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A FORMAL THEORY OF PHYSICAL NECESSITY

Abstract. A system HW of normal modal logic, developed by R. Bigelow & R. Pargetter is presented. Some formal issues concerning the system are examined, such as completeness, number of distinct modalities and relations to other systems. Some philosophical topics are also discussed. The Authors interpret the system HW as the system of physical (nomic) modalities. It is questioned, whether or not the system HW is justified to be claimed to be the logic of physical necessity. The answer seems to may be negative.

Keywords: modal logic, physical necessity

The objective of the present paper is to discuss some formal and philosophical issues concerning the system HW of modal logic, presented first by R. Pargetter and analysed by Pargetter and Bigelow [1, 258–262]. The system in question belongs to the set of normal modal logics and is claimed to formalize the concept of physical (nomic) necessity.

1. Axiomatics

The system HW is a normal modal logic. The language is standard and includes, beside classical symbols, i.e., negation $\neg$, conjunction $\wedge$, disjunction $\vee$, conditional $\rightarrow$ and equivalence $\equiv$, two unary propositional connectives, namely the physical necessity connective $\Box$ and the physical possibility connective $\Diamond$. So that, classical definition of a well formed formula should be

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supplemented with two conditions, claiming that, provided ϕ is a well formed formula, $(\Box\phi)$ and $(\Diamond\phi)$ are well formed formulas as well. In order to save brackets we adopt the usual convention concerning the biding force of the connectives. In the sequence $\Box$, $\Diamond$, $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\equiv$ the binding gets longer and longer. In all places PC is the classical propositional calculus, and letters i, j, k, n, m are natural numbers.

The modal connectives are mutually definable in the usual way, so we accept the definition:

(1) $(\Diamond\phi) \stackrel{\mathrm{df}}{=} (\neg\Box\neg\phi)$

The axioms of the presented system are:

(2) Any substitution of a PC-theorem is a HW-theorem.

(3) $\Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)$

(4) $\Box\phi \rightarrow \phi$

(5) $\Diamond\Box\Box\phi \rightarrow \Box\Box\phi$

(6) $\Diamond\Diamond\phi_1 \wedge \Diamond\Diamond\phi_2 \wedge \ldots \wedge \Diamond\Diamond\phi_n \rightarrow \Diamond(\Diamond\phi_1 \wedge \Diamond\phi_2 \wedge \ldots \wedge \Diamond\phi_n)$

The rules of Modus Ponens and Necessitation:

(7) $$\frac{\vdash (\phi \rightarrow \psi) \qquad \vdash \phi}{\vdash \psi}$$

(8) $$\frac{\vdash \phi}{\vdash (\Box\phi)}$$

are also accepted. More accurately is to say, the axiom (6) be rather a collection of infinitely many axioms.

It can be immediately seen, that HW is an extension of the well known system T of modal logic, where axioms (5) and (6) are specific for the system HW, and all other axioms and rules are adopted from the system T.

We will prove, as an example, some theorems of the system HW. In the following proofs we call T theorems and derived rules of the system T, and PC, theorems and derived rules of PC.

(9) $\Box\Box p \equiv \Diamond\Box\Box p$

1.	$p \rightarrow \Diamond p$	T
2.	$\Box\Box p \rightarrow \Diamond\Box\Box p$	$1 : p/\Box\Box p$
3.	$\Diamond\Box\Box p \rightarrow \Box\Box p$	(5)
	$\Box\Box p \equiv \Diamond\Box\Box p$	$2, 3 \times$ PC

(10) $\Box\Box p \equiv \Diamond\Diamond\Box\Box p$

1.	$\Box\Box p \equiv \Diamond\Box\Box p$	(9)
2.	$\Diamond\Box\Box p \equiv \Diamond\Diamond\Box\Box p$	$1 \times \mathsf{T}$
3.	$\Box\Box p \equiv \Diamond\Diamond\Box\Box p$	$1, 2 \times \mathsf{PC}$

(11) $\Diamond\Diamond p \equiv \Box\Diamond\Diamond p$

1.	$\Box\Box p \equiv \Diamond\Box\Box p$	(9)
2.	$\Box\Box p \equiv \neg\Diamond\Diamond\neg p$	T
3.	$\Diamond p \equiv \neg\Box\neg p$	T
4.	$\neg\Diamond\Diamond\neg p \equiv \neg\Box\neg\neg\Diamond\Diamond\neg p$	$1, 2, 3 \times \mathsf{T}$
5.	$\neg\Diamond\Diamond\neg\neg p \equiv \neg\Box\neg\neg\Diamond\Diamond\neg\neg p$	$4 : p/\neg p$
6.	$\neg\Diamond\Diamond p \equiv \neg\Box\Diamond\Diamond p$	$5 \times \mathsf{T}$
	$\Diamond\Diamond p \equiv \Box\Diamond\Diamond p$	$6 \times \mathsf{T}$

(12) $\Box\Box p \equiv \Box\Box\Box p$

1.	$\Box\Box p \to \Diamond\Diamond\Box\Box p$	(10)
2.	$\Diamond\Diamond\Box\Box p \equiv \Box\Diamond\Diamond\Box\Box p$	$(11) : p/\Box\Box p$
3.	$\Diamond\Diamond\Box\Box p \to \Box\Diamond\Diamond\Box\Box p$	$2 \times \mathsf{PC}$
4.	$\Diamond\Box\Box p \to \Box\Box p$	$(9) \times \mathsf{PC}$
5.	$\Box\Diamond\Diamond\Box\Box p \to \Box\Diamond\Box\Box p$	$4 \times \mathsf{T}$
6.	$\Box\Diamond\Box\Box p \to \Box\Box\Box p$	$4 \times \mathsf{T}$
7.	$\Box\Box p \to \Box\Box\Box p$	$1, 3, 5, 6 \times \mathsf{PC}$
8.	$\Box p \to p$	T
9.	$\Box\Box\Box p \to \Box\Box p$	$1 : p/\Box\Box p$
	$\Box\Box p \equiv \Box\Box\Box p$	$7, 9 \times \mathsf{PC}$

(13) $\Diamond\Diamond p \equiv \Diamond\Diamond\Diamond p$

1.	$\Box\Box p \equiv \Box\Box\Box p$	(12)
2.	$\Box\Box\neg p \equiv \Box\Box\Box\neg p$	$1 : p/\neg p$
3.	$\Box\Box p \equiv \neg\Diamond\Diamond\neg p$	T
4.	$\Box\Box\Box p \equiv \neg\Diamond\Diamond\Diamond\neg p$	T
5.	$\neg\Diamond\Diamond\neg\neg p \equiv \neg\Diamond\Diamond\Diamond\neg\neg p$	$2 \times 3, 4 \times \mathsf{T}$
6.	$\Diamond\Diamond\neg\neg p \equiv \Diamond\Diamond\Diamond\neg\neg p$	$5 \times \mathsf{PC}$
	$\Diamond\Diamond p \equiv \Diamond\Diamond\Diamond p$	$6 \times \mathsf{PC} \times \mathsf{T}$

The theorems proved are of some interest, because they are so called reduction laws, governing nested modalities. One can notice, the laws proved are similar to those provable in the system S5, but involve double modalities instead of single ones. It will be considered further later.

2. Formal Semantics

Formal semantics for the system HW is any relational model

$$\mathfrak{M} = \langle W, R, V\rangle$$

where W is a non empty set of possible worlds, R is an *accessibility relation* on W and V is a valuation function, mapping formulas into subsets of W, and so ascribing formulas their truth-values in possible worlds.

The accessibility relation R has two following features: reflexibility

(14) wRw

and, let us call it, *Heimson Property*

(15) $\exists v(uRv \wedge \forall w\; vRw)$

for all $u, v, w \in W$. The property (15) establishes, for any possible world u, there be a world v, called the *Heimson World* for the world v, such as all worlds are accessible from v. Consequently, in the model for the system HW there is accessibility in two steps between any two possible worlds. We will analyse the meaning of the Heimson Property and the reason to accept it later, when the interpretation of the system presented is discussed.

We say that a formula ϕ is true or false in a possible world $w \in W$, symbolically:

$$w \in V(\phi) \quad w \notin V(\phi)$$

or respectively in the canonical model

$$\phi \in w \quad \phi \notin w$$

The function V may ascribe any truth value to a propositional letter. Ascription of truth values to compound formulas satisfies the following conditions:

- $w \in V(\neg\phi)$ if and only if $w \notin V(\phi)$;
- $w \in V(\phi \wedge \psi)$ if and only if $w \in V(\phi)$ and $w \in V(\psi)$;
- $w \in V(\Box\psi)$ if and only if $\forall v \in W(wRv \to v \in V(\phi))$.

and respective conditions for other connectives, according to the classical definitions and Definition (1).

Models with accessibility relation R satisfying two features mentioned are called HW-models. The formulas valid in all the HW-models are called HW-valid.

The system presented in Section 1 is sound and complete with respect to the class of relational models described in the present section. It was first proved by Hughes in a personal communication with Pargetter [1, 260]. Nevertheless we provide an extra proof for the completeness result, achieved with very simple methods. We use the concept of *canonical model*, involving the Lindenbaum Theorem of Maximal and Consistent Extensions. We also use the tool of point-generated model. Although the proof presented is quite simple in the sense explicated, we think, it may be of some formal and philosophical interest. Some formal interest may be ascribed to the proof of Lemma 4 and the philosophical interest is connected with the fact that a deeper insight into mutual relations between points in the HW-model is important for the interpretation of the system and the ontology linked.

We definie a relation R^n, where $n \geqslant 1$.

(16)
$$\begin{aligned} xR^1y &\equiv xRy \\ xR^{n+1}y &\equiv \exists z(xR^nz \wedge zRy) \end{aligned}$$

We now can formulate the required lemmas.

Lemma 1. *In the canonical model of the system* HW *it is the case that* $xR^ny \rightarrow xR^2y$.

Proof. Let as assume, that xR^3y. We want to show, there is a z such as xRz and zRy. It is enough to show, that the set z of formulas such as

$$z \stackrel{\text{df}}{=} \{\phi : (\Box\phi) \in x\} \cup \{(\Diamond\phi) : \phi \in y\}$$

is consistent as an extension of the system HW. Let us assume, z be inconsistent. So there are formulas $\phi_1, \phi_2, \ldots, \phi_m, \Diamond\psi_1, \Diamond\psi_2, \ldots, \Diamond\psi_n$ such as

(17)
$$\begin{aligned} &\Box\phi_1, \Box\phi_2, \ldots, \Box\phi_m \in x \\ &\psi_1, \psi_2, \ldots, \psi_n \in y \\ &\phi_1, \phi_2, \ldots, \phi_m, \Diamond\psi_1, \Diamond\psi_2, \ldots, \Diamond\psi_n \vdash_{\mathsf{HW}} (\chi \wedge \neg\chi) \end{aligned}$$

Because of the T-theorem $\Box(p \wedge q) \equiv \Box p \wedge \Box q$ we can equally accept one formula ϕ and because of the T-theorem $\Diamond(p \wedge q) \rightarrow \Diamond p \wedge \Diamond q$ we can accept one formula $(\Diamond\psi)$, such as

(18)
$$\begin{aligned} &(\Box\phi) \in x \\ &\psi \in y \\ &\vdash_{\mathsf{HW}} (\phi \wedge \Diamond\psi \rightarrow \chi \wedge \neg\chi) \end{aligned}$$

It is so, because by classical logic $(\psi_1 \wedge \ldots \wedge \psi_n) \in y$, and so $\Diamond(\psi_1 \wedge \ldots \wedge \psi_n) \in x$. Now, from the third condition in (18) it follows that $\vdash_{\mathsf{HW}} \neg(\phi \wedge \Diamond\psi)$, and so $\vdash_{\mathsf{HW}} (\Diamond\psi \rightarrow \neg\phi)$, and so (by the system T) $\vdash_{\mathsf{HW}} (\Diamond\Diamond\psi \rightarrow \Diamond\neg\phi)$, and so

(19) $\vdash_{\mathsf{HW}} (\Diamond\Diamond\psi \rightarrow \neg\Box\phi)$

Furthermore, from (18) second condition and the assumption, that xR^3y it follows that $(\Diamond\Diamond\Diamond\psi) \in x$. But the formula (13) is a HW-theorem, so $(\Diamond\Diamond\psi) \in x$ but from it and (19) it follows that $(\neg\Box\phi) \in x$ however, this is incompatible with the first condition of (18), which ends this part of the proof.

Let us now assume by induction, that for some n, if xR^ny, then xR^2y, let us also assume, that $xR^{n+1}y$. It follows, there be such z, that xR^nz and zRy. It gives, by the inductive assumption, that xR^2z, and so xR^3y. But this gives, by the first part of the proof, that xR^2y. □

LEMMA 2. *In the canonical model of the system* HW *it is the case that* $yR^2x \rightarrow xR^2y$.

PROOF. Let us assume, that yR^2x. We will show, that

$$z = \{\phi : (\Box\phi) \in x\} \cup \{(\Diamond\psi) : \psi \in y\}$$

is consistent in HW. For indirect proof let us assume z be inconsistent. We have already shown, when proving Lemma 1, that it follows from it, that $\vdash_{\mathsf{HW}} \neg(\phi \wedge \Diamond\psi)$. Now, it follows from it, that $\vdash_{\mathsf{HW}} \phi \rightarrow \neg\Diamond\psi$ and so $\vdash_{\mathsf{HW}} \phi \rightarrow \Box\neg\psi$ and so (by the rules of the system T)

$$\vdash_{\mathsf{HW}} \Diamond\Diamond\Box\phi \rightarrow \Diamond\Diamond\Box\Box\neg\psi$$

but the formula (10) is a theorem of the system HW, so

(20) $\vdash_{\mathsf{HW}} \Diamond\Diamond\Box\phi \rightarrow \Box\Box\neg\psi$

Now, under the assumption, $(\Box\phi) \in x$ and yR^2x, so $(\Diamond\Diamond\Box\phi) \in y$. It follows from it and from (20), that $(\Box\Box\neg\psi) \in y$ and so $(\neg\psi) \in y$, but under the assumption $\psi \in y$. □

LEMMA 3. *In the canonical model of the system* HW *it is the case that* $yR^nx \rightarrow xR^2y$.

PROOF. Let us assume, that yR^nx. It follows from Lemma 1 that yR^2x. And by Lemma 2 that xR^2y. □

LEMMA 4. *In the canonical model of the system* HW *it is the case that* $\exists y\, xRy \wedge \forall z(xR^2z \rightarrow yRz)$

PROOF. Let us define for some established world w

$$\Omega \stackrel{\text{df}}{=} \{x \in W : wR^2x\}$$

Now, if $x \in \Omega$ and $\phi \in x$, then $(\Diamond\Diamond\phi) \in w$. Let us so define

$$\Lambda \stackrel{\text{df}}{=} \{\phi : (\exists x \in \Omega)\ \phi \in x\}$$

We are to show, that there exists such a world h, that wRh and for every $x \in \Omega$ it is the case that hRx. So we have to prove, that a set of formulas

$$h \stackrel{\text{df}}{=} \{\phi : (\Box\phi) \in w\} \cup \{(\Diamond\psi) : \psi \in \Lambda\}$$

is consistent in the system HW. Let us assume the set Λ be inconsistent. So there is a theorem of the system HW

$$\Diamond\psi_1 \wedge \cdots \wedge \Diamond\psi_n \rightarrow \neg\phi$$

where $\psi_1, \ldots, \psi_n \in \Lambda$ and $(\Box\phi) \in w$. But so the formula

$$\Diamond(\Diamond\psi_1 \wedge \cdots \wedge \Diamond\psi_n) \rightarrow \Diamond\neg\phi$$

and

$$\Diamond(\Diamond\psi_1 \wedge \cdots \wedge \Diamond\psi_n) \rightarrow \neg\Box\phi \tag{21}$$

are theorems. We have assumed $(\Diamond\Diamond\psi_1), \ldots, (\Diamond\Diamond\psi_n) \in w$, and so under the axiom (6) $(\Diamond(\Diamond\psi_1 \wedge \cdots \wedge \Diamond\psi_n)) \in w$. From this and from (21) it follows that $(\neg\Box\phi) \in w$ but we have assumed that $(\Box\phi) \in w$. □

We will now remind the definition of so called *point-generated* submodel of a canonical model.

DEFINITION 1. A w-generated submodel of a model $\mathfrak{M} = \langle W, R, V\rangle$ is the model $\mathfrak{M}' = \langle W', R', V'\rangle$ where

$$x \in W' \equiv x \in W \wedge \exists n\ wR^nx\,,$$
$$xR'y \equiv xRy \wedge x \in W' \wedge y \in W'\,,$$
$$V' = V \text{restricted in the domain to } W'\,.$$

LEMMA 5. *For all $w \in W'$ it is the case that*

$$W', w \models \phi \text{ if and only if } W, w \models \phi\,.$$

The proof of the lemma just formulated is well known and there is no need to repeat it. Now we can formulate The completeness theorem as follows

THEOREM 1 (completeness). *Every* HW*-valid formula is a theorem of the system* HW*.*

PROOF. Let ϕ not be a theorem of HW. We will show, there be a HW-model not satisfying ϕ. Hence ϕ is not theorem, it is not satisfied in the canonical model for the system HW. So there exists a world in canonical model which not satisfies ϕ. Let us call the world w, hence $\phi \notin w$. We will show that the w-generated submodel of the canonical model for the system HW is a HW-model. The relation R in the canonical model is reflexive, because of the presence of the axiom (4), which is, of course, the axiom (T). And so obviously reflexive is the relation R' because, under Definition 1, the relation R' shares formal properties of the relation R restricted to the set W'. So, we need to prove

$$wR^m x \to \exists y(xRy \land \forall z\ xR^n z \to yRz)$$

By lemmas 1 and 3 every world x in the w-generated part of the canonical model is such as $wR^2x \land xR^2w$, but it follows, that every two worlds in the model are such as xR^4y, but on the same lemmas it follows that

$$xR^2y \tag{22}$$

for every two worlds in the w-generated part of the canonical model. But under Lemma 4 it follows from (22) that for all x in the w-generated part of the canonical model for the system HW it is the case that

$$\exists y\ xRy \land \forall z\ yRz$$

which finishes the proof. □

3. HW and Some Other Modal Logics

The weakest normal modal logic is called K and consists assumptions (1), (2), (6), (7) and (8). By the addition of the formula (4) to the set of axioms we obtain the system T. We want to consider three systems stronger than T: B, S4 and S5. Their axiomatics consist from the assumptions of the system T and one specific axiom, which is called analogically to the respective system:

(B) $\phi \to \Box\Diamond\phi$

(S4) $\Box\phi \to \Box\Box\phi$

(S5) $\Diamond\phi \to \Box\Diamond\phi$

The system HW is a normal modal logic, so it includes the system K as its part. From the axioms presented it immediately follows that HW includes the system T as well. It may be easily proved, that T is a proper part of HW, because axioms (5) and (6) are not theorems of T. The system HW is itself included in S5 as its proper part, because the axiom (S5) of the system S5 is not a theorem of HW. There is no inclusion between HW and S4 or B in any direction. Those mutual relationships between modal systems are presented in Figure 1. It is a well known fact that the lattice presented in

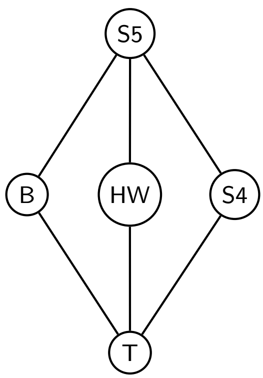

Figure 1. A lattice of some modal logics with HW

Figure 1 is not distributive, although the same lattice without the HW would be distributive.

As it was said, the relation of HW to the system T is obvious. To prove that every HW-theorem is an S5-theorem, one should notice, that the universal relation relation—such as uRw for any $u, w \in W$—satisfies the condition (15), defining the Heimson Property. The system S5 is sound and complete (inter alia) in the frame with universal accessibility relation, so any HW-valid formula is S5-valid. As regards other systems mentioned in Figure 1, neither the axiom (B), nor (S4) is a theorem of HW, and neither the axiom (5) nor (6) of the system HW is a theorem of the system B or S4.

As it was said, HW is a proper subset of S5. However, one can define the connectives of S5 in HW, but not otherwise. So, in a sense, S5 is included in HW.

Let $\Box_{\mathsf{HW}}$ and $\Diamond_{\mathsf{HW}}$ be the connectives of the system HW. If one adds the definitions

(23) $(\Box_{\mathsf{S5}}\phi) \stackrel{\mathrm{df}}{=} (\Box_{\mathsf{HW}}\Box_{\mathsf{HW}}\phi)$

(24) $(\Diamond_{\mathsf{S5}}\phi) \stackrel{\mathrm{df}}{=} (\Diamond_{\mathsf{HW}}\Diamond_{\mathsf{HW}}\phi)$

One can prove every theorems of S5 for the connectives $\Box_{\mathsf{S5}}$ and $\Diamond_{\mathsf{S5}}$. So it may be said, the system S5 be included in the system HW supplemented with definitions (23) and (24). The relation between HW and S5 is analogous to the relation between PC and the intuitionistic propositional calculus.

4. Distinct Modalities in HW

A modality is any unbroken finite sequence of symbols: $\neg$, $\Diamond$, $\Box$, including an empty sequence, which will be signified by $\times$. A variable representing modalities is $\#$. The number of connectives in a modality is called *length* of the modality. Modalities $\#_i$ and $\#_j$ are said to be *equivalent* in a system if and only if a formula $(\#_i \equiv \#_j)$ is a theorem of the system. Otherwise the modalities are called *distinct*. There are infinitely many distinct modalities in the system T, fourteen distinct modalities in the system S4 and six in the system S5. In a normal modal logic any modality is equivalent with a modality including no instances of the negation connective or only one instance and that at the beginning. The later modality is called a *standard* form. We shall assume from now on that all modalities are expressed in standard form.

There are infinitely many—more accurately $\aleph_0$—distinct modalities in the system HW. However, there are more reduction laws in the system in question than in the system T.

Actually, theorems (9), (11)–(13) are important reduction laws. Those theorems allow to remove or add equivalently any modality on the left side of any sequence $\Box\Box$ or $\Diamond\Diamond$.

Theorem 2. *Any modalities of the form $\#_i\Box\Box\#_k$ and $\#_j\Box\Box\#_k$ or of the form $\#_i\Diamond\Diamond\#_k$ and $\#_j\Diamond\Diamond\#_k$ are respectively equivalent in the system* HW.

Proof. Because the system HW is normal, the theorem follows from the reduction laws (9), (11), (12) and (13). □

We may show, there is no more relevant reduction laws in HW. First we will introduce some abbreviations. We will write

$$\Theta^n_\Box, \Theta^n_\Diamond$$

for an unbroken sequence of symbols $\Box$ and $\Diamond$, of the length n, beginning with—respectively—$\Box$ or $\Diamond$, such as immediately before and immediately after $\Box$ there are only $\Diamond$s or nothing, and immediately before and immediately after $\Diamond$ there are only $\Box$s or nothing, for example

$$\Theta^3_\Diamond = \Diamond\Box\Diamond$$
$$\Theta^4_\Box = \Box\Diamond\Box\Diamond$$

More accurate definition may be inductive.

$$\Theta^1_\Box \stackrel{\text{df}}{=} \Box$$
$$\Theta^2_\Box \stackrel{\text{df}}{=} \Box\Diamond$$
$$\Theta^{2k+1}_\Box \stackrel{\text{df}}{=} \Theta^{2k}_\Box\Box$$
$$\Theta^{2k+2}_\Box \stackrel{\text{df}}{=} \Theta^{2k+1}_\Box\Diamond$$

for any natural $k \geqslant 1$, and analogically, dually, for $\Theta^n_\Diamond$.

THEOREM 3. *No formula of the form*

(25) $\Diamond\Theta^n_\Diamond\phi \equiv \Theta^n_\Diamond\phi$

(26) $\Box\Theta^n_\Diamond\phi \equiv \Theta^n_\Diamond\phi$

(27) $\Diamond\Theta^n_\Box\phi \equiv \Theta^n_\Box\phi$

(28) $\Box\Theta^n_\Box\phi \equiv \Theta^n_\Box\phi$

is theorem of the system HW.

PROOF. The proof is inductive. If $n = 1$ formulas (25)–(28) are equivalent to axioms of the systems $S4$ or $S5$, so they cannot be theorems of HW. We will therefore assume, for an established n these formulas are not theorems of HW and show the same for $n + 1$.

Because of the assumption there exist HW-models including submodels defined as follows.

The formula (25) for $n + 1$ is not valid, if there exists a HW-model $\mathfrak{M} = \langle W, R, V\rangle$ including a submodel $\mathfrak{M}' = \langle W', R', V'\rangle$ satisfying following conditions:

$$w_1, w_2, w_3, w_4 \in W$$
$$w_1 R w_2, w_2 R w_3, w_2 R w_4$$
$$w_1 \in V(\Diamond\Theta^{n+1}_{\Diamond}\phi), w_1 \notin V(\Theta^{n+1}_{\Diamond}\phi)$$
$$w_2 \in V(\Theta^{n+1}_{\Diamond}\phi), w_2 \notin V(\Theta^{n}_{\Box}\phi)$$
$$w_3 \in V(\Theta^{n}_{\Box}\phi) w_4 \notin V(\Theta^{n-1}_{\Diamond}\phi)$$

But on the assumption that (27) is not valid for n, there exists a HW-model including a submodel $\mathfrak{M}'' = \langle W'', R'', V''\rangle$ satisfying

$$w_1, w_2, w_3 \in W$$
$$w_1 R w_2, w_1 R w_3$$
$$w_1 \in V(\Diamond\Theta^{n}_{\Box}\phi), w_1 \notin V(\Theta^{n}_{\Box}\phi)$$
$$w_2 \in V(\Theta^{n}_{\Box}\phi)$$
$$w_3 \notin V(\Theta^{n-1}_{\Diamond}\phi)$$

But w_2 and w_3 from $\mathfrak{M}''$ satisfy respectively w_3 and w_4 from $\mathfrak{M}'$. Furthermore w_2 from $\mathfrak{M}'$ may be the Heimson World for itself and for w_1, so the existence of a HW-model $\mathfrak{M}'$ is ensured.

In other cases the argument is analogous.

For (26) with $n+1$ $\mathfrak{M}'$ satisfies:

$$w_1, w_2, w_3, w_4 \in W$$
$$w_1 R w_2, w_1 R w_3, w_2 R w_4$$
$$w_1 \in V(\Theta^{n+1}_{\Diamond}\phi), w_1 \notin V(\Box\Theta^{n+1}_{\Diamond}\phi)$$
$$w_2 \notin V(\Theta^{n+1}_{\Diamond}\phi)$$
$$w_3 \in V(\Theta^{n}_{\Box}\phi)$$
$$w_4 \notin V(\Theta^{n}_{\Box}\phi)$$

But on the assumption that (28) is not valid with n, there exists a HW-model including a submodel $\mathfrak{M}'' = \langle W'', R'', V''\rangle$ satisfying

$$w_1, w_2 \in W$$
$$w_1 R w_2$$
$$w_1 \in V(\Theta^{n}_{\Box}\phi), w_1 \notin V(\Box\Theta^{n}_{\Box}\phi)$$
$$w_2 \in V(\Theta^{n-1}_{\Diamond}\phi), w_2 \notin V(\Theta^{n}_{\Box}\phi)$$

But w_1 and w_2 from $\mathfrak{M}''$ satisfy respectively w_3 and w_4 from $\mathfrak{M}'$. Furthermore w_1 from $\mathfrak{M}'$ may be the Heimson World for itself and w_2 from $\mathfrak{M}''$ may be the Heimson World for w_2 from $\mathfrak{M}'$, so the existence of a HW-model $\mathfrak{M}'$ is ensured.

It is easy to show analogous arguments for (27) and (28), however, it is not necessary, because in normal modal logics any formula of the form (27) or (28) is equivalent to an appropriate formula of the form (26) or (25) respectively. □

We will also use some simple lemmas.

Lemma 6. *Any two modalities $\#_i\Box$ ane $\#_j\Diamond$ not including the symbol $\neg$ are distinct in the system* HW.

This lemma obviously follows from the fact, that HW is a subset of S5. If $(\#_i\Box\phi \equiv \#_j\Diamond\phi)$, not including the symbol $\neg$, is a theorem of HW, then it is a theorem of S5. But—under the reduction laws of S5—in such case $(\Box\phi \equiv \Diamond\phi)$ is a theorem of S5, which is false.

Lemma 7. *Let k, n be natural numbers, and $k \geqslant 1$. No formula of the form*

$$\Theta^{2k}_{\Box}\Theta^{n}_{\Box}\phi \equiv \Theta^{n}_{\Box}\phi$$
$$\Theta^{2k}_{\Diamond}\Theta^{n}_{\Diamond}\phi \equiv \Theta^{n}_{\Diamond}\phi$$

is a theorem of the system HW.

The proof of this lemma is straightforward as well. If there is no HW-model falsifying formulas of the established forms, there is no HW-model falsifying an appropriate formula of the form

$$\Theta^{2k}_{\Box}\phi \equiv \phi$$
$$\Theta^{2k}_{\Diamond}\phi \equiv \phi$$

which is again false.

Lemma 8. *Let k, n be any natural numbers. No formula of the form*

$$\Diamond\Theta^{2k}_{\Box}\Theta^{n}_{\Box}\phi \equiv \Theta^{n}_{\Box}\phi \tag{29}$$
$$\Diamond\Diamond\Theta^{2k}_{\Box}\Theta^{n}_{\Box}\phi \equiv \Theta^{n}_{\Box}\phi \tag{30}$$
$$\Box\Theta^{2k}_{\Diamond}\Theta^{n}_{\Diamond}\phi \equiv \Theta^{n}_{\Diamond}\phi \tag{31}$$
$$\Box\Box\Theta^{2k}_{\Diamond}\Theta^{n}_{\Diamond}\phi \equiv \Theta^{n}_{\Diamond}\phi \tag{32}$$

is a theorem of the system HW.

Proof. Because HW is normal, it is again enough to prove the lemma for two first cases.

Suppose that $(\Diamond\Theta^{2k}_{\Box}\Theta^{n}_{\Box}\phi \equiv \Theta^{n}_{\Box}\phi)$ is a theorem, in that case so are: $(\Box\Diamond\Theta^{2k}_{\Box}\Theta^{n}_{\Box}\phi \equiv \Box\Theta^{n}_{\Box}\phi)$, $(\Box\Diamond\Box\Diamond\Theta^{2k}_{\Box}\Theta^{n}_{\Box}\phi \equiv \Box\Diamond\Box\Theta^{n}_{\Box}\phi)$ Under Theorem 2 there is a theorem $(\Diamond\Theta^{2k}_{\Box}\Theta^{n}_{\Box}\phi \equiv \Box\Theta^{n}_{\Box}\phi)$ and so $(\Box\Diamond\Box\Diamond\Theta^{2k}_{\Box}\Theta^{n}_{\Box}\phi \equiv \Box\Diamond\Theta^{2k}_{\Box}\Theta^{n}_{\Box}\phi)$, which contradicts Lemma 7.

Let us consider the second formula. Suppose $(\Diamond\Diamond\Theta^{2k}_{\Box}\Theta^{n}_{\Box}\phi \equiv \Theta^{n}_{\Box}\phi)$ is a theorem. In that case so are $(\Box\Diamond\Diamond\Theta^{2k}_{\Box}\Theta^{n}_{\Box}\phi \equiv \Box\Theta^{n}_{\Box}\phi)$ But under Theorem 2 there is a theorem $(\Diamond\Diamond\Theta^{2k}_{\Box}\Theta^{n}_{\Box}\phi \equiv \Box\Theta^{n}_{\Box}\phi)$ and so there is a theorem $(\Box\Theta^{n}_{\Box}\phi \equiv \Theta^{n}_{\Box}\phi)$ which contradicts Theorem 3.

For other two formulas the result follows from the two first and usual mutual definitions of modal symbols. □

The theorems and lemmas proved in this section allow us to tell, what exactly distinct modalities there are in the system HW. Any two sequences of symbols of the form $\Theta^{n}_{\Box}$, $\Box\Theta^{n}_{\Box}$, $\Diamond\Theta^{n}_{\Box}$, $\Theta^{n}_{\Diamond}$, $\Box\Theta^{n}_{\Diamond}$, $\Diamond\Theta^{n}_{\Diamond}$, for any natural n, including 0, either look exactly identical, or are distinct modalities in HW. There are no more distinct modalities in HW beside those mentioned and their negations. Anyway, as it was said, still, there are infinitely many distinct modalities in the system in question.

5. Interpretation

The philosophical reason to develop the system presented is its interpretation. The majority of modal logic was first concerned with the concept of *logical necessity*. Contemporary modal logicians often claim themselves not to be interested in any kind of modality at all. The subject of modal logic, they claim, is rather the scope of relational first order structures than any modality [2, xi–xii, xiv–xv]. The system HW turns back to modalities. It is supposed to formalize the concept of *physical* (*nomic*) *necessity*.

Logical necessity is a property characteristic of logical laws and physical (nomic) necessity is a property characteristic of physical laws. For example, the logical law of excluded middle

$$\phi \vee \neg\phi$$

is logically necessary. An example of physical modality can be provided with the relativistic mass equation

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{33}$$

The physically necessary formula (33) establishes—among other things—that it is *impossible* for an object with non-zero rest mass to move at the speed of light in vacuum. It is so because if any such object had moved at the speed of light, it would have had infinitely big mass, which is not *possible*, because no infinite force *could* be present in nature and such force must had been acting to move an infinite mass object [3, ch. 15]. So, it is logically necessary, that a given pen moves at the speed of light or does not move at the speed of light; but it is physically necessary that the pen doesn't move at the speed of light.

According to Bigelow and Pargetter there are three essential features of physical laws: they describe regularities (generalizations) of some sort, and they ascribe the special kind of necessity to these regularities, and they include idealizations, which make them vacuous in a sense (cf. [5] for further discussion).

A question then arises, what is the accurate meaning of those emphasized modal words of physical language. One should agree, that it is not the same concept, like in logical language, e.g., when one says, it is necessary that $(\phi \vee \neg\phi)$ or $m_0 = m_0$. The concept of logical modality is usually connected with the concept of consistency. A formula ϕ is logically necessary in a sense if and only if its negation $(\neg\phi)$ is not consistent. A formula (or a set of formulas) is consistent if and only if it has a model, if it is satisfiable.

But obviously physical laws—like (33)—are not necessary in the sense. There seems to be nothing inconsistent in the proposition about a non-zero rest mass object moving at the speed of light. Actually, in classical physics it is physically necessary, under physical laws, that if any object is acted by any force long enough, the object will move at the speed of light and faster. One cannot, say, know the result of Michelson-Morley experiment without the experiment, just from inconsistence of the opposite result.

On the other hand physical laws, like (33), and their consequences are not only a part of history of accidental behavior of physical objects. Physical laws do not just recount a coincidence. Scientists do not have to repeat, say, Michelson-Morley experiment in order to know, whether or not the speed of light depends on the motion of the source of light. Physical laws seem to recount a kind of physical structure of the world, or at least to approximate it.

Neither the view identifying physical and logical modalities, nor the one identifying the first kind of modal notions with truthfulness or just some sort of syntactical or pragmatic properties, seems to be correct. One should rather say, that logical necessity and physical necessity are two distinct concepts. Bigelow and Pargetter claim the physically necessary sentences are physical laws and their logical consequences [1, p. 214–227]. However, the concept of physical necessity is involved in the definition of physical law. The solution of the problem may be an axiomatic system providing the meaning of physical modal expressions.

Having so many systems of modal logic, one may ask, whether or not it is sensible to expect the formal analysis of nomic modalities be found among those systems.

The aim of formalization is now to find and describe the very difference between physically necessary sentences and logically necessary ones on one side, and accidental ones on the other. To achieve this Bigelow and Pargetter involve the ontology of possible worlds. They claim, the accidental truths hold at least in our world, logically necessary truths hold in all possible worlds—so they use so called leibnizian necessity—and physically necessary truths hold in our world and in all appropriately accessible worlds, but not necessarily in all worlds at all [1, p. 238]. They suugest as well, logical necessity, which—as we mentioned—is leibnizian, be formalized accurately in the system $\mathsf{S5}$. All they need is to find the formal similarities and differences between $\mathsf{S5}$ and the system constructed.

The main similarity involves the rule, that makes a scientist look for the truth. Physical laws should then be true sentences. Of course, in the history most propositions proposed as laws turned out to be false. However, they should at least reasonably approximate the truth. The first postulat is then physically necessary formulas be true. This means the accessibility relation in the model of physical nesessity should be reflexive. And so, the system of physical necessity should include the system T as its subset.

The other rule involved says, the physical laws could change, furthermore, they could be totally different than they actually are. There could be even no physical laws at all. That means, it could be the case any actual physical law could turn out to be only accidental truth. So in the model of physical necessity, for any possible world w, there should exist a world w' which is exactly the same like w, but there are no physical laws in w'. All physical laws of w are true in w', but only accidentally true. A question then arises, whether the possibility of any physical law being accidental is logical or physical. In the former case the law-free world just described should only exist, but in the latter case the world w' in question should not only exist, but also be accessible from the given world w. Bigelow and Pargetter recommend the second alternative to be accepted, but this seems to be only an assumption of theirs. Therefore, in the model of physical necessity, for any world w, there is to be a world w' accessible from w and such as there is no physical necessary formulas in w'. Such a world is called *Heimson World* for the given world w (that is way the system presented is called HW) [1, p. 238–245].

It may be noticed, if there are no necessitities in a world, there is no reason there should be any world not accessible from it. So the existence of the Heimson World for a world w may be represented as a world w' satisying two following conditions:

- w' is accessible from w, because any physcically necessary formula of w is—as it was said—true in w';
- any world is accessible from w', because no formula is physically necessary in w'.

These conditions, however, give us the formula (15), and so HW-model is claimed to be the model of physical necessity. But the system HW is sound and complete, so it is claimed to be the very system of physical necessity itself.

6. Discussion

Does system HW succeed in formalizing the concept of physical necessity? We suggest not.

We think, the analyses described in the previous section are deep and mostly correct. We agree that physically necessary formulas should be considered as true. And we find the construction of the concept of Heimson World brilliant, although it needs further discussion regarding the accessibility of the Heimson World, as it was mentioned. So, we think those assumptions of the system HW analysed are correct, or at least approximately correct.

However, there are some assumptions not analysed, in particular there is the system K involved, that is the weakest normal system of modal logic. Can it be justified that the concept of physical necessity satisfies the assumptions of the system K, that is the axiom (6) and the rule of necessitation (8)? Bigelow and Pargetter say, these assumption be *so basic*, that they are shared by any *normal* sense of modal connectives [1, 106]. However, is the concept of physical necessity normal? It can be answered, the system K is satisfied by any standard relational model. However, what makes us think, the model of physical necessity be standard relational model? We think, there are usually two reasons. The first one is formal simplicity and, one can even say, a kind of inner perfection of normal modal logic and their relational models. But, we claim, it is not any philosophical argument. The other reason involves the concept of physical necessities as relative necessities. If, as it was mentioned, physical necessities are simply logical consequences of the set of physical laws, then logical necessities are physical necessities, for logical necessities are logical consequences of any set of premises, including the empty set. But again, are physical necessities logical consequences of physical laws? In our paper [6] we argue, it is not necessary to accept that view. We sketch a theory, according to which the essential difference between logical and physical necessity is the difference of *truthmakers*.

Let us so consider an example (33) of physical necessity. As we said, under the equation it is physically impossible for an object with non-zero rest mass to move at the speed of light in vacuum. Let us write for it:

(34) $\Box\neg(v = c)$

for an established object. Futhermore, the equation (33) gives the reason for (34). Again, as it was said (34) obtains because if any such object had move at the speed of light, it would have had infinitely big mass:

(35) $\Box(v = c \rightarrow m = \infty)$

However, is well known, that the formula

(36) $\Box\neg\phi \rightarrow \Box(\phi \rightarrow \psi)$

which is one of, so called, paradoxes of the strict implication is a theorem of any normal modal logic. So it is a theorem of the system HW. The proof of that fact is obvious. Under the formula (36) both, (35) and

(37) $\Box(v = c \rightarrow m \neq \infty)$

obtain. We will try to argue, it is not acceptable for physical modalities.

Theorem (36) is a modal version of the classical law of Duns Scotus. The last law claims, any proposition to follow from a contradiction. As it was said, logical modalities are closely linked with the concept of consistency and inconsistency. But is so the case for physical modalities?

As it was said, an essential feature of the method of empirical sciences is to include *idealizations*. Physical laws describe some situations physically impossible, for example what would have happened, if any non-zero mass object had moved at the speed of light in vacuum. And there are physical laws that describe situation of the kind, furthermore the laws govern those situations with physical necessity. Those laws are—as each physical law—justified on empirical way, including even sophisticated experiments. When preparing experiments a scientist aims to create circumstances as similar to those physically impossible circumstances, that are to be described by the law, as he is able to. Hence, (34) (35) are true and (37) is false. The formula (37) cannot ever it be empirically confirmed.

To sum up, we claim, the adequate logic of physical (nomic) necessity is not any normal system of modal logic. Nevertheless, we think that the system HW is very interesting. It is interesting formally (and still not so involved). It is also philosophically interesting logic, because, even if we question the assumptions taken from the system K, there still are axioms (5) and (6) and important analyses suggesting the formulas in question to formalize important features of physical modalities.

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