

J. W. Degen  
J. M. Werner

## TOWARDS INTUITIONISTIC DYNAMIC LOGIC

**Abstract.** We propose the beginnings of an intuitionistic propositional dynamic logic, and describe several serious open problems.

*Keywords:* existential property; propositional dynamic logic; intuitionistic modal logic; completeness problem; failure of negative interpretation

### 1. Introduction

A theory  $T$  satisfies the Existential Property (E.P.) if from  $T \vdash \exists x \mathcal{F}[x]$  we can always get a term  $t$  such that  $T \vdash \mathcal{F}[t]$ . This property is violated by theories with classical logic as (e.g.) the classical validity of  $\exists x [D(x) \rightarrow \forall y D(y)]$  and the nonvalidity of  $D(a) \rightarrow \forall y D(y)$  show. In dynamic logic, the construct  $\langle \pi^* \rangle \varphi$  means there is a natural number  $k$  with  $\langle \pi^k \rangle \varphi$ , where  $\pi^k$  is the  $k$ -th iterate of the program  $\pi$ . Therefore,  $\langle \pi^* \rangle \varphi$  behaves as an existentially quantified sentence. If the surrounding logic is classical, the pertinent E.P. fails. For example, the schema  $\langle \rho^* \rangle (\langle \rho^* \rangle \varphi \rightarrow \varphi)$  is valid although  $\langle q^k \rangle (\langle q^* \rangle a \rightarrow a)$  is valid for no natural number  $k$  (where  $q$  is an atomic program and  $a$  a propositional letter). Can we restore the E.P. for dynamic logic if we make the surrounding logic intuitionistic?

We sketch in the present paper intuitionistic propositional dynamic logic, *iPDL*; let us remark, that the first-order extension of *iPDL* is definitely beyond present-day methods.

We have a set of propositional letters (including  $\perp = \text{false}$ ), and a set of program letters (or atomic programs). As usual we define by simultaneous induction *formulae* and *programs*:

1. Every propositional letter [program letter] is a formula [a program].
2. If  $\varphi$  und  $\psi$  are formulae,  $\pi$  and  $\rho$  programs, then
  - $\varphi \wedge \psi$ ,  $\varphi \vee \psi$ ,  $\varphi \rightarrow \psi$ ,  $\langle \pi \rangle \varphi$  and  $[\pi] \varphi$  are formulae and
  - $\pi$ ;  $\rho$ ,  $\pi \cup \rho$ ,  $\pi^*$  and  $\varphi?$  are programs.

We define  $\neg\varphi$  by  $\varphi \rightarrow \perp$ .

An intuitionistic dynamic Kripke model  $\mathfrak{K}$  is a triple  $\mathfrak{K} = (W^{\mathfrak{K}}, \preceq^{\mathfrak{K}}, m^{\mathfrak{K}})$  where  $W^{\mathfrak{K}}$  is a (e.g. non-empty) set of states,  $\preceq^{\mathfrak{K}} \subseteq W^{\mathfrak{K}} \times W^{\mathfrak{K}}$  is reflexive and transitive;  $\preceq^{\mathfrak{K}}$  will be used for the intuitionistic interpretation of  $\rightarrow$ . We have  $m^{\mathfrak{K}}(p) \subseteq W^{\mathfrak{K}} \times W^{\mathfrak{K}}$  for every atomic program  $p$ . For all propositional letters  $a$  we require  $m^{\mathfrak{K}}(a) \subseteq W^{\mathfrak{K}}$  to be the set of states where  $a$  is *true* with the condition that for every two states  $u, v \in W^{\mathfrak{K}}$   $u \preceq^{\mathfrak{K}} v$  and  $u \in m^{\mathfrak{K}}(a)$  implies  $v \in m^{\mathfrak{K}}(a)$ . The extension of  $m^{\mathfrak{K}}$  to complex programs and fromulae is defined as expected (see definition 3). For the sake of simplicity we will not distinguish between  $m^{\mathfrak{K}}$  and its extension.

We have the small model property for iPDL: if  $\varphi$  is satisfiable, then it is satisfiable in an intuitionistic Kripke model with at most  $2^{c|\varphi|}$  states,  $c$  being a universal constant. Furthermore, validity in iPDL is log-space reducible to validity in classical dynamic logic. (This is not true for intuitionistic versus classical (nonmodal) propositional logic unless PSPACE = co-NP.)

Our intuitionistic dynamic Kripke models exhibit several algebraic and combinatorial properties worth considering; and by them we can separate dynamic-logical laws which are classically equivalent. For example, the induction principle  $\varphi \wedge [\pi^*](\varphi \rightarrow [\pi]\varphi) \rightarrow [\pi^*]\varphi$  is intuitionistically valid; the classically equivalent schema  $\langle \pi^* \rangle \varphi \rightarrow \varphi \vee \langle \pi^* \rangle (\neg\varphi \wedge \langle \pi \rangle \varphi)$  is not intuitionistically valid.

Nevertheless, our models are still of a rather tentative character, and may be even the “wrong” models, for the following reasons.

- We have no nontrivial E.P. of the form:  $\Phi \models \langle p^* \rangle \varphi$  implies  $\Phi \models \langle p^k \rangle \varphi$  for some  $k$ . ( $\Phi$  a set of formulae.)
- Concerning a (correct and complete) calculus, the main obstacle is the impure rule for right  $\rightarrow$  introduction, namely

$$\frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \rightarrow \beta} \text{ (if all } \gamma \in \Gamma \text{ are monotonic)}$$

Here a formula  $\gamma$  is called monotonic if in every model  $\mathfrak{K}$  and any of its worlds  $u, v \in W^{\mathfrak{K}}$ :  $u \preceq^{\mathfrak{K}} v$ ,  $u \in m^{\mathfrak{K}}(\gamma)$  implies  $v \in m^{\mathfrak{K}}(\gamma)$ . Unfortunately, we have yet no syntactic characterization of monotonicity.

- There does not exist a good negative interpretation of classical dynamic Logic in intuitionistic dynamic Logic.

Let us conclude this introduction by some remarks about G. Fischer-Servi's paper [FS76]. The author asks the following important question (p. 141):

*Can we find a general criterion that will give us "the" intuitionistic analogue of some of the most usual modal systems?*

This quotation immediately continues with a sort of disclaimer:

*The problem as stated is of a technical nature and therefore the philosophical issues relating to the plausibility of an intuitionistic logic of modality will be in this context ignored.*

Despite some technically interesting developments, the paper [FS76] does neither address nor answer the following questions which we think are crucial w.r.t. intuitionistification of modal logics:

1. Is there a reasonable and uniform method  $\mathfrak{M}$  such that given a classical modal calculus  $C$ ,  $\mathfrak{M}(C)$  will be "the" intuitionistic counterpart of  $C$ ?
2. Which properties typical for intuitionistic systems should be satisfied by "the" intuitionistic counterpart of a classical modal system?

REMARK 1. *Typical properties of intuitionistic systems are*

- A) *the disjunctive and existential property, and*
- B) *a sort of negative translation of the classical version into the intuitionistic one.*

*It is especially A) and B) which are addressed in the present paper about classical versus intuitionistic dynamic logic (as a prominent modal logic nonexistent around 1976).*

## 2. Semantics

We define the classical Kripke models and its denotation function as in [HKT00].

DEFINITION 1. A *Kripke model*  $\mathfrak{K}$  (KM) is a pair  $\mathfrak{K} = (W_{\mathfrak{K}}, m_{\mathfrak{K}})$ , where

- $W_{\mathfrak{K}}$  a nonempty set of states
- $m_{\mathfrak{K}}$  : is a function with
  1. for every atomic propositional  $a$ :  $m_{\mathfrak{K}}(a) \subseteq W_{\mathfrak{K}}$
  2. for every atomic program  $p$ :  $m_{\mathfrak{K}}(p) \subseteq W_{\mathfrak{K}} \times W_{\mathfrak{K}}$

We construct the new class of models, by adding an accessibility relation.

DEFINITION 2. An *intuitionistic Kripke model*  $\mathfrak{K}$  (iKM) is a triple  $\mathfrak{K} = (W^{\mathfrak{K}}, \preceq^{\mathfrak{K}}, m^{\mathfrak{K}})$ , where

- $W^{\mathfrak{K}}$  a nonempty set of states
- $\preceq^{\mathfrak{K}} \subseteq W^{\mathfrak{K}} \times W^{\mathfrak{K}}$  the intuitionistic (reflexive and transitive) accessibility relation
- $m^{\mathfrak{K}}$  : is a function with
  1. for every atomic propositional  $a$ :  $m^{\mathfrak{K}}(a) \subseteq W^{\mathfrak{K}}$  monotonic in relation to  $\preceq^{\mathfrak{K}}$ ,  
this means:  $w \in m^{\mathfrak{K}}(a)$  and  $w \preceq^{\mathfrak{K}} w'$  implies  $w' \in m^{\mathfrak{K}}(a)$ .
  2. for every atomic program  $p$ :  $m^{\mathfrak{K}}(p) \subseteq W^{\mathfrak{K}} \times W^{\mathfrak{K}}$

We will also refer to the classical Kripke models as PDL-models and the just introduced intuitionistic models as iPDL-models.

The extension of the denotation functions  $m_{\mathfrak{K}}$  and  $m^{\mathfrak{K}}$  to complex expression is done recursively in the usual ways. For example the intuitionistic case is done as follows:

DEFINITION 3.

- $m^{\mathfrak{K}}(\pi \cup \rho) := m^{\mathfrak{K}}(\pi) \cup m^{\mathfrak{K}}(\rho)$
- $m^{\mathfrak{K}}(\pi; \rho) := m^{\mathfrak{K}}(\pi) \circ m^{\mathfrak{K}}(\rho)$  (composition of relations)
- $m^{\mathfrak{K}}(\pi^*) := m^{\mathfrak{K}}(\pi)^* = \bigcup_{i \in \mathbb{N}} m^{\mathfrak{K}}(\pi)^i$  (reflexive-transitive closure)
- $m^{\mathfrak{K}}(\varphi?) := \{(w, w) \mid w \in m^{\mathfrak{K}}(\varphi)\}$

- $m^{\mathfrak{K}}(\perp) := \emptyset$
- $m^{\mathfrak{K}}(\psi \wedge \eta) := m^{\mathfrak{K}}(\psi) \cap m^{\mathfrak{K}}(\eta)$
- $m^{\mathfrak{K}}(\psi \vee \eta) := m^{\mathfrak{K}}(\psi) \cup m^{\mathfrak{K}}(\eta)$
- $m^{\mathfrak{K}}([\pi]\psi) := \{w \in W^{\mathfrak{K}} \mid \{w' \mid (w, w') \in m^{\mathfrak{K}}(\pi)\} \subseteq m^{\mathfrak{K}}(\psi)\}$
- $m^{\mathfrak{K}}(\langle \pi \rangle \psi) := \{w \in W^{\mathfrak{K}} \mid \{w' \mid (w, w') \in m^{\mathfrak{K}}(\pi)\} \cap m^{\mathfrak{K}}(\psi) \neq \emptyset\}$
- $m^{\mathfrak{K}}(\psi \rightarrow \eta) := \{w \in W^{\mathfrak{K}} \mid \{w' \mid w \preceq^{\mathfrak{K}} w'\} \cap m^{\mathfrak{K}}(\psi) \subseteq m^{\mathfrak{K}}(\eta)\}$

The only difference to the classical case is the interpretation of the connector  $\rightarrow$ . To work with our new model class we also need a definition for *satisfiability*.

DEFINITION 4. A formula  $\varphi$  is *satisfiable* (in iPDL), if there are an iPDL-model  $\mathfrak{K}$  and a state  $w \in W^{\mathfrak{K}}$ , with  $w \in m^{\mathfrak{K}}(\varphi)$ .

In analogy to this we will call a formula  $\varphi$  to be satisfied in a state  $w$ , when  $w \in m^{\mathfrak{K}}(\varphi)$ .

DEFINITION 5. A formula  $\varphi$  is *valid* (in iPDL), if in every iPDL-model  $\mathfrak{K}$  and every state  $w \in W^{\mathfrak{K}}$   $w \in m^{\mathfrak{K}}(\varphi)$  holds. We say also that  $\varphi$  is intuitionistically valid.

DEFINITION 6. Let  $F$  be a set of formulae,  $\mathfrak{K}$  a iPDL-model and  $\varphi$  an arbitrary formula.

We say  $\mathfrak{K}$  *models*  $F$  (written  $\mathfrak{K} \models F$ ), if every formula  $\eta \in F$  is satisfied in every state  $w \in \mathfrak{K}$  ( $w \in m^{\mathfrak{K}}(\eta)$ ). For the sake of simplicity we omit the set braces if  $F$  contains just one formula.

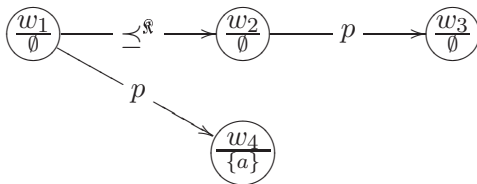
We say  $F$  *implies semantically*  $\varphi$  (written  $F \models \varphi$ ), if for every intuitionistic Kripke model  $\mathfrak{K}$  with  $\mathfrak{K} \models F$  also  $\mathfrak{K} \models \varphi$ .

### 3. Separation of classical and intuitionistic PDL

#### Separation of $[\ ]$ and $\langle \ \rangle$

In the classical PDL there is just one of the operators  $[\ ]$  and  $\langle \ \rangle$ . The other one is either given by the defining axiom  $[\pi]\varphi \leftrightarrow \neg\langle\pi\rangle\neg\varphi$  or  $\langle\pi\rangle\varphi \leftrightarrow \neg[\pi]\neg\varphi$ . In iPDL each of these axioms is invalid.

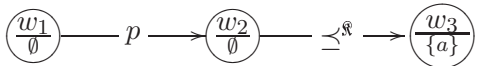
EXAMPLE 1. The shown model  $\mathfrak{K}$  is a counterexample to the validity of the formulae  $[p]a \rightarrow \neg\langle p \rangle\neg a$  (1).



(1) not satisfied in state  $w_1$ . Observe that the shown model also contradicts the formula  $\langle p \rangle a \rightarrow \neg[p]\neg a$  (2)

In contradiction to the falsity of (1) there is a intuitionistically valid first order formula, namely  $\forall x\varphi(x) \rightarrow \neg\exists x\neg\varphi(x)$ . If one interprets the possibilittas operator as existential quantifier and the necessittas operator as universal quantifier, the given formula seems to be reverted. The other direction of the axiom is also invalid:

EXAMPLE 2. Counter model against  $\neg[p]\neg a \rightarrow \langle p \rangle a$  and  $\neg\langle p \rangle\neg a \rightarrow [p]a$ :



Both formulae are not satisfied in state  $w_1$ .

### Induction principles

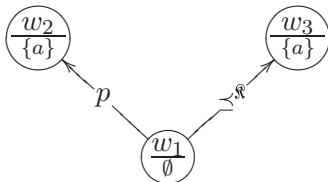
The Segerberg induction schema consists of the following two axioms

- $\varphi \wedge [ \pi^* ] (\varphi \rightarrow [ \pi ] \varphi) \rightarrow [ \pi^* ] \varphi$  (Necessittas or  $[ ]$  form)
- $\langle \pi^* \rangle \varphi \rightarrow \varphi \vee \langle \pi^* \rangle (\neg\varphi \wedge \langle \pi \rangle \varphi)$  (Possibilitas or  $\langle \rangle$  form)

As commonly known these two axioms are classically (in PDL) equivalent by dualisation. However this is not possible in iPDL.

LEMMA 1. *The Possibilitas form of the Segerberg induction is not valid.*

PROOF. We give a counterexample against the instance  $\langle p^* \rangle a \rightarrow a \vee \langle p^* \rangle ((a \rightarrow \perp) \wedge \langle p \rangle a)$ :



Again in state  $w_1$  the formula is invalidated. □

LEMMA 2. *The Necessitas form of Segerberg induction is valid.*

PROOF (SKETCH). One has to check that the proposition has the following equivalence: In every intuitionistic Kripke model  $\mathfrak{K}$  and on every state  $w \in W^{\mathfrak{K}}$

$$m^{\mathfrak{K}}(\varphi \wedge [\pi^*](\varphi \rightarrow [\pi]\varphi)) \subseteq m^{\mathfrak{K}}([\pi^*]\varphi)$$

By using  $m^{\mathfrak{K}}([\pi^*]\varphi) = \bigcap_{i \in \mathbb{N}} m^{\mathfrak{K}}([\pi^i]\varphi)$  it satisfies to prove

$$m^{\mathfrak{K}}(\varphi \wedge [\pi^*](\varphi \rightarrow [\pi]\varphi)) \subseteq m^{\mathfrak{K}}([\pi^k]\varphi)$$

This is done by induction on  $k$  using  $m^{\mathfrak{K}}([\pi^{k+1}]\varphi) = m^{\mathfrak{K}}([\pi^k][\pi]\varphi)$ . □

#### 4. Definability within iPDL

As in some versions of second order systems (see [Pra70]) several logic signs in iPDL can be eliminated for others. (Whereas in first order intuitionistic logic all logic signs are independent.)

LEMMA 3. *In every intuitionistic Kripke model  $\mathfrak{K} = (W^{\mathfrak{K}}, \leq^{\mathfrak{K}}, m^{\mathfrak{K}})$  holds*

1.  $m^{\mathfrak{K}}([\psi?]\perp) = W^{\mathfrak{K}} \setminus m^{\mathfrak{K}}(\psi)$
2.  $m^{\mathfrak{K}}(\langle \pi \rangle \varphi) = m^{\mathfrak{K}}([\pi][\varphi?]\perp)$
3.  $m^{\mathfrak{K}}(\psi \wedge \eta) = m^{\mathfrak{K}}(\langle \psi? \rangle \eta) = m^{\mathfrak{K}}([\psi?][\eta?]\perp)$
4.  $m^{\mathfrak{K}}(\psi \vee \eta) = m^{\mathfrak{K}}([\psi?]\perp \vee \eta)$
5.  $m^{\mathfrak{K}}([\pi \cup \rho]\varphi) = m^{\mathfrak{K}}([\pi]\varphi \wedge [\rho]\varphi) = m^{\mathfrak{K}}([\pi][\varphi?]\perp \vee [\rho][\varphi?]\perp)$
6.  $m^{\mathfrak{K}}(\langle \pi \cup \rho \rangle \varphi) = m^{\mathfrak{K}}(\langle \pi \rangle \varphi \vee \langle \rho \rangle \varphi) = m^{\mathfrak{K}}([\pi][\varphi?]\perp \vee [\rho][\varphi?]\perp)$

$$7. m^{\mathfrak{K}}(\langle \pi \rangle \langle \rho \rangle \varphi) = m^{\mathfrak{K}}(\langle \pi; \rho \rangle \varphi)$$

$$8. m^{\mathfrak{K}}([\pi][\rho]\varphi) = m^{\mathfrak{K}}([\pi; \rho]\varphi)$$

PROOF. The proof is straight forward by applying the definitions.  $\square$

As an immediate consequence of this lemma it is sufficient to assume that a formula  $\varphi$  contains only  $\perp$ ,  $\star$  (with arbitrary programs),  $\rightarrow$ ,  $[\ ]$  and  $?$  as logical connectors. An unpleasant implication of Lemma 3.1 is the intuitionistic validity of  $[\ a? \ ] \perp \vee a$ . This is of course very counterintuitive. Formally, this phenomenon destroys every hope to get a significant theorem on disjunctive and existential properties for iPDL.

## 5. Small model property

There is a direct way to the small model property for iPDL without using classical PDL-models as an intermediary step. This direct intuitionistic procedure would require the construction of a Fisher-Ladener-closure and the proof of a filtration lemma for iPDL. However, there is a shortcut using the classical theorem.

Throughout this chapter we fix an atomic program  $p$ , which will serve in the following lemmata as a means of translation of the intuitionistic accessibility relation.

DEFINITION 7. Let  $\varphi$  be a formula, which does not contain  $p$ . The *translation*  $\varphi^T$  is recursively defined as follows.

Let  $\psi, \eta$  be formulae and  $\pi, \rho, \sigma$  programs.

- If  $\varphi = \perp$ , then  $\varphi^T = \perp$
- If  $\varphi = a$  ( $a$  is a propositional letter), then  $\varphi^T = [p^*]a$
- If  $\varphi = \psi \rightarrow \eta$ , then  $\varphi^T = [p^*](\psi^T \rightarrow \eta^T)$
- If  $\varphi = [\pi]\eta$ , then  $\varphi^T = [\pi^T]\eta^T$
- If  $\pi = q$  is atomic, then  $\pi^T = q$
- If  $\pi = \rho; \sigma$ , then  $\pi^T = \rho^T; \sigma^T$
- If  $\pi = \rho \cup \sigma$ , then  $\pi^T = \rho^T \cup \sigma^T$
- If  $\pi = \eta?$ , then  $\pi^T = \eta^T?$
- If  $\pi = \rho^*$ , then  $\pi^T = (\rho^T)^*$



Let  $\mathfrak{K} = (W^{\mathfrak{K}}, \preceq^{\mathfrak{K}}, m^{\mathfrak{K}})$  be an intuitionistic Kripke model. The PDL-model  $\mathfrak{K}^T = (W_{\mathfrak{K}^T}, m_{\mathfrak{K}^T})$ , called the *translation* of  $\mathfrak{K}$ , is defined by

- $W_{\mathfrak{K}^T} := W^{\mathfrak{K}}$
- Let  $a$  be a propositional. Set  $m_{\mathfrak{K}^T}(a) := m^{\mathfrak{K}}(a)$
- Let  $q$  be an atomic program with  $p \neq q$ . Set  $m_{\mathfrak{K}^T}(q) := m^{\mathfrak{K}}(q)$ .
- Set  $m_{\mathfrak{K}^T}(p) := \preceq^{\mathfrak{K}}$ .

LEMMA 4. Let  $\varphi$  be a formula which does not contain  $p$ ,  $\mathfrak{K}$  a iPDL-model and  $w \in W^{\mathfrak{K}}$  a state.

Then  $w \in m^{\mathfrak{K}}(\varphi)$  if and only if  $w \in m_{\mathfrak{K}^T}(\varphi^T)$ .

PROOF. by induction on the structure of  $\varphi$ :

- Assume  $\varphi = \perp$ . This case is obvious.
- From  $w \in m^{\mathfrak{K}}(a)$  it follows per definitionem, that for each state  $w' \in W^{\mathfrak{K}}$  satisfying  $w \preceq^{\mathfrak{K}} w'$ , also  $w' \in m^{\mathfrak{K}}(a)$ . Since (also per definitionem)  $m_{\mathfrak{K}^T}(p) = \preceq^{\mathfrak{K}}$  is reflexive and transitive also  $m_{\mathfrak{K}^T}(p^*) = m_{\mathfrak{K}^T}(p) = \preceq^{\mathfrak{K}}$ . Hence we have for each state  $w' \in W_{\mathfrak{K}^T}$  with  $(w, w') \in m_{\mathfrak{K}^T}(p^*) = \preceq^{\mathfrak{K}}$  that  $w' \in m_{\mathfrak{K}^T}(a) = m^{\mathfrak{K}}(a)$ . It follows  $w \in m_{\mathfrak{K}^T}(\llbracket p^* \rrbracket a) = m_{\mathfrak{K}^T}(a^T)$ . Assume  $a \notin m^{\mathfrak{K}}(a)$  then, since  $m_{\mathfrak{K}^T}(a) = m^{\mathfrak{K}}(a)$  and  $p^*$  being reflexive, also  $w \notin m_{\mathfrak{K}^T}(\llbracket p^* \rrbracket a) = m_{\mathfrak{K}^T}(a^T)$ .
- Let  $\varphi^T = \llbracket p^* \rrbracket (\psi^T \rightarrow \eta^T)$  be translation of  $\psi \rightarrow \eta$ . Since the evaluation of  $\rightarrow$  is local in PDL, the proposition follows similary as above.
- The other cases are similar. □

DEFINITION 8. Let  $\mathfrak{K} = (W_{\mathfrak{K}}, m_{\mathfrak{K}})$  be a PDL-model. Then the iPDL-model  $\mathfrak{K}^P = (W^{\mathfrak{K}^P}, \preceq^{\mathfrak{K}^P}, m^{\mathfrak{K}^P})$ , called the *pullback* of  $\mathfrak{K}$ , is defined by:

- $W^{\mathfrak{K}^P} := W_{\mathfrak{K}}$
- For each atomic propositional  $a$ :  $m^{\mathfrak{K}^P}(a) := m_{\mathfrak{K}}(\llbracket p^* \rrbracket a)$
- For each atomic program  $q$ :  $m^{\mathfrak{K}^P}(q) := m_{\mathfrak{K}}(q)$
- $\preceq^{\mathfrak{K}^P} := (m_{\mathfrak{K}}(p))^*$

Notice that  $\mathfrak{K}^P = (W^{\mathfrak{K}^P}, \preceq^{\mathfrak{K}^P}, m^{\mathfrak{K}^P})$  is a well defined iPDL-model, since the only additional condition for the propositions, namely to be monotonic, is satisfied.

LEMMA 5. Let  $\varphi$  be a *iPDL*-formula not containing  $p$  and  $\mathfrak{R} = (W_{\mathfrak{R}}, m_{\mathfrak{R}})$  be a *PDL*-model. Then  $w \in m^{\mathfrak{R}^P}(\varphi)$  if and only if  $w \in m_{\mathfrak{R}}(\varphi^T)$

PROOF. By induction on the structure of  $\varphi$ .

- The cases where  $\varphi = \perp$  or  $\varphi = a$  are obvious.
- Let  $\varphi = \psi \rightarrow \eta$ . Assume  $w \in m_{\mathfrak{R}}(\varphi^T) = m_{\mathfrak{R}}(\llbracket p^* \rrbracket(\psi^T \rightarrow \eta^T))$ .  
Now apply  $\preceq^{\mathfrak{R}^P} := (m_{\mathfrak{R}}(p))^* = m_{\mathfrak{R}}(p^*)$  and I.H..  
Now assume  $w \notin m_{\mathfrak{R}}(\varphi^T)$ . This means there is a state  $w'$  with  $(w, w') \in m_{\mathfrak{R}}(p^*)$  and  $w' \notin m_{\mathfrak{R}}(\psi^T \rightarrow \eta^T)$ . This means  $w' \in m_{\mathfrak{R}}(\psi^T)$  and  $w' \notin m_{\mathfrak{R}}(\eta^T)$ . Hence by I.H.  $w' \in m^{\mathfrak{R}^P}(\psi)$  and  $w' \notin m^{\mathfrak{R}^P}(\eta)$ . Therefore  $w' \notin m^{\mathfrak{R}^P}(\psi \rightarrow \eta)$  and  $(w \preceq^{\mathfrak{R}^P} w') \wedge w \notin m^{\mathfrak{R}^P}(\psi \rightarrow \eta)$ .
- The other cases are similar. □

THEOREM 1. Every formula  $\varphi$ , which is satisfiable in a *iPDL*-model  $\mathfrak{R}$ , is also satisfiable in a model with less than  $2^{c \cdot |\varphi|}$  states, for some constant  $c$ .

PROOF. We construct the new model using the small model property from *PDL*.

Step 1) Choose a new (fresh) atomic program letter  $p$ , which is not contained in  $\varphi$ .

Step 2) Translate  $\varphi$  recursively into  $\varphi^T$

Notice that  $|\varphi^T| \leq c \cdot |\varphi|$

Step 3) Transform also the *iPDL*-model  $\mathfrak{R} = (W^{\mathfrak{R}}, \preceq^{\mathfrak{R}}, m^{\mathfrak{R}})$  into an *PDL*-model  $\mathfrak{R}^T = (W_{\mathfrak{R}^T}, m_{\mathfrak{R}^T})$

Notice that  $\mathfrak{R}^T$  is a *PDL*-model for  $\varphi^T$  by Lemma 4

Step 4) By the *small model theorem* of the *PDL* one can choose a (*PDL*-) model  $\mathfrak{L} = (W_{\mathfrak{L}}, m_{\mathfrak{L}})$ , which satisfies:

- There is a state  $u \in W_{\mathfrak{L}}$ , such that  $u \in m_{\mathfrak{L}}(\varphi^T)$
- $|W_{\mathfrak{L}}| \leq 2^{|\varphi^T|} \leq 2^{c \cdot |\varphi|}$

Step 5) Pull  $\mathfrak{L}$  back to an *iPDL*-model  $\mathfrak{L}^P = (W^{\mathfrak{L}^P}, \preceq^{\mathfrak{L}^P}, m^{\mathfrak{L}^P})$

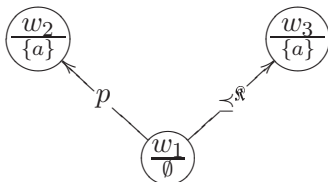
Notice that  $\mathfrak{L}^P$  satisfies  $u \in m^{\mathfrak{L}^P}(\varphi)$  by Lemma 5 □

As an immediate consequence of the above theorem we have the decidability of the validity problem. Moreover, this guarantees the existence of some sort of complete and correct calculus for iPDL. However, we were unable to construct a useful and structurally perspicuous calculus for iPDL.

## 6. Monotonic formulae

The program-free formulae, interpreted intuitionistically, are all monotonic. Unfortunately, this property is lost in iPDL for formulae with programs.

EXAMPLE 3. Let  $\mathfrak{K} = (W^{\mathfrak{K}}, \preceq^{\mathfrak{K}}, m^{\mathfrak{K}})$  be represented by



Observe that in state  $w_1$  the formula  $\langle p \rangle a$  is satisfied, but not in  $w_3$ .

The most critical consequence of the lack of monotonicity is that the  $\Rightarrow \rightarrow$  rule in Gentzen style sequent calculi is not correct.

LEMMA 6. *The rule*

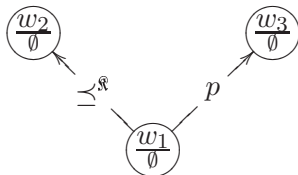
$$\frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \rightarrow \beta} (\Rightarrow \rightarrow)$$

*is not correct in iPDL.*

PROOF. It is easily checked that the sequent  $\langle p \rangle \top, \top \Rightarrow \langle p \rangle \top$  is valid.

We give a countermodel for the derived sequent  $\langle p \rangle \top \Rightarrow \top \rightarrow \langle p \rangle \top$ .

Let the intuitionistic Kripke model  $\mathfrak{K} = (W^{\mathfrak{K}}, \preceq^{\mathfrak{K}}, m^{\mathfrak{K}})$  be defined by



Notice that in state  $w_2$  the right side  $\top \rightarrow \langle p \rangle \top$  of the sequent is not satisfied. Thus (by the forced monotonicity of  $\rightarrow$ ) also  $w_1 \notin m^{\mathfrak{K}}(\top \rightarrow \langle p \rangle \top)$ . But obviously the left side of the sequent is satisfied in  $w_1$ . Therefore the sequent is not satisfied in  $w_1$ .  $\square$

The lack of monotonicity has, among others, the following more serious consequence.

**COROLLARY 2** (Failure of substitution). *The valid formulae from iPDL are not closed under substitution of atomic propositions for arbitrary formulae.*

**PROOF.** The formula  $a \rightarrow (b \rightarrow a)$  is valid in iPDL for atomic propositions  $a$  and  $b$ . As shown above the substitution  $\langle p \rangle \top \rightarrow (\top \rightarrow \langle p \rangle \top)$  is not.  $\square$

However, the  $\Rightarrow \rightarrow$  rule can partially be rescued just by adding a monotonicity restriction.

**LEMMA 7.** *The rule*

$$\frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \rightarrow \beta} (\Rightarrow \rightarrow \text{mono})$$

*is correct in iPDL if all of the formulae in  $\Gamma$  are monotonic.*

**PROOF.** This follows immediate from the definitions and the required monotonicity property.  $\square$

The major open problem with  $\Rightarrow \rightarrow$  mono is the fact that until now we have no syntactical criterion for the monotonicity.

**LEMMA 8.** *Let  $\varphi, \psi$  be arbitrary formulae.*

1. *Atomic propositions (including  $\perp$ ) are monotonic,*
2.  *$\varphi \rightarrow \psi$  is monotonic,*
3. *from  $\varphi$  and  $\psi$  monotonic follows, that  $\varphi \vee \psi$  is also monotonic,*
4. *from  $\varphi$  and  $\psi$  monotonic follows, that  $\varphi \wedge \psi$  is also monotonic.*

**PROOF.** Apply definitions.  $\square$

It follows immediately that all formulae which do not contain modal operators are monotonic. It is also straight-forward to prove that all intuitionistically correct inference rules from the (non dynamic) propositional calculus can be applied to monotonic formulae. Hence the valid formulae from iPDL are closed under substituting the atomic propositions for monotonic formulae.

### Existential property

One of the reasons to investigate the intuitionistic modification of PDL, as described, has been the hope to determine for any given iPDL-valid formula  $\langle \pi^* \rangle \eta$  a natural number  $k \in \mathbb{N}$  for which the formula  $\langle \pi^k \rangle \eta$  is valid. This hope was based on the existential properties of usual intuitionistic first order logics. Also, a version of the disjunctive property is interesting since  $\langle \pi \cup \rho \rangle \varphi$  should induce the validity of  $\langle \pi \rangle \varphi$  or  $\langle \rho \rangle \varphi$ . But in view of the shown equivalence between  $\langle \pi \cup \rho \rangle \varphi$  and  $\langle \pi \rangle \varphi \vee \langle \rho \rangle \varphi$  it is of minor interest.

As usual (compare [Tak87] or [TS00]) we try to construct a class of formulae  $\mathbb{H}$  from which to take subsets which define subclasses of the iPDL-model.

DEFINITION 9. The set  $\mathbb{H}$  of *Harrop formulae* is recursively defined as follows:

- every atomic proposition is in  $\mathbb{H}$
- if  $\psi, \eta \in \mathbb{H}$  then is also  $\psi \wedge \eta \in \mathbb{H}$
- if  $p$  is an atomic program and  $\varphi \in \mathbb{H}$  then is also  $[p] \varphi \in \mathbb{H}$
- if  $[\pi] \varphi, [\rho] \varphi \in \mathbb{H}$  then is also  $[\pi \cup \rho] \varphi \in \mathbb{H}$
- if  $[\pi][\rho] \varphi \in \mathbb{H}$  then is also  $[\pi; \rho] \varphi \in \mathbb{H}$
- if  $\varphi \in \mathbb{H}$  and  $\psi$  is a monotonic formula, then is also  $\psi \rightarrow \varphi \in \mathbb{H}$

This definition is already very strong as one can see in the following lemma's proof.

LEMMA 9. Let  $F \subseteq \mathbb{H}$  be a set of Harrop formulae,  $\pi$  an arbitrary program and  $\eta$  an arbitrary formula, then holds: if  $F \models \langle \pi^* \rangle \eta$  and  $\eta$  is monotonic, then there exists a number  $k \in \mathbb{N}$  with  $F \models \langle \pi^k \rangle \eta$ .

PROOF. We will prove, that  $k = 0$ .

The claim of the lemma is then equivalent to:

If  $F \models \langle \pi^* \rangle \eta$  and  $\eta$  is monotonic, then  $F \models \eta$ .

Assume in contradiction that  $F \not\models \eta$ . This means that there is a iPDL-model  $\mathfrak{R}$  with  $\mathfrak{R} \models F$ , but  $w \notin m^{\mathfrak{R}}(\eta)$  for a state  $w \in W^{\mathfrak{R}}$ . We choose a fresh state  $w' \notin W^{\mathfrak{R}}$  and construct a new iPDL-model by inserting  $w'$  as follows.

- (1)  $W^{\mathfrak{K}} := W^{\mathfrak{R}} \cup \{w'\}$
- (2) for every atomic proposition  $a$  with  $w \notin m^{\mathfrak{R}}(a)$ :  $m^{\mathfrak{K}}(a) = m^{\mathfrak{R}}(a)$
- (3) for every atomic proposition  $a$  with  $w \in m^{\mathfrak{R}}(a)$ :  $m^{\mathfrak{K}}(a) = m^{\mathfrak{R}}(a) \cup \{w'\}$

- (4) for every atomic program  $p$ :  $m^{\mathfrak{E}}(p) = m^{\mathfrak{S}}(p)$   
 (remark that no atomic program leads to or leaves  $w'$ ).
- (5)  $\preceq^{\mathfrak{E}} := \preceq^{\mathfrak{S}} \cup \{(w', w)\} \cup \{(w', w'') \mid (w, w'') \in \preceq^{\mathfrak{S}}\}$

Claim 1: for every formula  $\varphi$  and every state  $u \in W^{\mathfrak{S}}$  holds:  $u \in m^{\mathfrak{S}}(\varphi)$  iff.  $u \in m^{\mathfrak{E}}(\varphi)$ .

This can easily be seen by induction on the structure of formulae and the fact that the new state  $w'$  can not be seen from any "old" state  $u$ . Neither by a atomic program nor the accessibility relation.

Claim 2:  $\mathfrak{E} \models F$ .

From claim 1 it follows that it is sufficient to prove the following:

For every  $\varphi \in \mathbb{H}$  holds: if  $w \in m^{\mathfrak{E}}(\varphi)$  then  $w' \in m^{\mathfrak{E}}(\varphi)$ .

This is also proven by induction on the structure of Harrop formulae.

*Induction basis:*

- Let  $a$  be an atomic proposition (in  $\mathbb{H}$ ). This case follows from construction step (3).
- Let  $\llbracket p \rrbracket \varphi \in \mathbb{H}$ . Since  $\{u \mid (w', u) \in m^{\mathfrak{E}}(p)\} = \emptyset$  it follows that  $w' \in m^{\mathfrak{E}}(\llbracket p \rrbracket \varphi)$

*Induction hypothesis (IH):* The claim shall hold for  $\psi_0, \psi_1, \llbracket \pi \rrbracket \psi_2, \llbracket \rho \rrbracket \psi_2 \in \mathbb{H}$ .

*Induction step:*

- Case  $\varphi = \psi_0 \wedge \psi_1$ : Assume  $w \in m^{\mathfrak{E}}(\psi_0 \wedge \psi_1)$ . then  $w \in m^{\mathfrak{E}}(\psi_0)$  and  $w \in m^{\mathfrak{E}}(\psi_1)$ . Now use IH and compound the statements.
- Case  $\varphi = \psi \rightarrow \psi_0$  for a monotonic formula  $\psi$ : Assume  $w \in m^{\mathfrak{E}}(\psi \rightarrow \psi_0)$ .

There are two cases:

- $w' \notin m^{\mathfrak{E}}(\psi)$ : Then it is easy to see  $w' \in m^{\mathfrak{E}}(\varphi)$ .
- $w' \in m^{\mathfrak{E}}(\psi)$ : Since  $\psi$  is monotonic follows  $w \in m^{\mathfrak{E}}(\psi)$ . Hence follows  $w \in m^{\mathfrak{E}}(\psi_0)$ . By IH follows also  $w' \in m^{\mathfrak{E}}(\psi_0)$  and therefore  $w' \in m^{\mathfrak{E}}(\varphi)$ .

- The cases  $\varphi = \llbracket \psi \cup \rho \rrbracket \psi_2$  and  $\llbracket \psi; \rho \rrbracket \psi$  follow straight from Lemma 3.

Because of  $F \models \langle \pi^* \rangle \eta$  and  $\mathfrak{E} \models F$  it follows that  $w' \in m^{\mathfrak{S}}(\langle \pi^* \rangle \eta)$ .

But since  $\{u \mid (w', u) \in m^{\mathfrak{E}}(p)\} = \emptyset$  for every atomic program  $p$  the state  $w'$  can not be left by  $\pi$ . Hence  $w' \in m^{\mathfrak{S}}(\eta)$  holds, which contradicts by monotonicity of  $\eta$  the demand  $w' \notin m^{\mathfrak{S}}(\eta)$ .  $\square$

By slightly modifying this proof one gets a similar disjunctive property as well.

LEMMA 10. *Let  $F \subseteq \mathbb{H}$  be a set of Harrop formulae and  $\eta$  and  $\psi$  monotonic formulae.*

*If  $F \models \eta \vee \psi$ , then we have  $F \models \eta$  or  $F \models \psi$ .*

Whether the given restrictions on the notion of Harrop formulae together with the requirement of monotonicity can be made more liberal we do not know. It is however clear that they can not be removed completely.

EXAMPLE 4. Let  $F := \{ \llbracket a? \rrbracket \perp \rightarrow b, \llbracket b? \rrbracket \perp \rightarrow a \}$ .

Claim: It holds  $F \models a \vee b$ .

Let  $\mathfrak{K}$  be an iPDL model with  $\mathfrak{K} \models F$ .

Let  $w \in W^{\mathfrak{K}}$  be a state. Assume  $w \notin m^{\mathfrak{K}}(a)$ . Hence  $w \in m^{\mathfrak{K}}(\llbracket a? \rrbracket \perp)$ . Since  $w \in m^{\mathfrak{K}}(\llbracket a? \rrbracket \perp \rightarrow b)$  holds by assumption, therefore we have  $w \in m^{\mathfrak{K}}(b)$ .

Never the less neither  $F \models a$ , nor  $F \models b$ , holds. This is shown in the following iPDL-model

$$\left( \frac{w_0}{\{a\}} \right) \qquad \left( \frac{w_1}{\{b\}} \right)$$

Observe that this is obviously a model for  $F$ , but neither for  $a$  nor for  $b$

## 7. Modal translation of PDL into iPDL

Usually in intuitionistic calculi there is a possibility for transforming classical tautologoids into intuitionistic ones. It turns out that there exists a simple way to realize such a translation. This translation is based on the fact that the test-operator in combination with the necessitas-operator behaves in iPDL like a classical implication.

DEFINITION 10. The *modal translation*  $\overline{\varphi}$  of a formula  $\varphi$ , resp.  $\overline{\pi}$  of a program  $\pi$ , is recursively defined, as follows:

Let  $\varphi, \eta, \psi$  be formulae and  $\pi, \rho, \sigma$  be programs

- $\overline{\perp} := \perp$ .
- for atomic propositions  $a$ :  $\overline{a} := a$
- for atomic programs  $p$ :  $\overline{p} := p$

- let  $\varphi \equiv \psi \wedge \eta$ : then  $\overline{\varphi} := \overline{\psi} \wedge \overline{\eta}$ .
- let  $\varphi \equiv \psi \vee \eta$ : then  $\overline{\varphi} := \overline{\psi} \vee \overline{\eta}$ .
- let  $\varphi \equiv \psi \rightarrow \eta$ : then  $\overline{\varphi} := \llbracket \overline{\psi}^? \rrbracket \overline{\eta}$ .
- let  $\pi \equiv \rho; \sigma$ : then  $\overline{\pi} := \overline{\rho}; \overline{\sigma}$ .
- let  $\pi \equiv \rho \cup \sigma$ : then  $\overline{\pi} := \overline{\rho} \cup \overline{\sigma}$ .
- let  $\pi \equiv \rho^*$ : then  $\overline{\pi} := (\overline{\rho})^*$ .
- let  $\pi \equiv \eta^?$ : then  $\overline{\pi} := \overline{\eta}^?$ .
- let  $\varphi \equiv \llbracket \rho \rrbracket \eta$ : then  $\overline{\varphi} := \llbracket \overline{\rho} \rrbracket \overline{\eta}$ .
- let  $\varphi \equiv \langle \rho \rangle \eta$ : then  $\overline{\varphi} := \langle \overline{\rho} \rangle \overline{\eta}$ .

The given modal translation leaves all connectors fixed except one. Whereas the usual negative translations leave the implication identical, it is to note that our modal translation changes  $\varphi \rightarrow \psi$  into  $\llbracket \varphi^? \rrbracket \psi$  which is quite different from identity.

**THEOREM 3.** *Let  $\varphi$  be a formula. Then  $\varphi$  is valid in PDL iff  $\overline{\varphi}$  is valid in iPDL.*

**PROOF.** It is easy to see that the formulae  $\llbracket \varphi^? \rrbracket \psi$  and  $\varphi \rightarrow \psi$  are equivalent in PDL. Therefore the validity of  $\varphi$  in PDL implies the validity of  $\overline{\varphi}$  in PDL.

Since  $\overline{\varphi}$  does not contain any implications, it does not relate to the accessibility relation on its evaluation. Thus  $\overline{\varphi}$  behaves in every iPDL-model exactly as in a PDL-model and therefore it is valid.  $\square$

The major disadvantage of the modal translation is a certain lack of conceptual purity: it introduces extra modalities.

## 8. On negative translations

We treat only test-free formulae and programs, and consider the following double negation translation (of a usual brand):

**DEFINITION 11.** The *negative translation*  $\varphi'$  of a formula  $\varphi$  (which does not contain tests) and  $\pi'$  of a program  $\pi$  is defined by recursion as follows:

- For atomic propositions  $a$ :  $a' := \neg\neg a$  ( $\perp' = \perp$ )
- For atomic programs  $p$ :  $p' := p$



- $(\varphi \wedge \psi)' = \varphi' \wedge \psi'$
- $(\varphi \vee \psi)' = \neg(\neg\varphi' \wedge \neg\psi')$
- $(\varphi \rightarrow \psi)' = \varphi' \rightarrow \psi'$
- $(\pi; \rho)' = \pi'; \rho'$
- $(\pi \cup \rho)' = \pi' \cup \rho'$
- $(\pi^*)' = (\pi')^*$
- $([\pi] \varphi)' = \neg\neg[\pi'] \varphi'$
- $(\langle \pi \rangle \varphi)' = \neg\neg\neg[\pi'] \neg\varphi'$

LEMMA 11.  $\varphi \leftrightarrow \varphi'$  is classically valid.

PROOF. By induction on the structure of formulae. □

DEFINITION 12. The *Segerberg system* (without test) is defined as follows.

Axioms:

1. Tautologoids of propositional logic
2.  $\langle \pi \rangle \varphi \leftrightarrow \neg[\pi] \neg\varphi$
3.  $[\pi](\varphi \rightarrow \psi) \rightarrow ([\pi]\varphi \rightarrow [\pi]\psi)$
4.  $[\pi](\varphi \wedge \psi) \leftrightarrow ([\pi]\varphi \wedge [\pi]\psi)$
5.  $[\pi \cup \rho]\varphi \leftrightarrow [\pi]\varphi \wedge [\rho]\varphi$
6.  $[\pi; \rho]\varphi \leftrightarrow [\pi][\rho]\varphi$
7.  $\varphi \wedge [\pi][\pi^*]\varphi \leftrightarrow [\pi^*]\varphi$
8.  $\varphi \wedge [\pi^*](\varphi \rightarrow [\pi]\varphi) \rightarrow [\pi^*]\varphi$

Deduction rules:

$$\frac{\varphi \quad \varphi \rightarrow \psi}{\psi} \text{ (M.P.)} \qquad \frac{\varphi}{[\pi]\varphi} \text{ (Nec.)}$$

Observe that it follows from Lemma 8, that all formulae  $\varphi'$  are monotonic.

We would like to prove a result of the form:

*For all (test-free) formulae  $\varphi$ :  $\varphi$  is classically valid iff  $\varphi'$  is intuitionistically valid.*

Unfortunately, we were not able to achieve this result. Nevertheless the following lemma is a good exercise of the use of monotonicity.

LEMMA 12.  $\varphi' \leftrightarrow \neg\neg\varphi'$  is intuitionistically valid for arbitrary formulae  $\varphi$ .

PROOF. By induction on the number of outermost signs  $\wedge$ ,  $\vee$ ,  $\rightarrow$ ,  $[ \ ]$  and  $\langle \ \rangle$ .

1. Let  $a$  be an atomic propositional:  $\neg\neg a \leftrightarrow \neg\neg\neg\neg a$  is intuitionistically valid, since  $\neg\varphi \leftrightarrow \neg\neg\neg\varphi$  is.
2.  $(\varphi \wedge \psi)' = \varphi' \wedge \psi'$ . By I.H. we have  $\varphi' \leftrightarrow \neg\neg\varphi'$  and  $\psi' \leftrightarrow \neg\neg\psi'$ . The direction  $(\varphi \rightarrow \psi)' \rightarrow \neg\neg(\varphi' \rightarrow \psi')$  is trivial. For the other direction: since all  $\varphi'$  and  $\psi'$  are monotonic we have after easy intuitionistic derivations and applying I.H.  $\neg\neg(\varphi \wedge \psi)' \rightarrow \neg\neg\varphi' \rightarrow \varphi'$  and  $\neg\neg(\varphi \wedge \psi)' \rightarrow \neg\neg\psi' \rightarrow \psi'$ , hence we have  $\neg\neg(\varphi \wedge \psi)' \rightarrow (\varphi \wedge \psi)'$ .
3.  $(\varphi \vee \psi)' = \neg(\neg\varphi' \wedge \neg\psi')$ . We do not need the I.H. since  $\neg\neg\neg(\neg\varphi' \wedge \neg\psi') \leftrightarrow \neg(\neg\varphi' \wedge \neg\psi')$  is intuitionistically valid.
4. We need to show  $(\varphi' \rightarrow \psi') \leftrightarrow \neg\neg(\varphi' \rightarrow \psi')$ . The formula  $(\neg\neg(\varphi' \rightarrow \psi') \wedge \varphi') \rightarrow \neg\neg\psi'$  is intuitionistically valid. By I.H. we have  $\neg\neg\psi' \rightarrow \psi'$ , hence  $\neg\neg(\varphi' \rightarrow \psi') \rightarrow (\varphi' \rightarrow \psi')$ .
5.  $([ \ \pi \ ] \varphi)' = \neg\neg[ \ \pi' \ ] \varphi'$ . The direction  $\neg\neg[ \ \pi' \ ] \varphi' \rightarrow \neg\neg\neg\neg[ \ \pi' \ ] \varphi'$  is trivial. The other direction follows from the validity of  $\neg\neg\neg a \rightarrow \neg a$  and substituting  $a$  by the monotonic formula  $\neg[ \ \pi' \ ] \varphi'$ .
6.  $(\langle \ \pi \ \rangle \varphi)' = \neg\neg\neg[ \ \pi' \ ] \neg\varphi'$ . This case is again immediate from the intuitionistic validity of  $\neg a \leftrightarrow \neg\neg\neg a$  and  $\neg\neg[ \ \pi' \ ] \neg\varphi'$  being monotonic.

□

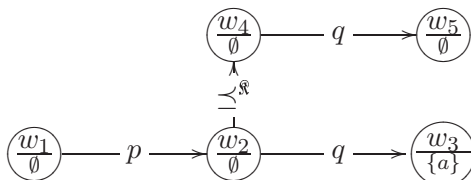
The first two schemata of the Segerberg system translate into intuitionistically valid formulae.

- Let  $\varphi$  a tautologoid. We write  $\varphi$  and its subformulae as a tree, with the leaves being atomic propositions or formulae beginning with  $[ \ ]$  or  $\langle \ \rangle$ . These two types of leaves we call *relative atoms*. In  $\varphi'$  the relative atoms are changed to monotonic formulae. That  $\varphi'$  is intuitionistically valid can be seen by the usual negative interpretation.
- $(\langle \ \pi \ \rangle \varphi \leftrightarrow \neg[ \ \pi \ ] \neg\varphi)' = \neg\neg\neg[ \ \pi' \ ] \neg\varphi' \leftrightarrow \neg\neg\neg[ \ \pi' \ ] \neg\varphi'$

Also the rules M.P. and Nec. translate into intuitionistically correct inferences.

Unfortunately, the negative translation of the Segerberg axiom 6 has an instance which is not intuitionistically valid.

EXAMPLE 5. We falsify the instance  $\neg\neg[p; q] \neg\neg a \leftrightarrow \neg\neg[p] \neg\neg[q] \neg\neg a$  in the following model



Notice that in state  $w_1$  the left hand side of the formula is satisfied. The right hand side is less obvious. From  $w_5 \notin m^s(a)$  and no other world can be seen via  $\preceq^s$  from  $w_5$  follows that  $w_5 \notin m^s(\neg\neg a)$ . Thus  $w_4 \notin m^s([q] \neg\neg a)$  and  $w_4 \notin m^s(\neg\neg[q] \neg\neg a)$ . Since the formula  $\neg\neg[q] \neg\neg a$  is monotonic and  $w_2 \preceq^s w_4$  follows  $w_2 \notin m^s(\neg\neg[q] \neg\neg a)$ . Thus  $w_1 \notin m^s([p] \neg\neg[q] \neg\neg a)$  and  $w_1 \notin m^s(\neg\neg[p] \neg\neg[q] \neg\neg a)$ . This falsifies the given instance.

We leave the translations of the axioms 3, 4, 5, 7 and 8 as an exercise to the reader for analogous treatment.

REMARK 2. Surprisingly, all Segerberg axioms 4 to 8 are intuitionistically valid.

## References

- [FS76] Gisele Fischer-Servi, “On modal logic with an intuitionistic base”, *Studia Logica* 36(4) (1976), 141–149.
- [HKT00] D. Harel, D. Kozen, and J. Tiuryn, *Dynamic Logic*, The MIT Press, London, 2000.
- [Pra70] Dag Prawitz, “Some results for intuitionistic logic with second order quantifiers”, in J. Myhill et al (eds.), *Intuitionism and Proof Theory, Proc. of the Summer Conference at Buffalo 1968*, North-Holland, 1970, pp. 259–269.
- [Tak87] G. Takeuti, *Proof Theory*, North-Holland, second edition, 1987.

- [TS00] A.S. Troelstra and H. Schwichtenberg, *Basic Proof Theory*, Cambridge University Press, second edition, 2000.

J.W. DEGEN AND J.M. WERNER

System Simulation

Friedrich-Alexander Universität Erlangen-Nürnberg

Cauerstr. 6, 91058 Erlangen, Germany

`degen@informatik.uni-erlangen.de`

and

`johannes.werner@informatik.stud.uni-erlangen.de`