logics including contractionless positive logic of entailment E plus a22, a28 and a29 (cf. §5 below). The proof in [7] is based upon the ternary relational semantics defined in [10]. Moreover, the logics in which (Asser) is admissible are supposed to be sound and complete in respect of a certain class of models definable in the said semantics. The results in the present paper much improve those in [7] in two respects:

[1] In general: "$A_1, \dots A_n \;/\; B$" stands for "from $A_1, \dots, A_n$ to infer B".

1. They do not depend on any particular semantics and so, neither on the possible soundness or completeness of the logics.
2. The spectrum of logics in which (Asser) is admissible is here considerably wider than that in [7].

The distinction between derivable and admissible rules is central to the paper. So, in Section 2, we recall a series of well-known definitions previous to those of "derivable" and "admissible" rule in order to establish the terminology clearly. The equivalence of δ and (Asser) is proved at the end of the section. Given this equivalence, we shall from then on concentrate on (Asser), which presents a structure simpler than that of δ. In Section 3, the basic positive logic where δ is admissible, $\mathrm{B_{a+}}$, is defined. Then, some derivable rules of B_{a+} are proved. In Section 4, these rules are used to show that (Asser) is admissible in $\mathrm{B_{a+}}$ and a wealth of its extensions. In Section 5, some of these extensions are exemplified. Anderson and Belnap remark that the fact stated in their theorem cited above is "a lucky accident", the reason being that the proof depends on a particular structure of the axioms and rules of E. And so is it the case with the proof of the admissibility of (Asser) in Section 4. In this sense, in Section 6, two natural logics with $\mathrm{a_0}$ as one of the axioms, a sublogic and an extension of E in which (Asser) is not admissible are defined. Finally, in Section 7, the admissibility of (Asser) and that of rule K in Lewis' S3, S4 and S5 are briefly discussed.

2. Equivalence of (Asser) and δ

The logics to be considered in this paper are propositional logics formulated in the Hilbert-style way. We shall begin by defining the concept of a (propositional) logic (cf. [10], p. 286).

Definition 1 (Propositional logic). A *propositional logic S* is a quadruple ($\mathcal{L}$, $\mathcal{A}$, $\mathcal{R}$, $\mathcal{T}$) where (i) $\mathcal{L}$ is a propositional language (ii) $\mathcal{A}$ is an effectively specified set of axioms (or axiom schemes) (iii) $\mathcal{R}$ is an effectively specified set of (primitive) rules of derivation, and (iv) $\mathcal{T}$ is the set of theorems derivable from $\mathcal{A}$ using zero or more applications of rules from $\mathcal{R}$.

More precisely "proof" and "theorem" are defined as follows.

Definition 2 (Proof). A *proof* is a sequence $B_1, \ldots, B_n$ of wff of $\mathcal{L}$ such that for each B_i, i $(1 \leqslant i \leqslant n)$, either (1) $B_i \in \mathcal{A}$ or else (2) B_i is the result of applying one of the rules of $\mathcal{R}$ to one or more previous wff in the sequence.

Definition 3 (Theorem). A is a *theorem* of S ($\vdash_S A$) iff there is a proof $\langle B_1, \ldots, B_n \rangle$ and A is B_i $(1 \leqslant i \leqslant n)$.

So, $\mathcal{A}$ is a distinguished set of wff of $\mathcal{L}$ (or a distinguished set of schemes of wff) and $\mathcal{T}$ is the smallest class of wff containing $\mathcal{A}$ and closed under $\mathcal{R}$. From now on, by a "logic" we refer to a propositional logic.

We now need the concepts of "derivable rule" and "admissible rule". We follow Anderson and Belnap in [2].

Definition 4 (Derivable rule). "A rule from $A_1, \ldots, A_n$ to infer B is derivable when it is possible to proceed from the premises to the conclusion with the help of axioms and primitive rules alone" ([2], pp. 53–54).

Definition 5 (Admissible rule). A rule from $A_1, \ldots, A_n$ to infer B is *admissible* if "whenever there is a proof of the premises, there is a proof of the conclusion". ([2], p. 54).

Of course, in any propositional logic S, every derivable rule in S is admissible in S.

Next, a sufficient condition for a logic to be included in another one is defined.

Definition 6 (Extension of a logic). Let $S = (\mathcal{L}, \mathcal{A}, \mathcal{R}, \mathcal{T})$ and $S' = (\mathcal{L}', \mathcal{A}', \mathcal{R}', \mathcal{T}')$ be logics. Then, S' is an *extension* of S, if $\mathcal{L} \subseteq \mathcal{L}'$, $\mathcal{A} \subseteq \mathcal{T}'$ and any rule from $\mathcal{R}$ is derivable in S'. Then $\mathcal{T} \subseteq \mathcal{T}'$ and all derivable rules in S are derivable in S'.

The rest of this section is dedicated to prove that (Asser) and δ are equivalent in almost any logic. I shall begin by defining a very weak implicative logic. But before doing that we recall the notion of an "implicative formula", which is central in the development of this paper.

Definition 7 (Implicative formula). Let $\mathcal{L}$ be a propositional language in which $\rightarrow$ (conditional) is one of the connectives. Then, A is an *implicative formula* iff A is of the form $B \rightarrow C$, where B and C are wff.

Then, an implicative logic is a logic with $\to$ as the sole connective. That is, a logic in which the members in the sets $\mathcal{A}$ and $\mathcal{T}$ are exclusively implicative formulas. The weak logic referred to above is defined as follows.

DEFINITION 8 (The logic $\mathrm{S}_{\mathrm{a}\delta\mathrm{m}}$). The propositional language of $\mathrm{S}_{\mathrm{a}\delta\mathrm{m}}$ consists of a set of propositional variables and $\to$ as the sole connective. The set of wff is defined in the customary way. The axiom (actually, axiom scheme) and rule are the following:

Axiom A1. $\quad A \to A$

Rule Transitivity (Trans). $\quad A \to B,\ B \to C \ /\ A \to C$

The label "$\mathrm{S}_{\mathrm{a}\delta\mathrm{m}}$" stands for the minimal (m) logic in which δ and Assertion (a) are equivalent.

According to definitions 1, 2 and 3, the (primitive) rules are instructions that can be applied to wff appearing in proofs in order to obtain other wff.

Now, for every extension of $\mathrm{S}_{\mathrm{a}\delta\mathrm{m}}$ (cf. Definition 6), it is proved that:

PROPOSITION 1. *Let* $\mathrm{ES}_{\mathrm{a}\delta\mathrm{m}}$ *be an extension of* $\mathrm{S}_{\mathrm{a}\delta\mathrm{m}}$. *Then,* (Asser) *is derivable* (*resp. admissible*) *in* $\mathrm{ES}_{\mathrm{a}\delta\mathrm{m}}$ *iff* δ *is derivable* (*resp. admissible*) *in* $\mathrm{ES}_{\mathrm{a}\delta\mathrm{m}}$.

PROOF. (a) "$\Rightarrow$" Suppose that (Asser) is derivable in $\mathrm{ES}_{\mathrm{a}\delta\mathrm{m}}$. Then:

1. B	Hyp.
2. $A \to (B \to C)$	Hyp.
3. $(B \to C) \to C$	from 1 by (Asser)
4. $A \to C$	from 2 and 3, by (Trans)

"$\Leftarrow$" Suppose that δ is derivable in $\mathrm{ES}_{\mathrm{a}\delta\mathrm{m}}$. Then:

1. A	Hyp.
2. $(A \to B) \to (A \to B)$	A1
3. $(A \to B) \to B$	from 1 and 2, by δ

(b) A proof for admissibility of these rules in $\mathrm{ES}_{\mathrm{a}\delta\mathrm{m}}$ is analogous. $\dashv$

Remark 1. Notice that if (Asser) (resp. δ) is admissible in $\mathrm{ES}_{\mathrm{a}\delta\mathrm{m}}$, it does not follow from Proposition 1 that δ (resp. (Asser)) is derivable in $\mathrm{ES}_{\mathrm{a}\delta\mathrm{m}}$.

Finally, we remark that, given that (Asser) and δ are equivalent in any logic which is an extension of $\mathrm{S}_{\mathrm{a}\delta\mathrm{m}}$, we shall concentrate from now on the rule (Asser), whose structure is simpler than that of δ.

3. The logic B_{a+}

In this section, the logic B_{a+} is defined. The label "B_{a+}" stands for: "Basic positive logic in which rule (Asser) is admissible. Then, some theorems and derivable rules of B_{a+} are proved.

DEFINITION 9 (The logic B_{a+}). The propositional language of B_{a+} consists of a denumerable set of propositional variables, and the connectives $\rightarrow$ (conditional) and $\wedge$ (conjunction). The set of wff is defined in the customary way. The axioms (actually, axioms schemes) and rules are the following.
Axioms:
A1. $A \rightarrow A$
A2. $(A \wedge B) \rightarrow A$
A3. $(A \wedge B) \rightarrow B$
A4. $[[(A \rightarrow A) \wedge (B \rightarrow B)] \rightarrow C] \rightarrow C$

Rules:
Modus ponens (MP). $A,\ A \rightarrow B\ /\ B$
Adjunction (Adj). $A,\ B\ /\ A \wedge B$
Suffixing (Suf). $A \rightarrow B\ /\ (B \rightarrow C) \rightarrow (A \rightarrow C)$
Conditional Adjunction (cAdj). $A \rightarrow B,\ A \rightarrow C\ /\ A \rightarrow (B \wedge C)$

By (Suf) and (MP), we obtain:

PROPOSITION 2. *The rule* (Trans) *is derivable in* B_{a+}. *So* B_{a+} *is an extension of* $S_{a\delta\mathrm{m}}$

PROOF. 1. $A \rightarrow B$ Hyp.
2. $B \rightarrow C$ Hyp.
3. $(B \rightarrow C) \rightarrow (A \rightarrow C)$ from 1 by (Suf)
4. $A \rightarrow C$ from 2 and 3, by (MP) ⊣

The following theorem of B_{a+} and derivable rules in B_{a+} are used in proving that (Asser) (and so, δ) is admissible in B_{a+} and a wealth of its extensions.

T1. $[(A \rightarrow A) \rightarrow B] \rightarrow B$

PROOF. By A2 and two applications of (Suf)
1. $\{[[(A \rightarrow A) \wedge (A \rightarrow A)] \rightarrow B] \rightarrow B\} \rightarrow [[(A \rightarrow A) \rightarrow B] \rightarrow B]$.

Then T1 follows from 1, by A4 and (MP). ⊣

LEMMA 1. *The following rules are derivable in* B_{a+}:

R1. $A \to B$ / $[(A \to B) \to C] \to C$
R2. $A \to C,\ B \to D$ / $(A \wedge B) \to (C \wedge D)$
R3. $(A \to A) \to A,\ [(A \to B) \to (A \to B)] \to (A \to B)$ / $(B \to B) \to B$
R4. $(A \to A) \to A,\ (B \to B) \to B$ / $[(A \wedge B) \to (A \wedge B)] \to (A \wedge B)$

PROOF. For R1:

1. $A \to B$ — Hyp.
2. $[[(B \to B) \to C] \to C] \to [[(A \to B) \to C] \to C]$ — from 1, by three applications of (Suf)
3. $[(B \to B) \to C] \to C$ — T1
4. $[(A \to B) \to C] \to C$ — from 2 and 4, by (MP)

For R2:

1. $A \to C$ — Hyp.
2. $B \to D$ — Hyp.
3. $(A \wedge B) \to A$ — A2
4. $(A \wedge B) \to C$ — from 1 and 3, by (Trans)
5. $(A \wedge B) \to B$ — A3
6. $(A \wedge B) \to D$ — from 2 and 5, by (Trans)
7. $(A \wedge B) \to (C \wedge D)$ — from 4 and 5, by (cAdj)

For R3:

1. $(A \to A) \to A$ — Hyp.
2. $[(A \to B) \to (A \to B)] \to (A \to B)$ — Hyp.
3. $(A \to B) \to (A \to B)$ — A1
4. $A \to B$ — from 2 and 3, by (MP)
5. $(B \to B) \to (A \to B)$ — from 4 by (Suf)
6. $(A \to B) \to [(A \to A) \to B]$ — from 1 by (Suf)
7. $[(A \to A) \to B] \to B$ — T1
8. $(A \to B) \to B$ — from 6 and 7, by (Trans)
9. $(B \to B) \to B$ — from 5 and 8, by (Trans)

For R4:

1. $(A \to A) \to A$ — Hyp.
2. $(B \to B) \to B$ — Hyp.
3. $[(A \to A) \wedge (B \to B)] \to (A \wedge B)$ — from 1 and 2, by R2
4. $\{[[(A \to A) \wedge (B \to B)] \to (A \wedge B)] \to (A \wedge B)\} \to$ $[[(A \wedge B) \to (A \wedge B)] \to (A \wedge B)]$ — from 3 by two applications of (Suf)

5. $[[(A \to A) \wedge (B \to B)] \to (A \wedge B)] \to (A \wedge B)$ A4
6. $[(A \wedge B) \to (A \wedge B)] \to (A \wedge B)$ from 4 and 5, by (MP) $\dashv$

4. Admissibility of (Asser)

Let L be a propositional language containing the conditional connective. As in [2], the necessity operator can be defined as follows:

Definition 10 (The necessity operator). For any wff A of L we put:

$$\Box A =_{\mathrm{df}} (A \to A) \to A$$

($\Box$ is the *necessity operator*).

Now, we consider any extension $\mathrm{EB_{a+}}$ of $\mathrm{B_{a+}}$. We set:

Definition 11 (Rule preserving necessity). Let $R = \{(A_1, \dots, A_n; B) : A_1, \dots, A_n, B \in L\}$ be a (primitive or derivable) rule of $\mathrm{EB_{a+}}$. It is said that R *preserves necessity* in $\mathrm{EB_{a+}}$, if for any $A_1, \dots, A_n, B \in L$: if $A_1, \dots, A_n \ / \ B$ and $\vdash_{\mathrm{EB_{a+}}} \Box A_1$, $\vdash_{\mathrm{EB_{a+}}} \Box A_2$, $\dots$, $\vdash_{\mathrm{EB_{a+}}} \Box A_n$, then $\vdash_{\mathrm{EB_{a+}}} \Box B$.

Remark 2. Concerning the proofs to follow, we remark that any theorem (resp. derivable rule) of $\mathrm{B_{a+}}$ is also a theorem (resp. derivable rule) of any extension $\mathrm{EB_{a+}}$ of $\mathrm{B_{a+}}$. This is not, of course, necessarily the case for admissible rules of $\mathrm{B_{a+}}$ in general.

Now notice that, respectively by A1, (MP), R1, R3 and R4 (see Lemma 1), we obtain:

Lemma 2. *Let* $\mathrm{EB_{a+}}$ *be an extension of* $\mathrm{B_{a+}}$. *Then the following rules*

$$\Box A \ / \ A$$
$$A \to B \ / \ \Box(A \to B)$$
$$\Box A,\ \Box(A \to B) \ / \ \Box B$$
$$\Box A,\ \Box B \ / \ \Box(A \wedge B)$$

are derivable in $\mathrm{EB_{a+}}$. *Consequently, the rules* (MP) *and* (Adj) *preserve necessity in* $\mathrm{EB_{a+}}$.

We say that a rule is *implicative*, if all conclusions of this rule are implicative formulas. For these rule we obtain:

LEMMA 3. *Let* $\mathrm{EB_{a+}}$ *be an extension of* $\mathrm{B_{a+}}$. *Then every admissible implicative rule of* $\mathrm{EB_{a+}}$ *preserve necessity in* $\mathrm{EB_{a+}}$.

PROOF. Let R be an admissible implicative rule of $\mathrm{EB_{a+}}$, $(A_1, \dots A_n/ B \to C)$ belong to R and $\vdash_{\mathrm{EB_{a+}}} \Box A_1$, $\dots$, $\vdash_{\mathrm{EB_{a+}}} \Box A_n$. Then, by Lemma 2, $\vdash_{\mathrm{EB_{a+}}} A_n$, $\dots$, $\vdash_{\mathrm{EB_{a+}}} A_n$. Hence $\vdash_{\mathrm{EB_{a+}}} B \to C$. And finally, $\vdash_{\mathrm{EB_{a+}}} \Box(B \to C)$, by Lemma 2. ⊣

Consequently, it is obtained:

COROLLARY 1. *Let* $\mathrm{EB_{a+}}$ *be an extension of* $\mathrm{B_{a+}}$. *Then the rules* (Suf) *and* (cAdj) *preserve necessity in* $\mathrm{EB_{a+}}$.

LEMMA 4. *Let* $\mathrm{EB_{a+}}$ *be an extension of* $\mathrm{B_{a+}}$ *in which all axioms are implicative formulas and all primitive rules* (*except* (MP) *and* (Adj)) *are implicative. Then the following rule "demodalizer"*

dm. $\quad A \,/\, \Box A \qquad$ (*i.e.* $A \,/\, (A \to A) \to A$)

is admissible in $\mathrm{EB_{a+}}$,[2] *i.e., for any formula* A,

$$\text{if } \vdash_{\mathrm{EB_{a+}}} A \text{ then } \vdash_{\mathrm{EB_{a+}}} \Box A.$$

PROOF. Let $\vdash_{\mathrm{EB_{a+}}} A$. Induction on the length of the proof of A. 1. A is an axiom. So A is of the form $B \to C$ and $\vdash_{\mathrm{EB_{a+}}} \Box(B \to C)$, by Lemma 2 (R1). 2. A has been derived by some rule. Then $\vdash_{\mathrm{EB_{a+}}} \Box A$, by inductive hypothesis and lemmas 2 and 3. ⊣

Finally, it is proved:

THEOREM 1. *Let* $\mathrm{EB_{a+}}$ *be an extension of* $\mathrm{B_{a+}}$ *in which all axioms are implicative formulas and all primitive rules* (*except* (MP) *and* (Adj)) *are implicative. Then* (Asser) *is admissible in* $\mathrm{EB_{a+}}$.

PROOF. Let $\vdash_{\mathrm{EB_{a+}}} A$. Then $\vdash_{\mathrm{EB_{a+}}} (A \to A) \to A$, by Lemma 4. Hence, by (Suf), $\vdash_{\mathrm{EB_{a+}}} (A \to B) \to [(A \to A) \to B]$. Whence, by T1 and (Trans), $\vdash_{\mathrm{EB_{a+}}} (A \to B) \to B$ as was to be proved. ⊣

Immediate by propositions 1 and 2, and Theorem 1 we obtain:

COROLLARY 2. *Let* $\mathrm{EB_{a+}}$ *be an extension of* $\mathrm{B_{a+}}$ *of the sort described in Theorem 1. Then rule* δ *is admissible in* $\mathrm{EB_{a+}}$.

COROLLARY 3. (Asser), *and so,* δ *are admissible in* $\mathrm{B_{a+}}$.

[2] In the next section we show that the rule (dm) is not derivable in $\mathrm{EB_{a+}}$. Consequently, also (Asser) and δ are not derivable in $\mathrm{EB_{a+}}$ (see Proposition 4).

5. Some extensions of $\mathrm{B_{a+}}$ in which (Asser) is admissible

We shall describe a wealth — but we do not try to be exhaustive — of extensions of $\mathrm{B_{a+}}$ in which (Asser), though not derivable, is admissible. As it follows from Corollary 1, in any of these extensions the rule δ is admissible as well.

Consider the following axiom schemes and rules of derivation:

a1. $(B \to C) \to [(A \to B) \to (A \to C)]$
a2. $(A \to B) \to [(B \to C) \to (A \to C)]$
a3. $[A \to (A \to B)] \to (A \to B)$
a4. $[A \to (B \to C)] \to [(A \to B) \to (A \to C)]$
a5. $(A \to B) \to [[A \to (B \to C)] \to (A \to C)]$
a6. $(A \to B) \to [(A \to B) \to (A \to B)]$
a7. $[(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)]$
a8. $[(A \to B) \wedge (B \to C)] \to (A \to C)$
a9. $[A \wedge (A \to B)] \to B$
a10. $[A \to (B \to C)] \to [(A \wedge B) \to C]$
a11. $A \to (A \vee B)$
a12. $B \to (A \vee B)$
a13. $[(A \to C) \wedge (B \to C)] \to [(A \vee B) \to C]$
a14. $[A \wedge (B \vee C)] \to [(A \wedge B) \vee (A \wedge C)]$
a15. $A \to \neg\neg A$
a16. $\neg\neg A \to A$
a17. $(A \to \neg A) \to \neg A$
a18. $(\neg A \to A) \to A$
a19. $(A \to B) \to [(A \to \neg B) \to \neg A]$
a20. $(\neg A \to B) \to [(\neg A \to \neg B) \to A]$
a21. $(\neg A \to B) \to [(A \to B) \to B]$
a22. $(A \to B) \to (\neg B \to \neg A)$
a23. $(A \to \neg B) \to (B \to \neg A)$
a24. $(\neg A \to B) \to (\neg B \to A)$
a25. $(\neg A \to \neg B) \to (B \to A)$
a26. $(A \to B) \to \neg(A \wedge \neg B)$
a27. $(A \to B) \to (\neg A \vee B)$
a28. $\neg(A \wedge B) \to (\neg A \vee \neg B)$
a29. $(\neg A \wedge \neg B) \to \neg(A \vee B)$
a30. $(A \to B) \to [(A \wedge C) \to (B \wedge C)]$
a31. $(A \to B) \to [(A \vee C) \to (B \vee C)]$
a32. $(A \to B) \to [(A \to C) \to [A \to (B \wedge C)]]$

$\to$	0	1	2	3	$\neg$
0	2	2	2	2	3
1*	0	2	2	2	2
2*	0	0	2	2	1
3*	0	0	0	2	0

$\wedge$	0	1	2	3
0	0	0	0	0
1*	0	1	1	1
2*	0	1	2	2
3*	0	1	2	3

$\vee$	0	1	2	3
0	0	1	2	3
1*	1	1	2	3
2*	2	2	2	3
3*	3	3	3	3

Table 1. The matrix MI

a33. $(A \to C) \to [(B \to C) \to [(A \vee B) \to C]]$
a34. $(B \to A) \to (A \to A)$
a35. $(A \to B) \to (A \to A)$
a36. $[[(A \to B) \to C] \to (A \to B)] \to (A \to B)$
a37. $(B \to C) \to (A \to A)$
a38. $[(A \wedge B) \to C] \to [(A \to C) \vee (B \to C)]$
a39. $[A \to (B \vee C)] \to [(A \to B) \vee (A \to C)]$

r1. $B \to C \ / \ (A \to B) \to (A \to C)$
r2. $A \to (A \to B) \ / \ A \to B$
r3. $A \to B \ / \ (A \wedge C) \to (B \wedge C)$
r4. $A \to B \ / \ (A \vee C) \to (B \vee C)$
r5. $A \to (B \to C) \ / \ (A \wedge C) \to B$
r6. $A \to C, \ B \to C \ / \ (A \vee B) \to C$
r7. $A \to B \ / \ \neg B \to \neg A$
r8. $A \to \neg B \ / \ B \to \neg A$
r9. $\neg A \to B \ / \ \neg B \to A$
r10. $\neg A \to \neg B \ / \ B \to A$
r11. $A \to B \ / \ (A \to \neg B) \to \neg A$
r12. $\neg A \to B \ / \ (\neg A \to \neg B) \to A$
r13. $\neg A \to B \ / \ (A \to B) \to B$

Consider now the logical matrix MI from Table 1, where all values but 0 are designated.[3] We have:

PROPOSITION 3. *Let* $\mathrm{EB}_{\mathrm{a}+}$ *be any extension of* $\mathrm{B}_{\mathrm{a}+}$ *formulated with any selection of* a1–a39 *and* r1–r13*. Then* $\mathrm{EB}_{\mathrm{a}+}$ *is verified by* MI.

PROOF. Left to the reader. In addition to the axioms and rules of $\mathrm{B}_{\mathrm{a}+}$, MI satisfy a1–a39 and r1–r13. ⊣

But this same set can be used to show that (Asser) is not derivable in any of the logics verified by MI.

[3] On the concept of logical matrix and related concepts see, e.g., [8].

Proposition 4. *Let* $\mathrm{EB_{a+}}$ *be any extension of* $\mathrm{B_{a+}}$ *formulated as in Proposition 3. Then the rule* (dm) *is not derivable in* $\mathrm{EB_{a+}}$. *In consequence, also the rule* (Asser) *is not derivable in* $\mathrm{EB_{a+}}$.

Proof. By Proposition 3, $\mathrm{EB_{a+}}$ is verified by MI, but (Asser) is falsified, when $v(A) = v(B) = 1$. Actually, MI falsifies the rule (dm), when $v(A) = 1$. (Notice that δ is of course falsified: when $v(A) = 2$, $v(B) = v(C) = 1$). ⊣

Although (Asser) is not derivable in the logics described in Proposition 3, it is proved:

Proposition 5. *Let* $\mathrm{EB_{a+}}$ *be as in Proposition 3. Then* (Asser) (*and so* δ) *is admissible in* $\mathrm{EB_{a+}}$.

Proof. Given the structure of a1–a39 and r1–r13, Proposition 5 follows by Theorem 1. ⊣

Let us next have a look at some relevant logics derivable from A1–A4, a1–a39, (MP), (Adj), (Suf), (cAdj) and r1–r13. First of all, notice that a31–a39 are not derivable in the logic of relevant implication R; and that a31–a37 are not theorems of R-Mingle (see [1] about these logics). Then, Routley and Meyer's basic positive logic $\mathrm{B_+}$ (cf. [10]) is axiomatized with A1–A3, a7, a11, a12, a13, a14 and (MP), (Adj), (Suf) and r1. And Sylvan and Plumwood's $\mathrm{B_M}$ (cf. [11]; $\mathrm{B_M}$ is the basic logic in Routley-Meyer's ternary relational semantics) is formulated by adding a28, a29 and r7 to $\mathrm{B_+}$. Now, let $\mathrm{BA4_+}$ and $\mathrm{BA4_M}$ be the result of adding A4 to $\mathrm{B_+}$ and $\mathrm{B_M}$, respectively. Then, as we know by Proposition 5, (Asser) is admissible in $\mathrm{BA4_+}$ and $\mathrm{BA4_M}$. Moreover, (Asser) is admissible in any extension of $\mathrm{BA4_+}$ or $\mathrm{BA4_M}$ formulable with any selection of a1–a39 and r1–r13. The logic of Entailment E or the logic E-Mingle, EM, are among theses extensions (cf. [2, 3] about these logics). Leaving aside questions of independence, the logic E can be axiomatized by adding to $\mathrm{BA4_+}$ a2, a3, a17, a23 and a24. And the logic EM is the result of adding a6 to E.

Finally, we remark that it is possible to define a number of relevant logics not included in R-Mingle, RM, and in which (Asser) is admissible, but not derivable. As is known, axioms a34, a35 and a36 are not provable in RM, but as shown in [9], they can be used for axiomatizing logics with the variable-sharing property, i.e., relevant logics in the minimal sense of the term.

$\rightarrow$	0	1	2	3	$\neg$
0	3	3	3	3	3
1*	0	3	3	3	2
2*	0	0	3	3	1
3*	0	0	0	3	0

$\wedge$	0	1	2	3
0	0	0	0	0
1*	0	1	1	1
2*	0	1	2	2
3*	0	1	2	3

$\vee$	0	1	2	3
0	0	1	2	3
1*	1	1	2	3
2*	2	2	2	3
3*	3	3	3	3

Table 2. The matrix MII

6. Two natural extensions of $\mathrm{B}_{\mathrm{a}+}$ in which (Asser) is not admissible

Concerning their theorem that all theorems of E are probably necessary, Anderson and Belnap remark that this fact is a "lucky accident" ([1], p. 236). And indeed it is. As it is the case in the proof of the admissibility of (Asser) provided above, that of Anderson and Belnap's theorem depends on a particular structure of the axioms and rules of derivation. This particular structure has to preserve necessity in the proofs. If any of the axioms and (or) rules of a given logic S does not conform to this structure, it may be the case that (Asser) is not admissible in S. We shall provide a couple of examples: a sublogic, on the one hand, and an extension, on the other, of the logic of entailment E.

Let us designate by E_{sr} the result of restricting a17 in the axiomatization of E given above to the rule of "specialized reductio"

sr. $\quad A \rightarrow \neg A \;/\; \neg A$

Consider now the logical matrix MII from Table 2 (where all values but 0 are designated): We have:

Proposition 6. *The matrix* MII *verifies* E_{sr}, *i.e., it satisfies all axioms and rules of* E_{sr}.

Proof. It is left to the reader. ⊣

Notice that the following formula (scheme):

nc. $\quad \neg(A \wedge \neg A)$

is a theorem of E_{sr}. Indeed:

1. $(A \wedge \neg A) \rightarrow A$ — A2
2. $(A \wedge \neg A) \rightarrow \neg A$ — A3
3. $[(A \wedge \neg A) \rightarrow \neg A] \rightarrow [A \rightarrow \neg(A \wedge \neg A)$ — a23
4. $A \rightarrow \neg(A \wedge \neg A)$ — from 2 and 3, by (MP)

5. $[(A \to \neg(A \wedge \neg A)] \to [(A \wedge \neg A) \to \neg(A \wedge \neg A)]$ from 1 by (Suf)
6. $(A \wedge \neg A) \to \neg(A \wedge \neg A)$ from 4 and 5, by (MP)
7. $\neg(A \wedge \neg A)$ from 6 by (sr)

PROPOSITION 7. *The rule* (Asser) *is not admissible in* $\mathrm{E_{sr}}$.

PROOF. $\neg(A \wedge \neg A)$ is a theorem of $\mathrm{E_{sr}}$, but $[\neg(A \wedge \neg A) \to B] \to B$ is not a theorem of $\mathrm{E_{sr}}$: it is falsified when $v(A) = 1$ and $v(B) = 2$ in the matrix MII. ⊣

On the other hand, consider the connexive logic that results from adding Aristotle's thesis

Aris. $\neg(A \to \neg A)$

to the logic E (cf. [4] and references there). We shall here refer by $\mathrm{E_{Aris}}$ to this extension of E. We have (cf. Proposition 3):

PROPOSITION 8. *The matrix* MI *verifies* $\mathrm{E_{Aris}}$.

PROOF. It is left to the reader. ⊣

And, finally,

PROPOSITION 9. *The rule* (Asser) *is not admissible in* $\mathrm{E_{Aris}}$.

PROOF. $[\neg(A \to \neg A) \to B] \to B$ is not a theorem of $\mathrm{E_{Aris}}$: it is falsified when $v(A) = 0$ and $v(B) = 1$. ⊣

7. Admissibility of (Asser) in Lewis' modal logics S3, S4 and S5

In some sense, Lewis' modal logics S3, S4 and S5 are, from a mere syntactical point of view (that of Definition 6), a natural extension of the spectrum of relevant logics whose more important points are B, TW, T, E and EM (see [10] about these logics). So, the aim of this section is to briefly discuss the admissibility of (Asser) and that of rule K in S3, S4 and S5.

Lewis' S3 can be axiomatized by adding a37 to E (cf. the axiomatization of E in Section 4 and [6]). Therefore, by Proposition 3, we have:

PROPOSITION 10. MI *verifies* S3.

And also, by Proposition 5, we obtain:

PROPOSITION 11. (Asser) *is admissible in* S3.

$\to$	0	1	2	3	$\neg$
0	3	3	3	3	3
1	0	3	0	3	2
2^*	0	0	3	3	1
3^*	0	0	0	3	0

$\wedge$	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2^*	0	0	2	2
3^*	0	1	2	3

$\vee$	0	1	2	3
0	0	1	2	3
1	1	1	3	3
2^*	2	3	2	3
3^*	3	3	3	3

Table 3. The matrix MIII

Actually, MI verifies some of the extensions of S3 in which (Asser) is admissible: a36, a38 and a39 are not theorems of S3. But, on the other hand, MI does not however verify Lewis' S4, which is the result of adding the axiom

a40. $B \to (A \to A)$

to E (cf. [6]); set $v(B) = 3$, $v(A) = 2$).

So, we shall provide a logical matrix verifying Lewis' logics S4 and S5 in order to show that, though not derivable, (Asser) is admissible in these logics. Consider the logical matrix MIII from Table 3, where 2 and 3 are designated values. We have:

PROPOSITION 12. MIII *verifies Lewis' logic* S5 (*and so,* S4).

PROOF. Left to the reader. In addition to B_{a+}, MIII verifies a1–a37, a40 and r1–r13. But, as pointed out above, S4 is axiomatized by adding a40 to E; and S5, by adding a36 to S4 (cf. [6]). ⊣

On the other hand, it is proved:

PROPOSITION 13. *Let S be a logic verified by* MIII. *Then* (Asser) *is not derivable in S* (*so,* (Asser) *is not derivable in* S5).

PROOF. Set $v(A) = v(B) = 2$. Then $v[(A \to B) \to B] = 0$. ⊣

But, given that a40 is an implicative formula, by Theorem 1, we have:

PROPOSITION 14. (Asser) *is admissible in* S4 *and* S5.

The section is ended with a couple of remarks on rule K. As it is known, rule K is the following

K. $A \ / \ B \to A$

We prove:

PROPOSITION 15. *Let* EB_{a+} *be a logic verified by* MI. *Then, the rule* K *is not admissible in* EB_{a+}.

PROOF. Notice that $A \to A$ is a theorem of $\mathrm{EB}_{\mathrm{a+}}$. But $B \to (A \to A)$ is not a theorem of $\mathrm{EB}_{\mathrm{a+}}$, since — as remarked above — it is not verified by MI. ⊣

So, K is admissible neither in S3 nor in any of its extensions verified by MI. Nevertheless, it is proved:

PROPOSITION 16. *Let* $\mathrm{EB}_{\mathrm{a+}}$ *be as in Theorem 1. Moreover, let* a40 *be an axiom (or theorem) of* $\mathrm{EB}_{\mathrm{a+}}$. *Then* K *is admissible in* $\mathrm{EB}_{\mathrm{a+}}$. *Therefore,* K *is admissible in* S4 *and* S5.

PROOF. Suppose that $\vdash_{\mathrm{EB}_{\mathrm{a+}}} A$. By Theorem 1, (Asser) is admissible in $\mathrm{EB}_{\mathrm{a+}}$. So, $\vdash_{\mathrm{EB}_{\mathrm{a+}}} (A \to A) \to A$. But, by a40, $\vdash_{\mathrm{EB}_{\mathrm{a+}}} B \to (A \to A)$. So, $\vdash_{\mathrm{EB}_{\mathrm{a+}}} B \to A$, by (Trans). ⊣

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